



On toric ideals arising from signed graphs

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Received: 21 August 2019 / Accepted: 18 June 2020 / Published online: 7 September 2020
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Abstract

A *signed graph* is a pair (G, τ) of a graph G and its sign τ , where a *sign* τ is a function from $\{(e, v) \mid e \in E(G), v \in V(G), v \in e\}$ to $\{1, -1\}$. Note that graphs or digraphs are special cases of signed graphs. In this paper, we study the toric ideal $I_{(G, \tau)}$ associated with a signed graph (G, τ) , and the results of the paper give a unified idea to explain some known results on the toric ideals of a graph or a digraph. We characterize all primitive binomials of $I_{(G, \tau)}$ and then focus on the complete intersection property. More precisely, we find a complete list of graphs G such that $I_{(G, \tau)}$ is a complete intersection for every sign τ .

Keywords Signed graph · Toric ideal · Primitive element · Complete intersection

Mathematics Subject Classification 14M25 · 05C22 · 05C25

1 Introduction

Throughout the paper, a graph means a finite simple graph. A finite graph allowed to have a multiple edge or a loop is called a multigraph. For a graph G , we set $V(G) = \{v_1, \dots, v_n\}$, $E(G) = \{e_1, \dots, e_m\}$ and $\mathfrak{e} = (e_1, \dots, e_m)$ unless otherwise specified. For a positive integer n , we denote $\{1, \dots, n\}$ by $[n]$. For an integer vector \mathbb{b} , \mathbb{b}^+ (resp. \mathbb{b}^-) means the vector whose i th entry is $\max\{b_i, 0\}$ (resp. $-\min\{b_i, 0\}$). For an integer vector $\mathfrak{x} = (x_1, \dots, x_m)$, $\mathfrak{e}^{\mathfrak{x}}$ means a monomial $e_1^{x_1} e_2^{x_2} \dots e_m^{x_m}$.¹

¹ Throughout the paper, to denote a vector, we use \mathfrak{a} , \mathbb{b} , \mathfrak{c} , etc. The standard bold type letters (**a**, **b**, **c**, etc.) are for walks in a graph.

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Let $K[e_1, \dots, e_m]$ be a polynomial ring in m variables over a field K . For an $n \times m$ integer matrix A without zero columns, the ideal

$$I_A = \left\langle e^{\mathbf{b}^+} - e^{\mathbf{b}^-} \in K[e_1, \dots, e_m] \mid \mathbf{b} \in \mathbb{Z}^m \text{ and } A\mathbf{b} = 0 \right\rangle$$

is called the *toric ideal* associated with A . It is well known that a toric ideal is a prime binomial ideal. For more details about toric ideals and related topics, see [11, 25].

A (homogeneous) toric ideal not only defines a projective toric variety (see [6, 24]), but also provides wide applications in other areas, such as algebraic statistics, dynamical system, hypergeometric differential equations, toric geometry, and graph theory, see [5, 14, 25]. Toric ideals arising from various kinds of combinatorial objects have been widely studied by many researchers, see [12, 13, 19, 20] for some recent results. Particularly, the toric ideal of a graph or a digraph, which is the toric ideal associated with its vertex-edge incidence matrix, has been an interesting topic (see [2–4, 7, 8, 17, 18, 21, 22]).

A major line of research on toric ideal arising from a combinatorial object focuses on a ‘special’ set of binomials of the ideal (giving a combinatorial interpretation). Among them, the set of primitive binomials, which is known to form the Graver basis, was studied widely related to a problem initiated by Sturmfels, called true degree problem. (See [23, 25, 26, 28] for detail.) For a toric ideal I_A , an irreducible binomial $B = B^+ - B^-$ of I_A is *primitive* if there exists no other binomial $B_0 = B_0^+ - B_0^-$ such that $B_0^+ | B^+$ and $B_0^- | B^-$. For the toric ideal of a graph, the primitive binomials and some other important binomials were characterized in [22]. The primitive binomials of the toric ideal of a digraph are nicely stated in [8, 9]. See Sect. 2.2.1 for the primitive binomials of the toric ideal of a graph/digraph.

Another important research direction on a toric ideal is about the complete intersection property. A toric ideal I_A associated with an $n \times m$ integer matrix A has the height $\text{ht}(I_A) = m - \text{rank}(A)$. We say I_A is a *complete intersection* if it is generated by $\text{ht}(I_A)$ elements (see [25]). A complete intersection toric ideal was first studied by Herzog in [10], and it is known that the Hilbert series of the corresponding quotient ring R/I can be computed easily when I is a complete intersection. The complete intersection property of the toric ideal from a combinatorial object was also investigated by many researchers, see [1, 2, 7–9, 15, 16, 21, 27]. We summarize some known results on the toric ideals of graphs/digraphs in Sect. 2.2.2.

In this paper, we consider toric ideals of signed graphs, as a generalization of graphs and digraphs. An *incidence* of a graph G is a pair (e, v) of an edge e and a vertex v such that v is an endpoint of e . A *sign* τ of G is a function from the set of all incidences to the set $\{1, -1\}$, and a *signed graph* is a pair (G, τ) of a graph G and its sign τ . For a signed graph (G, τ) with n vertices and m edges, the *incidence matrix* $A(G, \tau)$ of (G, τ) is an $n \times m$ matrix whose rows are labeled by the vertices v_1, \dots, v_n and columns are labeled by the edges e_1, \dots, e_m such that $[A(G, \tau)]_{ij} = \tau(e_j, v_i)$ if v_i is incident to e_j , and $[A(G, \tau)]_{ij} = 0$ otherwise. With an abuse of notation, we often consider the codomain of τ is $\{+, -\}$. See Fig. 1 for an example.

We remark that if a sign τ is a constant function, then (G, τ) is just a graph. If a sign τ satisfies that $\tau(e, u)\tau(e, v) = -1$ for each edge $e = uv$, then (G, τ) is equal to

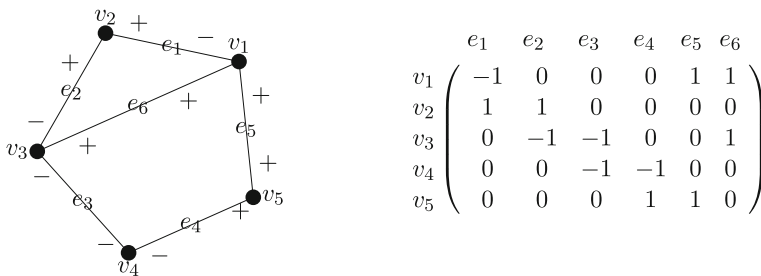


Fig. 1 A signed graph (G, τ) and its incidence matrix $A(G, \tau)$

a digraph. The *toric ideal of a signed graph* (G, τ) , denoted by $I_{(G, \tau)}$, is the toric ideal associated with the incidence matrix $A(G, \tau)$ in the polynomial ring $K[e_1, \dots, e_m]$ over a field K .

As long as the authors are aware, the toric ideal of a signed graph is firstly considered in this paper, and so we start our research from a fundamental question on generators. We completely characterize the primitive binomials of $I_{(G, \tau)}$ of a signed graph (G, τ) with graph theory language. This gives a way to explain the previous results on graphs/digraphs in a unified idea. The latter part of the paper sheds light on the complete intersection property of $I_{(G, \tau)}$. We give a necessary and sufficient condition for a graph G to have a complete intersection $I_{(G, \tau)}$ for every sign τ , see Theorems 3.9 and 3.10. We emphasize that this result is more than unifying the results of graphs/digraphs in [2, 8], since there are infinitely many graphs G such that the toric ideal of G and its every orientation are complete intersections but $I_{(G, \tau)}$ is not a complete intersection for some sign τ (see Sect. 3). Lastly, we find a full list of such graphs without assuming 2-connectedness, see Theorem 3.9.

2 Preliminaries

This section gives some basic notion and terminology in graphs and then summarizes some known results on toric ideals of graphs and digraphs. In addition, we explain how to define binomials from walks in a signed graph, which generate $I_{(G, \tau)}$.

2.1 Basic notion for walks in a graph

For a graph G , let $\mathbf{w} : v_{i_1} e_{j_1} \cdots e_{j_t} v_{i_{t+1}}$ be a walk or $(v_{i_1}, v_{i_{t+1}})$ -walk, which is an alternating sequence of vertices v_i 's and edges e_j 's where $e_{j_\ell} = v_{i_\ell} v_{i_{\ell+1}}$ for each $\ell \in [t]$. We call v_{i_ℓ} (resp. e_{j_ℓ}) the ℓ th vertex (resp. edge) term of \mathbf{w} . A vertex term is said to be *internal* if it is neither first nor last. We let $V(\mathbf{w})$ be the set of vertex terms of \mathbf{w} and $E(\mathbf{w})$ be the multiset of the edge terms in \mathbf{w} . We denote the multigraph with the vertex set $V(\mathbf{w})$ and the edge set $E(\mathbf{w})$ by $[\mathbf{w}]$. The underlying simple graph of $[\mathbf{w}]$ is a subgraph of G , but $[\mathbf{w}]$ may not be a subgraph of G by multiple edges. See Fig. 2.

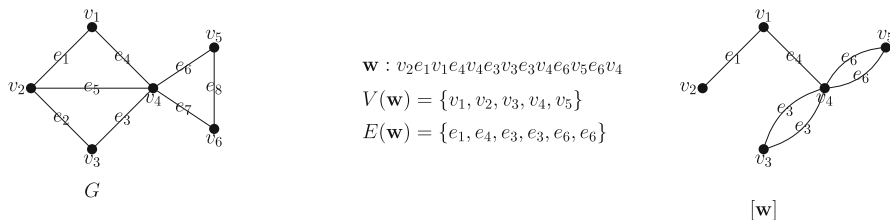


Fig. 2 A walk \mathbf{w} in a graph G and $V(\mathbf{w})$, $E(\mathbf{w})$, and $[\mathbf{w}]$

The *length* of \mathbf{w} is the number of edge terms in \mathbf{w} . A *subwalk* of \mathbf{w} is a subsequence of \mathbf{w} which is a walk, and a *section* is a subwalk consisting of consecutive terms of \mathbf{w} . When we consider a subwalk or a section of a closed walk \mathbf{w} , the terms are considered cyclically so that the last edge term is consecutive to the first vertex term.

For two walks \mathbf{w} and \mathbf{w}' in a graph, if the last vertex term of \mathbf{w} and the first vertex term of \mathbf{w}' are equal, then we denote by $\mathbf{w} + \mathbf{w}'$ the walk going through \mathbf{w} and then \mathbf{w}' . If $\mathbf{w}_0, \dots, \mathbf{w}_k$ are sections of a walk \mathbf{w} such that $\mathbf{w} = \mathbf{w}_0 + \dots + \mathbf{w}_k$, then we call this form a *section decomposition* of \mathbf{w} . If every \mathbf{w}_i is a nontrivial walk, then we say it is nontrivial.

The walk obtained by reading a walk \mathbf{w} in the reverse order is denoted by \mathbf{w}^{-1} . For a closed walk $\mathbf{w} : v_{i_1}e_{j_1} \cdots e_{j_t}v_{i_1}$ ($t \geq 2$), we say a vertex $v \in V(\mathbf{w})$ is *repeated* if v appears in $v_{i_1}e_{j_1}v_{i_2} \cdots v_{i_t}$ ($e_{j_t}v_{i_1}$ is deleted from \mathbf{w}) at least two times. Note that if a closed walk \mathbf{w} has a repeated vertex $v \in V(\mathbf{w})$, then \mathbf{w} has a nontrivial section decomposition $\mathbf{w}_0 + \mathbf{w}_1$ for some closed walks \mathbf{w}_0 and \mathbf{w}_1 (whose first vertex terms are v).

2.2 Toric ideals of graphs and digraphs

Recall that the *toric ideal* I_G of a graph (resp. digraph) G is the toric ideal $I_{(G, \tau)}$ when τ is a constant function (resp. $\tau(e, u)\tau(e, v) = -1$ for every edge $e = uv$). In this section, we summarize some previous results on the toric ideals of graphs/digraphs, which will be used in this paper.

2.2.1 Primitive binomials

Let A be an $n \times m$ matrix without zero columns. An irreducible binomial of I_A has a form of $e^{\mathbf{b}^+} - e^{\mathbf{b}^-}$ for some $\mathbf{b} \in \mathbb{Z}^m$ with $A\mathbf{b} = \mathbf{0}$. An irreducible binomial $e^{\mathbf{b}^+} - e^{\mathbf{b}^-}$ is called *primitive* if there exists no other binomial $e^{\mathbf{c}^+} - e^{\mathbf{c}^-}$ such that $e^{\mathbf{c}^+} | e^{\mathbf{b}^+}$ and $e^{\mathbf{c}^-} | e^{\mathbf{b}^-}$.

For a closed nontrivial walk $\mathbf{w} : v_{i_1}e_{j_1} \cdots v_{i_{2t}}e_{j_{2t}}v_{i_1}$ of even length in a graph G , let $B_{\mathbf{w}} = B^+ - B^-$, where $B^+ = e_{j_1}e_{j_3} \cdots e_{j_{2t-1}}$ and $B^- = e_{j_2}e_{j_4} \cdots e_{j_{2t}}$. Here, the same closed walk can be written in different ways but associated binomials differ only in the sign. It is observed that (see [29]) I_G is generated by those binomials $B_{\mathbf{w}}$. A necessary condition for the primitive binomials was firstly studied in [18] and a necessary and sufficient condition was established in [22] as follows. When two

graphs G and G' contain cliques K and K' of size k , respectively, a graph obtained from G and G' by identifying K and K' is called a k -clique sum of G and G' .

Theorem 2.1 [22, Theorem 3.2] *For a closed walk \mathbf{w} of even length in a graph, the binomial $B_{\mathbf{w}}$ is primitive if and only if the following hold:*

- (i) *The multigraph $[\mathbf{w}]$ is constructed by 1-clique sums of cycles of length at least two such that every vertex of $[\mathbf{w}]$ belongs to at most two cycles.*
- (ii) *For every nontrivial section decomposition $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$ into two closed walks \mathbf{w}_0 and \mathbf{w}_1 , the length of each \mathbf{w}_i is odd.*

In [22], other important sets of binomials in I_G were also characterized and we omit them here as it is not related to our main purposes.

Let G be a graph and D be its orientation. The primitive binomials of I_D are much more simply described. For every cycle \mathbf{w} of G , we define $B_{\mathbf{w}} = B^+ - B^-$, where B^+ is the product of the clockwise oriented edges and B^- is the product of the other edges.

Theorem 2.2 [8, Proposition 2] *For a graph G , let D be its orientation. The primitive binomials of I_D are binomials $B_{\mathbf{w}}$ associated with cycles \mathbf{w} of G .*

2.2.2 The complete intersection property

Recall that the toric ideal I_A is a *complete intersection* if it can be generated by $\text{ht}(I)$ elements, where $\text{ht}(I)$ is the *height* of I . It also holds that $\text{ht}(I_A) = m - \text{rank}(A)$. For a connected graph G with n vertices and m edges, I_G is a complete intersection if and only if it is generated by $r(G)$ binomials, where

$$r(G) = \begin{cases} m - n + 1 & \text{if } G \text{ is bipartite,} \\ m - n & \text{otherwise.} \end{cases}$$

For a disconnected graph G , I_G is a *complete intersection* if every connected component of G has a complete intersection toric ideal. Let \mathcal{G}^{ci} be the set of all graphs with complete intersection toric ideals. The bipartite graphs in \mathcal{G}^{ci} are nicely characterized as follows.

Theorem 2.3 [9, Corollary 3.4] *For a connected bipartite graph G , $G \in \mathcal{G}^{ci}$ if and only if G is a ring graph. Here, a ring graph is a graph whose nonedge block is constructed by 2-clique sums of cycles.*

A pseudo-code to check whether a graph is in \mathcal{G}^{ci} or not is given in [2], see Algorithm 2.4. The algorithm relies on inductive idea, which may not give an exact description on structures of graphs in \mathcal{G}^{ci} . Instead, useful structural properties are provided in [2, 27], and some are listed in Theorem 2.5. For a graph H , the number of connected components which are bipartite is denoted by $b(H)$.

Algorithm 2.4 ([2])Input: a simple graph G Output: TRUE if $G \in \mathcal{G}^{ci}$ or FALSE otherwise $H := G; \mathcal{B} := \emptyset$ **while** $\exists v \in V(H)$ with $\deg_H(v) \leq 2$ **do** **if** $\deg_H(v) = 2$ and $b(H - v) = b(H)$ **then** $W := \{v\} \cup N_H(v) \cup \{u \in V(H) \mid b(H - \{u, v\}) > b(H - u)\}$ **if** not exists a closed walk \mathbf{w} of even length such that $V(\mathbf{w}) = W$ **then** **return** FALSE **end if** Let \mathbf{w} be a shortest closed walk of even length such that $V(\mathbf{w}) =$ W . $\mathcal{B} := \mathcal{B} \cup \{B_{\mathbf{w}}\}$ **end if** $H := H - v$ **end while**Let H_1, \dots, H_s be the connected components of H .**if** exists i such that H_i is neither odd band nor even Möbius band **then** **return** FALSE**end if**Let \mathcal{B}_i be a minimal generating set of I_{H_i} for each $1 \leq i \leq s$.**if** I_G is generated by $\mathcal{B} \cup \mathcal{B}_1 \cup \dots \cup \mathcal{B}_s$ **then** **return** TRUE**end if****return** FALSE

Theorem 2.5 [2, Theorem 3.6, Corollary 3.9, Lemma 6.2], [27, Theorem 3.1, Corollary 5.6] *Let G be a connected graph in \mathcal{G}^{ci} . Then, the following hold:*

- (i) *If G is not bipartite, then $2|E(G)| \leq 3|V(G)| - \sum_{v \in V(G)} b(G - v)$.*
- (ii) *G has no $K_{2,3}$ as a subgraph.*
- (iii) *If G is 2-connected and has two cycles C and C' of odd length sharing exactly one vertex v , then there is an edge e not incident to v which connects C and C' .*
- (iv) *If G is 2-connected and has disjoint two cycles C and C' of odd length, then there are two disjoint edges e_1 and e_2 such that each e_i connects C and C' .*
- (v) *G has at most two nonbipartite blocks.*
- (vi) *Every induced subgraph of G belongs to \mathcal{G}^{ci} .*

In [2], 3-regular graphs with complete intersection toric ideals are characterized, see Theorem 2.6. Instead of giving the definitions of bands or Möbius bands, we note that the complete graph K_4 is an even Möbius band. For the definitions, see [2, Definition 4.2].

Theorem 2.6 [2, Theorem 4.4] *For a 3-regular connected graph G , $G \in \mathcal{G}^{ci}$ if and only if it is an odd band or an even Möbius band.*

The following lists results on digraphs. For a connected graph G with n vertices and m edges, and an orientation D of G , I_D is a (binomial) complete intersection if and only if it is generated by $r(D) = m - n + 1$ binomials. For an orientation D of a disconnected graph, I_D is a complete intersection if the toric ideal of the digraph restricted to every connected component of G is a complete intersection. Let \mathcal{G}^{cio} be the set of graphs G such that I_D is generated a complete intersection for every orientation D of G .

Theorem 2.7 [8, Theorems 4 and 6, Corollary 4] *Let G be a connected graph.*

- (i) $G \in \mathcal{G}^{cio}$ if and only if G is constructed by clique sums of complete graphs and/or cycles.
- (ii) Every induced subgraph of a graph in \mathcal{G}^{cio} belongs to \mathcal{G}^{cio} .

2.3 Even-signed walks in (G, τ) and their associated binomials of $I_{(G, \tau)}$

In this subsection, we explain how to define binomials associated with closed walks in a signed graph, and this will play a key role in Sect. 3.

Definition 2.8 Let $\mathbf{w} : v_{i_1} e_{j_1} v_{i_2} \cdots e_{j_t} v_{i_{t+1}}$ be a walk in a signed graph (G, τ) of length at least two. An internal vertex term v_{i_ℓ} of \mathbf{w} is *unbalanced* if $\tau(e_{j_{\ell-1}}, v_{i_\ell})\tau(e_{j_\ell}, v_{i_\ell}) = 1$. As long as \mathbf{w} is closed, we say v_{i_1} (or $v_{i_{t+1}}$) is *unbalanced* if $\tau(e_{j_t}, v_{i_1})\tau(e_{j_1}, v_{i_1}) = 1$. We define $\mu(\mathbf{w}) = (-1)^k$, where k is the number of unbalanced vertex terms of \mathbf{w} . We also say \mathbf{w} is *even-signed* if $\mu(\mathbf{w}) = 1$, and \mathbf{w} is *odd-signed* if $\mu(\mathbf{w}) = -1$.

Throughout the paper, a walk/cycle with odd/even number of edge terms is said to be a walk/cycle of odd/even length. A *triangle* means a cycle of length three.

A *balanced section* \mathbf{w}_0 of a walk \mathbf{w} is a maximal section of \mathbf{w} such that \mathbf{w}_0 has no internal unbalanced vertex term. If $\mathbf{w}_0 + \cdots + \mathbf{w}_k$ is a section decomposition of \mathbf{w} such that each \mathbf{w}_i is a balanced section, then this form is called a *balanced section decomposition* of \mathbf{w} .

For a closed walk \mathbf{w} , if it has no unbalanced vertex term, then it is even-signed and has exactly one balanced section which is itself. Otherwise, its balanced section decomposition $\mathbf{w}_0 + \cdots + \mathbf{w}_k$ is also unique up to cyclic permutations. So, by choosing an unbalanced vertex as the first vertex term properly, we can denote by $\mathbf{w} = \mathbf{w}_0 + \cdots + \mathbf{w}_k$.

Example 2.9 Consider a signed graph (G, τ) in Fig. 3, and its two closed walks \mathbf{w} and \mathbf{w}' , where

$$\mathbf{w} : v_3 e_3 v_4 e_4 v_5 e_5 v_1 e_6 v_3 e_2 v_2 e_1 v_1 e_6 v_3, \quad \mathbf{w}' : v_1 e_1 v_2 e_2 v_3 e_6 v_1 e_5 v_5 e_5 v_1.$$

For a closed walk \mathbf{w} , the 2nd, 3rd, 4th, and 6th vertex terms are the unbalanced vertex terms, which implies that \mathbf{w} is an even-signed walk in (G, τ) and the balanced sections are

$$\mathbf{w}_0 : v_2 e_1 v_1 e_6 v_3 e_3 v_4, \quad \mathbf{w}_1 : v_4 e_4 v_5, \quad \mathbf{w}_2 : v_5 e_5 v_1, \quad \mathbf{w}_3 : v_1 e_6 v_3 e_2 v_2.$$

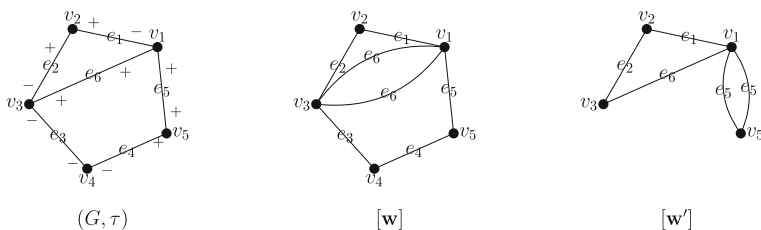


Fig. 3 A signed graph (G, τ) and two closed walks \mathbf{w} and \mathbf{w}'

The walk has a balanced section decomposition $\mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$. For the closed walk \mathbf{w}' , there are three unbalanced vertex terms, the 2nd, 4th, and 5th vertex terms, which implies that \mathbf{w}' is odd-signed in (G, τ) and its balanced sections are

$$\mathbf{w}'_0 : v_5 e_5 v_1 e_1 v_2, \quad \mathbf{w}'_1 : v_2 e_2 v_3 e_6 v_1, \quad \mathbf{w}'_2 : v_1 e_5 v_5,$$

and $\mathbf{w}'_0 + \mathbf{w}'_1 + \mathbf{w}'_2$ is a balanced section decomposition of \mathbf{w}' . Note that by taking the first vertex term of \mathbf{w} or \mathbf{w}' properly, we can write $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ or $\mathbf{w}' = \mathbf{w}'_0 + \mathbf{w}'_1 + \mathbf{w}'_2$.

Definition 2.10 Let \mathbf{w} be an even-signed closed walk in a signed graph, and $\mathbf{w}_0 + \mathbf{w}_1 + \dots + \mathbf{w}_{2k-1}$ be its balanced section decomposition ($k \geq 1$). The *binomial* $B_{\mathbf{w}}$ associated with \mathbf{w} is $B_{\mathbf{w}} = B_{\mathbf{w}}^+ - B_{\mathbf{w}}^-$ where

$$B_{\mathbf{w}}^+ = \prod_{i:\text{even}} \prod_{e \in E(\mathbf{w}_i)} e \quad \text{and} \quad B_{\mathbf{w}}^- = \prod_{i:\text{odd}} \prod_{e \in E(\mathbf{w}_i)} e.$$

If \mathbf{w} has no unbalanced vertex term, then it is defined by $B_{\mathbf{w}}^+ = \prod_{e \in E(\mathbf{w})} e$ and $B_{\mathbf{w}}^- = 1$.

Since \mathbf{w} has an even number of unbalanced vertex terms, its binomial is unique up to sign. That is, the binomial is either $B_{\mathbf{w}}$ or $-B_{\mathbf{w}}$ according to its balanced section decomposition. For the even-signed closed walk \mathbf{w} in (G, τ) in Example 2.9, $B_{\mathbf{w}} = e_2 e_4 e_6 - e_1 e_3 e_5 e_6$ (one may say $B_{\mathbf{w}} = e_1 e_3 e_5 e_6 - e_2 e_4 e_6$).

Observation 2.11 If \mathbf{w} is an even-signed closed walk in a signed graph (G, τ) , then $B_{\mathbf{w}} \in I_{(G, \tau)}$.

Proof Let $\mathbf{w} : v_{i_1} e_{j_1} \dots v_{i_t} e_{j_t} v_{i_1}$ be an even-signed closed walk in (G, τ) . We may assume that the first vertex term is unbalanced, and let $\mathbf{w} = \mathbf{w}_0 + \dots + \mathbf{w}_{2k-1}$ be a balanced section decomposition of \mathbf{w} . For each edge term e_{j_ℓ} , we let $\kappa(e_{j_\ell}) = (-1)^s$ if e_{j_ℓ} belongs to the section \mathbf{w}_s .

Let $\mathbb{b} = (b_e)_{e \in E(G)}$ be a vector such that $b_e = f^+(e) - f^-(e)$ for every edge e , where

$$f^+(e) = |\{\ell \mid e_{j_\ell} = e \text{ and } \kappa(e_{j_\ell}) = 1\}|, \quad f^-(e) = |\{\ell \mid e_{j_\ell} = e \text{ and } \kappa(e_{j_\ell}) = -1\}|.$$

Then, the entry of $A\mathbb{b}$ corresponding to a vertex v is

$$\begin{aligned} \sum_{e:v \in e} (f^+(e) - f^-(e))\tau(e, v) &= \sum_{e:v \in e} \sum_{\ell: e_{j_\ell} = e} \kappa(e_{j_\ell})\tau(e, v) \\ &= \sum_{\ell: v_{i_\ell} = v} (\kappa(e_{j_{\ell-1}})\tau(e_{j_{\ell-1}}, v) + \kappa(e_{j_\ell})\tau(e_{j_\ell}, v)). \end{aligned}$$

If $\kappa(e_{j_\ell}) = \kappa(e_{j_{\ell-1}})$, then $\tau(e_{j_{\ell-1}}, v) = -\tau(e_{j_\ell}, v)$, and if $\kappa(e_{j_\ell}) \neq \kappa(e_{j_{\ell-1}})$, then $\tau(e_{j_{\ell-1}}, v) = \tau(e_{j_\ell}, v)$. In both cases, the sum $\kappa(e_{j_{\ell-1}})\tau(e_{j_{\ell-1}}, v) + \kappa(e_{j_\ell})\tau(e_{j_\ell}, v)$ is 0. This implies that $e^{\mathbb{b}^+} - e^{\mathbb{b}^-} \in I_{(G, \tau)}$.

On the other hand, from the definition, $e^{\mathbb{x}}(e^{\mathbb{b}^+} - e^{\mathbb{b}^-}) = B_{\mathbf{w}}$, where $\mathbb{x} = (x_e)_{e \in E(G)}$ is the vector such that $x_e = \min\{f^+(e), f^-(e)\}$ for every edge e . Thus, $B_{\mathbf{w}}$ is an element of $I_{(G, \tau)}$. \square

It seems natural to have the following proposition from the definition of $I_{(G, \tau)}$.

Proposition 2.12 *If (G, τ) is a signed graph, then the toric ideal $I_{(G, \tau)}$ is generated by*

$$\{B_{\mathbf{w}} \mid \mathbf{w} \text{ is an even-signed closed walk in } (G, \tau)\}.$$

The above proposition immediately follows from Observation 2.11 and Proposition 2.13, and we leave the proof detail of Proposition 2.13 in Appendix.

Proposition 2.13 *Let (G, τ) be a signed graph, and $\mathbb{b} = (b_e)_{e \in E(G)}$ be a nonzero integer vector such that $A\mathbb{b} = \mathbb{0}$, where $A = A(G, \tau)$. If we denote by $(G_{\mathbb{b}}, \tau_{\mathbb{b}})$ the signed multigraph induced by $|b_e|$ copies of e for every edge e with its sign copied, then each connected component D of $G_{\mathbb{b}}$ has an Eulerian \mathbf{w}_D which is an even-signed closed walk in (G, τ) and*

$$e^{\mathbb{b}^+} - e^{\mathbb{b}^-} = \prod_{\substack{D: \text{connected} \\ \text{component of } G_{\mathbb{b}}}} B_{\mathbf{w}_D}^+ - \prod_{\substack{D: \text{connected} \\ \text{component of } G_{\mathbb{b}}}} B_{\mathbf{w}_D}^-.$$

3 The main results

In this section, we state the main results of the paper. Section 3.1 focuses on the primitive binomials of $I_{(G, \tau)}$, and Sect. 3.2 gives characterizations of graphs G with a complete intersection $I_{(G, \tau)}$ for every sign τ .

3.1 Primitive binomials of $I_{(G, \tau)}$

We characterize all primitive binomials in $I_{(G, \tau)}$.

Theorem 3.1 *For an even-signed closed walk \mathbf{w} in a signed graph, $B_{\mathbf{w}}$ is primitive if and only if the following hold:*

Fig. 4 The multigraph $[\mathbf{w}]$, where \mathbf{w} is an even-signed closed walk in a signed graph (G, τ) . If (G, τ) has no even-signed cycle, then \mathbf{w} is primitive

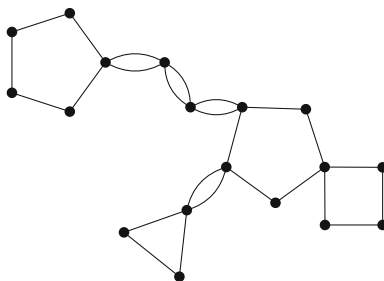
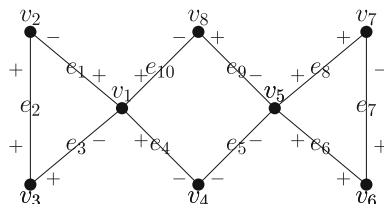


Fig. 5 A signed graph (G, τ)



- (i) The multigraph $[\mathbf{w}]$ is constructed by 1-clique sums of cycles of length at least two such that every vertex of $[\mathbf{w}]$ belongs to at most two cycles.
- (ii) For every nontrivial section decomposition $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$ into two closed walks \mathbf{w}_0 and \mathbf{w}_1 , each \mathbf{w}_i is odd-signed in (G, τ) .

A proof of Theorem 3.1 is given in Sect. 5.1. We often say an even-signed closed walk \mathbf{w} in a signed graph (G, τ) is *primitive* if $B_{\mathbf{w}}$ is primitive in $I_{(G, \tau)}$. Figure 4 shows an image of the multigraph $[\mathbf{w}]$ for a primitive walk \mathbf{w} . Note that if \mathbf{w} is a primitive walk, then every cut vertex of $[\mathbf{w}]$ decomposes $[\mathbf{w}]$ into two parts, and each part corresponds to an odd-signed closed walk in (G, τ) .²

Example 3.2 Consider a signed graph (G, τ) in Fig. 5. Let \mathbf{w} be a walk in (G, τ) defined by

$$\mathbf{w} : v_1 e_1 v_2 e_2 v_3 e_3 v_1 e_4 v_4 e_5 v_5 e_6 v_6 e_7 v_7 e_8 v_5 e_9 v_8 e_{10} v_1.$$

It has four balanced sections, and so \mathbf{w} is an even-signed closed walk in (G, τ) . See the vertex v_5 which is repeated in \mathbf{w} . Then, \mathbf{w} has a nontrivial section decomposition $\mathbf{w}_0 + \mathbf{w}_1$ and each of \mathbf{w}_0 and \mathbf{w}_1 is an even-signed closed walk, where $\mathbf{w}_0 : v_5 e_6 v_6 e_7 v_7 e_8 v_5$ and $\mathbf{w}_1 : v_5 e_9 v_8 e_{10} v_1 e_1 v_2 e_2 v_3 e_3 v_1 e_4 v_4 e_5 v_5$. Thus, its associated binomial $B_{\mathbf{w}}$ is not primitive by Theorem 3.1(ii).

It is easy to see that Theorem 3.1 is a generalization of Theorem 2.1. Moreover, if you consider an orientation of a cycle as a signed graph, then it is an even-signed closed walk. Thus, every digraph has no odd-signed closed walk, which implies that Theorem 3.1 is also a generalization of Theorem 2.2.

² In the toric ideals of graphs, this was explained with a notion of ‘sink’ of a block, see [22].

3.2 The complete intersection property of the toric ideal $I_{(G, \tau)}$

We compute the rank of the incidence matrix of a connected signed graph first.

Proposition 3.3 *Let (G, τ) be a connected signed graph. Then,*

$$\text{rank}(A(G, \tau)) = \begin{cases} |V(G)| - 1 & \text{if there is no odd-signed closed walk in } (G, \tau), \\ |V(G)| & \text{otherwise.} \end{cases}$$

See Appendix for the proof of Proposition 3.3. Due to the proposition and from the fact that $\text{ht}(I_{(G, \tau)}) = m - \text{rank}(A(G, \tau))$, we define the following.

Definition 3.4 Let (G, τ) be a connected signed graph. We say (G, τ) is a *complete intersection* when $I_{(G, \tau)}$ is a (binomial) complete intersection, i.e., $I_{(G, \tau)}$ is generated by $r(G, \tau)$ binomials, where

$$r(G, \tau) = \begin{cases} |E(G)| - |V(G)| + 1 & \text{if there is no odd-signed closed walk in } (G, \tau), \\ |E(G)| - |V(G)| & \text{otherwise.} \end{cases}$$

For a disconnected signed graph, it is said to be a *complete intersection* if every connected component is a complete intersection. Let \mathcal{G}^{cis} be the set of all graphs G such that (G, τ) is a (binomial) complete intersection for every sign τ . It is natural to ask which graphs are in \mathcal{G}^{cis} , and we start from basic observations.

Proposition 3.5 *Every block of a graph in \mathcal{G}^{cis} belongs to \mathcal{G}^{cis} .*

Proof Let G be a graph in \mathcal{G}^{cis} having a block H such that $H \notin \mathcal{G}^{cis}$. We may assume that G is connected. Then, there is a sign τ of H such that $I_{(H, \tau)}$ cannot be generated by $r(H, \tau)$ binomials. Let t be the minimum number of binomials which generate $I_{(H, \tau)}$. Then, $t > r(H, \tau)$.

Let τ' be the sign of G such that $\tau'(e, v) = \tau(e, v)$ for every $e \in E(H)$ and $\tau'(e, v)\tau'(e, w) = -1$ for every $e = vw \in E(G) \setminus E(H)$. Let $X = \{v \in V(G) \mid v \text{ is contained in a block other than } H\}$. Since $G \in \mathcal{G}^{cis} \subset \mathcal{G}^{cio}$, it follows that $G[X] \in \mathcal{G}^{cio}$ by Theorem 2.7(ii). Note that $(G[X], \tau'|_X)$ can be understood as a digraph, and so $(G[X], \tau'|_X)$ has no odd-signed cycle. Thus,

$$r(G, \tau') = r(H, \tau) + (|E(G[X])| - |X| + c) = r(H, \tau) + (|E(G)| - |E(H)| - |X| + c), \quad (3.1)$$

where $c = |V(H) \cap X|$. Moreover, since $G[X]$ has at least c components, we need at least $|E(G)| - |E(H)| - |X| + c$ binomials to generate $I_{(G[X], \tau'|_X)}$. Hence, in order to generate $I_{(G, \tau')}$, we need at least $t + |E(G)| - |E(H)| - |X| + c$ binomials. Since $t + |E(G)| - |E(H)| - |X| + c > r(G, \tau')$ by (3.1), we reach a contradiction to the fact that $G \in \mathcal{G}^{cis}$. \square

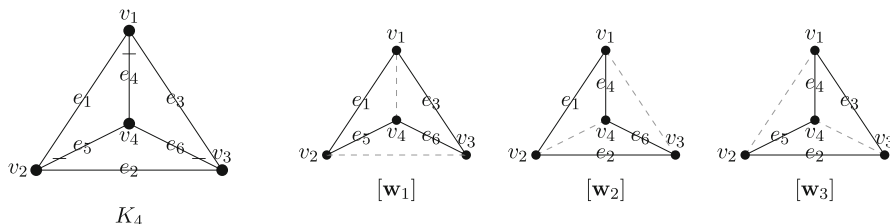


Fig. 6 A complete graph K_4 and its three closed walks

Proposition 3.6 *Let H be either a cycle or a K_2 . For a connected graph G , let G' be a 1- or 2-clique sum of G and H . Let τ' be a sign of G' such that $\tau = \tau'|_{V(G)}$. Suppose that H is even-signed in (G', τ') when H is a cycle. Then, $I_{(G', \tau')}$ is a complete intersection if and only if $I_{(G, \tau)}$ is a complete intersection.*

Proof Suppose that H is a cycle. It is clear that $r(G', \tau') = r(G, \tau) + 1$. Since C is a primitive walk in (G', τ') , its associated binomial generates $I_{(G', \tau')}$ together with a generating set of $I_{(G, \tau)}$. If $H = K_2$, then $r(G', \tau') = r(G, \tau)$ for every sign τ' of G' and the primitive walks of (G, τ) and those of (G', τ') are the same. Thus, the proposition holds. \square

Proposition 3.6 implies that for a graph G , if $G \notin \mathcal{G}^{cis}$, then a graph constructed by clique sums of G and cycles/ K_2 is not in \mathcal{G}^{cis} . Thus, the following holds.

Corollary 3.7 *For a graph $G \in \mathcal{G}^{cis}$ and an induced subgraph H of G , if G can be constructed by clique sums of H and cycles/ K_2 , then H belongs to \mathcal{G}^{cis} .*

Observation 3.8 *A graph in \mathcal{G}^{cis} is K_4 -free.*

Proof First, we show that $K_4 \notin \mathcal{G}^{cis}$. Following the labeling in Fig. 6, let τ be a sign of K_4 such that

$$\tau(e_4, v_1) = \tau(e_5, v_2) = \tau(e_6, v_3) = -1,$$

and all the others have sign 1. By Theorem 3.1, there are only three primitive walks $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, which are defined as Fig. 6.³ Then, we have $B_{\mathbf{w}_1} = e_1e_5 - e_3e_6$, $B_{\mathbf{w}_2} = e_1e_4 - e_2e_6$, and $B_{\mathbf{w}_3} = e_2e_5 - e_3e_4$. Note that each of three cannot be generated by the others. However, $r(K_4, \tau) = 2$, since (K_4, τ) has odd-signed cycles. Thus, $I_{(K_4, \tau)}$ is not a complete intersection.

Suppose that there is a graph G in \mathcal{G}^{cis} having K_4 as a subgraph. We take such G as a smallest one. By Proposition 3.5, a block of G with K_4 is in \mathcal{G}^{cis} . Thus, G is 2-connected. Moreover, by the above argument, $G \neq K_4$. Since $G \in \mathcal{G}^{cio}$, G is constructed by clique sums of cycles and/or complete graphs by Theorem 2.7(i). Since G is 2-connected, it follows that every clique sum to construct G is not a 1-clique sum. From the fact that $G \in \mathcal{G}^{ci}$, by Theorem 2.5(ii), it follows that every

³ We note that Observation 3.8 is not used in the proof of Theorem 3.1. Moreover, in Fig. 6 (also in the following figures of the paper), we use dashed lines and gray color to draw the rest part of the graph not belonging to $[\mathbf{w}_i]$ together to distinguish the walks easily.

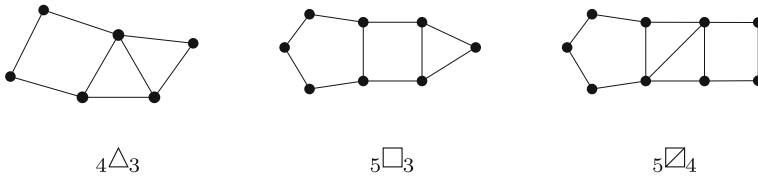


Fig. 7 Some graphs in Theorem 3.9

clique sum to construct G is a 2-clique sum. If G is constructed by clique sums of exactly one K_4 and cycles, then by Proposition 3.6 and the fact that $K_4 \notin \mathcal{G}^{cis}$, it follows that $G \notin \mathcal{G}^{cis}$, a contradiction. Thus, G has at least two K_4 , say K and K' . By Theorem 2.5(v), $G[K \cup K']$ is in \mathcal{G}^{ci} . By applying Theorem 2.5(i) to $G[K \cup K']$, we know that $G[K \cup K']$ is a disjoint union of K and K' . By Theorem 2.5(iv), every two vertex disjoint cycles of length three from K and K' are connected by two disjoint edges, which is a contradiction. \square

We remark that \mathcal{G}^{cis} is a subset of $\mathcal{G}^{ci} \cap \mathcal{G}^{cio}$ by definitions, and so Observation 3.8 tells us from Theorem 2.7(i) that the (connected) graphs in \mathcal{G}^{cis} are constructed by clique sums of cycles and/or K_2 . Among those graphs, we completely characterize all graphs in \mathcal{G}^{cis} . The following considers only 2-connected graphs in \mathcal{G}^{cis} , and its proof is given in Sect. 5.2.

Theorem 3.9 *For a 2-connected graph G with at least three vertices, $G \in \mathcal{G}^{cis}$ if and only if G is one of (G1)–(G5) for some $m, n \geq 3$ (see Fig. 7):*

- (G1) A cycle C_n ;
- (G2) A 2-clique sum of two cycles C_n and C_m ;
- (G3) $m\Delta_n$: the graph obtained from C_3 by gluing C_m and C_n to two distinct edges of C_3 using 2-clique sum, respectively;
- (G4) $m\Box_n$: the graph obtained from C_4 by gluing C_m and C_n to two opposite edges of C_4 using 2-clique sum, respectively;
- (G5) $m\Box_n$: the graph obtained from $m\Box_n$ by adding a diagonal edge of the middle C_4 .

Now we characterize all graphs in \mathcal{G}^{cis} . The proof of Theorem 3.10 is given in Sect. 5.3, and see Fig. 8 for some graphs described in the theorem.

Theorem 3.10 *For a graph G , G is in \mathcal{G}^{cis} if and only if every connected component G' of G is one of the following:*

- (i) G' is a tree.
- (ii) G' has exactly one nonedge block and it is isomorphic to one of (G1)–(G5).
- (iii) G' has exactly two nonedge blocks B and B' , each of which is isomorphic to (G1) or (G2). When B is (G2), the vertex v of B closest to B' is on a triangle of B and $\deg_B(v) = 2$.

We remark that from the structures of the graphs in Theorem 3.10, it follows that every induced subgraph of a graph in \mathcal{G}^{cis} belongs to \mathcal{G}^{cis} .

Now, we finish the section by noting that it is not difficult to find graphs in $\mathcal{G}^{ci} \cap \mathcal{G}^{cio}$ which are not in \mathcal{G}^{cis} . A reader may already notice that K_4 is such an example by

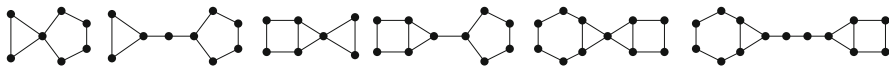


Fig. 8 Examples of graphs in \mathcal{G}^{cis} satisfying Theorem 3.10(iii)

Theorems 2.6 and 2.7 (i), and Observation 3.8. The following properties not only are helpful to understand Example 3.12 but also may give an idea to find graphs in $(\mathcal{G}^{ci} \cap \mathcal{G}^{cio}) \setminus \mathcal{G}^{cis}$.

Let $\mathbf{p} : v_0 e_1 v_1 \cdots e_t v_t$ be a path in G of length at least two such that $v_0 v_t \notin E(G)$ and $\deg_G(v_i) = 2$ for each $i \in [t-1]$ as depicted in Fig. 22. We call such path an *ear* of G , and we denote by G/\mathbf{p} the graph obtained from G by deleting the vertices v_1, \dots, v_{t-1} and adding an edge between v_0 and v_t . We sometimes call G/\mathbf{p} a *contraction* of G by \mathbf{p} . The proof of the following proposition is given in Appendix.

Proposition 3.11 *Let $\mathbf{p} : v_0 e_1 v_1 \cdots e_t v_t$ ($t \geq 2$) be an ear of a graph G . Then, the following hold:*

- (i) *If $G \in \mathcal{G}^{cis}$, then $G/\mathbf{p} \in \mathcal{G}^{cis}$. (Equivalently, if a graph is not in \mathcal{G}^{cis} , then its subdivision is not in \mathcal{G}^{cis} .)*
- (ii) *If $t \geq 3$ and $G/\mathbf{q} \in \mathcal{G}^{cis}$, where $\mathbf{q} = \mathbf{p} - v_t$, then $G \in \mathcal{G}^{cis}$. (Equivalently, if $G \notin \mathcal{G}^{cis}$, then the graph obtained by contracting an edge $e = uv$ with $\deg_G(u) = \deg_G(v) = 2$ is not in \mathcal{G}^{cis} .)*

Example 3.12 Let G be the graph in Fig. 9. Then, $G \notin \mathcal{G}^{cis}$ and $G \in \mathcal{G}^{ci} \cap \mathcal{G}^{cio}$.

First, we consider the graph G_0 in Fig. 9. Note that if $G_0 \notin \mathcal{G}^{ci}$ then $G_0 \notin \mathcal{G}^{cis}$, which also implies that by Proposition 3.11(i), $G \notin \mathcal{G}^{cis}$. Thus, it is sufficient to show that $G_0 \notin \mathcal{G}^{ci}$ by Algorithm 2.4. Note that for each $i \in \{1, 3, 5\}$, vertex v_i of G_0 has degree two and $b(G_0 - v_i) = b(G_0)$. We apply the algorithm to G_0 with v_1 . Then, $W = \{v_1, v_2, v_4, v_6\}$ and a shortest closed walk \mathbf{w}_1 of even length with $V(\mathbf{w}_1) = W$ is a cycle of length 4. Its associated binomial is $B_{\mathbf{w}_1} = e_1 e_8 - e_6 e_7$. Similarly, by considering the vertex v_3 and v_5 one by one, finally, we have $\mathcal{B} = \{B_{\mathbf{w}_1}, B_{\mathbf{w}_2}, B_{\mathbf{w}_3}\}$ where $B_{\mathbf{w}_2} = e_2 e_8 - e_3 e_9$ and $B_{\mathbf{w}_3} = e_4 e_9 - e_5 e_7$. It remains to check if $I_G = \langle \mathcal{B} \rangle$. However, $B_{\mathbf{w}_4}$ cannot be generated by \mathcal{B} , where $B_{\mathbf{w}_4} = e_1 e_3 e_5 - e_2 e_4 e_6$ is the primitive binomial associated with $\mathbf{w}_4 : v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_1$. Thus, Algorithm 2.4 returns FALSE, as a desired one.

Now, we will show that $G \in \mathcal{G}^{ci} \cap \mathcal{G}^{cio}$. Since G is constructed by clique sums of cycles, $G \in \mathcal{G}^{cio}$ by Theorem 2.7(i). It remains to check that $G \in \mathcal{G}^{ci}$. Note that G has exactly six primitive walks $\mathbf{x}_1 \sim \mathbf{x}_6$, defined as Fig. 9. Then, one can check from Corollary 4.5 that

$$B_{\mathbf{x}_4} \in \langle B_{\mathbf{x}_1}, B_{\mathbf{x}_2} \rangle, \quad B_{\mathbf{x}_5} \in \langle B_{\mathbf{x}_1}, B_{\mathbf{x}_3} \rangle, \quad B_{\mathbf{x}_6} \in \langle B_{\mathbf{x}_2}, B_{\mathbf{x}_3} \rangle.$$

Thus, $I_G = \langle B_{\mathbf{x}_1}, B_{\mathbf{x}_2}, B_{\mathbf{x}_3} \rangle$, which implies that I_G is a complete intersection.

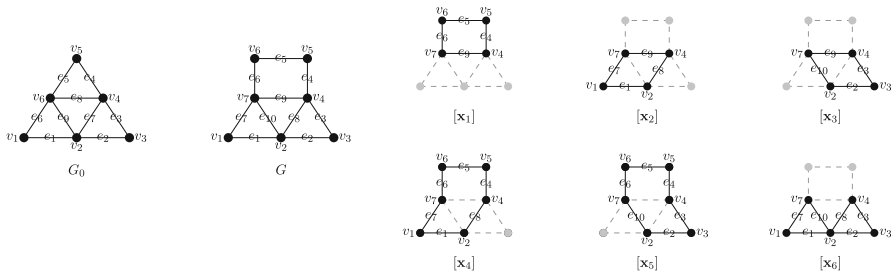


Fig. 9 Graphs G_0 , G and primitive walks in G

4 Properties of walks in a signed graph

In this section, we investigate properties of even-signed closed walks in a signed graph (G, τ) , which play an important role in the following section.

Lemma 4.1 *In a signed graph (G, τ) , for two closed walks \mathbf{w} and \mathbf{w}' sharing a vertex, $\mu(\mathbf{w} + \mathbf{w}') = \mu(\mathbf{w})\mu(\mathbf{w}')$.*

Proof If \mathbf{w} or \mathbf{w}' is trivial, then it is clear. Suppose that both are nontrivial. Let $\mathbf{w} : ue_{j_1}v_{i_2} \cdots e_{j_t}u$ and $\mathbf{w}' : ue'_{j_1}v'_{i_2} \cdots e'_{j_r}u$. Since $\tau(e_{j_1}, u)\tau(e_{j_t}, u) = 1$ (resp. $\tau(e'_{j_1}, u)\tau(e'_{j_r}, u) = 1$) means that u is an unbalanced vertex term of \mathbf{w} (resp. \mathbf{w}'), $\mu(\mathbf{w} + \mathbf{w}')$ is equal to

$$\begin{aligned} & (-\tau(e_{j_1}, u)\tau(e_{j_t}, u))(-\tau(e'_{j_1}, u)\tau(e'_{j_r}, u))\mu(\mathbf{w})\mu(\mathbf{w}') \\ & (-\tau(e_{j_t}, u)\tau(e'_{j_r}, u))(-\tau(e'_{j_1}, u)\tau(e_{j_1}, u)) = \mu(\mathbf{w})\mu(\mathbf{w}'). \end{aligned}$$

□

Lemma 4.2 *Let \mathbf{w} be a (u, v) -walk in a signed graph (G, τ) . For any two (v, u) -walks \mathbf{w}_1 and \mathbf{w}_2 ,*

$$\mu(\mathbf{w}_1 + \mathbf{w}_2^{-1}) = \mu(\mathbf{w} + \mathbf{w}_1)\mu(\mathbf{w} + \mathbf{w}_2).$$

Proof If $u = v$, then the lemma holds, since we have the following from Lemma 4.1:

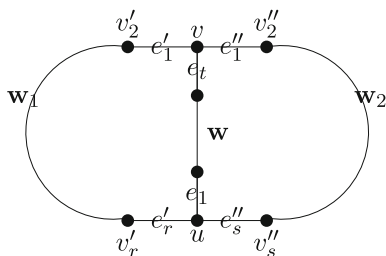
$$\begin{aligned} \mu(\mathbf{w}_1 + \mathbf{w}_2^{-1}) &= \mu(\mathbf{w}_1)\mu(\mathbf{w}_2^{-1}) = \mu(\mathbf{w}_1)\mu(\mathbf{w}_2) = \mu(\mathbf{w})\mu(\mathbf{w}_1)\mu(\mathbf{w})\mu(\mathbf{w}_2) \\ &= \mu(\mathbf{w} + \mathbf{w}_1)\mu(\mathbf{w} + \mathbf{w}_2). \end{aligned}$$

Suppose that u and v are distinct. Then, each of \mathbf{w} , \mathbf{w}_1 , and \mathbf{w}_2 is nontrivial. Without loss of generality, it is enough to consider the case when

$$\mathbf{w} : ue_1v_2 \cdots e_tv, \quad \mathbf{w}_1 : ve'_1v'_2 \cdots e'_ru, \quad \mathbf{w}_2 : ve''_1v''_2 \cdots e''_su$$

as depicted in Fig. 10.

Fig. 10 An illustration for the proof of Lemma 4.2



Note that $\tau(e'_r, u)\tau(e''_s, u) = 1$ (resp. $\tau(e'_1 v)\tau(e''_1, v) = 1$) means that u (resp. v) is a new unbalanced vertex term of $\mathbf{w}_1 + \mathbf{w}_2^{-1}$. Thus,

$$\mu(\mathbf{w}_1 + \mathbf{w}_2^{-1}) = \mu(\mathbf{w}_1)(-\tau(e'_r, u)\tau(e''_s, u))\mu(\mathbf{w}_2)(-\tau(e'_1, v)\tau(e''_1, v)).$$

Likewise,

$$\begin{aligned}\mu(\mathbf{w} + \mathbf{w}_1) &= \mu(\mathbf{w})(-\tau(e_t, v)\tau(e'_1, v))\mu(\mathbf{w}_1)(-\tau(e'_r, u)\tau(e_1, u)), \\ \mu(\mathbf{w} + \mathbf{w}_2) &= \mu(\mathbf{w})(-\tau(e_t, v)\tau(e''_1, v))\mu(\mathbf{w}_2)(-\tau(e''_s, u)\tau(e_1, u)).\end{aligned}$$

Therefore, $\mu(\mathbf{w}_1 + \mathbf{w}_2^{-1}) = \mu(\mathbf{w}_1)\mu(\mathbf{w}_2)\tau(e'_r, u)\tau(e''_s, u)\tau(e'_1, v)\tau(e''_1, v) = \mu(\mathbf{w} + \mathbf{w}_1)\mu(\mathbf{w} + \mathbf{w}_2)$. \square

Lemma 4.3 *Let \mathbf{w} and \mathbf{w}' be two even-signed closed walks in a signed graph, whose first vertex terms are the same. Then (by taking $B_{\mathbf{w}}$ and $B_{\mathbf{w}'}$ properly) $B_{\mathbf{w}+\mathbf{w}'}^+ = B_{\mathbf{w}}^+ B_{\mathbf{w}'}^+$ and $B_{\mathbf{w}+\mathbf{w}'}^- = B_{\mathbf{w}}^- B_{\mathbf{w}'}^-$.*

Proof Let $\mathbf{w} = \mathbf{w}_0 + \cdots + \mathbf{w}_r$ and $\mathbf{w}' = \mathbf{w}'_0 + \cdots + \mathbf{w}'_s$ be balanced section decompositions of \mathbf{w} and \mathbf{w}' for some odd integers r and s . By Lemma 4.1, $\mathbf{w}_r + \mathbf{w}'_0$ is a balanced section of $\mathbf{w} + \mathbf{w}'$ if and only if $\mathbf{w}'_s + \mathbf{w}_0$ is a balanced section of $\mathbf{w} + \mathbf{w}'$. First, suppose that $\mathbf{w}_r + \mathbf{w}'_0$ is not a balanced section of $\mathbf{w} + \mathbf{w}'$. Then, $\mathbf{w}_0 + \cdots + \mathbf{w}_r + \mathbf{w}'_0 + \cdots + \mathbf{w}'_s$ is a balanced section decomposition of $\mathbf{w} + \mathbf{w}'$. By definition, it follows that $B_{\mathbf{w}}^+ B_{\mathbf{w}'}^+ = B_{\mathbf{w}+\mathbf{w}'}^+$ and $B_{\mathbf{w}}^- B_{\mathbf{w}'}^- = B_{\mathbf{w}+\mathbf{w}'}^-$. Suppose that $\mathbf{w}_r + \mathbf{w}'_0$ is a balanced section of $\mathbf{w} + \mathbf{w}'$. Letting $\mathbf{x} = \mathbf{w}'_s + \mathbf{w}_0$ and $\mathbf{y} = \mathbf{w}_r + \mathbf{w}'_0$, we have a balanced section decomposition of $\mathbf{w} + \mathbf{w}'$,

$$\mathbf{x} + \mathbf{w}_1 + \cdots + \mathbf{w}_{r-1} + \mathbf{y} + \mathbf{w}'_1 + \cdots + \mathbf{w}'_{s-1}.$$

Then, we can obtain that $B_{\mathbf{w}}^+ B_{\mathbf{w}'}^+ = B_{\mathbf{w}+\mathbf{w}'}^+$ and $B_{\mathbf{w}}^- B_{\mathbf{w}'}^- = B_{\mathbf{w}+\mathbf{w}'}^-$ by redefining $B_{\mathbf{w}}$ and $B_{\mathbf{w}'}$ properly. \square

Lemma 4.4 *Let \mathbf{w} be a (u, v) -walk in a signed graph without unbalanced vertex term. For two (v, u) -walks \mathbf{w}_1 and \mathbf{w}_2 , if $\mathbf{w} + \mathbf{w}_i$ is even-signed for each $i = 1, 2$, then $B_{\mathbf{w}_1+\mathbf{w}_2^{-1}}$ belongs to the ideal $\langle B_{\mathbf{w}+\mathbf{w}_1}, B_{\mathbf{w}+\mathbf{w}_2} \rangle$.*

Proof For simplicity, let \mathbf{a} , \mathbf{b} , \mathbf{c} , and \mathbf{d} be closed walks such that

$$\mathbf{a} = \mathbf{w} + \mathbf{w}_1, \mathbf{b} = \mathbf{w}_2^{-1} + \mathbf{w}^{-1}, \mathbf{c} = \mathbf{w}_1 + \mathbf{w}_2^{-1}, \mathbf{d} = \mathbf{w}^{-1} + \mathbf{w}.$$

Note that each of the four walks are even-signed (\mathbf{a} and \mathbf{b} are even-signed by the assumptions, \mathbf{c} and \mathbf{d} are even-signed by Lemma 4.2). Moreover, the first vertex terms of \mathbf{a} and \mathbf{b} are the same as u , and the first vertex terms of \mathbf{c} and \mathbf{d} are the same as v .

We consider two closed walks $\mathbf{a} + \mathbf{b}$ and $\mathbf{c} + \mathbf{d}$. Since $\mathbf{a} + \mathbf{b} = \mathbf{w} + \mathbf{w}_1 + (\mathbf{w}_2)^{-1} + \mathbf{w}^{-1}$ and $\mathbf{c} + \mathbf{d} = \mathbf{w}_1 + (\mathbf{w}_2)^{-1} + \mathbf{w}^{-1} + \mathbf{w}$, they are the same walk and so $\pm B_{\mathbf{a}+\mathbf{b}} = B_{\mathbf{c}+\mathbf{d}}$. By Lemma 4.3, by taking binomials associated with four even-signed closed walks \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} properly, we have

$$B_{\mathbf{a}+\mathbf{b}}^+ = B_{\mathbf{a}}^+ B_{\mathbf{b}}^+, \quad B_{\mathbf{a}+\mathbf{b}}^- = B_{\mathbf{a}}^- B_{\mathbf{b}}^-, \quad B_{\mathbf{c}+\mathbf{d}}^+ = B_{\mathbf{c}}^+ B_{\mathbf{d}}^+, \quad B_{\mathbf{c}+\mathbf{d}}^- = B_{\mathbf{c}}^- B_{\mathbf{d}}^-.$$

Note that since \mathbf{w} has no unbalanced vertex term, $B_{\mathbf{d}}^+ = B_{\mathbf{d}}^-$, and we let $X := B_{\mathbf{d}}^+$. Hence,

$$B_{\mathbf{c}} = B_{\mathbf{c}}^+ - B_{\mathbf{c}}^- = \frac{B_{\mathbf{c}+\mathbf{d}}^+}{B_{\mathbf{d}}^+} - \frac{B_{\mathbf{c}+\mathbf{d}}^-}{B_{\mathbf{d}}^-} = \frac{1}{X} (B_{\mathbf{c}+\mathbf{d}}^+ - B_{\mathbf{c}+\mathbf{d}}^-) = \frac{1}{X} B_{\mathbf{c}+\mathbf{d}} = \pm \frac{1}{X} B_{\mathbf{a}+\mathbf{b}} = \pm \frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^+}{X} \mp \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X}.$$

We may assume that $B_{\mathbf{c}} = \frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^+}{X} - \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X}$. (The other case is similar.) Thus, X divides both $B_{\mathbf{a}}^+ B_{\mathbf{b}}^+$ and $B_{\mathbf{a}}^- B_{\mathbf{b}}^-$. Moreover, since \mathbf{w} has no unbalanced vertex term, X divides one of $B_{\mathbf{a}}^+$ and $B_{\mathbf{a}}^-$, and one of $B_{\mathbf{b}}^+$ and $B_{\mathbf{b}}^-$. Thus, X divides either $B_{\mathbf{a}}^+$ and $B_{\mathbf{b}}^-$, or $B_{\mathbf{a}}^-$ and $B_{\mathbf{b}}^+$. If X divides $B_{\mathbf{a}}^+$ and $B_{\mathbf{b}}^-$, then

$$\begin{aligned} B_{\mathbf{c}} &= \frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^+}{X} + \left(-\frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^-}{X} + \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X} \right) - \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X} \\ &= \left(\frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^+}{X} - \frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^-}{X} \right) + \left(\frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^-}{X} - \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X} \right) = \frac{B_{\mathbf{a}}^+}{X} B_{\mathbf{b}} + \frac{B_{\mathbf{b}}^-}{X} B_{\mathbf{a}}. \end{aligned}$$

Similarly, if X divides $B_{\mathbf{a}}^-$ and $B_{\mathbf{b}}^+$, then

$$B_{\mathbf{c}} = \frac{B_{\mathbf{a}}^+ B_{\mathbf{b}}^+}{X} + \left(-\frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^+}{X} + \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X} \right) - \frac{B_{\mathbf{a}}^- B_{\mathbf{b}}^-}{X} = \frac{B_{\mathbf{a}}^-}{X} B_{\mathbf{b}} + \frac{B_{\mathbf{b}}^+}{X} B_{\mathbf{a}}.$$

In any case, $B_{\mathbf{c}}$ belongs to the ideal $\langle B_{\mathbf{a}}, B_{\mathbf{b}} \rangle$, a desired conclusion. \square

The following is from Lemma 4.4 by considering cases where \mathbf{w} is a walk of length one.

Corollary 4.5 *Let \mathbf{w}_1 and \mathbf{w}_2 be two even-signed closed walks in a signed graph, starting with u , e , v for an edge $e = uv$. Then, $B_{\mathbf{w}} \in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_2} \rangle$, where $\mathbf{w}_i = uev + \mathbf{w}'_i$ for $i = 1, 2$ and $\mathbf{w}' = \mathbf{w}'_1 + \mathbf{w}'_2^{-1}$.*

The following lemma may fail if we drop the assumption on oddness of sign of \mathbf{w} or \mathbf{w}' .

Lemma 4.6 *Let \mathbf{w} and \mathbf{w}' be two odd-signed closed walks in a signed graph (G, τ) , whose first vertex terms are the same. Then, $\mathbf{w} + \mathbf{w}'$ and $\mathbf{w}^{-1} + \mathbf{w}'$ are even-signed closed walks in (G, τ) and $B_{\mathbf{w}^{-1}+\mathbf{w}'} = \pm B_{\mathbf{w}+\mathbf{w}'}$.*

Proof Note that $\mathbf{w} + \mathbf{w}'$ and $\mathbf{w}^{-1} + \mathbf{w}'$ are even-signed by Lemma 4.1, since $\mu(\mathbf{w}^{-1}) = \mu(\mathbf{w}) = \mu(\mathbf{w}') = -1$, $\mu(\mathbf{w} + \mathbf{w}') = \mu(\mathbf{w})\mu(\mathbf{w}') = 1$, and $\mu(\mathbf{w}^{-1} + \mathbf{w}') = \mu(\mathbf{w}^{-1})\mu(\mathbf{w}') = 1$. Let $\mathbf{w}_0 + \cdots + \mathbf{w}_{2r}$ be a balanced section decomposition of \mathbf{w} for some nonnegative integer r . We assume that \mathbf{w}_0 contains the first vertex term v . Then, \mathbf{y} is a nontrivial walk and \mathbf{x} may be trivial, and $\mathbf{w} + \mathbf{w}'$ and $\mathbf{w}^{-1} + \mathbf{w}'$ have the following section decompositions (the parts $\mathbf{w}_1 + \cdots + \mathbf{w}_{2r}$ and $\mathbf{w}_{2r}^{-1} + \cdots + \mathbf{w}_1$ are dropped if $r = 0$):

$$\begin{aligned}\mathbf{w} + \mathbf{w}' &= \mathbf{y} + \mathbf{w}_1 + \cdots + \mathbf{w}_{2r} + \mathbf{x} + \mathbf{w}', \\ \mathbf{w}^{-1} + \mathbf{w}' &= \mathbf{x}^{-1} + \mathbf{w}_{2r}^{-1} + \cdots + \mathbf{w}_1^{-1} + \mathbf{y}^{-1} + \mathbf{w}'.\end{aligned}$$

Then, we make a binomial $B_{\mathbf{w}+\mathbf{w}'} = B^+ - B^-$ by putting the edges in $E(\mathbf{y})$ to B^+ , the edges in $E(\mathbf{w}_1)$ to B^- , the edges in $E(\mathbf{w}_2)$ to B^+ , and so on. Let U^+ and U^- be the (multi)set so that

$$B^+ = \prod_{e \in U^+} e \quad \text{and} \quad B^- = \prod_{e \in U^-} e.$$

Then, by a way to make the binomial $B_{\mathbf{w}+\mathbf{w}'}$,

$$U^+ \supset E(\mathbf{y}) \cup \left(\bigcup_{i>0: \text{ even}} E(\mathbf{w}_i) \right), \quad U^- \supset \left(\bigcup_{i: \text{ odd}} E(\mathbf{w}_i) \right) \cup E(\mathbf{x}).$$

Similarly, we also make a binomial $B_{\mathbf{w}^{-1}+\mathbf{w}'} = B'^+ - B'^-$ by putting the edges in $E(\mathbf{x}^{-1})$ to B'^- , the edges in $E(\mathbf{w}_{2r}^{-1})$ to B'^+ , and so on. Let W^+ and W^- be the (multi)set so that

$$B'^+ = \prod_{e \in W^+} e \quad \text{and} \quad B'^- = \prod_{e \in W^-} e.$$

Then, by a way to make the binomial $B_{\mathbf{w}^{-1}+\mathbf{w}'}$,

$$W^+ \supset E(\mathbf{y}) \cup \left(\bigcup_{i>0: \text{ even}} E(\mathbf{w}_i) \right), \quad W^- \supset \left(\bigcup_{i: \text{ odd}} E(\mathbf{w}_i) \right) \cup E(\mathbf{x}).$$

Note that it is sufficient to show that for the first edge term e'_{j_1} of \mathbf{w}' , $e'_{j_1} \in U^+$ if and only if $e'_{j_1} \in W^+$. In the following, let e_{j_1} be the first edge term of \mathbf{w} (i.e., the first edge term of \mathbf{y}), and e_{j_*} be the last edge term of \mathbf{w} . We note $e_{j_1} \in E(\mathbf{y}) \subset W^+$.

(Case 1) Suppose that \mathbf{x} is nontrivial. Then, e_{j_*} is the last edge term of \mathbf{x} and $\mathbf{x} + \mathbf{y}$ is a balanced section of \mathbf{w} . Thus,

$$\tau(e_{j_*}, v) = -\tau(e_{j_1}, v). \quad (4.1)$$

We also note that $e_{j_*} \in E(\mathbf{x}) \subset U^-$. Then, where the second biconditional is from (4.1),

$$e'_{j_1} \in U^+ \Leftrightarrow \tau(e_{j_*}, v) = \tau(e'_{j_1}, v) \Leftrightarrow -\tau(e_{j_1}, v) = \tau(e'_{j_1}, v) \Leftrightarrow e'_{j_1} \in W^+.$$

(Case 2) Suppose that \mathbf{x} is trivial. Then, the first vertex term v of \mathbf{w} is unbalanced, and so

$$\tau(e_{j_*}, v) = \tau(e_{j_1}, v). \quad (4.2)$$

We also note that $e_{j_*} \in E(\mathbf{w}_{2r}) \subset U^+$. Then, where the second biconditional is from (4.2),

$$e'_{j_1} \in U^+ \Leftrightarrow \tau(e_{j_*}, v) = -\tau(e'_{j_1}, v) \Leftrightarrow \tau(e_{j_1}, v) = -\tau(e'_{j_1}, v) \Leftrightarrow e'_{j_1} \in W^+.$$

□

5 Proofs of the main results

5.1 Proof of Theorem 3.1

Proof of Theorem 3.1 The following is directly derived from Lemma 4.3.

Claim 5.1 *For an even-signed closed walk \mathbf{w} in a signed graph, if $B_{\mathbf{w}}$ is primitive, then \mathbf{w} has no proper nontrivial section that is an even-signed closed walk.*

Let $\mathbf{w} : v_{i_1}e_{j_1}v_{i_2} \cdots v_{i_r}e_{j_r}v_{i_1}$ be an even-signed closed walk in a signed graph (G, τ) . First, we show the ‘only if’ part. Suppose that $B_{\mathbf{w}}$ is primitive in $I_{(G, \tau)}$. Note that (ii) holds by Claim 5.1, and so we will show (i).

Claim 5.2 *Let $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ be a nontrivial section decomposition of \mathbf{w} . Then, at least one of $\mathbf{w}_0 + \mathbf{w}_1$ and $\mathbf{w}_1 + \mathbf{w}_2$ is not a closed walk.*

Proof Suppose to contrary that each of $\mathbf{w}_0 + \mathbf{w}_1$ and $\mathbf{w}_1 + \mathbf{w}_2$ is a closed walk. First, we will show that both $\mathbf{w}_3 + \mathbf{w}_0$ and $\mathbf{w}_3 + \mathbf{w}_1^{-1}$ are odd-signed closed walks in (G, τ) . Let v be the first vertex term of \mathbf{w}_0 and u be the first vertex term of \mathbf{w}_1 . Then, \mathbf{w}_3 is a (u, v) -walk, and each of \mathbf{w}_0 and \mathbf{w}_1^{-1} is a (v, u) -walk. Hence, both $\mathbf{w}_3 + \mathbf{w}_0$ and $\mathbf{w}_3 + \mathbf{w}_1^{-1}$ are closed walks. Then, $\mathbf{w}_3 + \mathbf{w}_0$ is clearly a proper closed section of \mathbf{w} , and so it is odd-signed by Claim 5.1. In a closed walk $\mathbf{w}^* = \mathbf{w}_1^{-1} + \mathbf{w}_0^{-1} + \mathbf{w}_2 + \mathbf{w}_3$, note that $\mathbf{w}_3 + \mathbf{w}_1^{-1}$ is a proper closed section. By Lemma 4.6, \mathbf{w}^* is also an even closed

walk such that $B_{\mathbf{w}} = B_{\mathbf{w}^*}$, and so $B_{\mathbf{w}^*}$ is also primitive. It follows from Claim 5.1 that $\mathbf{w}_3 + \mathbf{w}_1^{-1}$ is odd-signed in (G, τ) . Then,

$$\mu(\mathbf{w}_0 + \mathbf{w}_1) = \mu(\mathbf{w}_0 + (\mathbf{w}_1^{-1})^{-1}) = \mu(\mathbf{w}_3 + \mathbf{w}_0)\mu(\mathbf{w}_3 + \mathbf{w}_1^{-1}) = 1,$$

where the second equality is from Lemma 4.2 and the last one is from the fact that both $\mathbf{w}_3 + \mathbf{w}_0$ and $\mathbf{w}_3 + \mathbf{w}_1^{-1}$ are odd-signed. Hence, $\mathbf{w}_0 + \mathbf{w}_1$ is a proper even-signed closed section of \mathbf{w} , a contradiction to Claim 5.1. \square

Claim 5.3 *Let v be a vertex repeated in \mathbf{w} . Then, v is repeated exactly twice in \mathbf{w} and is a cut vertex of $[\mathbf{w}]$ such that $[\mathbf{w}] - v$ has exactly two connected components and each block of $[\mathbf{w}]$ is a cycle of length at least two.*

Proof We may assume that the first vertex term is v . First, we claim that there is no vertex repeated more than twice. Suppose that there are k and ℓ such that $1 < k < \ell \leq r$ and $v = v_{i_k} = v_{i_\ell}$. Let $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1 + \mathbf{w}_2$ where each \mathbf{w}_i is a nontrivial closed walk whose first vertex term is v . Then, $\mu(\mathbf{w}_0)\mu(\mathbf{w}_1)\mu(\mathbf{w}_2) = \mu(\mathbf{w}) = 1$ by Lemma 4.1, which implies that at least one of \mathbf{w}_0 , \mathbf{w}_1 , and \mathbf{w}_2 is an even-signed closed walk in (G, τ) , and say \mathbf{w}_0 . This contradicts to Claim 5.1. Hence, v is repeated twice and so we let $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$ where \mathbf{w}_i is a nontrivial closed walk whose first vertex term is v .

By Claim 5.2, every internal vertex term u of \mathbf{w}_0 is not appeared in \mathbf{w}_1 at all. Hence v is a cut vertex of $[\mathbf{w}]$. Moreover, since v is repeated twice, $[\mathbf{w}] - v$ has exactly two connected components and every block of $[\mathbf{w}]$ containing v contains exactly two edges incident to v . Thus, each block is a cycle. \square

By Claim 5.3, it is clear that every vertex belongs to at most two blocks, which implies (i).

We show the ‘if’ part. Suppose that an even-signed closed walk \mathbf{w} in (G, τ) satisfies (i) and (ii).

Claim 5.4 *For each edge $e \in E(\mathbf{w})$, e divides exactly one of $B_{\mathbf{w}}^+$ and $B_{\mathbf{w}}^-$.*

Proof Suppose to contrary that an edge $e \in E(\mathbf{w})$ divides both $B_{\mathbf{w}}^+$ and $B_{\mathbf{w}}^-$. Then, e is repeated in \mathbf{w} and so e is on a cycle of length two in $[\mathbf{w}]$ by (i). Moreover, \mathbf{w} has at least two balanced sections, and let $\mathbf{w}_0 + \cdots + \mathbf{w}_{2k-1}$ be a balanced section decomposition of \mathbf{w} . Then, we may assume that both \mathbf{w}_0 and \mathbf{w}_{2i-1} contain e for some $i \in [k]$. Then, there are section decompositions $\mathbf{w}_0 = \mathbf{x}_0 + \mathbf{y}_0$ and $\mathbf{w}_{2i-1} = \mathbf{x}_{2i-1} + \mathbf{y}_{2i-1}$ such that the first edge term of \mathbf{y}_0 and the last edge term of \mathbf{x}_{2i-1} are e . Consider the section \mathbf{w}' of \mathbf{w} so that the first and the last edge terms are e . Then, by the structure of $[\mathbf{w}]$ from (i) and (ii), \mathbf{w}' is a closed walk and $\mathbf{w}' = \mathbf{y}_0 + \mathbf{w}_1 + \cdots + \mathbf{w}_{2i-2} + \mathbf{x}_{2i-1}$ is a balanced section decomposition ($\mathbf{w}_1 + \cdots + \mathbf{w}_{2i-2}$ is dropped if $i = 1$). Hence, \mathbf{w}' is an even-signed closed section of \mathbf{w} , a contradiction to (ii). \square

Let $A = A(G, \tau)$. By Claim 5.4, from the same way in (the proof of) Observation 2.11, we can find an integer vector $\mathbb{b} = (b_e)_{e \in E(G)}$ such that $\mathbb{e}^{\mathbb{b}^+} - \mathbb{e}^{\mathbb{b}^-} = B_{\mathbf{w}}$ and $G_{\mathbb{b}} = [\mathbf{w}]$ ($G_{\mathbb{b}}$ is the multigraph in Proposition 2.13). Suppose to contrary that $B_{\mathbf{w}}$ is not primitive. Then, there is a binomial $\mathbb{e}^{\mathbb{c}^+} - \mathbb{e}^{\mathbb{c}^-}$ in $I_{(G, \tau)}$ (for some $\mathbb{c} = (c_e)_{e \in E(G)}$, other than \mathbb{b})

such that $e^{c^+} | e^{b^+}$, $e^{c^-} | e^{b^-}$. It also holds $A_c = \emptyset$. By Proposition 2.13, each connected component of G_c has an even-signed Eulerian. Now consider two multigraphs G_b and G_c . Note that G_c is a proper subgraph of G_b . By the condition (i) on G_b , each block B of G_b is a cycle and so each block of G_c is also a block of G_b . Thus, G_c is made by taking some blocks of G_b .

Let $d = (d_e)_{e \in E(G)}$ be a vector such that $d_e = b_e - c_e$ for every edge e . By definition,

$$e^{d^+} = \frac{e^{b^+}}{e^{c^+}} \quad \text{and} \quad e^{d^-} = \frac{e^{b^-}}{e^{c^-}},$$

and G_d is the graph obtained from G_b by deleting the edges of blocks of G_c . Take a nontrivial connected component D of G_d . Since $A_d = \emptyset$, by Proposition 2.13, D has an even-signed Eulerian w_D . However, w_D is a nontrivial section of w , which is an even-signed closed walk in (G, τ) . This is a contradiction to (ii). \square

5.2 Proof of Theorem 3.9

We often use the fact that a graph in \mathcal{G}^{cis} satisfies all statements in Theorem 2.5, since $\mathcal{G}^{cis} \subset \mathcal{G}^{ci}$.

Proof of Theorem 3.9 We show the ‘only if’ part first. Suppose to contrary that G is a 2-connected graph in \mathcal{G}^{cis} , none of (G1)–(G5) in Theorem 3.9. Since $\mathcal{G}^{cis} \subset \mathcal{G}^{cio}$, by Theorem 2.7(i) and Observation 3.8 it follows that G is constructed by clique sums of cycles. Note that since G is 2-connected, 1-clique sum cannot be done to make G . Thus, G is constructed by 2-clique sums of cycles. By Corollary 3.7, every induced subgraph which is constructed by 2-clique sums of cycles belongs to \mathcal{G}^{cis} . Not to be (G1) or (G2), G is constructed by clique sums of at least three cycles.

Claim 5.5 *For each edge e , there are at most two induced cycles containing e .*

Proof of Claim 5.5 Suppose that there are three induced cycles $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ of G , containing the edge e . Let $H = G[V(C^{(1)}) \cup V(C^{(2)}) \cup V(C^{(3)})]$. Since H is constructed by 2-clique sums of cycles, $H \in \mathcal{G}^{cis}$ by Corollary 3.7. Moreover, all vertices of H except the endpoints of e have degree two in the graph H . By Proposition 3.11(i), by contracting ears of H , we obtain a graph $H^* \in \mathcal{G}^{cis}$, which is a 2-clique sum of three triangles at one edge. But H^* contains $K_{2,3}$, a contradiction to Theorem 2.5(ii).

Claim 5.6 *An induced cycle of G shares an edge with at most two induced cycles.*

Proof For an induced cycle C of G , suppose that there are three induced cycles $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ of G , each of which shares an edge with C . Then, for each $i \in [3]$ there is a unique edge e_i which belongs to both C and $C^{(i)}$. By Claim 5.5, e_1, e_2, e_3 are distinct. Now let $H = G[V(C) \cup V(C^{(1)}) \cup V(C^{(2)}) \cup V(C^{(3)})]$, and then $H \in \mathcal{G}^{cis}$ by Corollary 3.7. In addition, all vertices of H , except the endpoints of e_i ’s, have degree two in the graph H . By Proposition 3.11(i), by contracting ears of H , we obtain a graph

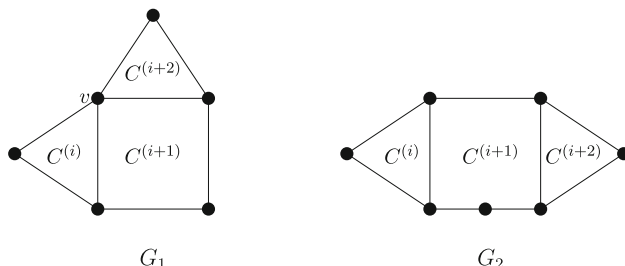


Fig. 11 Some graphs not in \mathcal{G}^{cis}

$H^* \in \mathcal{G}^{cis}$. Note that H^* is 2-connected with three triangles. By Theorem 2.5(iii) and (iv), it follows that H^* must be the graph G_0 in Fig. 9. By Example 3.12, $G_0 \notin \mathcal{G}^{ci}$ and so $G_0 \notin \mathcal{G}^{cis}$, a contradiction. \square

By Claims 5.5 and 5.6, there are induced cycles $C^{(1)}, C^{(2)}, \dots, C^{(m)}$ (for some $m \geq 3$) such that G is constructed by 2-clique sums of those m cycles, where for each $i \in [m-1]$, $C^{(i)}$ and $C^{(i+1)}$ share an edge e_i . Note that e_1, \dots, e_{m-1} are distinct. For each $i \in [m-2]$ and $\ell \in \{2, \dots, m-i+1\}$, let $H_\ell^{(i)} = G[V(C^{(i)}) \cup V(C^{(i+1)}) \cup \dots \cup V(C^{(i+\ell-1)})]$. Note that $H_\ell^{(i)} \in \mathcal{G}^{cis}$ by Corollary 3.7. In addition, Proposition 3.11(i) says that by contracting two ears of $H_\ell^{(i)}$ lying on the cycles $C^{(i)}$ and $C^{(i+\ell-1)}$, we obtain a graph $F_\ell^{(i)}$ in \mathcal{G}^{cis} . In $F_\ell^{(i)}$, the cycles corresponding to $C^{(i)}$ and $C^{(i+\ell-1)}$ are triangles.

Claim 5.7 *Let $i \in [m-2]$. Then, $C^{(i+1)}$ has length at most four. Moreover, if $C^{(i)}, C^{(i+1)}$, and $C^{(i+2)}$ share a vertex v , then $C^{(i+1)}$ is a triangle and there is no more induced cycle containing v .*

Proof We firstly show the ‘moreover’ part. Suppose that $C^{(i)}, C^{(i+1)}$, and $C^{(i+2)}$ share a vertex v . If $C^{(i+1)}$ is not a triangle, then by contracting ears of $F_3^{(i)}$ properly, we can obtain G_1 in Fig. 11, and note that $G_1 \notin \mathcal{G}^{ci}$ by Theorem 2.5(iii). Hence, $C^{(i+1)}$ is a triangle.

Suppose that there is another induced cycle $C^{(j)}$ containing the vertex v . Since G is 2-connected, we may assume that $j = i+3$. By the above argument, both $C^{(i+1)}$ and $C^{(i+2)}$ are triangles. Then, $F_4^{(i)}$ has four triangles and so $|V(F_4^{(i)})| = 6$ and $|E(F_4^{(i)})| = 9$. By deleting the vertex v , it becomes a bipartite graph, and so $\sum_x b(F_4^{(i)} - x) \geq 1$. Applying Theorem 2.5(i), we have $2|E(F_4^{(i)})| < 3|V(F_4^{(i)})|$, a contradiction.

It remains to show that $C^{(i+1)}$ has length at most 4. Suppose that $C^{(i+1)}$ has length at least 5. Then, by the previous argument, $C^{(i)}, C^{(i+1)}$, and $C^{(i+2)}$ do not share one common vertex. By contracting ears of $F_3^{(i)}$ properly, we obtain G_2 in Fig. 11. By Theorem 2.5(iv), $G_2 \notin \mathcal{G}^{ci}$, a contradiction. \square

Suppose that $C^{(2)}$ is a cycle of at least length 4. By Claim 5.7, $C^{(2)}$ has length 4. By the moreover part of Claim 5.7, $C^{(1)}$ and $C^{(3)}$ do not share a vertex. Not to be

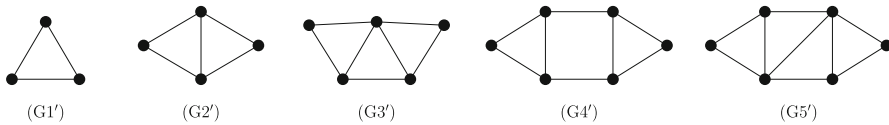


Fig. 12 Graphs $(G1')$ – $(G5')$

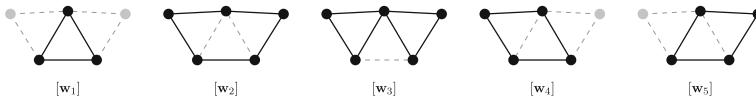


Fig. 13 All possible primitive walks in (G, τ) when G is $(G3')$

$(G4)$, $m \geq 4$, and so consider $F_4^{(1)}$. Then, the triangles corresponding to $C^{(1)}$ and $C^{(4)}$ in $F_4^{(1)}$ are disjoint by Claim 5.7. Then, there are no two disjoint edges connecting those two triangles, which is a contradiction to Theorem 2.5(iv). Suppose that $C^{(2)}$ is a triangle. By Claim 5.7, not to be $(G5)$, $m \geq 5$. Consider $F_5^{(1)}$ and then there are no two disjoint edges connecting two triangles corresponding to $C^{(1)}$ and $C^{(5)}$ in $F_5^{(1)}$, a contradiction to Theorem 2.5(iv).

Now we prove the ‘if’ part of Theorem 3.9. By Proposition 3.11(ii), if (Gi') is in \mathcal{G}^{cis} then (Gi) is in \mathcal{G}^{cis} , where $(G1')$ – $(G5')$ are shown in Fig. 12. Let G be one of $(G1')$ – $(G5')$, and τ be its sign.

$(G1')$ and $(G2')$ It is trivial that the graph $(G1')$ is in \mathcal{G}^{cis} , since either $I_{(G,\tau)} = \{0\}$ and the triangle \mathbf{w} is odd-signed in (G, τ) , or $I_{(G,\tau)} = \langle B_{\mathbf{w}} \rangle$ and \mathbf{w} is even-signed in (G, τ) . Suppose that G is $(G2')$. Let \mathbf{w} and \mathbf{w}' be two triangles of G . If one of \mathbf{w} and \mathbf{w}' is even-signed in (G, τ) , then $I_{(G,\tau)}$ is a complete intersection by Proposition 3.6. Suppose that \mathbf{w} and \mathbf{w}' are odd-signed in (G, τ) . Then, (G, τ) has only one primitive walk, by Theorem 3.1, which is the cycle \mathbf{w}'' of length four. Thus, $I_{(G,\tau)} = \langle B_{\mathbf{w}''} \rangle$, and so $I_{(G,\tau)}$ is a complete intersection.

$(G3')$ Suppose that G is $(G3')$. If one of triangles having a vertex of degree two is even-signed in (G, τ) , then $I_{(G,\tau)}$ is a complete intersection by Proposition 3.6, since we already show that the graph in $(G2')$ is in \mathcal{G}^{cis} . Hence, suppose that both triangles having a vertex of degree two are odd-signed in (G, τ) . We will find $2(= r(G, \tau))$ binomials which generate $I_{(G,\tau)}$.

By Theorem 3.1, the set of primitive walks is a subset of $\{\mathbf{w}_1, \dots, \mathbf{w}_5\}$, where $\mathbf{w}_1, \dots, \mathbf{w}_5$ are the closed walks in Fig. 13. If the triangle \mathbf{w}_1 is even-signed in (G, τ) , then G has only three primitive walks $\mathbf{w}_1, \mathbf{w}_2$ and \mathbf{w}_3 , and $B_{\mathbf{w}_3} \in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_2} \rangle$ by Corollary 4.5. If \mathbf{w}_1 is odd-signed in (G, τ) , then G has only three primitive walks $\mathbf{w}_3, \mathbf{w}_4$ and \mathbf{w}_5 , and $B_{\mathbf{w}_3} \in \langle B_{\mathbf{w}_4}, B_{\mathbf{w}_5} \rangle$ by Corollary 4.5.

$(G4')$ Suppose that G is $(G4')$. Similar to previous case, by Proposition 3.6, we may assume that two triangles are odd-signed in (G, τ) . Then, we will find $2(= r(G, \tau))$ binomials which generate $I_{(G,\tau)}$. Note that the six walks defined as Fig. 14 are all possible primitive walks.

If \mathbf{w}_1 is even-signed in (G, τ) , then (G, τ) has only four primitive walks $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and \mathbf{w}_4 , and $B_{\mathbf{w}_3}, B_{\mathbf{w}_4} \in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_2} \rangle$ by Corollary 4.5. If \mathbf{w}_1 is odd-signed in

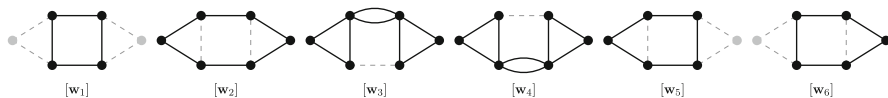


Fig. 14 All possible primitive walks in (G, τ) when G is $(G4')$

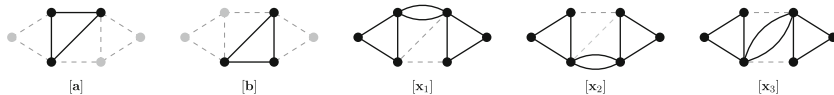


Fig. 15 Five closed walks in G when G is $(G5')$, where $x_1 \sim x_3$ are even-signed in (G, τ)

$\mu(a), \mu(b)$	Primitive walks $w_1 \sim w_6$
$\mu(a) = \mu(b) = 1$	
$\mu(a) = -1, \mu(b) = 1$	
$\mu(a) = \mu(b) = -1$	

Fig. 16 Six primitive walks when G is $(G5')$

(G, τ) , then (G, τ) has only four primitive walks w_3, w_4, w_5 and w_6 , and $B_{w_3}, B_{w_4} \in \langle B_{w_5}, B_{w_6} \rangle$ by Corollary 4.5.

(G5') Suppose that G is $(G5')$. Similar to previous case, by Proposition 3.6, we may assume that two triangles having a vertex of degree two are odd-signed in (G, τ) . We will find $3(= r(G, \tau))$ binomials which generate $I_{(G, \tau)}$. Consider closed walks a, b, x_1, x_2 , and x_3 , defined as Fig. 15.

We consider cases according to $\mu(a)$ and $\mu(b)$, and then, in each case we will define six walks $w_1 \sim w_6$ as Fig. 16.⁴ Then, (G, τ) has only 9 primitive walks, $w_1, \dots, w_6, x_1, x_2$, and x_3 , and then $B_{w_1}, B_{w_2}, B_{w_3}$ generate $I_{(G, \tau)}$, since it follows from Corollary 4.5 that

$$B_{w_4} \in \langle B_{w_1}, B_{w_2} \rangle, \quad B_{w_5} \in \langle B_{w_1}, B_{w_3} \rangle, \quad B_{w_6} \in \langle B_{w_2}, B_{w_3} \rangle, \\ B_{x_1}, B_{x_2} \in \langle B_{w_3}, B_{w_4} \rangle, \quad B_{x_3} \in \langle B_{w_1}, B_{w_6} \rangle.$$

□

⁴ The case where $\mu(a) = 1$ and $\mu(b) = -1$ is similar to the second case of Fig. 16.

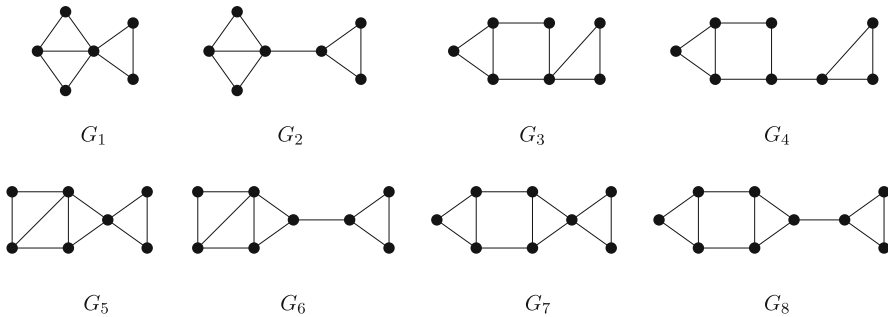


Fig. 17 Eight graphs $G_1 \sim G_8$ whose toric ideals are not complete intersections

5.3 Proof of Theorem 3.10

First, we note that each of $G_1 \sim G_8$ in Fig. 17 does not belong to \mathcal{G}^{ci} by Algorithm 2.4.⁵ Therefore, those eight graphs are not in \mathcal{G}^{cis} .

Proof of Theorem 3.10 It is sufficient to consider only connected graphs. We first show the ‘only if’ part. Suppose that G is a connected graph in \mathcal{G}^{cis} . For a nonedge block F of G , F is one of (G1)–(G5) by Proposition 3.5 and Theorem 3.9. If G has at most one nonedge block, then (i) or (ii) holds. Now suppose that G has at least two nonedge blocks.

Claim 5.8 Suppose that F_1 and F_2 are two nonedge blocks of G such that F_1 is not (G1). Let \mathbf{p} be a shortest path from a vertex of F_1 to a vertex of F_2 , and $C^{(i)}$ an induced cycle of F_i having a vertex of \mathbf{p} . Then, F_1 is (G2), the cycle $C^{(1)}$ is a triangle, and $\deg_{F_1}(v_1) = 2$, where $v_1 = V(F_1) \cap V(\mathbf{p})$.

Proof Since F_1 is not (G1), we can take another induced cycle $C^{(0)}$ of F_1 which shares an edge with $C^{(1)}$. Let $H = G[V(C^{(0)}) \cup V(C^{(1)}) \cup V(C^{(2)}) \cup V(\mathbf{p})]$. Note that H is in \mathcal{G}^{cis} by Corollary 3.7. Using Proposition 3.11(i), by contracting ears, we obtain a graph H^* in \mathcal{G}^{cis} . Let $C_*^{(i)}$ and \mathbf{p}^* be the cycle and the path of H^* corresponding to $C^{(i)}$ and \mathbf{p} , respectively. Note that \mathbf{p}^* is a path of length at most one. Since the graphs G_1 and G_2 in Fig. 17 are not in \mathcal{G}^{cis} , it follows that v_1 has degree two in $H^*[V(C_*^{(0)}) \cup V(C_*^{(1)})]$.

From the fact that the graphs G_3 and G_4 in Fig. 17 are not in \mathcal{G}^{cis} , together with Proposition 3.11(i), the cycle $C^{(1)}$ must be a triangle. Similarly, from the fact that the

⁵ For the graph G_3 , [2, Example 4.10], it was shown that the toric ideal I_G is not a complete intersection by using the algorithm. Fix G_i for some in $i \in \{1, 2, 4, 5, 6, 7, 8\}$. Since G_i is not bipartite, we need at least $|E(G_i)| - |V(G_i)|$ nontrivial binomials to generate the ideal I_{G_i} . Let v be a vertex of degree two on the block isomorphic to K_3 . Note that $b(G_i - v) = b(G_i) = 0$, and let us apply Algorithm 2.4 starting from the vertex v . If $i = 4$, then there is no such closed walk \mathbf{w} of even length with $V(\mathbf{w}) = W$ and so the algorithm returns FALSE. Otherwise, the binomial associated with any shortest closed walk \mathbf{w} of even length with $V(\mathbf{w}) = W$ is trivial. In the remaining process, we consider the graph $G'_i = G_i - v$, which is not bipartite. Since $|E(G'_i)| - |V(G'_i)| = |E(G_i)| - |V(G_i)| - 1$, at most $|E(G_i)| - |V(G_i)| - 1$ nontrivial binomials are obtained through the algorithm, and those cannot generate I_{G_i} . Consequently, the algorithm returns FALSE.

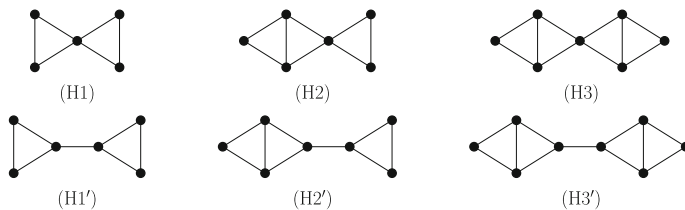


Fig. 18 Graphs (H1)–(H3) and (H1')–(H3')

graphs G_5 , G_6 , G_7 , and G_8 in Fig. 17 are not in \mathcal{G}^{cis} , it follows that F_1 must be (G_2) , and therefore, $\deg_{F_1}(v_1) = 2$. \square

By Claim 5.8, it is sufficient to show that G has at most two nonedge blocks. Suppose to contrary that G has three nonedge blocks F_1 , F_2 , F_3 . Without loss of generality, we may assume that the distance between F_1 and F_2 is maximum among the distances between two of F_1 , F_2 and F_3 . Let \mathbf{p} be a shortest path between F_1 and F_2 . For each $i \in \{1, 2\}$, we take an induced cycle $C^{(i)}$ of F_i having a vertex of \mathbf{p} . Let H be a smallest induced connected subgraph of G containing $V(C^{(1)}) \cup V(C^{(2)}) \cup V(\mathbf{p}) \cup V(F_3)$. Note that $H \in \mathcal{G}^{cis}$ by Corollary 3.7. In addition, there is a vertex w_i such that $\deg_H(w_i) = 2$ and $w_i \in C^{(i)}$ for each $i = 1, 2$. By Proposition 3.11(ii), by subdividing edges incident to w_1 and w_2 properly so that $C^{(1)}$ and $C^{(2)}$ become cycles of odd length, we can obtain a new graph H' in \mathcal{G}^{cis} with two nonbipartite blocks. If F_3 is not bipartite, then H' has three nonbipartite blocks, and so $H' \notin \mathcal{G}^{ci}$ by Theorem 2.5(v), a contradiction. Thus, F_3 is bipartite. Then, we can find a vertex $w_3 \in F_3$ with $\deg_{H'}(w_3) = 2$ and let H'' be the graph obtained from H' by subdividing an edge incident to w_3 once. Then, H'' has three nonbipartite blocks and so $H'' \notin \mathcal{G}^{ci}$ by Theorem 2.5(v). On the other hand, $H' \in \mathcal{G}^{cis}$ by Proposition 3.11(ii), and we reach a contradiction.

Now we show the ‘only if’ part. Note that for every sign τ of G , any primitive walk in (G, τ) does not contain a pendent edge e by Theorem 3.1. Thus, each graph satisfying (i) or (ii) is in \mathcal{G}^{cis} by Theorem 3.9. Consider graphs satisfying (iii). Together with Proposition 3.11(ii), it is sufficient to show that each of the six graphs (H1)–(H3) and (H1')–(H3') in Fig. 18 is in \mathcal{G}^{cis} . We consider (H1)–(H3) first. Let G be one of (H1)–(H3), and τ be its sign.

(H1) Suppose that G is (H1). Let \mathbf{w} and \mathbf{w}' be two triangles in G . By Proposition 3.6, it is sufficient to suppose that each of \mathbf{w} and \mathbf{w}' is odd-signed in (G, τ) . Then, there is only one primitive walk $\mathbf{w} + \mathbf{w}'$ and $r(G, \tau) = 1$, which implies that (G, τ) is a complete intersection.

(H2) Suppose that G is (H2). Let \mathbf{a} be the triangle without a vertex of degree two. Since we already show that (G_2') in Fig. 12 and (H1) are in \mathcal{G}^{cis} , by Proposition 3.6, we may assume that the two triangles other than \mathbf{a} are odd-signed in (G, τ) . We will find $2(= r(G, \tau))$ binomials which generate $I_{(G, \tau)}$. By Theorem 3.1, the six walks defined as Fig. 19 are all possible primitive walks.

If \mathbf{a} is even-signed in (G, τ) , then (G, τ) has only four primitive walks \mathbf{w}_1 , \mathbf{w}_2 , \mathbf{w}_3 , and \mathbf{w}_4 , and $B_{\mathbf{w}_3}, B_{\mathbf{w}_4} \in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_2} \rangle$ by Corollary 4.5. If \mathbf{a} is odd-signed in (G, τ) , then (G, τ) has only four primitive walks \mathbf{w}_3 , \mathbf{w}_4 , \mathbf{w}_5 , and \mathbf{w}_6 , and $B_{\mathbf{w}_3}, B_{\mathbf{w}_4} \in \langle B_{\mathbf{w}_5}, B_{\mathbf{w}_6} \rangle$ by Corollary 4.5.

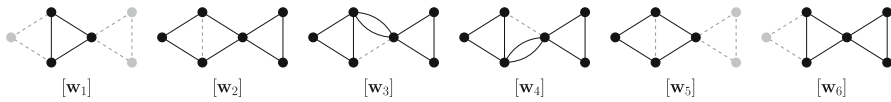


Fig. 19 All possible primitive walks in (G, τ) when G is (H2)

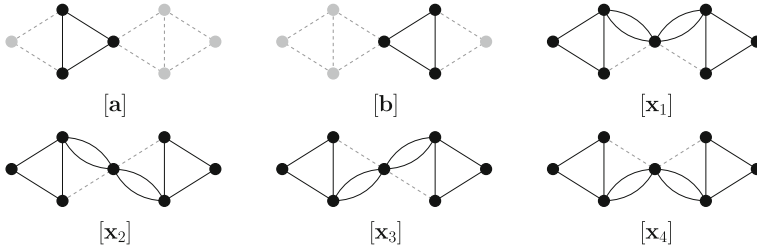


Fig. 20 Six closed walks in G when G is (H3), where $x_1 \sim x_4$ are even-signed in (G, τ)

(H3) Suppose that G is (H3). Since we already show that (H2) is in \mathcal{G}^{cis} , by Proposition 3.6, we may assume that both triangles having a vertex of degree two are odd-signed in (G, τ) . Then, we will find $3(= r(G, \tau))$ binomials which generate $I_{(G, \tau)}$. Consider six walks $\mathbf{a}, \mathbf{b}, \mathbf{x}_1 \sim \mathbf{x}_4$ defined as Fig. 20.

Now we divide cases according to $\mu(\mathbf{a})$ and $\mu(\mathbf{b})$. In each case, we define eight walks $\mathbf{w}_1 \sim \mathbf{w}_8$ as Fig. 21.⁶ Then, (G, τ) has only 12 primitive walks, $\mathbf{w}_1, \dots, \mathbf{w}_8, \mathbf{x}_1, \dots, \mathbf{x}_4$, and then $B_{\mathbf{w}_1}, B_{\mathbf{w}_2}, B_{\mathbf{w}_3}$ generate $I_{(G, \tau)}$, since it follows from Corollary 4.5 that

$$\begin{aligned} B_{\mathbf{w}_4} &\in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_2} \rangle, & B_{\mathbf{w}_5}, B_{\mathbf{w}_6} &\in \langle B_{\mathbf{w}_1}, B_{\mathbf{w}_3} \rangle, & B_{\mathbf{w}_7}, B_{\mathbf{w}_8} &\in \langle B_{\mathbf{w}_2}, B_{\mathbf{w}_3} \rangle, \\ B_{\mathbf{x}_1}, B_{\mathbf{x}_2} &\in \langle B_{\mathbf{w}_2}, B_{\mathbf{w}_6} \rangle, & B_{\mathbf{x}_3}, B_{\mathbf{x}_4} &\in \langle B_{\mathbf{w}_2}, B_{\mathbf{w}_5} \rangle. \end{aligned}$$

Let (G, τ) and (G, τ') be such that G is (Hi), G' is (Hi') ($i \in \{1, 2, 3\}$), and the sign coincides on a cycle. Then, $r(G, \tau) = r(G', \tau')$ and the primitive walks are also corresponding. Similar to the argument of (Hi), we can conclude that (Hi') is in \mathcal{G}^{cis} . \square

⁶ The case where $\mu(\mathbf{a}) = -1$ and $\mu(\mathbf{b}) = 1$ is similar to the second case of Fig. 21.

$\mu(\mathbf{a}), \mu(\mathbf{b})$	Primitive walks $\mathbf{w}_1 \sim \mathbf{w}_8$							
$\mu(\mathbf{a}) = \mu(\mathbf{b}) = 1$								
$\mu(\mathbf{a}) = 1, \mu(\mathbf{b}) = -1$								
$\mu(\mathbf{a}) = \mu(\mathbf{b}) = -1$								

Fig. 21 Eight primitive walks when G is (H3)

Acknowledgements The authors would like to thank the anonymous reviewers for their insightful comments and suggestions. JiSun Huh was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (NRF-2020R1C1C1A01008524). Sangwook Kim was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2017R1D1A3B03031839). Boram Park was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (NRF-2018R1C1B6003577).

Appendix

Proof of Propositions 2.13

Proof of Proposition 2.13 Recall the definition of a signed multigraph $(G_{\mathbf{b}}, \tau_{\mathbf{b}})$, that is, $G_{\mathbf{b}}$ is induced by $|b_e|$ copies of e for every edge e , and $\tau_{\mathbf{b}}(e, v) = \tau(e', v)$ if e is a copy of $e' \in E(G)$. For simplicity, we let $H = G_{\mathbf{b}}$.

Let $E_H^{++}(v)$ (resp. $E_H^{+-}(v)$) be the (multi)set of edges e of H incident to v with $b_e > 0$ and $\tau_{\mathbf{b}}(e, v) = 1$ (resp. $\tau_{\mathbf{b}}(e, v) = -1$). Similarly, let $E_H^{-+}(v)$ (resp. $E_H^{--}(v)$) be the (multi)set of edges e of H incident to v if $b_e < 0$ and $\tau_{\mathbf{b}}(e, v) = 1$ (resp. $\tau_{\mathbf{b}}(e, v) = -1$). Note that $\deg_H(v) = |E_H^{++}(v)| + |E_H^{+-}(v)| + |E_H^{-+}(v)| + |E_H^{--}(v)|$, where the size $|M|$ of a multiset M counts multiplicity.

Claim A.1 For every vertex v of G ,

$$|E_H^{++}(v)| + |E_H^{--}(v)| = |E_H^{+-}(v)| + |E_H^{-+}(v)|.$$

Proof of Claim A.1 Let v be a vertex in H . The entry of $\mathbb{A}\mathbb{b}$ corresponding to v is $\sum_{e \in E_G(v)} b_e \tau(e, v) = 0$, where $E_G(v)$ denotes the set of all edges of G incident to v . Note that

$$\sum_{e \in E_G(v)} b_e \tau(e, v) = 0 \Leftrightarrow \sum_{\substack{e \in E_G(v) \\ \tau(e, v) = 1}} b_e = \sum_{\substack{e \in E_G(v) \\ \tau(e, v) = -1}} b_e,$$

and by definition,

$$\sum_{\substack{e \in E_G(v) \\ \tau(e, v) = 1}} b_e = |E_H^{++}(v)| - |E_H^{-+}(v)|, \quad \sum_{\substack{e \in E_G(v) \\ \tau(e, v) = -1}} b_e = |E_H^{+-}(v)| - |E_H^{--}(v)|.$$

Thus, $|E_H^{++}(v)| - |E_H^{-+}(v)| = |E_H^{+-}(v)| - |E_H^{--}(v)|$, and so the claim holds. \square

For simplicity, let

$$E_H^+(v) = E_H^{++}(v) \cup E_H^{--}(v), \quad E_H^-(v) = E_H^{+-}(v) \cup E_H^{-+}(v).$$

Take any edge $e_{j_1} \in E(H)$ from a nontrivial connected component D of H , say v_{i_1} and v_{i_2} are the endpoints, and let \mathbf{w}_1 be the walk $v_{i_1} e_{j_1} v_{i_2}$. Without loss of generality, we may assume $e_{j_1} \in E_H^+(v_{i_2})$ (other cases are similar). By Claim A.1, we can take an edge $e_{j_2} \in E_H^-(v_{i_2})$, say the endpoint of e_{j_2} other than v_{i_2} is v_{i_3} , and then we have a walk $\mathbf{w}_2 : v_{i_1} e_{j_1} v_{i_2} e_{j_2} v_{i_3}$ so that two edge terms incident to v_{i_2} belong to $E_H^+(v_{i_2})$ and $E_H^-(v_{i_2})$, respectively. We choose a walk repeatedly by a same way. To be precise, suppose that a walk $\mathbf{w}_\ell : v_{i_1} e_{j_1} v_{i_2} \cdots v_{i_\ell} e_{j_\ell} v_{i_{\ell+1}}$ is selected. Then, repeat the following process (§) until no more edge can be selected.

- (§) If $e_{j_\ell} \in E_H^+(v_{i_{\ell+1}})$, then we choose an edge $e_{j_{\ell+1}} \in E_H^-(v_{i_{\ell+1}}) - \{e_{j_1}, \dots, e_{j_\ell}\}$ (as long as it is not empty), and if $e_{j_\ell} \in E_H^-(v_{i_{\ell+1}})$, then we choose an edge $e_{j_{\ell+1}} \in E_H^+(v_{i_{\ell+1}}) - \{e_{j_1}, \dots, e_{j_\ell}\}$ (as long as it is not empty), and then let $v_{i_{\ell+2}}$ be the other endpoint of $e_{j_{\ell+1}}$ and set $\mathbf{w}_{\ell+1} : v_{i_1} e_{j_1} v_{i_2} \cdots v_{i_\ell} e_{j_\ell} v_{i_{\ell+1}} e_{j_{\ell+1}} v_{i_{\ell+2}}$.

Let \mathbf{w}_D be the walk lastly obtained. Since every edge of D is selected at most once at each step in the process, the length of \mathbf{w}_D is bounded by the number of edges in D . We choose such \mathbf{w}_D as long as possible (maximizing its length).

Claim A.2 For each connected component D of H , the walk \mathbf{w}_D is an Eulerian of D and it is an even-signed closed walk in (G, τ) .

Proof For simplicity, we denote \mathbf{w}_D by \mathbf{w} , and let $\mathbf{w} : v_{i_1} e_{j_1} \cdots e_{j_i} v_{i_{i+1}}$. Without loss of generality, we assume that $e_{j_i} \in E_H^+(v_{i_{i+1}})$. First, we show that \mathbf{w} is closed. Suppose that $v_{i_1} \neq v_{i_{i+1}}$. Let I be the set of indices ℓ for the vertices v_{i_ℓ} of \mathbf{w} such that

$v_{i_\ell} = v_{i_{t+1}}$. Clearly, $t + 1 \in I$ and $1 \notin I$. By the way of choosing the walk, for each $\ell \in I \setminus \{t + 1\}$, one of $e_{j_{\ell-1}}$ and e_{j_ℓ} counts 1 of one of the sets $E_H^+(v_{i_\ell})$ and $E_H^-(v_{i_\ell})$, and the other edge counts 1 of the other set. Hence,

$$|E_H^+(v_{i_{t+1}}) \cap \{e_{j_1}, \dots, e_{j_{t-1}}\}| = |E_H^-(v_{i_{t+1}}) \cap \{e_{j_1}, \dots, e_{j_{t-1}}\}|.$$

Since $e_{j_t} \in E_H^+(v_{i_{t+1}})$, by Claim A.1, it follows that $E_H^-(v_{i_{t+1}}) - \{e_{j_1}, \dots, e_{j_t}\} \neq \emptyset$. Thus, we reach a contradiction that \mathbf{w}_t is a longest one. Thus, $v_{i_1} = v_{i_{t+1}}$, which means \mathbf{w} is a closed walk.

Suppose that there is an edge of D not covered by the closed walk \mathbf{w} . Let D' be the graph obtained from D by deleting the edges of $[\mathbf{w}]$. Then, by the choice of \mathbf{w} , it follows that

$$\forall v \in V(D), \quad |E_D^+(v)| = |E_D^-(v)|.$$

By taking a nontrivial connected component of D' , we can proceed the same argument in (§) to obtain a closed walk \mathbf{w}' . Since both \mathbf{w} and \mathbf{w}' are closed, we may assume that both walks start at the vertex v_{i_1} . Then, the closed walk $\mathbf{w} + \mathbf{w}'$ is a longer closed walk which can be obtained from the procedure (§), a contradiction. Hence, \mathbf{w} is an Eulerian of D .

It remains to show that \mathbf{w} is an even-signed closed walk in (G, τ) . From the definition, it is clear that \mathbf{w} is a closed walk in G , and so it is sufficient to show that \mathbf{w} has an even number of unbalanced vertex terms. Note that if the ℓ th vertex term v_{i_ℓ} is unbalanced, then for the three consecutive terms $e_{j_{\ell-1}}v_{i_\ell}e_{j_\ell}$ of \mathbf{w} , one of the four holds: (1) $e_{j_{\ell-1}} \in E_H^{++}(v_{i_\ell})$ and $e_{j_\ell} \in E_H^{+-}(v_{i_\ell})$; (2) $e_{j_\ell} \in E_H^{++}(v_{i_\ell})$ and $e_{j_{\ell-1}} \in E_H^{+-}(v_{i_\ell})$; (3) $e_{j_{\ell-1}} \in E_H^{-+}(v_{i_\ell})$ and $e_{j_\ell} \in E_H^{--}(v_{i_\ell})$; (4) $e_{j_\ell} \in E_H^{-+}(v_{i_\ell})$ and $e_{j_{\ell-1}} \in E_H^{--}(v_{i_\ell})$. Then, the number of unbalanced vertex terms of \mathbf{w} is

$$\sum_{v \in V(D)} (|E_H^{++}(v)| - |E_H^{+-}(v)| + |E_H^{-+}(v)| - |E_H^{--}(v)|) = \sum_{v \in V(D)} 2|E_H^{++}(v)| - |E_H^{+-}(v)|,$$

where the equality is from Claim A.1. Hence, \mathbf{w} is even-signed. \square

By Claim A.2, we consider a walk $\mathbf{w}_D : v_{i_1}e_{j_1}v_{i_2} \cdots v_{i_t}e_{j_t}v_{i_{t+1}}$ and its associated binomial $B_{\mathbf{w}_D}$ for a fixed connected component D of H . To complete the proof, it is sufficient to show that

$$\mathfrak{e}^{\mathbb{b}^+} = \prod_{\substack{D: \text{connected} \\ \text{component of } H}} B_{\mathbf{w}_D}^+, \quad \text{and} \quad \mathfrak{e}^{\mathbb{b}^-} = \prod_{\substack{D: \text{connected} \\ \text{component of } H}} B_{\mathbf{w}_D}^-.$$

Claim A.3 For any $\ell \in [t]$, $b_{e_{j_{\ell-1}}}b_{e_{j_\ell}} < 0$ if and only if the ℓ th vertex term v_{i_ℓ} is unbalanced.

Proof of Claim A.3 Without loss of generality, we may assume that $b_{e_{j_\ell}} > 0$. First, we suppose that $b_{e_{j_{\ell-1}}} > 0$. Then, $e_{j_{\ell-1}} \in E_H^{+-}(v_{i_\ell}) \cup E_H^{++}(v_{i_\ell})$. More precisely, if $e_{j_{\ell-1}} \in E_H^{+-}(v_{i_\ell})$ then $e_{j_\ell} \in E_H^{++}(v_{i_\ell})$, and if $e_{j_{\ell-1}} \in E_H^{++}(v_{i_\ell})$ then $e_{j_\ell} \in E_H^{+-}(v_{i_\ell})$. Thus,

$\tau(e_{j_{\ell-1}}, v_{i_\ell})\tau(e_{j_\ell}, v_{i_\ell}) = -1$, which implies that the vertex term v_{i_ℓ} is not unbalanced. On the other hand, if we suppose that $b_{e_{j_{\ell-1}}} < 0$, then $e_{j_{\ell-1}} \in E_H^{++}(v_{i_\ell}) \cup E_H^{--}(v_{i_\ell})$. More precisely, if $e_{j_{\ell-1}} \in E_H^{++}(v_{i_\ell})$ then $e_{j_\ell} \in E_H^{++}(v_{i_\ell})$, and if $e_{j_{\ell-1}} \in E_H^{--}(v_{i_\ell})$ then $e_{j_\ell} \in E_H^{--}(v_{i_\ell})$. Thus, $\tau(e_{j_{\ell-1}}, v_{i_\ell})\tau(e_{j_\ell}, v_{i_\ell}) = 1$, which implies that v_{i_ℓ} is unbalanced.

Take any edge e of H . We assume that $b_e > 0$, and the other case is similar. Let D be the connected component of H containing the edge e . We will show that the power of e in $B_{\mathbf{w}_D}^+$ is equal to b_e . If there is no unbalanced vertex term in \mathbf{w}_D , then $B_{\mathbf{w}_D} = B_{\mathbf{w}_D}^+ - 1$ and so clearly it holds. Suppose that \mathbf{w}_D has an unbalanced vertex term. Then, let $\mathbf{w}_D = \mathbf{w}_0 + \cdots + \mathbf{w}_{2k-1}$ be a balanced section decomposition of \mathbf{w}_D so that the first edge term e_{j_1} is in \mathbf{w}_0 if $b_{e_{j_1}} > 0$ and e_{j_1} is in \mathbf{w}_1 if $b_{e_{j_1}} < 0$. By Claim A.3, for every edge term e_{j_ℓ} of \mathbf{w}_D , e_{j_ℓ} is in some $(2s)$ th section if $b_{e_{j_\ell}} > 0$, and e_{j_ℓ} is in some $(2s+1)$ th section if $b_{e_{j_\ell}} < 0$, which completes the proof. \square

Proof of Proposition 3.3

Lemma A.4 *For an odd-signed closed walk \mathbf{w} , $[\mathbf{w}]$ contains a cycle of G , which is odd-signed in (G, τ) .*

Proof We show it by the induction on the length of the walk. If it has length at most three, then it is trivial. Suppose that the lemma holds for any odd-signed walk of length less than ℓ ($\ell > 3$). Let \mathbf{w} be an odd-signed walk of length $(\ell + 1)$. If there is no repeated vertex in \mathbf{w} , then $[\mathbf{w}]$ is a cycle. Suppose that there is a repeated vertex in \mathbf{w} . If v is repeated in \mathbf{w} , then we let $\mathbf{w} = \mathbf{w}_0 + \mathbf{w}_1$ so that each \mathbf{w}_i is a closed nontrivial walk with first vertex term v , and then we have $-1 = \mu(\mathbf{w}) = \mu(\mathbf{w}_0)\mu(\mathbf{w}_1)$ by Lemma 4.1, which implies that one of \mathbf{w}_0 and \mathbf{w}_1 is an odd-signed closed walk, say \mathbf{w}_0 . Since \mathbf{w}_0 is a proper subwalk of \mathbf{w} , by the induction hypothesis, $[\mathbf{w}_0]$ contains a cycle which is odd-signed in (G, τ) and so $[\mathbf{w}]$ does. \square

Proof of Proposition 3.3 Let $A = A(G, \tau)$ and $A(e)$ be the column of A corresponding to an edge e . Take a spanning tree T of G . Then clearly, the submatrix obtained by the columns corresponding to the edges of T has the rank $|V(G)| - 1$. Thus, $\text{rank}(A) \geq |V(G)| - 1$.

Suppose to contrary that (G, τ) contains no odd-signed closed walk and there are $|V(G)|$ linearly independent columns. Let A' be the submatrix induced by those columns. Then, the subgraph of G induced by the edges corresponding to the columns of A' has $|V(G)|$ edges and so it contains a cycle C . By the assumption that (G, τ) has no odd-signed cycle, C is an even-signed cycle and we let C have a balanced section decomposition $\mathbf{w}_0 + \cdots + \mathbf{w}_r$. Without loss of generality, we assume that $\mathbf{w}_i : v_1^i e_1^i \cdots v_{a_i}^i e_{a_i}^i v_{a_i+1}^i$ for each i . If $r = 0$, then it is easy to see that $\sum_{e \in E(C)} A(e) = \mathbf{0}$, a contradiction to the fact that $\{A(e) \mid e \in E(C)\}$ are linearly independent. Suppose that $r > 0$. Then, each v_1^i is unbalanced. Since $v_{a_i+1}^i = v_1^{i+1}$, $\tau(e_1^i, v_{a_i+1}^i) = \tau(e_1^{i+1}, v_1^{i+1})$. Then, it follows that

$$\sum_{i=0}^r (-1)^i \sum_{j=1}^{a_i} A(e_j^i) = \mathbf{0}.$$

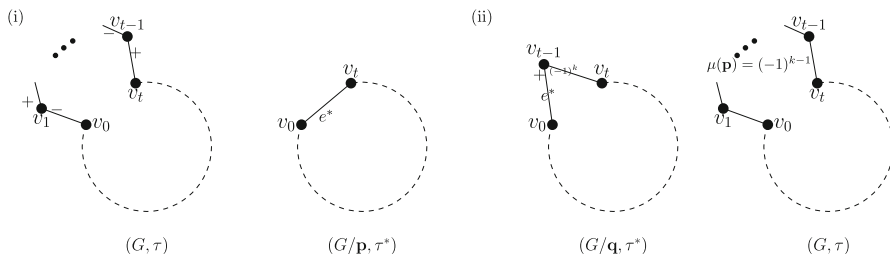


Fig. 22 An illustration for the proof of Proposition 3.11

This is a contradiction to the fact that $\{A(e) \mid e \in E(C)\}$ are linearly independent.

Suppose that (G, τ) contains an odd-signed closed walk \mathbf{w} , and then (G, τ) contains an odd-signed cycle C by Lemma A.4. We take a unicyclic spanning subgraph H of G , containing the cycle C . Let A' be the submatrix of A induced by the columns corresponding to the edges of H . Note that A' is an $|V(G)| \times |V(G)|$ matrix and by permuting lines, we may assume that $A' = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ where A_{ii} 's are square matrices, the rows and the columns corresponding to A_{11} are the vertices and the edges of the cycle C , and A_{22} is a upper triangular matrix without zero diagonal element. Since $\det(A') = \det(A_{11}) \det(A_{22})$ and $\det(A_{22}) \neq 0$, it is sufficient to show that the columns of the submatrix A_{11} are linearly independent.

Let $C : v_1 e_1 v_2 \dots v_r e_r v_1$. Suppose that $\sum_{i=1}^r c_i A(e_i) = 0$ for some constant c_i . Since the entry of $\sum_{i=1}^r c_i A(e_i)$ corresponding to a vertex v_j ($j \in [r]$) is equal to $c_{j-1} \tau(e_{j-1}, v_j) + c_j \tau(e_j, v_j)$, where the indices are modulo r , we have $c_{j-1} \tau(e_{j-1}, v_j) + c_j \tau(e_j, v_j) = 0$. If a vertex term v_j of C is not unbalanced then $c_{j-1} = c_j$, and if a vertex term v_i is unbalanced then $c_{j-1} = -c_j$. Since C has an odd number of unbalanced vertex terms, then it follows that $c_1 = c_2 = \dots = c_r = 0$.

□

Proof of Proposition 3.11

Proof of Proposition 3.11 Note that G/\mathbf{p} or G/\mathbf{q} in the cases is a simple graph. For every sign τ of G , every primitive walk \mathbf{w} in (G, τ) containing an edge of the path \mathbf{p} contains all edges of \mathbf{p} by Theorem 3.1. In the following, let G^* be G/\mathbf{p} or G/\mathbf{q} (according to the cases), and let e^* be its newly added edge. To show (i), suppose that $G \in \mathcal{G}^{cis}$. Take a sign τ^* of G^* , and define a sign τ of G as follows, and see Fig. 22.

$$\tau(e, y) = \begin{cases} \tau^*(e, y) & \text{if } e \in E(G) \setminus E(\mathbf{p}), \\ \tau^*(e^*, v_0) & \text{if } (e, y) = (v_0 v_1, v_0), \\ \tau^*(e^*, v_t) & \text{if } (e, y) = (v_{t-1} v_t, v_t), \\ 1 & \text{if } (e, y) = (v_i v_{i+1}, v_i) \text{ for some } i \in [t-1], \\ -1 & \text{if } (e, y) = (v_{i-1} v_i, v_i) \text{ for some } i \in [t-1]. \end{cases}$$

For a closed walk \mathbf{w} , \mathbf{w} is even-signed in (G, τ) if and only if the walk \mathbf{w}^* obtained from \mathbf{w} by contracting the ear \mathbf{p} is even-signed in (G^*, τ^*) . Thus, $r(G, \tau) = r(G^*, \tau^*)$ and there are s primitive binomials of $I_{(G, \tau)}$ generating $I_{(G, \tau)}$ if and only if there are s primitive binomials of $I_{(G^*, \tau^*)}$ generating $I_{(G^*, \tau^*)}$. Therefore, $G/\mathbf{p} \in \mathcal{G}^{cis}$ since $G \in \mathcal{G}^{cis}$.

To show (ii), take a sign τ of G . Consider the balanced section decomposition of \mathbf{p} , and let k be the number of balanced sections of \mathbf{p} . Define a sign τ^* of G^* as follows, and see Fig. 22.

$$\tau^*(e, y) = \begin{cases} \tau(e, y) & \text{if } e \notin \{e^*, v_{t-1}v_t\}, \\ \tau(v_0v_1, v_0) & \text{if } (e, y) = (e^*, v_0), \\ \tau(v_{t-1}v_t, v_t) & \text{if } (e, y) = (v_{t-1}v_t, v_t), \\ 1 & \text{if } (e, y) = (e^*, v_{t-1}), \\ (-1)^k & \text{if } (e, y) = (v_{t-1}v_t, v_{t-1}). \end{cases}$$

For a closed walk \mathbf{w} , \mathbf{w} is even-signed in (G, τ) if and only if the walk \mathbf{w}^* obtained from \mathbf{w} by contracting the ear \mathbf{q} is even-signed in (G^*, τ^*) . Similar to the argument of (i), we have $G \in \mathcal{G}^{cis}$ since $G/\mathbf{q} \in \mathcal{G}^{cis}$. \square

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