Correction to “Faithful permutation representations of toroidal regular maps”

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1 Introduction

In [1], we claim that we give a complete list of the possible degrees of a faithful transitive permutation representation of the groups of the toroidal regular maps. Indeed, the list given for type $\{4, 4\}$ is complete; however, recently we were surprised with the existence of exceptional degrees for the map $\{3, 6\}$. After, we struggled to find the reason why there were some degrees missing in our classification. The fact is that there is a gap in one proof, having consequences in two of our main theorems. Our goal is to fill in that gap.

In what follows, let G be the group of symmetries of a toroidal regular map of type $\{4, 4\}$ or $\{3, 6\}$, and suppose G is represented as a faithful transitive permutation representation group of degree n . Let T be the translation group generated by unitary independent translations of order s defining the groups of $\{4, 4\}_s$ or $\{3, 6\}_s$ for $s \in \{(s, 0), (s, s)\}$ (see [1]). We recall the following results.

Lemma 1.1 [1] [Lemma 1.1] *If T is transitive, then $n = s^2$.*

An immediate consequence of Lemma 1.1 is that if $n \neq s^2$, T is intransitive. Note that T is a normal subgroup of G that is a direct product of two cyclic groups of order s . Consider that α and β are the actions of the generators of T on a block and let $K := \langle \alpha, \beta \rangle$. In [1], we consider the cases where K is a cyclic group or a direct product, but these cases do not cover all possibilities. In what follows, we complete all the proofs of [1] where this has any effect.

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2 The size of a T -orbit

In what follows, $o(x)$ denotes the order of x .

Lemma 2.1 *If $B := |K : \langle \alpha \rangle|$ and $C := |K : \langle \beta \rangle|$, then the size of a T -orbit is $k = ds$ where $d = \gcd(B, C)$.*

Proof Consider that α and β are the actions of the generators of T on a block of size k . Then, $K := \langle \alpha, \beta \rangle$, $A := o(\alpha)$, $B := |K : \langle \alpha \rangle|$ and $C := |K : \langle \beta \rangle|$. The order of K is AB and acts regularly on the block; hence, $k = AB$. As α and β commute, we have the following.

$$K/\langle \alpha \rangle = \{\langle \alpha \rangle, \langle \alpha \rangle \beta, \langle \alpha \rangle \beta^2, \dots, \langle \alpha \rangle \beta^{B-1}\} \text{ and} \\ K/\langle \beta \rangle = \{\langle \beta \rangle, \langle \beta \rangle \alpha, \langle \beta \rangle \alpha^2, \dots, \langle \beta \rangle \alpha^{C-1}\}.$$

Thus, B divides $o(\beta)$ and C divides $o(\alpha) = A$. Let $D := A/C$. As $k = AB = o(\beta)C$, we have $o(\beta) = DB$. Now $s = \text{lcm}(o(\alpha), o(\beta)) = \text{lcm}(CD, DB) = D \text{lcm}(C, B)$ and $k = AB = DCB = D \text{lcm}(C, B) \gcd(C, B) = s \gcd(C, B)$. To conclude the proof, consider $d = \gcd(C, B)$. $\square$

The item (i) of the following lemma is a consequence of Lemma 2.1.

Lemma 2.2 [1] [Lemma 2.2] *If $n \neq s^2$, then G is embedded into $S_k \wr S_m$ with $n = km$ ($m, k > 1$) and*

- (i) $k = ab$ where $s = \text{lcm}(a, b)$ and,
- (ii) m is a divisor of $\frac{|G|}{s^2}$.

From now on, we use m for number of blocks (the T -orbits) and k size of a T -orbit.

Consider first the regular map $\{4, 4\}_{(s,0)}$. In [1], the existence of a faithful transitive permutation representation of degree $n = mab$ was guaranteed whenever $s = \text{lcm}(a, b)$ and m is a divisor of $\frac{|G|}{s^2} = 4$. Moreover, we proved that the degrees of $\{4, 4\}_{(s,s)}$ are the degrees of $\{4, 4\}_{(s,0)}$ multiplied by 2. Consequently, there are no changes on our results for $\{4, 4\}_{(s,s)}$. Anyway we take this opportunity to restate the classification of all possible degrees for the regular maps of type $\{4, 4\}$.

Theorem 2.3 *Let $s > 1$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{4, 4\}_s$ are*

- (1) $2ds, 4ds$ and $8ds$ for any divisor d of s , for $\mathbf{s} = (s, 0)$;
- (2) $4ds, 8ds$ and $16ds$ for any divisor d of s , for $\mathbf{s} = (s, s)$.

Proof Thus, this is a consequence of Theorems 4.2 and 4.4 of [1]. Note that if $k = ab$ and $s = \text{lcm}(a, b)$, then $k = ab = \gcd(a, b)s$; thus, $k = ds$ for some divisor d of s . Conversely, if $k = ds$ with d being a divisor of s , then $k = ab$ for some integers a and b such that $s = \text{lcm}(a, b)$. $\square$

For the map $\{3, 6\}_{(s,0)}$, we did not find faithful transitive permutation representations of degrees $2ab$ and $4ab$ when both a and b are different from s . Indeed, the only degrees found, when T is intransitive, were $2s^2, 4s^2$ and the indexes of the core-free subgroups given in the following proposition.

Proposition 2.4 [1] [Proposition 2.4] *Let G be the group of $\{3, 6\}_{(s,0)}$ ($s \geq 2$).*

- (1) *$H = \langle u^a, v^b \rangle$ is core-free and $|G : H| = 12ab$, where $s = \text{lcm}(a, b)$.*
- (2) *If $s = \text{lcm}(a, b)$ and $s \neq ab$ then $H = \langle u^a, v^b \rangle \rtimes \langle \rho_0 \rho_2 \rangle$ is core-free and $|G : H| = 6ab$.*
- (3) *If d is a divisor of s then $H = \langle u^d \rangle \rtimes \langle \rho_0, \rho_2 \rangle$ and $H' = \langle u^d \rangle \rtimes \langle \rho_0 \rho_2 \rangle$ are core-free. Moreover, $|G : H| = 3ds$ and $|G : H'| = 6ds$.*

Thus, for $m \in \{3, 6, 12\}$ there are no gaps to be filled. We only need to consider the case $m \in \{2, 4\}$.

In Sect. 3, we consider $m \in \{2, 4\}$ for the map $\{3, 6\}_{(s,0)}$. In Sect. 4, we determine the core-free subgroups corresponding to the degrees found in the previous section. Finally, in Sect. 5, we give a classification of all possible degrees for $\{3, 6\}$, including the degrees of $\{3, 6\}_{(s,s)}$.

3 Cases $m = 2$ and $m = 4$

Let ρ_0, ρ_1 and ρ_2 be the distinguished generators of the group of automorphisms G of the map $\{3, 6\}_{(s,0)}$. Consider that $T = \langle u, v \rangle$ with $u = \rho_0(\rho_1 \rho_2)^2 \rho_1$ and $v = u^{\rho_1} = (\rho_0 \rho_1 \rho_2)^2$, and let $t = u^{-1}v$. The actions of u, v and t on the block i are denoted by u_i, v_i and t_i , respectively. In addition $K = \langle u_1, v_1 \rangle$, that is, the action of T on the block 1.

Let us first prove a result to be used in both cases, $m = 2$ and $m = 4$.

Proposition 3.1 *Let q be an odd number. The modular equation*

$$x^2 - x + 1 \equiv 0 \pmod{q}$$

has a solution if and only if all prime divisors p of q are such that $p \equiv 1 \pmod{6}$.

Proof Let 2^* be an inverse of 2 modulo q (which exists because q is odd). Then, the equation $x^2 - x + 1 \equiv 0 \pmod{q}$, which is equivalent to $(x - 2^*)^2 \equiv (-3)(2^*)^2 \pmod{q}$, has a solution if and only if -3 is a quadratic residue modulo p for every prime divisor of q [2]. In addition, -3 is a quadratic residue modulo p for every prime divisor of q if and only if $p \equiv 1 \pmod{3}$, or, as by assumption p is odd, $p \equiv 1 \pmod{6}$. $\square$

Proposition 3.2 *If $m = 2$, then $k = sd$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \pmod{6}$.*

Proof Let $m = 2$. The following graphs represent all the possible block actions determined in [1].

In any of the three cases $o(u_1) = o(u_2) = o(v_1) = o(v_2) = o(t_1) = o(t_2) = s$. Let $d = |K : \langle u_1 \rangle| = |K : \langle v_1 \rangle|$, then $k = sd$. Assume $d \neq s$ since the degree $2s^2$ is already known to exist.

Case 1 Let $u_1^d = v_1^j$. Conjugating by ρ_0 , we obtain $u_1^{-d} = t_1^j = u_1^{-j} v_1^j = u_1^{d-j}$; hence, $j \equiv 2d \pmod s$. Now conjugating the equation $u_1^d = v_1^{2d}$ by ρ_2 , we obtain $u_2^d = t_2^{-2d} = u_2^{2d} v_2^{-2d}$; thus, $u_2^d = v_2^{2d}$, which gives $u^d = v^{2d}$, a contradiction.

Case 2 Let $u_1^d = v_1^j$. Conjugating by ρ_2 , we obtain $u_1^d = t_1^{-j} = u_1^j v_1^{-j} = u_1^{j-d}$; hence, $j \equiv 2d \pmod s$. Now conjugating the equation $u_1^d = v_1^{2d}$ by $\rho_1 \rho_0$, we obtain $t_1^d = u_1^{-2d}$; thus, $v_1^d = u_1^{-d} = v_1^{-2d}$. Hence, $3d \equiv 0 \pmod s$, and therefore, $u_1^d = v_1^{-d}$ and, conjugating by ρ_1 , $v_2^d = u_2^{-d}$. Consequently, $(u_1 u_2)^d = (v_1 v_2)^{-d}$, a contradiction.

Case 3 Let $u_1^d = v_1^j$. As $|K : \langle u_1 \rangle| = d$, j must be a multiple of d , say $j = \alpha d$. From the equality $u_1^d = v_1^{\alpha d}$ and as $o(u_1) = o(v_1)$, we have $\gcd(\alpha, s/d) = 1$. Conjugating the equation $u_1^d = v_1^{\alpha d}$ by $\rho_0 \rho_1$, we obtain $v_1^{-d} = t_1^{-\alpha d} = u_1^{\alpha d} v_1^{-\alpha d}$. Thus, $u_1^{\alpha d} = v_1^{\alpha d - d}$. Consequently $v_1^{\alpha^2 d} = v_1^{\alpha d - d}$, which implies $d(\alpha^2 - \alpha + 1) \equiv 0 \pmod s$, or equivalently $\alpha^2 - \alpha + 1 \equiv 0 \pmod{(s/d)}$. The rest follows from Proposition 3.1. $\square$

Proposition 3.3 *If $m = 4$, then $k = sd$ where d is a divisor of s and all prime divisors p of s/d are such that $p \equiv 1 \pmod 6$.*

Proof Using GAP [3], it can be checked that there is only one possibility for the action of G given by the following graph.

Let Δ_i denote the block i , $i \in \{1, \dots, 4\}$, as follows:

$$\begin{aligned}\Delta_2 &= \Delta_1 \rho_0 = \Delta_1 \rho_1, \quad \Delta_3 = \Delta_1 \rho_2 \text{ and} \\ \Delta_4 &= \Delta_3 \rho_0 = \Delta_3 \rho_1\end{aligned}$$

We have $o(u_i) = s$ for $i \in \{1, 2, 3, 4\}$, and the same holds for $o(v_i)$ and $o(t_i)$.

Let $|K : \langle u_1 \rangle| = |K : \langle v_1 \rangle| = d$. Assume that $d \neq s$. Analogously to Proposition 3.2, we may write $u_1 = v_1^{\alpha d}$ with $\gcd(\alpha, s/d) = 1$. Then, conjugating by $\rho_0 \rho_1$, we get $v_1^{(\alpha-1)d} = v_1^{\alpha^2 d}$, which implies $\alpha^2 - \alpha + 1 \equiv 0 \pmod{(s/d)}$. The rest follows from Proposition 3.1. $\square$

4 Core-free subgroups

The following proposition gives core-free subgroups that will guaranty the existence of the degrees obtained in Propositions 3.2 and 3.3.

Proposition 4.1 *Let d be a divisor of s . Suppose that there exists α , coprime with s/d , such that $\alpha^2 - \alpha + 1 \equiv 0 \pmod{(s/d)}$. Then, $\langle (v^{-\alpha} u)^d, \rho_0 \rho_1 \rangle$ and $\langle (v^{-\alpha} u)^d, \rho_1 \rho_2 \rangle$ are core-free subgroups of G with indexes $4sd$ and $2sd$, respectively.*

Proof Let us consider $H := \langle (v^{-\alpha}u)^d, \rho_0\rho_1 \rangle$. We have $(v^{-\alpha}u)^{\rho_0\rho_1} = t^\alpha v^{-1} = v^{\alpha-1}u^{-\alpha} = v^{\alpha^2}u^{-\alpha} = (v^{-\alpha}u)^{-\alpha}$. Hence, $H = \langle (v^{-\alpha}u)^d \rangle \rtimes \langle \rho_0\rho_1 \rangle$. Furthermore, $|H| = 3s/d$, and therefore, $|G : H| = 4sd$.

Now let us prove that H is core-free. Suppose that $H \cap H^{\rho_0}$ is nontrivial. Then, there exist $i, i' \in \{0, \dots, s/d - 1\}$ and $j, j' \in \{0, 1, 2\}$ such that $(v^{-\alpha}u)^{id}(\rho_0\rho_1)^j = (t^{-\alpha}u^{-1})^{i'd}(\rho_0\rho_1)^{j'}$. Clearly, this is only possible when $j = j', id = i'd$ and $(\alpha - 1)id \equiv id \pmod{s}$, or equivalently $\alpha^2 i \equiv i \pmod{(s/d)}$. As $\alpha^2 - \alpha + 1 \equiv 0 \pmod{(s/d)}$, α is a cubic root of -1 modulo (s/d) , $i \equiv -\alpha i \pmod{(s/d)}$. Consequently, $i \equiv -2i \pmod{(s/d)}$. As, by Proposition 3.1, 3 does not divide (s/d) , we obtain $i = 0$. Consequently, $H \cap H^{\rho_0} \leq \langle \rho_0\rho_1 \rangle$. Now suppose that $\langle \rho_0\rho_1 \rangle \cap H^{\rho_2}$ is nontrivial. Then, either $(\rho_1\rho_2)^2, (\rho_0\rho_1\rho_2\rho_0)^2, v$ or t belongs to $\langle t^\alpha u \rangle$, which is not possible.

For $H := \langle (v^{-\alpha}u)^d, \rho_1\rho_2 \rangle$, the proof is analogous. $\square$

5 The degrees of the regular maps of type $\{3, 6\}$

The results given previously have consequences on Theorems 5.6 and 5.7 of [1]. The correct statement of these results should be as follows:

Theorem 5.1 *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3, 6\}_{(s,0)}$ are $n = s^2$ and the following.*

- (1) $3ds, 6ds$ or $12ds$ for any divisor d of s ,
- (2) $2ds$ and $4ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to 1 mod 6.

Proof This is a consequence of results included in the previous sections. $\square$

Let us now consider the map $\{3, 6\}_{(s,s)}$. We first recall the following.

Corollary 5.2 [1] [Corollary 5.2] *If n is a degree of $\{3, 6\}_{(s,0)}$ (resp. $\{3, 6\}_{(s,s)}$), then $3n$ is a degree of $\{3, 6\}_{(s,s)}$ (resp. $\{3, 6\}_{(3s,0)}$);*

Theorem 5.3 *Let $s \geq 2$. The degrees of a faithful transitive permutation representation of a toroidal regular map of type $\{3, 6\}_{(s,s)}$ are $n = 3s^2$ and the following.*

- (1) $9ds, 18ds$ or $36ds$ for any divisor d of s ,
- (2) $6ds$ and $12ds$ if and only if d is a divisor of s and all prime divisors of s/d are equal to 1 mod 6.

Proof By Corollary 5.2, all the degrees given on Theorem 5.1 multiplied by 3 are degrees for the map $\{3, 6\}_{(s,s)}$. Moreover, the degrees of $\{3, 6\}_{(s,s)}$ multiplied by 3 are degrees of $\{3, 6\}_{(3s,0)}$. Hence, by Theorem 5.1, the set of degrees of $\{3, 6\}_{(s,s)}$ must be contained in

$$\{3s^2, 3\delta s, 6\delta s, 12\delta s\}$$

with δ being any divisor of $3s$, or in

$$\{2\delta s, 4\delta s\}.$$

if and only if δ is a divisor of $3s$ and all prime divisors of $3s/\delta$ are equal to 1 mod 6.

If all prime factors of $3s/\delta$ are 1 mod 6, δ must be divisible by 3. Say $\delta = 3d$, thus $\{2\delta s, 4\delta s\} = \{6ds, 12ds\}$ where d is a divisor of s and all prime divisors of s/d are equal to 1 mod 6. These degrees are in (2).

Let us now prove that the degrees $3\delta s$, $6\delta s$ and $12\delta s$ correspond to the ones listed in (1). We need only to prove that δ is divisible by 3.

Consider now that G is the group of $\{3, 6\}_{(3s,0)}$ and K is the action of T on block 1, where $T = \langle u, v \rangle$ is the translation group of order $(3s)^2$. The group of $\{3, 6\}_{(s,s)}$ is a factorization of the group of $\{3, 6\}_{(3s,0)}$ by $\langle (uv)^s \rangle$. A faithful transitive permutation representation of $\{3, 6\}_{(3s,0)}$ on n points corresponds to a permutation representation of $\{3, 6\}_{(s,s)}$ on $n/3$ triples of points of the form

$$\{x, x(uv)^s, x(uv)^{2s}\}.$$

Let $B := |K : \langle u_1 \rangle|$ and $C := |K : \langle v_1 \rangle|$, where u_1 and v_1 are the action of u and v on block 1. By Lemma 2.1, $\delta = \gcd(B, C)$.

Suppose that B divides s . Then, $(u_1 v_1)^s = u_1^{is}$ for some integer $i \in \{0, 1, 2\}$. In this case, the triples of points are as follows:

$$\{x, x u_1^{is}, x u_1^{2is}\},$$

for x in block 1. But then both u^s and v^s fix these triples. By conjugation with ρ_0 , ρ_1 and ρ_2 , we get that the order of u and v in $\{3, 6\}_{(s,s)}$ is at most s , a contradiction. Thus, B does not divide s . In particular, 3 must divide B . Analogously, 3 divides C , and therefore, $\delta = \gcd(B, C)$ is divisible by 3, as wanted. $\square$

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