



Oriented matroid structures from realized root systems

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Abstract

This paper investigates the question of uniqueness of the reduced oriented matroid structure arising from root systems of a Coxeter system in real vector spaces. We settle the question for finite Coxeter systems, irreducible affine Weyl groups and all rank three Coxeter systems. In these cases, the oriented matroid structure is unique unless W is of type \tilde{A}_n , $n \geq 3$, in which case there are three possibilities.

Keywords Oriented matroid · Affine Weyl group · Root system · Coxeter group

1 Introduction

Given a Coxeter system (W, S) , one can associate to it different root systems Φ in a real vector space, parameterized by (possibly non-integral) generalized Cartan matrices (NGCMs for short). Such root systems play crucial roles in understanding various mathematical structures, particularly those arising from Lie theory. They naturally determine oriented matroids in the sense of [6]. One may ask if different (possibly non-reduced) root system realizations yield non-isomorphic reduced oriented matroid structures when transferred to the abstract root system $T \times \{\pm 1\}$ with natural W -action, where T is the set of reflections in W (see [1, 3]). The transfer is defined as follows. Let Φ be a realized root system with the oriented matroidal closure operator $\text{cone}_\Phi(\Gamma) = \text{cone}(\Gamma) \cap \Phi$, $\Gamma \subseteq \Phi$. Here $\text{cone}(\Gamma) = \{\sum_{i \in I} k_i v_i \mid v_i \in \Gamma \cup \{0\}, k_i \in \mathbb{R}_{\geq 0}, |I| < \infty\}$. We have a canonical $W \times \{\pm 1\}$ -equivariant surjection $\theta: \Phi \rightarrow T \times \{\pm 1\}$ given by

$$\epsilon \alpha \mapsto (s_\alpha, \epsilon), \quad \alpha \in \Phi^+, \epsilon \in \{\pm 1\}.$$

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Then we transfer cone_Φ to an oriented matroid closure operator c_Φ on $T \times \{\pm 1\}$ given by $c_\Phi(A) = \theta(\text{cone}_\Phi(\theta^{-1}A))$ for $A \subseteq T \times \{\pm 1\}$ and consider the corresponding (reduced) oriented matroid structure on $T \times \{\pm 1\}$.

In this paper we show that for a finite Coxeter system, an irreducible affine Weyl group or a rank 3 Coxeter system, this induced oriented matroid structure is independent of the realized root system Φ except for type $\tilde{A}_n, n \geq 3$. For an affine Weyl group of type $\tilde{A}_n, n \geq 3$, there are 3 different structures. Type \tilde{A}_n is the most interesting case and is the main focus of this paper. We show that after rescaling, realized root systems of type \tilde{A}_n can be parameterized by a positive real number v . We then give an explicit general form of the values of chirotope maps as a (Laurent polynomial) function of v . For rank three Coxeter systems, we prove the uniqueness by an argument involving homotopies of root systems. Finally, we give an example, touching on our motivations for this work, illustrating the fact that reflection orders ([1]) do not relate well to the geometry of realized root systems, and ask how they relate to possibly non-realizable oriented matroid structures on root systems.

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2 Coxeter groups, realized root systems and oriented matroids

In this section, we collect basic concepts of Coxeter groups and oriented matroids.

2.1 Oriented matroids

There are many equivalent axioms of oriented matroids. In this paper we emphasize the one describing an oriented matroid as an involuted set with a closure operator as given in [6,17]. An oriented matroid is a triple $(E, *, cx)$ where E is a set with an involution map $* : E \rightarrow E$ (i.e., $x^{**} = x, x \neq x^*$) and cx a closure operator on E such that (1) if $x \in cx(X)$ there exists a finite set $Y \subseteq X$ such that $x \in cx(Y)$, (2) $cx(X)^* = cx(X^*)$, (3) if $x \in cx(X \cup \{x^*\})$ then $x \in cx(X)$, (4) if $x \in cx(X \cup \{y^*\})$ and $x \notin cx(X)$ then $y \in cx(X \setminus \{y\} \cup \{x^*\})$.

For each oriented matroid $(E, *, cx)$, define $\bar{E} = E / \sim$ where the equivalence relation is that $x \sim y$ if and only if $x^* = y$ or $x = y$. Let $\bar{cx}(X) = cx(X \cup X^*) / \sim$. Then (\bar{E}, \bar{cx}) is an unoriented matroid. Then rank of the oriented matroid of $(E, *, cx)$ is defined to be the rank of the unoriented matroid (\bar{E}, \bar{cx}) .

Let V be a vector space. Let Γ be a non-empty set such that $\Gamma = -\Gamma$ and $0 \notin \Gamma$. Then $(\Gamma, -, \text{cone}_\Gamma)$ is an oriented matroid (where cone_Γ is defined in the previous section) with rank being $\dim(\text{span}(\Gamma))$. Such an oriented matroid is said to be realizable. It is known that not every oriented matroid is realizable.

It is known (cf. [2, Exercise 3.13]) that a finite rank oriented matroid is determined up to isomorphism by the so-called chirotope map $\chi : E^r \rightarrow \{+, -, 0\}$, up to a sign. If $(E, *, cx)$ is realizable of finite rank r then the chirotope map is given by $\chi(e_1, e_2, \dots, e_r) = \text{sgn}(\det(e_1, e_2, \dots, e_r))$, i.e., the sign of the determinant of the $r \times r$ matrix formed by these r vectors (regarded as column vectors with respect to

a chosen basis of the ambient space). The oriented matroid determines two chirotope maps, depending on the orientation of the chosen basis and differing only by a sign.

2.2 Coxeter systems

Let W be a group and S be a set of generators of W such that every element of S has order 2 in W . For $s, s' \in S$, denote $m(ss')$ the order of ss' . We call (W, S) a Coxeter system if S and the relations $(ss')^{m(ss')}$ with $s, s' \in S, m(ss') \neq \infty$ form a presentation of W . W is called a Coxeter group. The elements of S are called simple reflections. Call $|S|$ the rank of the Coxeter system (W, S) . In this paper, we assume unless otherwise stated that $|S|$ is finite, but such Coxeter systems may contain infinite rank reflection subgroups.

Given a Coxeter system (W, S) we associate to it a Coxeter graph in the following way: the set of vertices is S and $s, r \in S$ have an edge between them if $m(rs) \geq 3$ (including $m(rs) = \infty$). If $m(rs) \geq 4$ we label the edge with that number. If the Coxeter graph is connected, (W, S) (and the Coxeter group W) is called irreducible and it is called reducible otherwise. An element of W is called a reflection if it is conjugate to some $s \in S$. The set of reflections is denoted T . Let $w \in W$. If $w = s_1s_2 \dots s_k, s_i \in S$ with k minimal, we say the length of w , denoted $l(w)$, is k .

There are complete classifications by Coxeter graphs of finite irreducible Coxeter systems and of a family of infinite Coxeter systems, called irreducible affine Weyl groups, arising from affine reflections in Euclidean spaces. For details, see [1,3,18].

Recall that an R -indexed Coxeter matrix is an $R \times R$ -indexed family $(m_{r,s})_{r,s \in R}$ such that $m_{r,r} = 1$ for $r \in R$ and $m_{r,s} = m_{s,r} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ for $r \neq s$ in R . Associated to a Coxeter system (W, S) , there is an S -indexed Coxeter matrix $(m_{r,s})_{r,s \in S}$ determined by $m_{r,s} := m(rs) \in \mathbb{N}_{\geq 1} \cup \{\infty\}$. An isomorphism $(W, S) \rightarrow (W', S')$ of Coxeter systems is a group isomorphism $W \rightarrow W'$ which restricts to a bijection $S \rightarrow S'$. If $(W, S), (W', S')$ are Coxeter systems, a map $f: S \rightarrow S'$ extends to an isomorphism $(W, S) \rightarrow (W', S')$ if and only if it is a bijection which preserves the Coxeter matrices, in the sense that $m(rs) = m(f(r)f(s))$ for all $r, s \in S$.

Let $T' \subseteq T$, the set of reflections. Then the subgroup W' of W generated by T' is also a Coxeter group with the Coxeter system (W', S') where $S' = \chi(W') := \{t \in T \cap W' \mid l(t't) > l(t), \forall t' \in T \cap W' \setminus \{t\}\}$. It can be shown that if $|T'| = 2$, W' is a dihedral group. Such a group is called a dihedral reflection subgroup. Partially order the set of all dihedral reflection subgroups of W under inclusion. The maximal elements of this poset are called maximal dihedral reflection subgroups.

2.3 Realized root systems

Following [7,13,16,20,21], we define a realized root system datum to be data $D = (V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ such that:

- (1) V, V^\vee are \mathbb{R} -vector space with positively independent subsets Π, Π^\vee , respectively (where a subset Γ of a real vector space is said to be positively independent if there do not exist some $n \in \mathbb{Z}^+$, pairwise distinct $\gamma_1, \dots, \gamma_n$ in Γ and strictly positive scalars c_1, \dots, c_n such that $\sum_{i=1}^n c_i \gamma_i = 0$),

- (2) $\iota : \Pi \rightarrow \Pi^\vee$ is a bijection; denote $\iota(\alpha)$ by α^\vee ,
- (3) $(-, -) : V \times V^\vee \rightarrow \mathbb{R}$ is a bilinear pairing such that $(\alpha, \alpha^\vee) = 2$ for all $\alpha \in \Pi$,
- (4) $(\alpha, \beta^\vee) \leq 0$, for all $\alpha \neq \beta$ in Π ,
- (5) $(\alpha, \beta^\vee) = 0$ if and only if $(\beta, \alpha^\vee) = 0$, for all $\alpha \neq \beta$ in Π ,
- (6) $(\alpha, \beta^\vee)(\beta, \alpha^\vee) \in \{4 \cos^2 \frac{\pi}{m} | m \in \mathbb{N}, m \geq 2\} \cup [4, \infty)$, for all $\alpha \neq \beta$ in Π .
 We recall the main properties (see op. cit.). For $\alpha \in \Pi$ define $s_\alpha : V \rightarrow V$ by $v \mapsto v - (v, \alpha^\vee)\alpha$. Let \widehat{W} be the subgroup of $GL_{\mathbb{R}}(V)$ generated by $\widehat{S} := \{s_\alpha | \alpha \in \Pi\}$.
- (7) The map $\alpha \mapsto s_\alpha : \Pi \rightarrow \widehat{S}$ is injective. Further, $(\widehat{W}, \widehat{S})$ is a Coxeter system with Coxeter matrix $(m'_{r,s})_{r,s \in \widehat{S}}$ given by $m'_{s_\alpha, s_\alpha} = 2$ for $\alpha \in \Pi$, $m'_{s_\alpha, s_\beta} = m$ if $(\alpha, \beta^\vee)(\beta, \alpha^\vee) = 4 \cos^2 \frac{\pi}{m}$ where $m \in \mathbb{N}_{\geq 2}$, and $m'_{s_\alpha, s_\beta} = \infty$ if $(\alpha, \beta^\vee)(\beta, \alpha^\vee) \geq 4$.

By a realized root system datum of a Coxeter system (W, S) , we mean a realized root system datum D in the above sense together with a specified isomorphism of Coxeter systems $(W, S) \rightarrow (\widehat{W}, \widehat{S})$, which we generally write (and regard) as an identification even in situations where several realized root system data of (W, S) are under simultaneous consideration. Fix a realized root system datum of (W, S) .

Π is called the set of simple roots. $\Phi := \widehat{W}\Pi$ is called the set of roots. $\Phi^+ := \Phi \cap \text{cone}(\Pi)$ is called the set of positive roots. $\Phi^- := -\Phi^+$ is called the set of negative roots. It can be shown that $\Phi = \Phi^+ \dot{\cup} \Phi^-$ (disjoint union). (See for example [21, Section 3] and [13, Section 2] for proofs of this and other non-trivial facts mentioned without proof in this paragraph.) Similarly, one defines $\Phi^\vee, (\Phi^\vee)^\pm, \widehat{S}^\vee, \widehat{W}^\vee$, etc. We call Φ^\vee the set of coroots. There is a natural identification of $W = \widehat{W}$ with \widehat{W}^\vee . Also, ι extends to a W -equivariant bijection $\iota : \Phi \rightarrow \Phi^\vee$, which we denote as $\alpha \mapsto \alpha^\vee$ for $\alpha \in \Phi$, satisfying $w\alpha^\vee = c^{-1}\beta^\vee$ if $w\alpha = c\beta$ with $\alpha, \beta \in \Phi$ and $c \in \mathbb{R}$. When restricted to Φ^+ , ι gives a bijection between Φ^+ and $(\Phi^\vee)^+$. Then for $\alpha \in \Phi$ we can define $s_\alpha : V \rightarrow V$ whose action follows the same formula as when $\alpha \in \Pi$. One can prove that $ws_\alpha w^{-1} = s_{w(\alpha)}$ for all $w \in W$. We call $((\alpha, \beta^\vee))_{\alpha, \beta \in \Pi}$ a non-integral generalized Cartan matrix (abbreviated NGCM).

In general, a matrix $A = (c_{\alpha, \beta})_{\alpha, \beta \in \Pi}$ is a NGCM of a Coxeter system with Coxeter matrix $(m_{\alpha, \beta})_{\alpha, \beta \in \Pi}$ if and only if the following conditions hold for all $\alpha, \beta \in \Pi$. First, $c_{\alpha, \alpha} = 2$. Second, if $\alpha \neq \beta$, then $c_{\alpha, \beta} \leq 0$ with equality if and only if $m_{\alpha, \beta} = 2$. Third, if $m_{\alpha, \beta} \neq \infty, 1$, then $c_{\alpha, \beta}c_{\beta, \alpha} = 4 \cos^2 \frac{\pi}{m_{\alpha, \beta}}$. Finally, if $m_{\alpha, \beta} = \infty$, then $c_{\alpha, \beta}c_{\beta, \alpha} \geq 4$.

For simplicity, we refer to Φ above as a realized root system of (W, S) . In general the realized root system thus constructed is not reduced, i.e., there might be roots $\beta = c\alpha$ where $c \neq 1$ and $c \in \mathbb{R}_{>0}$. However, recall from [1,3] that there is a W -action on $T \times \{\pm 1\}$ given by $w(t, \epsilon) = (wtw^{-1}, \eta(w, t)\epsilon)$ where $\eta(w, t) = -1$ if $l(tw^{-1}) < l(w^{-1})$ and $\eta(w, t) = 1$ otherwise. This W -action commutes with the surjection $\pi : \Phi \rightarrow T \times \{\pm 1\} : \epsilon\alpha \mapsto (s_\alpha, \epsilon), \alpha \in \Phi^+, \epsilon \in \{\pm 1\}$.

Using π to transfer the oriented matroid $M = (\Phi, -, \text{cone}_\Phi)$ to $T \times \{\pm 1\}$ we have the following:

Definition 2.4 The oriented matroid $M' = (T \times \{\pm 1\}, -, c = c_\Phi)$ where for $A \subseteq T \times \{\pm 1\}, c(A) := \pi(\text{cone}_\Phi(\pi^{-1}A))$ is called the reduced oriented matroid structure from the realized root system Φ .

If in the above construction of realized root system for (W, S) , we have $(\alpha, \beta^\vee) = -2 \cos \frac{\pi}{m(s_\alpha s_\beta)}$ for all $\alpha, \beta \in \Pi$ (so $(\alpha, \beta^\vee) = -2$ if $m(s_\alpha s_\beta) = \infty$) and Π, Π^\vee are linearly independent, the realized root system is said to be a standard one.

2.5 Rescaling

Let $D = (V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ be a realized root system datum associated to a Coxeter system (W, S) . Define $D' := (V, V^\vee, (-, -), \Delta, \Delta^\vee, \iota')$ where $\Delta = \{d_\alpha \alpha \mid \alpha \in \Pi\}$, $\Delta^\vee = \{d_\alpha^{-1} \alpha \mid \alpha \in \Pi\}$, $d_\alpha \in \mathbb{R}_{>0}$ for $\alpha \in \Pi$ and $\iota'(d_\alpha \alpha) = d_\alpha^{-1} \alpha^\vee$.

Lemma 2.6 *D' above is also a realized root system datum of the Coxeter system (W, S) . Further, D and D' determine the same oriented matroid structure on T .*

Proof It is straightforward to check that D' is also a realized root system datum. In particular one notes that $s_\alpha = s_{d_\alpha \alpha}$ and $w(d_\alpha \alpha) = d_\alpha w(\alpha)$. The associated Coxeter system (\tilde{W}, \tilde{S}) of D' is equal to that of D and is therefore canonically identified with (W, S) . The above also shows the set of rays spanned by the positive roots for D and D' are exactly the same. Hence the resulting reduced oriented matroid on $T \times \{\pm 1\}$ is clearly the same for D and D' , by the chirotope characterization of oriented matroids. □

From now on, we shall say that D' in the above lemma is a realized root system datum obtained by rescaling D . We observe that if (W, S) is of rank at most two, then any realized root system datum has linearly independent Π and Π^\vee , and can be rescaled to one with a symmetric NGCM. It is well known in this case from explicit formulae for the roots (see for example [10,11] or [21]) that the oriented matroid structure on $T \times \{\pm 1\}$ is independent of choice of realized root system datum.

2.7 Convex geometry of realized root systems

Since oriented matroids provide an abstract notion of convexity, it is appropriate to discuss here some characterizations of realized root systems in terms of convex geometry. The results may be viewed as formalizations of the idea that realized root system data constitute essentially the most general natural context for root systems in real vector spaces, in which one has the standard convex geometric properties of root systems, such as decomposition of the root and coroot systems into positive and negative subsystems spanned as cones by simple roots and simple coroots, and spanning salient convex cones. We first recall relevant terminology.

Let V be a \mathbb{R} -vector space with dual space $V^* := \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$. By a pointed, convex cone C in V , hereafter called just a cone, we mean a subset C of V such that $C = \text{cone}(C)$, or equivalently, $C = \text{cone}(\Gamma)$ for some $\Gamma \subseteq V$.

Let C be a cone in V . We say that C is salient if $C \cap -C = \{0\}$. In these terms, a subset Γ of V is positively independent if $0 \notin \Gamma$ and $\text{cone}(\Gamma)$ is a salient cone. An extreme ray R of C is a subset R of C of the form $R = \text{cone}(\{\alpha\})$ for some $0 \neq \alpha \in C$ such that $x, y \in C$ and $x + y \in R$ implies $x, y \in R$. Note that if C is not salient, it has no extreme rays. We say that a subset Γ of C is a set of representatives of the extreme

rays of C if the map $\alpha \mapsto \text{cone}(\{\alpha\})$, for $\alpha \in \Gamma$, defines a bijection from Γ to the set of extreme rays of C ; this implies in particular that $0 \notin \Gamma$, that C is salient if $\Gamma \neq \emptyset$, and that $\text{cone}(\{\alpha\}) \cap \Gamma = \{\alpha\}$ for all $\alpha \in \Gamma$.

The following facts are mostly well known and all are easily checked; details are omitted.

Proposition 2.8 *Let $\Gamma \subseteq V$. Then the following conditions are equivalent:*

- (i) *If $\sum_{\gamma \in \Gamma} c_\gamma \gamma = 0$ where the scalars $c_\gamma \in \mathbb{R}$ are zero for all but finitely many $\gamma \in \Gamma$, then $|\{\gamma \in \Gamma | c_\gamma < 0\}| > 1$.*
- (ii) *$0 \notin \Gamma$ and for all distinct α, β in Γ , $\{\alpha, \beta\}$ is linearly independent and $\text{cone}(\Gamma) \cap \text{span}(\{\alpha, \beta\}) = \text{cone}(\{\alpha, \beta\})$.*
- (iii) *Γ is positively independent and $\gamma \notin \text{cone}(\Gamma \setminus \{\gamma\})$ for all $\gamma \in \Gamma$.*
- (iv) *Γ is positively independent and $\text{cone}(\Gamma') \neq \text{cone}(\Gamma)$ for all $\Gamma' \subsetneq \Gamma$.*
- (v) *Γ is a set of representatives of the extreme rays of some cone in V .*
- (vi) *Γ is a set of representatives of the extreme rays of $\text{cone}(\Gamma)$.*

If these conditions hold, we say that Γ is strongly positively independent.

Proposition 2.9 *Suppose that $\Gamma \subseteq V$ is positively independent and for each $\gamma \in \Gamma$, there is some element $\gamma' \in V^*$ such that $\gamma'(\gamma) > 0$ and $\gamma'(\Gamma \setminus \{\gamma\}) \subseteq \mathbb{R}_{\leq 0}$. Then Γ is strongly positively independent.*

Proof This follows directly from characterization Proposition 2.8(iii) of strong linear independence. □

Corollary 2.10 *If $(V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ is a realized root system datum, then Π and Π^\vee are strongly positively independent.*

Proof This follows immediately from 2.3(1),(3),(4) and Proposition 2.9. □

2.11. For the formulation of the next result, suppose that V and V^\vee are \mathbb{R} vector spaces and $(-|-) : V \times V^\vee \rightarrow \mathbb{R}$ is a \mathbb{R} -bilinear pairing. Suppose given a subset Π of V and for each $\alpha \in \Pi$, some element $\alpha^\vee \in V^\vee$ such that $(\alpha|\alpha^\vee) = 2$. Let $\Pi^\vee := \{\alpha^\vee | \alpha \in \Pi\}$ and let $\iota : \Pi \rightarrow \Pi^\vee$ be the surjective map given by $\iota(\alpha) = \alpha^\vee$ for $\alpha \in \Pi$. Let $s_\alpha \in \text{GL}(V)$ denote the reflection of V given by $v \mapsto v - (v|\alpha^\vee)\alpha$ for $v \in V$ and let $W \subseteq \text{GL}(V)$ denote the reflection group on V generated by the set $S := \{s_\alpha | \alpha \in \Pi\}$ of these reflections. Define $\Phi := W\Pi$, $C := \text{cone}(\Pi)$ and $\Phi^+ := \Phi \cap C$. Let $D = (V, V^\vee, (-|-), \Pi, \Pi^\vee, \iota)$.

Also, let U be a \mathbb{R} -vector space with a basis Δ equipped with a bijection $\iota' : \Pi \rightarrow \Delta$. Write $\alpha' := \iota'(\alpha)$ for $\alpha \in \Pi$. Let $\theta : U \rightarrow V^\vee$ be the linear map such that $\theta(\alpha') = \alpha^\vee$ for all $\alpha \in \Pi$ and define a bilinear form $(-|-)' : V \times U \rightarrow \mathbb{R}$ by $(v|u)' = (v|\theta(u))$. Let $D' = (V, U, (-|-)', \Pi, \Delta, \iota')$.

Proposition 2.12 *Assume above that $\Pi \subseteq V$ is strongly positively independent (this holds if Π is positively independent and $(\alpha|\beta^\vee) \leq 0$ for all $\alpha \neq \beta$ in Π). Then*

- (a) *One has $\Phi = \Phi^+ \cup -\Phi^+$ if and only if 2.3(4)–(6) hold.*

- (b) Assume that 2.3(4)–(6) hold and that Π^\vee is positively independent. Then D is a realized root system datum with root system Φ , set of positive roots Φ^+ and (W, S) is the Coxeter system associated to D . Further, $\Phi^\vee = \Phi^{\vee+} \cup -\Phi^{\vee+}$ where Φ^\vee is the coroot system of D .
- (c) Assume that 2.3(4)–(6) hold. Then D' is a realized root system datum with root system Φ , set of positive roots Φ^+ , a linearly independent set Δ of simple coroots and associated Coxeter system (W, S) .

Remark One may view (b) as an assertion that local conditions 2.3(4)–(6), on values of pairings of roots with coroots, in the definition of a realized root system datum can be replaced by the global condition $\Phi = \Phi^+ \cup -\Phi^+$, on convexity properties of the root system, provided Π is strongly positively independent. Similar results go back at least to [8,10,11].

Proof Assume first that 2.3(4)–(6) hold. Then it is easily checked from the definitions that, under the assumptions of (b) and (c), respectively, D and D' are realized root system data. Then (b)–(c) hold by previously described properties of realized root system data. In particular, one has $\Phi = \Phi^+ \cup -\Phi^+$ from (c). This proves (b)–(c) and one direction of (a).

Conversely, assume as in (a) that $\Phi = \Phi^+ \cup -\Phi^+$. We verify 2.3(4)–(6). Let $\alpha \neq \beta \in \Pi$ and $W' := \langle s_\alpha, s_\beta \rangle \subseteq W$. Let $C' := \text{cone}(\{\alpha, \beta\})$. By the assumption that $\Phi = \Phi^+ \cup -\Phi^+$ and the characterization of strong positive independence in Proposition 2.8(ii), we have $W'\{\alpha, \beta\} \subseteq C' \cup -C'$. Since $s_\alpha(\beta) = \beta - (\beta|\alpha^\vee)\alpha \in C' \cup -C'$, it follows that $(\beta|\alpha^\vee) \leq 0$. By symmetry, $(\alpha|\beta^\vee) \leq 0$ as well. Suppose that $(\beta|\alpha^\vee) = 0$. Then $s_\alpha(\beta) = \beta$, so $s_\alpha(s_\beta(\alpha)) = s_\alpha(\beta - (\alpha|\beta^\vee)\alpha) = \beta + (\alpha|\beta^\vee)\alpha \in C' \cup -C'$, which implies $(\alpha|\beta^\vee) \geq 0$ and thus that $(\alpha|\beta^\vee) = 0$. This shows that 2.3(4)–(5) hold. Now one may choose $c \in \mathbb{R}_{>0}$ so that $d := c^{-1}(\alpha|\beta^\vee) = c(\beta|\alpha^\vee)$. Let $\beta' := c\beta$. Then $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\beta') = \beta' - d\alpha$, $s_\beta(\beta') = -\beta'$ and $s_\beta(\alpha) = \alpha - d\beta'$. Recall that $W'\{\alpha, \beta'\} \subseteq C' \cup -C'$. Well-known calculations for reflection representations of dihedral groups (see [11, Lemma 4.1] or, for more detail, [10, (2.1)–(2.4)]) imply that $d \in \{-2 \cos \frac{\pi}{m} | m \in \mathbb{N}_{\geq 2}\} \cup (-\infty, -2]$. Since $(\alpha|\beta^\vee)(\beta|\alpha^\vee) = d^2$, 2.3(6) follows. This completes the proof of (a), and of the proposition. \square

Remark A seemingly more general notion of dual root systems Ψ and Ψ^\vee for which simple roots and coroots span as usual the extreme rays of salient cones $\text{cone}(\Psi^+)$ and $\text{cone}(\Psi^{\vee+})$ is as follows. Suppose again that V and V^\vee are \mathbb{R} vector spaces and $(-|-) : V \times V^\vee \rightarrow \mathbb{R}$ is a \mathbb{R} -bilinear pairing. Assume given an index set I and families $(\alpha_i)_{i \in I}$, $(\beta_i)_{i \in I}$ of vectors in V and V^\vee , respectively, such that $(\alpha_i|\beta_i) = 2$ for all $i \in I$. Assume that $\Delta := \{\alpha_i | i \in I\}$ and $\Delta^\vee := \{\beta_i | i \in I\}$ are positively independent. Define $C := \text{cone}(\Delta)$ and $C^\vee := \text{cone}(\Delta^\vee)$. Suppose for each $i \in I$ that $\text{cone}(\{\alpha_i\})$ is an extreme ray of C and that $\text{cone}(\{\beta_i\})$ is an extreme ray of C^\vee .

For $i \in I$, define $s_i \in \text{GL}(V)$ by $s_i(v) = v - (v|\beta_i)\alpha_i$ for $v \in V$, and $s_i^\vee \in \text{GL}(V^\vee)$ by $s_i^\vee(u) = u - (\alpha_i|u)\beta_i$ for $u \in V^\vee$. Let $S := \{s_i | i \in I\}$ (resp. $S^\vee := \{s_i^\vee | i \in I\}$). Define reflection groups $W := \langle S \rangle \subseteq \text{GL}(V)$ and $W^\vee := \langle S^\vee \rangle \subseteq \text{GL}(V^\vee)$. Let $\Psi := W\Delta$, $\Psi^\vee := W^\vee\Delta^\vee$, $\Psi^+ = \Psi \cap C$ and $\Psi^{\vee+} := \Psi^\vee \cap C^\vee$.

Assume further that $\Psi = \Psi^+ \cup -\Psi^+$ and $\Psi^\vee = \Psi^{\vee+} \cup -\Psi^{\vee+}$. Then the following can be proved. Firstly, (W, S) (resp. (W^\vee, S^\vee)) is a Coxeter system. Secondly, there

is a group isomorphism $W \rightarrow W^\vee$ such that $s_i \mapsto s_i^\vee$ for all $i \in I$ and thus we may identify W^\vee with W . Thirdly, there is a W -equivariant bijection $\Psi \xrightarrow{\cong} \Psi^\vee$ (denoted as $\alpha \mapsto \alpha^\vee$), determined by $\alpha_i \mapsto \beta_i$ for $i \in I$, and it restricts to bijections $\Psi^+ \xrightarrow{\cong} \Psi^{\vee+}$ and $\Delta \mapsto \Delta^\vee$. Finally, the reflection representations of W and W^\vee on V and V^\vee , and root system Ψ and coroot system Ψ^\vee , arise from an associated realized root system datum in the following way.

There exists $\Pi \subseteq \Delta$ such that, if one lets $\Pi^\vee := \{\alpha^\vee \mid \alpha \in \Pi\}$ and $\iota: \Pi \rightarrow \Pi^\vee$ be the map $\alpha \mapsto \alpha^\vee$, then $D := (V, V^\vee, (-|-, \Pi, \Pi^\vee, \iota)$ is a realized root system datum with associated Coxeter system (W, S) with $C = \text{cone}(\Pi)$ and $C^\vee = \text{cone}(\Pi^\vee)$. Furthermore, one can define the roots, coroots, positive roots and positive coroots of D by $\Phi := W\Pi, \Phi^\vee := W\Pi^\vee, \Phi^+ := \Phi \cap C$ and $\Phi^{\vee+} := \Phi^\vee \cap C^\vee$. The bijection $\Psi \xrightarrow{\cong} \Psi^\vee$ restricts to the standard W -equivariant bijection $\Phi \xrightarrow{\cong} \Phi^\vee$. If $\alpha \in \Psi$ (resp., Ψ_+, Δ) then there exists $\beta \in \Phi$ (resp., Φ^+, Π) and $c \in \mathbb{R}_{>0}$ such that $\alpha = c\beta$ (so also $\alpha^\vee = c^{-1}\beta^\vee$).

Thus, the seemingly more general framework considered in this remark leads only to a notion essentially equivalent to that of a realized root system datum.

3 Coxeter systems with a forest as Coxeter graph

Henceforward in this paper, we shall assume for convenience, unless otherwise stated, that if $D := (V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ is any realized root system datum considered, then Π spans V, Π^\vee spans V^\vee , and Π is finite. The last assumption makes it possible to describe the reduced oriented matroid structure M' from the root system Φ of D by its chirotope map. Note that for $w \in W$, where (W, S) is the Coxeter system associated to D , we have $\det(w) = (-1)^{\ell(w)}$.

In this section we show that if a Coxeter system has the Coxeter graph of a forest without infinite bonds, then all realized root systems with linearly independent simple roots induce the same oriented matroid structure on $T \times \{1, -1\}$.

Definition 3.1 A realized root system datum $(V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ is said to have a symmetric NGCM if $(\alpha, \beta^\vee) = (\beta, \alpha^\vee)$ for all $\alpha, \beta \in \Pi$.

Proposition 3.2 Let $(V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ be a realized root system datum of the Coxeter system (W, S) whose Coxeter graph is a forest. Then the root system Φ can be rescaled to a realized root system having a symmetric NGCM.

Proof It suffices to check the assertion for (W, S) having the Coxeter graph of a tree. Note that Coxeter graph can be considered as a graph with vertices set being Π . Denote $d(\alpha, \beta)$ the distance between α and β in the Coxeter graph for $\alpha, \beta \in \Pi$. We pick and designate an element $\alpha_0 \in \Pi$ as the root of the tree. To prove the assertion, we use induction. Let $\Pi_k = \{\alpha \in \Pi \mid d(\alpha_0, \alpha) \leq k\} \subseteq \Pi$. Assume that we can rescale $(V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ to $(V, V^\vee, (-, -), \Delta, \Delta^\vee, \iota')$ where $\Delta = \Delta_k \cup (\Pi \setminus \Pi_k), \Delta^\vee = \Delta_k^\vee \cup (\Pi^\vee \setminus \Pi_k^\vee), \Delta_k = \{d_\alpha \alpha \mid \alpha \in \Pi_k\}, \Delta_k^\vee = \{d_\alpha^{-1} \alpha^\vee \mid \alpha \in \Pi_k\}$ and $(d_\alpha \alpha, d_\beta^{-1} \beta^\vee) = (d_\beta \beta, d_\alpha^{-1} \alpha^\vee)$ for all $\alpha, \beta \in \Pi_k$. Take any α such that $d(\alpha, \alpha_0) = k + 1$. Then one has some unique $\alpha' \in \Pi$ such that $d(\alpha', \alpha_0) = k$ and $d(\alpha', \alpha) = 1$.

Let $d_{\alpha'}\alpha' \in \Delta_k$. Denote $d_{\alpha} = d_{\alpha'}\sqrt{\frac{(\alpha', \alpha^{\vee})}{(\alpha, (\alpha')^{\vee})}}$. Then let $\Xi = \Delta_k \cup \{d_{\alpha}\alpha \mid d(\alpha, \alpha_0) = k + 1\} \cup (\Pi \setminus \Pi_{k+1})$ and $\Xi^{\vee} = \Delta_k^{\vee} \cup \{d_{\alpha}^{-1}\alpha^{\vee} \mid d(\alpha, \alpha_0) = k + 1\} \cup (\Pi^{\vee} \setminus \Pi_{k+1}^{\vee})$. Let $l'' = l'$ when acting on $\Xi \setminus \{d_{\alpha}\alpha \mid d(\alpha, \alpha_0) = k + 1\}$ and $l''(d_{\alpha}\alpha) = d_{\alpha}^{-1}\alpha^{\vee}$ otherwise. Then $(V, V^{\vee}, (-, -), \Xi, \Xi^{\vee}, l'')$ is a rescaled realized root system datum having the property that $(d_{\alpha}\alpha, d_{\beta}^{-1}\beta^{\vee}) = (d_{\beta}\beta, d_{\alpha}^{-1}\alpha^{\vee})$ for all $\alpha, \beta \in \Pi_{k+1}$. For sufficiently large k , this gives the desired conclusion since S is assumed finite. \square

Corollary 3.3 *If the Coxeter graph of (W, S) is a forest and has no infinite bonds (i.e., the order of ss' is finite for any $s, s' \in S$), then all realized root systems of (W, S) with linearly independent simple roots give the same oriented matroid structure on $T \times \{\pm 1\}$.*

Proof By Proposition 3.2, $(V, V^{\vee}, (-, -), \Pi, \Pi^{\vee}, l)$ can be rescaled to a realized root system datum having symmetric generalized Cartan matrix and the rescaling does not affect the oriented matroid structure by Lemma 2.6. Combining this with the fact the Coxeter graph has no infinite bonds, one can assume that the realized root system is the standard one. The assertion follows. \square

3.4 Linearly dependent simple roots

We now discuss the relationship between realized root systems with positively independent simple roots and those with linearly independent simple roots (compare [10], [19, §6.1]). Suppose given a realized root system datum $D := (V, V^{\vee}, (-, -), \Pi, \Pi^{\vee}, l)$ in which Π is linearly independent. Let $U \subseteq V$ be a subspace of the left radical of the bilinear form $(-, -)$ such that $U \cap \text{cone}(\Pi) = 0$. Let $V' := V/U$ and $\pi: V \rightarrow V'$ be the canonical surjection. The restriction of π to Π is injective and $\Pi' = \pi(\Pi)$, is positively independent, by the assumptions on U . Define a bijection $l': \Pi' \rightarrow \Pi^{\vee}$ by $l'(\pi(\alpha)) = l(\alpha)$ and a bilinear form $(-, -)': V' \times V^{\vee} \rightarrow \mathbb{R}$ by $(\pi(v), v') := (v, v')$ for $v \in V$ and $v' \in V^{\vee}$. Then $D' := (V', V^{\vee}, (-, -)', \Pi', \Pi^{\vee}, l')$ is a realized root system datum with the same NGCM as D . The Coxeter system attached to D' is therefore canonically isomorphic to (W, S) , by an isomorphisms mapping $s_{\alpha} \mapsto s_{\pi(\alpha)}$ for all $\alpha \in \Pi$. The definitions show that

$$\pi(s_{\alpha}(v)) = \pi(v - (v, \alpha^{\vee})\alpha) = \pi(v) - (\pi(v), \alpha^{\vee})\pi(\alpha) = s_{\pi(\alpha)}(\pi(v))$$

for $\alpha \in \Pi$ and $v \in V$, and it follows readily that π is W -equivariant. This implies the root system for D' is $\Phi' = \pi(\Phi)$. It is easily seen that any realized root system is isomorphic to one obtained in this way from a realized root system with linearly independent simple roots (compare [10, 3.5]). Moreover, rescaling D simply has the effect of rescaling D' . We remark that the oriented matroid structures on $T \times \{\pm 1\}$ arising by transfer of those on Φ and Φ' have ranks $|S|$ and $|S| - \dim(U)$, so are not the same if $U \neq 0$. Analogous facts to those above also apply to the “dual” root system Φ^{\vee} in V^{\vee} , so for study of Φ and Φ' above, there would be no loss of generality in assuming from the outset that Π^{\vee} is linearly independent.

If under the assumptions of the previous corollary, the standard realized root system datum D is such that there is no non-trivial subspace U of V satisfying the above

assumptions, then D' above must be isomorphic to D , and it follows that the assumption in the hypotheses of the Corollary that the simple roots are linearly independent can be omitted. It is well known that for realized root systems of finite Coxeter groups, the (left) radical of the form for the standard realized root system datum D is zero, and for irreducible affine Weyl groups, the left radical is one-dimensional, spanned by a vector lying in $\text{cone}(\Pi)$. Thus, no non-trivial subspace U as above exists in either case. Therefore we have

Corollary 3.5 *For a finite Coxeter system or an irreducible affine Weyl group which is not of type \tilde{A}_n , all realized root systems give the same oriented matroid structure on $T \times \{\pm 1\}$.*

For any Coxeter system with two or more components which are affine Weyl groups, the left radical of the bilinear form for the standard realized root system datum D contains a non-trivial subspace which meets the cone spanned by the positive roots only at zero, and so the oriented matroid structures from realized root systems of such a Coxeter system are not unique.

4 Realized root systems of type \tilde{A}_n

4.1 Realized root systems of type \tilde{A}_n

In this section we deal with the case that the Coxeter system (W, S) is of type \tilde{A}_n , $(n \geq 2)$. We give explicitly a formula for values of the chirotope map and show that for $n \geq 3$ there are three different oriented matroid structures from realized root systems. We take $S = \{s_0, s_1, s_2, \dots, s_n\}$ and

$$W = \langle S | s_0^2 = s_1^2 = \dots = s_n^2 = e, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i = 0, 1, \dots, n - 1, \\ s_0 s_n s_0 = s_n s_0 s_n, \quad s_i s_j = s_j s_i \text{ if } 0 \leq i < j \leq n \text{ and } 1 < j - i < n \rangle.$$

Then W admits a realized root system as in the previous section. We assume that $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ spans V and must have

$$(\alpha_i, \alpha_{i+1}^\vee) = a_i, \quad (\alpha_{i+1}, \alpha_i^\vee) = \frac{1}{a_i}, \quad 0 \leq i \leq n - 1 \\ (\alpha_n, \alpha_0^\vee) = a_n, \quad (\alpha_0, \alpha_n^\vee) = \frac{1}{a_n} \\ (\alpha_i, \alpha_i^\vee) = 2, \\ (\alpha_i, \alpha_j^\vee) = 0 \text{ otherwise}$$

for certain negative real numbers a_0, \dots, a_n . By Lemma 2.6 if we rescale the roots by various positive numbers, it does not change the oriented matroid structure of Φ . Take $v = \sqrt{|a_0 a_1 a_2 \dots a_n|}$. We replace $\alpha_i, 2 \leq i \leq n$ with $|a_1 a_2 \dots a_{i-1}| \alpha_i, \alpha_i^\vee, 2 \leq i \leq n$ with $\frac{1}{|a_1 a_2 \dots a_{i-1}|} \alpha_i^\vee, \alpha_0$ with $\frac{|a_1 a_2 \dots a_n|}{v} \alpha_0$ and α_0^\vee with $\frac{v}{|a_1 a_2 \dots a_n|} \alpha_0^\vee$.

Thus without loss of generality, we assume that

$$\begin{aligned}
 (\alpha_0, \alpha_1^\vee) &= -v, (\alpha_1, \alpha_0^\vee) = -\frac{1}{v} \\
 (\alpha_i, \alpha_{i+1}^\vee) &= -1, (\alpha_{i+1}, \alpha_i^\vee) = -1, 1 \leq i \leq n - 1 \\
 (\alpha_n, \alpha_0^\vee) &= -v, (\alpha_0, \alpha_n^\vee) = -\frac{1}{v} \\
 (\alpha_i, \alpha_i^\vee) &= 2,
 \end{aligned}$$

and $(\alpha_i, \alpha_j^\vee) = 0$ otherwise.

One can check that the NGCM $((\alpha_i, \alpha_j^\vee))_{i,j=0,\dots,n}$ has determinant $-(v - v^{-1})^2$. It is well known that for $v = 1$ (see for example [18]), the left radical of the form $(-, -)$ is one-dimensional, spanned by $\alpha_0 + \dots + \alpha_n \in \text{cone}(\Pi)$, and the determinant computation shows the (left or right) radical is zero if $v \neq 1$. It follows from the discussion at the end of Sect. 3 that any realized root system of type \tilde{A}_n has linearly independent simple roots and simple coroots, and is isomorphic to one of those considered above for some (in fact, uniquely determined) positive real number v .

We remark that distinct positive real scalars v afford non-isomorphic representations of W on the linear span of Π , and these may all be viewed as specializations of a generic reflection representation of W on a free $\mathbb{R}[v, v^{-1}]$ -module (or even $\mathbb{Z}[v, v^{-1}]$ -module) where v is an indeterminate, with basis $\alpha_0, \dots, \alpha_n$ (see [10, Chapter 2]). The formulae we give for $v \in \mathbb{R}$ apply equally well in this generic setting, with v regarded instead as an indeterminate.

For $1 \leq i \leq j \leq n$ we denote

$$\begin{aligned}
 \alpha'_{i,j} &= \alpha_0 + v(\alpha_1 + \alpha_2 + \dots + \alpha_{i-1}) + v^{-1}(\alpha_{j+1} + \alpha_{j+2} + \dots + \alpha_n) \\
 &= s_{\alpha_{j+1}} s_{\alpha_{j+2}} \dots s_{\alpha_n} s_{\alpha_{i-1}} s_{\alpha_{i-2}} \dots s_{\alpha_1}(\alpha_0)
 \end{aligned}$$

and denote

$$\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \dots + \alpha_j = s_{\alpha_j} \dots s_{\alpha_{i+2}} s_{\alpha_{i+1}}(\alpha_i).$$

For all $n \in \mathbb{Z}$, define $c_n := \frac{v^n - v^{-n}}{v - v^{-1}}$ if $v \neq 1$ and $c_n := n$ if $v = 1$. We have $c_0 = 0$, $c_1 = 1$, $c_{-n} = -c_n$ and if $n > 0$, then $c_n = v^{n-1} + v^{n-3} + \dots + v^{-n+1}$. (If v is regarded as indeterminate, then c_n is known as a Gaussian (or quantum) integer).

The following identity is well known and easily verified.

Lemma 4.2 $c_{n+1}c_{m+1} - c_n c_m = c_{n+m+1}$.

Next we describe the root system explicitly.

Proposition 4.3 $\Phi = \{\pm v^m(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) \mid m, k \in \mathbb{Z}, 1 \leq i \leq j \leq n\}$.

Proof We first show that if α is a root then $v^m\alpha$, where $m \in \mathbb{Z}$, is a root. It suffices to show that for α simple and $m = \pm 1$. For α_0 we have $s_{\alpha_1} s_{\alpha_0}(\alpha_1) = v^{-1}\alpha_0$ and $s_{\alpha_n} s_{\alpha_0}(\alpha_n) = v\alpha_0$. And we have $s_{\alpha_0} s_{\alpha_1}(\alpha_0) = v\alpha_1$, $s_{\alpha_1} s_{\alpha_2}(v\alpha_1) =$

$v\alpha_2, \dots, s_{\alpha_{n-1}}s_{\alpha_n}(v\alpha_{n-1}) = v\alpha_n$. $s_{\alpha_0}s_{\alpha_n}(\alpha_0) = v^{-1}\alpha_n$ and $s_{\alpha_n}s_{\alpha_{n-1}}(v^{-1}\alpha_n) = v^{-1}\alpha_{n-1}, \dots, s_{\alpha_2}s_{\alpha_1}(v^{-1}\alpha_2) = v^{-1}\alpha_1$.

Since $(s_{\alpha_j} \dots s_{\alpha_{i+2}}s_{\alpha_{i+1}}(\alpha_i))^\vee = s_{\alpha_j} \dots s_{\alpha_{i+2}}s_{\alpha_{i+1}}(\alpha_i^\vee)$ we have $\alpha_{i,j}^\vee = \alpha_i^\vee + \alpha_{i+1}^\vee + \dots + \alpha_j^\vee$. Similarly one sees that $(\alpha'_{i,j})^\vee = \alpha_0^\vee + v^{-1}(\alpha_1^\vee + \alpha_2^\vee + \dots + \alpha_{i-1}^\vee) + v(\alpha_{j+1}^\vee + \alpha_{j+2}^\vee + \dots + \alpha_n^\vee)$. One checks that $s_{\alpha_{i,j}}(\alpha'_{i,j}) = c_1\alpha'_{i,j} + c_2\alpha_{i,j}$ and $s_{\alpha'_{i,j}}(\alpha_{i,j}) = c_1\alpha_{i,j} + c_2\alpha'_{i,j} (= -(c_{-1}\alpha_{i,j} + c_{-2}\alpha'_{i,j}))$. Then an inductive argument (together with Lemma 4.2) shows that by acting by elements from the (infinite) dihedral subgroup $\langle s_{\alpha_{i,j}}, s_{\alpha'_{i,j}} \rangle$ on the roots $\alpha_{i,j}, \alpha'_{i,j}$ one gets the set $\{\pm(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) | k \in \mathbb{Z}\}$.

Therefore we see $\Phi \supseteq \{\pm v^m(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) | m, k \in \mathbb{Z}, 1 \leq i \leq j \leq n\}$. To finish the proof one only needs to check that the right-hand side is stable under the action of simple reflections. This may be done by easy calculations, noting the following cases:

$$\begin{aligned} s_{\alpha_p}(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) &= c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}, & p \neq i-1, i, j, j+1 \\ s_{\alpha_{i-1}}(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) &= c_{k+1}\alpha_{i-1,j} + c_k\alpha'_{i-1,j}, & i \geq 2, \\ s_{\alpha_i}(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) &= c_{k+1}\alpha_{i+1,j} + c_k\alpha'_{i+1,j}, & i \neq j, \\ s_{\alpha_i}(c_{k+1}\alpha_{i,i} + c_k\alpha'_{i,i}) &= -(c_{-k+1}\alpha_{i,i} + c_{-k}\alpha'_{i,i}), \\ s_{\alpha_j}(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) &= c_{k+1}\alpha_{i,j-1} + c_k\alpha'_{i,j-1}, & i \neq j, \\ s_{\alpha_{j+1}}(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) &= c_{k+1}\alpha_{i,j+1} + c_k\alpha'_{i,j+1}, & j \leq n-1, \\ s_{\alpha_0}(c_{k+1}\alpha_{1,j} + c_k\alpha'_{1,j}) &= -v^{-1}(c_{-k}\alpha_{j+1,n} + c_{-k-1}\alpha'_{j+1,n}), & j \leq n-1, \\ s_{\alpha_0}(c_{k+1}\alpha_{1,n} + c_k\alpha'_{1,n}) &= -(c_{-k-1}\alpha_{1,n} + c_{-k-2}\alpha'_{1,n}), \\ s_{\alpha_0}(c_{k+1}\alpha_{j,n} + c_k\alpha'_{j,n}) &= -v(c_{-k}\alpha_{1,j-1} + c_{-k-1}\alpha'_{1,j-1}), & j \geq 2. \end{aligned}$$

□

We shall denote Φ as $\Phi_{(n,v)}$ if necessary to indicate its dependence on n and v . Now we show that for a root $\alpha \in \Phi$ and integer m , $(v^m\alpha)^\vee = v^{-m}\alpha^\vee$. Note that $\Pi \subset W\{\alpha_0, \alpha_n\}$ and therefore $\Phi = W\{\alpha_0, \alpha_n\}$. Suppose that the identity holds for roots $\alpha_i, i = 0, n$. Then for any $w(\alpha_i) \in \Phi$, $(v^m w(\alpha_i))^\vee = (w(v^m\alpha_i))^\vee = w((v^m\alpha_i)^\vee) = w(v^{-m}\alpha_i^\vee) = v^{-m}(w(\alpha_i)^\vee)$ thanks to the W -equivariance of ι . Now one checks that for $m > 0$, $(v^m\alpha_0)^\vee = ((s_{\alpha_n}s_{\alpha_0})^m(\alpha_n))^\vee = (s_{\alpha_n}s_{\alpha_0})^m(\alpha_n^\vee) = v^{-m}\alpha_0^\vee$ and $(v^{-m}\alpha_0)^\vee = ((s_{\alpha_1}s_{\alpha_0})^m\alpha_1)^\vee = (s_{\alpha_1}s_{\alpha_0})^m(\alpha_1^\vee) = v^m\alpha_0^\vee$. Similarly one can verify that $(v^m\alpha_n)^\vee = v^{-m}\alpha_n^\vee$. Therefore we have $s_{v^m\alpha} = s_\alpha$. We say that two roots α, β in Φ are associate, written $\alpha \sim \beta$, if one of them is a power of v times the other. The reflections of W correspond bijectively to the \sim -equivalence classes of positive roots.

Lemma 4.4 *The infinite maximal dihedral reflection subgroups of W are the subgroups $W_{i,j} := \langle s_{\alpha_{i,j}}, s_{\alpha'_{i,j}} \rangle$ for $1 \leq i \leq j \leq n$. The set of roots of Φ for which the corresponding reflection lies in $W_{i,j}$ is*

$$\Phi_{i,j} := \{ \eta v^m(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j}) | \eta \in \{\pm 1\}, m, k \in \mathbb{Z} \}.$$

One has $\Phi = \bigcup_{1 \leq i \leq j \leq n} \Phi_{i,j}$.

Proof We first assume that $v = 1$. In this case, the realized root system is the standard reduced root system as defined in [1] or [18]. We denote $\delta = \alpha_0 + \alpha_1 + \dots + \alpha_n$. Then $(\delta, \gamma^\vee) = (\gamma, \delta^\vee) = 0$ for all $\gamma \in \Phi$. When $v = 1$ it is well known (see for example [9], though it also follows from Proposition 4.3 with $v = 1$) that the set of positive roots is

$$\{\alpha_{i,j} + k\delta \mid 1 \leq i \leq j \leq n, k \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha_{i,j} + k\delta \mid 1 \leq i \leq j \leq n, k \in \mathbb{Z}_{> 0}\}.$$

Consider the dihedral reflection subgroup $W_{i,j} = \langle s_{\alpha_{i,j}}, s_{\delta - \alpha_{i,j}} \rangle$. Since we have $(\alpha_{i,j}, (\delta - \alpha_{i,j})^\vee) = -2$, the group is infinite and $\Phi_{i,j} := \{\alpha \in \Phi \mid s_\alpha \in W_{i,j}\} = \{\alpha_{i,j} + k\delta \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha_{i,j} + k\delta \mid k \in \mathbb{Z}_{> 0}\}$. One checks that $\text{span}(\Pi_{i,j}) \cap \Phi = \Phi_{i,j}$. This guarantees that $W_{i,j}$ is maximal by Remark 3.2 of [15]. For $\{i, j\} \neq \{p, q\}$, the dihedral reflection subgroups $\langle s_{\epsilon_1 \alpha_{i,j} + t\delta}, s_{\epsilon_2 \alpha_{p,q} + r\delta} \rangle, \epsilon_1, \epsilon_2 \in \{1, -1\}, t, r \in \mathbb{Z}$ are finite. Hence the groups $W_{i,j}$ exhaust all maximal dihedral reflection subgroups.

Note $s_{\delta - \alpha_{i,j}}$ corresponds to $s_{\alpha_{j+1}} s_{\alpha_{j+2}} \dots s_{\alpha_n} s_{\alpha_{i-1}} s_{\alpha_{i-2}} \dots s_{\alpha_1}(\alpha_0)$. For arbitrary v the latter is $s'_{\alpha'_{i,j}}$. For any $v, s_{\alpha_{i,j}}$ corresponds to $s_{\alpha_j} \dots s_{\alpha_{i+1}}(\alpha_i)$. Since the maximal dihedral reflection subgroups are independent of v , this proves the first assertion. The formula for $\Phi_{i,j}$ follows since its right-hand side is the set of all roots in Φ in the plane spanned by $\alpha_{i,j}$ and $\alpha'_{i,j}$, while, from the preceding proof, $W_{i,j}\{\alpha_{i,j}, \alpha'_{i,j}\}$ consists of the roots $\pm(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j})$. The final claim is clear from Proposition 4.3. \square

4.5 Chirotopes for \tilde{A}_n

Our next goal is to compute the values of the chirotope map of the oriented matroid $(\Phi, -, \text{cone}_\Phi)$ by calculating the determinant of $n + 1$ roots $\gamma_0, \dots, \gamma_n \in \Phi$. More precisely, we calculate the determinant of the linear operator which maps $\alpha_j \mapsto \gamma_j$ for $j = 0, \dots, n$. Regard γ_j as a column vector, with rows indexed by $0, \dots, n$, with the entry in its i -th row being the coefficient of α_i in γ_j . Let $(\gamma_0, \dots, \gamma_n)$ be the $(n + 1) \times (n + 1)$ -matrix, with rows and columns indexed by $0, \dots, n$, whose j -th column is γ_j (viewed as column vector); this matrix represents the above linear transformation with respect to the ordered basis $\alpha_0, \dots, \alpha_n$ of V , and we seek to determine $\det(\gamma_0, \gamma_1, \dots, \gamma_n)$. For later use, observe that for $1 \leq i \leq j \leq n, m, k \in \mathbb{Z}$ and $\eta \in \{\pm 1\}$, (the coordinate vector of the) root $\eta v^m(c_{k+1}\alpha_{i,j} + c_k\alpha'_{i,j})$ is

$$\eta v^m(c_k, v c_k, \dots, v c_k, c_{k+1}, \dots, c_{k+1}, v^{-1} c_k, \dots, v^{-1} c_k)^T \tag{4.5.1}$$

where T denotes transpose, the entry c_k is in row 0, the first c_{k+1} is in row i and the last c_{k+1} is in row j . The main result we shall establish is the following:

Theorem 4.6 *Suppose $\gamma_0, \gamma_1, \dots, \gamma_n$ are $n + 1$ roots of W of type $\tilde{A}_n, n \geq 2$. Regard γ_i as coordinate vectors as above. Then $\det(\gamma_0, \gamma_1, \dots, \gamma_n)$ is either 0 or of the form*

$$\mu v^l (v - v^{-1})^{m-1} \prod_{k=1}^m c_{h_k}$$

where $\mu \in \{1, -1\}, m \geq 1$ and $h_1, \dots, h_m \in \mathbb{Z}_+$.

More precisely, this result holds (with v regarded as an indeterminate) for the generic reflection representation over $\mathbb{R}[v, v^{-1}]$. It would be equivalent to just require $h_i \in \mathbb{Z}$ since $c_n = -c_n$ and $c_0 = 0$.

Observe that if there is a permutation σ of $\{0, \dots, n\}$ with $\text{sgn}(\sigma) = \epsilon \in \{\pm 1\}$, an element $w \in W$, integers $\eta_i \in \{\pm 1\}$, and integers m_i such that with $\gamma'_i = \eta_i v^{m_i} w(\gamma_{\sigma(i)})$ for $i = 0, \dots, n$, then

$$\det(\gamma'_0, \dots, \gamma'_n) = (-1)^{l(w)} \epsilon \eta_0 \dots \eta_n v^{m_0 + \dots + m_n} \det(\gamma_0, \dots, \gamma_n), \tag{4.6.1}$$

so validity of the theorem for $\gamma_0, \dots, \gamma_n$ is equivalent to its validity for $\gamma'_0, \dots, \gamma'_n$.

We begin the proof with a series of results to deal with the cases in which some maximal dihedral root subsystems $\Phi_{i,j}$ contain exactly two of the roots.

Proposition 4.7 *Let $A = (a_{ij})$ be a $n \times n$ matrix with $n \geq 2$. Assume that the i -th column is of the form*

$$(0, \dots, 0, x_i, 0, \dots, 0, -1, 0, \dots, 0)^T$$

for $i = 1, \dots, n$. Then $\det(A)$ is either 0 or

$$\mu \prod(x_i) \prod(x_{j_1} x_{j_2} \dots x_{j_l} - x_{k_1} x_{k_2} \dots x_{k_m})$$

where $\mu \in \{1, -1\}$, each x_q appears at most once in this formula and each l, m appearing is positive.

Proof We prove this by induction. If $n = 2$ the matrix can only be

$$\begin{pmatrix} x_1 & x_2 \\ -1 & -1 \end{pmatrix}$$

whose determinant is $x_2 - x_1$.

We claim that after suitably permuting the columns and rows this matrix must contain a $p \times p$ -block of the form

$$\begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & a_p \\ a'_1 & a_2 & 0 & \dots & 0 & 0 \\ 0 & a'_2 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{p-1} & 0 \\ 0 & 0 & 0 & \dots & a'_{p-1} & a'_p \end{pmatrix}$$

where $p \geq 2$ and $\{a_q, a'_q\} = \{x_{j_q}, -1\}$ for $q = 1, \dots, p$.

To see this, consider a loopless graph (with parallel edges allowed) with vertex set $\{1, \dots, n\}$ and n edges, joining k_i, l_i for $i = 1, \dots, n$ where $k_i \neq l_i$ and the entries $a_{k_i, i}, a_{l_i, i}$ of A are nonzero. Such a graph must have a circuit, i.e., distinct edges with endpoints of the form

$$\{t_1, t_2\}, \{t_2, t_3\}, \dots, \{t_{p-1}, t_p\}, \{t_1, t_p\}$$

where $p \geq 2$. For the above block, we suitably permute the rows and columns of the submatrix of A with rows indexed by t_1, \dots, t_p and columns corresponding to the above p edges.

The above submatrix has determinant $a_1 \dots a_{p-1} a'_p - (-1)^p a'_1 \dots a'_{p-1} a_p$. Note that if a_1, \dots, a_{p-1} (resp., a'_1, \dots, a'_{p-1}) are all -1 then so is a_p (resp., a'_p). It follows the determinant is of the form $\mu_1(x_{j_1}x_{j_2} \dots x_{j_l} - x_{k_1}x_{k_2} \dots x_{k_m})$ where $l + m = p$, $0 < l < p$, $\mu_1 \in \{1, -1\}$ and $j_1, \dots, j_l, k_1, \dots, k_m$ are pairwise distinct.

After suitably permuting the columns and rows, A is of the following form

$$\begin{pmatrix} A_1 & A_2 \\ 0 & B \end{pmatrix}$$

with A_1 having the above form and the column of B are determined by deleting certain entries from the columns of A with index unequal to $j_1, \dots, j_l, k_1, \dots, k_m$. Then $\det(A) = \mu'_1 \det(B)(x_{j_1}x_{j_2} \dots x_{j_l} - x_{k_1}x_{k_2} \dots x_{k_m})$ with $\mu'_1 \in \{1, -1\}$.

If B has a column consisting of all zeroes, then $\det(B) = 0$ and we are done. So we assume this is not the case. Now we define a sequence of matrices $\{B_i\}_{i=0}^\infty$. Let $B_0 = B$. If every column of B_i has two nonzero entries or B_i is a 1-by-1 matrix or $B_i = 0$, then $B_{i+1} = B_i$. Otherwise if column q is the first column from the left which has only one nonzero entry and it is the entry $a_{q,v}$, then B_{i+1} is obtained by removing q -th row and v -th column from B_i . Then eventually we always have $B_j = B_{j+1} = \dots$ and we denote $B_\infty = B_j$. Then it is clear that $\det(B) = \eta \prod_{i \in K} x_i \det(B_\infty)$, $\eta \in \{1, -1\}$.

B_∞ must be a matrix of one of the following forms:

- (1) a zero matrix
- (2) a one-by-one matrix with entry -1 or x_l for some l
- (3) a matrix with the same property as in the lemma.

Then the lemma follows by induction. □

Corollary 4.8 *Let A be a $n \times n$ matrix with $n \geq 2$ and i -th column of the form*

$$(0, \dots, 0, x_i, 0, \dots, 0, -x_i^{-1}, 0, \dots, 0)^T$$

for $i = 1, \dots, n$. Then $\det(A)$ is either 0 or

$$\mu \frac{1}{x_1 x_2 \dots x_n} \prod x_p^2 \prod_{k=1}^m (\prod_{i \in I_k} x_i^2 - \prod_{j \in J_k} x_j^2)$$

where $\mu \in \{1, -1\}$ and the product $\prod x_p^2 \prod_{k=1}^m (\prod_{i \in I_k} x_i^2 - \prod_{j \in J_k} x_j^2)$ contains each x_q at most once.

Corollary 4.9 *Let A be a $n \times n$ matrix with $n \geq 2$. Each column is of the form*

$$(0, \dots, 0, v^{a_i}, 0, \dots, 0, -v^{-a_i}, 0, \dots, 0)^T.$$

Then $\det(A)$ is either 0 or

$$\mu v^l (v - v^{-1})^m \prod_{k=1}^m c_{h_k}$$

where $\mu \in \{1, -1\}$ and $m \in \mathbb{N}$.

Proof Corollary 4.8 shows that the determinant is zero or

$$\begin{aligned} & \mu v^l \prod_{k=1}^m (v^{2A_k} - v^{2B_k}) \\ &= \mu v^l \prod_{k=1}^m v^{A_k+B_k} (v^{-B_k+A_k} - v^{-A_k+B_k}) \\ &= \mu v^l \prod_{k=1}^m \frac{v - v^{-1}}{v - v^{-1}} (v^{-B_k+A_k} - v^{-A_k+B_k}) \end{aligned}$$

Clearly the assertion holds for any $v > 0$ even though the second equality above requires $v \neq 1$. □

Lemma 4.10 Theorem 4.6 holds in the case that there exist $i \neq j$ in $\{0, \dots, n\}$ such that $\gamma_i = \alpha_0$ and $\gamma_j = c_{k_j}\alpha_0 + c_{k_j+1}(\alpha_1 + \dots + \alpha_n)$ for some $k_j \in \mathbb{Z}$.

Proof Using (4.6.1), we may assume $\gamma_0 = \alpha_0$ and $\gamma_n = c_{k_n}\alpha_0 + c_{k_n+1}(\alpha_1 + \dots + \alpha_n)$.

We look at the $n \times n$ lower-right block A of the matrix $(\gamma_0, \gamma_1, \dots, \gamma_n)$. Equation (4.5.1) shows that after factoring out a power of v and ± 1 , each column is of the form

$$(v c_{k_i}, \dots, v c_{k_i}, c_{k_i+1}, \dots, c_{k_i+1}, v^{-1} c_{k_i}, \dots, v^{-1} c_{k_i})^T.$$

The last column is

$$(c_{k_n+1}, \dots, c_{k_n+1}).$$

In the subsequent computation we assume $v \neq 1$ and $c_t = \frac{v^t - v^{-t}}{v - v^{-1}}$. The results apply for $v = 1$ as well because it is a removable discontinuity of $\frac{v^t - v^{-t}}{v - v^{-1}}$.

Multiplying each column by $v - v^{-1}$ gives a matrix B . With columns of the form

$$(v(v^{k_i} - v^{-k_i}), \dots, v(v^{k_i} - v^{-k_i}), v^{k_i+1} - v^{-k_i-1}, \dots, v^{k_i+1} - v^{-k_i-1}, v^{-1}(v^{k_i} - v^{-k_i}), \dots, v^{-1}(v^{k_i} - v^{-k_i}))^T.$$

In particular the last column is of the form

$$(v^{k_n+1} - v^{-k_n-1}, \dots, v^{k_n+1} - v^{-k_n-1})^T.$$

And we have $\det(\gamma_0, \gamma_1, \dots, \gamma_n) = \det(A) = (v - v^{-1})^{-n} \det(B)$

Now we compute the determinant of B . To do this, we subtract the first row from each of the rows below. There are three cases:

- (1) the last column is

$$((v^{k_n+1} - v^{-1-k_n}), 0, \dots, 0)^T$$

(2) the first entry of the column is $v^{k_i+1} - v^{-k_i-1}$

Then the column will become

$$(v^{k_i+1} - v^{-k_i-1}, 0, \dots, 0, v^{k_i}(v^{-1} - v), \dots, v^{k_i}(v^{-1} - v))^T$$

(3) the first entry of the column is $v(v^{k_i} - v^{-k_i})$

Then this column will become

$$(v(v^{k_i} - v^{-k_i}), 0, \dots, 0, v^{-k_i}(v - v^{-1}), \dots, v^{-k_i}(v - v^{-1}), (v^{k_i} - v^{-k_i})(v^{-1} - v), \dots, (v^{k_i} - v^{-k_i})(v^{-1} - v))^T$$

Now we examine the lower left $(n - 1) \times (n - 1)$ block of B (denoted C). $\det(B) = \rho(v^{k_n+1} - v^{-1-k_n}) \det(C)$ where $\rho \in \{1, -1\}$. So $\det(A) = \rho(v^{k_n+1} - v^{-1-k_n})(v - v^{-1})^{-n} \det(C)$.

We factor $v - v^{-1}$ from each column and obtain a $(n - 1) \times (n - 1)$ matrix D whose columns are either

$$(0, \dots, 0, v^{k_i}, \dots, v^{k_i})^T$$

or

$$(0, \dots, 0, v^{-k_i}, \dots, v^{-k_i}, -(v^{k_i} - v^{-k_i}), \dots, -(v^{k_i} - v^{-k_i}))^T.$$

$$\det(A) = \rho(v^{k_n+1} - v^{-1-k_n})(v - v^{-1})^{-1} \det(D) = \rho c_{k_n+1} \det(D).$$

Next we consecutively subtract the 1st, 2nd, 3rd ...row from the rows below and transform D into a matrix with columns of the form either

$$(0, \dots, 0, v^{k_i}, 0, \dots, 0)^T$$

or

$$(0, \dots, 0, v^{-k_i}, 0, \dots, 0, -v^{k_i}, 0 \dots, 0)^T.$$

Expanding the resulting determinant down any column containing a single entry until no such columns remain, we obtain a matrix to which Corollary 4.9 applies, proving the desired formula. □

Lemma 4.11 Theorem 4.6 holds in the case that there are i, j with $1 \leq i \leq j \leq n$ and $p \neq q$ in $\{0, \dots, n\}$ such $\gamma_p, \gamma_q \in \Phi_{i,j}$.

Proof Using (4.6.1), we may without loss of generality replace the roots γ_i by their conjugates by some $w \in W$, so one of the two roots in $\Phi_{i,j}$ is an associate of α_0 . Then by Lemma 4.4, $(i, j) = (1, n)$, both roots lie in $\Phi_{1,n}$ and the other one must be associate, up to sign, to a root $c_k\alpha_0 + c_{k+1}(\alpha_1 + \dots + \alpha_n)$ for some integer k . The lemma now follows from Lemma 4.10. □

We next state the last lemma needed for the inductive proof of Theorem 4.6. Recall that we identify roots with their coordinate vectors with respect to the (naturally ordered) simple roots.

Lemma 4.12 *Assume $n \geq 3$ and $1 \leq i \leq n$. Let $\gamma \in \Phi \setminus \Phi_{i,i}$. Regard γ as a column vector with rows indexed by $0, \dots, n$. Let β be the column vector obtained by deleting the i -th row of γ . Then β is the coordinate vector for some root of the root system $\Phi_{(n-1,v)}$ of type \tilde{A}_{n-1} and parameter v .*

Proof Deletion of the entry from row i of the coordinate vector for $\gamma \in \Phi$ as in (4.5.1) gives a coordinate vector for a root in type \tilde{A}_{n-1} unless there is a single entry c_{k+1} and it occurs in row i , which, by Lemma 4.4, occurs only when $\gamma \in \Phi_{i,i}$. \square

We now state and prove the following more precise version of Theorem 4.6

Theorem 4.13 *Suppose $\gamma_0, \gamma_1, \dots, \gamma_n$ are $n + 1$ roots of W of type $\tilde{A}_n, n \geq 2$. Regard γ_i as coordinate vectors as above. Then $d := \det(\gamma_0, \gamma_1, \dots, \gamma_n)$ is either 0 or of the form*

$$\mu v^l (v - v^{-1})^{m-1} \prod_{k=1}^m c_{h_k}$$

where $\mu \in \{1, -1\}, m \geq 1$ and $h_1, \dots, h_m \in \mathbb{Z}_+$. Further, if $d \neq 0$ and $n = 2$, then $m = 1$, while if $n \geq 3$, there exist roots $\gamma_0, \dots, \gamma_n$ for which d is of the form displayed above with $m > 1$.

Proof We prove the theorem by induction. Let $n = 2$. If there are two roots whose corresponding reflections are contained in the same maximal dihedral reflection subgroup then Lemma 4.11 ensures the determinant has the desired form. But we still have to show that the power $m - 1$ of $(v - v^{-1})$ is zero. Using (4.6.1), we may assume the first root is $\alpha_0 \in \Phi_{1,2}$ and the second is $c_k \alpha_0 + c_{k+1} (\alpha_1 + \alpha_2) \in \Phi_{1,2}$. If the third is in $\Phi_{1,2}$, then $d = 0$. It suffices to show the determinants of the following matrices are of the required form:

$$A = \begin{pmatrix} 1 & c_k & c_{k'} \\ 0 & c_{k+1} & v c_{k'} \\ 0 & c_{k+1} & c_{k'+1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & c_k & c_{k'} \\ 0 & c_{k+1} & c_{k'+1} \\ 0 & c_{k+1} & v^{-1} c_{k'} \end{pmatrix}$$

For A , note that $v c_{k'} - c_{k'+1} = -v^{-k'}$ so $\det(A) = v^{-k'} c_{k'+1}$. For B , note that $v^{-1} c_{k'} - c_{k'+1} = -v^{k'}$ and $\det(B) = -v^{k'} c_{k+1}$.

Now assume, still with $n = 2$, that there are no two roots with corresponding reflections in the same maximal dihedral reflection subgroup. By conjugation one can assume one of the roots is α_0 . Then by (4.6.1), $\det(\gamma_0, \gamma_1, \gamma_2)$ is the product of a power of $v, \pm 1$ and the determinant of

$$\begin{pmatrix} 1 & c_k & c_{k'} \\ 0 & c_{k+1} & v c_{k'} \\ 0 & v^{-1} c_k & c_{k'+1} \end{pmatrix}.$$

By Lemma 4.2, it is of the form $\mu v^l c_{hd}$, proving this case.

Next, we consider the general case ($n \geq 3$). To show d is of the required form, we may assume by Lemma 4.11 that there are no two roots whose reflections are contained in the same maximal dihedral reflection subgroup. Conjugating γ_n suitably if necessary, we may assume using (4.6.1) that γ_n is equal to a simple root α_i where $1 \leq i \leq n$. By the assumption, none of $\gamma_0, \dots, \gamma_{n-1}$ is in $\Phi_{i,i}$. Expanding the determinant $\det(\gamma_0, \dots, \gamma_n)$ down the n -th column shows $d = \pm \det(\gamma'_0, \dots, \gamma'_{n-1})$ where γ'_j is obtained by deleting the i -th row of γ_j . By Lemma 4.12, each γ'_i for $i = 0, \dots, n - 1$ is the coordinate vector with respect to the (ordered) simple roots of a root of the root system Φ' of type A_{n-1} with parameter v . Hence d has the desired form by induction.

It remains to give an example to show $m > 1$ can occur in case $n \geq 4$. Let $\gamma_0 := c_2\alpha_{1,1} + c_1\alpha'_{1,1} = \alpha_0 + (v + v^{-1})\alpha_1 + v^{-1}(\alpha_2 + \dots + \alpha_n)$, $\gamma_i := \alpha_i$ for $i = 1, \dots, n - 2$, $\gamma_{n-1} := c_2\alpha_{n,n} + \alpha'_{n,n} = \alpha_0 + v\alpha_1 + \dots + v\alpha_{n-1} + (v + v^{-1})\alpha_n$ and $\gamma_n := \alpha_n$. A trivial calculation shows that $d = v - v^{-1}$ in this case. \square

Theorem 4.14 *If (W, S) is a finite Coxeter system or W is an irreducible affine Weyl group which is not of type $\tilde{A}_n, n \geq 3$, then all of its realizable root systems give the same oriented matroid structure. If W is of type $\tilde{A}_n, n \geq 3$, there are three oriented matroid structures from its realizable root systems.*

Proof For finite types and affine types except \tilde{A}_n where $n \geq 1$, this follows from Corollary 3.5. It has already been remarked that the result is well known for W of rank at most two, in particular for \tilde{A}_1 . For \tilde{A}_n where $n \geq 2$, the simple roots of $T \times \{\pm 1\}$ are $\alpha'_j := (s_j, 1)$ for $j = 0, \dots, n$. We consider the chirotope $\chi_{(n,v)}$ on $T \times \{\pm 1\}$ obtained by transfer from $\Phi_{(n,v)}$, with sign normalized by $\chi_{(n,v)}(\alpha'_0, \dots, \alpha'_n) = +$. The values of the chirotope $\chi_{(n,v)}$ in general are given by $\chi_{(n,v)}(w_0(\alpha'_{i_0}), \dots, w_n(\alpha'_{i_n})) = \text{sgn}(\det(w_0(\alpha_{i_0}), \dots, w_n(\alpha_{i_n})))$ for $w_j \in W$ and $i_j \in \{0, \dots, n\}$ for $j = 0, \dots, n$. Theorem 4.13 ensures that $\chi_{(2,v)}$ is independent of v , and that for $n \geq 3$ and $v, v' \in \mathbb{R}_{>0}$, one has $\chi_{(n,v)} = \chi_{(n,v')}$ if and only if $\text{sgn}(v - 1) = \text{sgn}(v' - 1)$. \square

5 Rank 3 Coxeter systems

In this section we show that for a given rank 3 Coxeter system all oriented matroid structures from realized root systems are the same using an argument involving homotopies of root systems. This has already been briefly sketched in a remark in [14]. We here elaborate the proof.

Lemma 5.1 *Consider a NGCM as a vector in $\mathbb{R}^{n \times n}$. Then the space of all NGCMs in $\mathbb{R}^{n \times n}$ associated to a fixed Coxeter system of rank n is path connected.*

Proof Let $A = (c_{\alpha,\beta})_{\alpha,\beta=1}^n$ and $B = (c'_{\alpha,\beta})_{\alpha,\beta=1}^n$ be two NGCMs in $\mathbb{R}^{n \times n}$ corresponding to the same Coxeter system (W, S) of rank n . Define a map $\phi: [0, 1] \rightarrow \mathbb{R}^{n \times n}$ as follows: for $t \in [0, 1]$, $\phi(t) = (c_{\alpha,\beta}^{(t)})_{\alpha,\beta=1}^n$ where $c_{\alpha,\beta}^{(t)} = 2$ if $\alpha = \beta$ and otherwise

$$c_{\alpha,\beta}^{(t)} = -|c'_{\alpha,\beta}|^t |c_{\alpha,\beta}|^{1-t}$$

and we interpret $0^t = 0$ for all $t \in [0, 1]$. It is easy to see from above that ϕ is continuous, $\phi(t)$ is a NGCM for all $t \in [0, 1]$, and $\phi(0) = A, \phi(1) = B$. \square

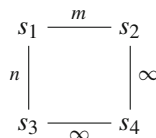
Theorem 5.2 *Let (W, S) be a rank 3 Coxeter system. Then all oriented matroid structures from realized root systems of (W, S) are equal.*

Proof If a realized root system of a Coxeter system (W', S') spans a subspace of dimension $d \leq 2$, then $|S'| = d$. Hence the simple roots (and coroots) of any realized root system of (W, S) are linearly independent, and the realized root system is completely determined by its NGCM. Let A, B be two NGCMs determining two realized root systems of (W, S) . Then by Lemma 5.1 we have a continuous map $\theta : [0, 1] \rightarrow \mathbb{R}^{3 \times 3}$ such that each $\theta(t)$ is a NGCM of (W, S) and $\theta(0) = A$ and $\theta(1) = B$. Let $\gamma_1, \gamma_2, \gamma_3 \in \Pi, w_1, w_2, w_3 \in W$. Let $(w_i(\gamma_i))_t, 1 \leq i \leq 3$ be the root $w_i(\alpha_i)$ in the realized root system having NGCM $\phi(t)$. Now we show that if $\det((w_1(\gamma_1))_0, (w_2(\gamma_2))_0, (w_3(\gamma_3))_0) = 0$ then $\det((w_1(\gamma_1))_t, (w_2(\gamma_2))_t, (w_3(\gamma_3))_t) = 0$ for any $0 \leq t \leq 1$. Note that the assumption implies that $(w_i(\gamma_i))_0, 1 \leq i \leq 3$ are linearly dependent. This is equivalent to the condition that either we have some $w_i(\alpha_i)_0 = kw_j(\alpha_j)_0, i, j \in \{1, 2, 3\}$ or $(w_i(\gamma_i))_0, 1 \leq i \leq 3$ span a plane. But the former is equivalent to that $\pi(w_i(\alpha_i)_0) (= \pi(w_i(\alpha_i)_t))$ is $\pi(w_j(\alpha_j)_0) (= \pi(w_j(\alpha_j)_t))$ or its negative, which is independent of the realized root system. The latter is equivalent to the reflection in $\pi(w_i(\alpha_i)_0) (= \pi(w_i(\alpha_i)_t)), 1 \leq i \leq 3$ being contained in a maximal dihedral reflection subgroup. This property is again independent of the realized root system. Hence we established the claim. Consequently as a continuous function in $t, \det((w_1(\gamma_1))_t, (w_2(\gamma_2))_t, (w_3(\gamma_3))_t)$ is either always positive, always zero or always negative. \square

6 Further examples and remarks

6.1 Oriented matroid Structures from symmetric NGCMs

In this subsection, we give an example showing that for a fixed Coxeter system, even for its realized root systems with symmetric NGCMs and linearly independent simple roots, the corresponding oriented matroid structures on the abstract root system may differ. Fix a Coxeter system (W, S) with $S = \{s_1, s_2, s_3, s_4\}$ and Coxeter graph



where $m, n \in \mathbb{N}_{\geq 3} \cup \{\infty\}$.

We define realized root system datum D as follows: $D = (V, V^\vee, (-, -), \Pi, \Pi^\vee, \iota)$ where $\Pi = \{\alpha_i | i = 1, \dots, 4\}$ and the associated NGCM $C = (c_{i,j})_{i,j=1}^4$, where $c_{i,j} = (\alpha_i, \alpha_j^\vee)$, is given by

$$C = \begin{pmatrix} 2 & -a & -b & 0 \\ -a & 2 & 0 & -c \\ -b & 0 & 2 & -d \\ 0 & -c & -d & 2 \end{pmatrix}.$$

where $a = 2 \cos \frac{\pi}{m}$, $b = 2 \cos \frac{\pi}{n}$ and $c, d \geq 2$.

The oriented matroid structure of the realized root system is determined by $\chi : \Phi^4 \rightarrow \{+, -, 0\}$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto \text{sgn}(\det(\gamma_1, \gamma_2, \gamma_3, \gamma_4))$ where $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is a 4×4 matrix with γ_i is regarded as the column vector with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

One trivially has $\chi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = +$, so these chirotopes (for varying c, d) cannot differ by a sign. One has $s_{\alpha_1} s_{\alpha_4}(\alpha_2) = a\alpha_1 + \alpha_2 + c\alpha_4$ and $s_{\alpha_1} s_{\alpha_4}(\alpha_3) = b\alpha_1 + \alpha_3 + d\alpha_4$. Hence

$$\chi(s_{\alpha_1} s_{\alpha_4}(\alpha_2), \alpha_2, \alpha_3, s_{\alpha_1} s_{\alpha_4}(\alpha_3)) = \text{sgn} \begin{vmatrix} a & 0 & 0 & b \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ c & 0 & 0 & d \end{vmatrix} = \text{sgn}(ad - bc).$$

Since $a, b > 0$ are fixed and $c, d \geq 2$ are arbitrary, any value of the sign in $\{+, -, 0\}$ is possible as c, d vary, so there are at least three possible chirotopes χ arising this way which are distinct even up to sign.

6.2 A 2-closure biclosed set which is not cone-biclosed

Let Φ be a realized root system of Coxeter system (W, S) with simple roots Π . We introduce the 2-closure operator $c_{2,\Phi}$ on Φ . A subset Γ of Φ is said to be $c_{2,\Phi}$ -closed if for any $\alpha, \beta \in \Gamma$, one has $\text{cone}(\{\alpha, \beta\}) \cap \Phi \subseteq \Gamma$. The $c_{2,\Phi}$ -closure of a set is the intersection of all $c_{2,\Phi}$ -closed sets containing it.

We can transfer $c_{2,\Phi}$ -closure to a closure operator c_2 on the abstract root system $T \times \{1, -1\}$ in the same way as we transferred matroid closure operators, by setting $c_2(A) = \pi(c_{2,\Phi}(\pi^{-1}A))$ for $A \subseteq T \times \{\pm 1\}$. It can be shown (using for instance [14, Lemma 4.3(a)] and [13, 1.35–1.36 and Proposition 3.1(c)]) that this closure operator c_2 on $T \times \{1, -1\}$ is independent of the realized root system Φ . It is studied for its role in the description of the meet and join of weak order of a Coxeter group, completion of weak order and its relation to reflection orders and their initial sections. See for example [1, 14].

Generally, for any closure operator c on a set X and any subset Y of X , we say that $Z \subseteq Y$ is c -biclosed (in Y) if Z and $Y \setminus Z$ are both c -closed. Let $\mathcal{B}_c(Y)$ denote the set of all c -biclosed subsets of Y . In particular, this defines $\mathcal{B}_{c_2}(T \times \{1\})$ and $\mathcal{B}_{c_\Phi}(T \times \{1\})$ where c_Φ is the reduced closure operator on $T \times \{\pm 1\}$ obtained by transfer of the oriented matroid closure operator cone_Φ of some realized root system Φ of (W, S) . Let $\Psi^+ := T \times \{1\}$ be the abstract positive system in Ψ . We shall call elements of $\mathcal{B}_{c_2}(\Psi^+)$ (resp., $\mathcal{B}_c(\Psi^+)$) for a fixed realized root system Φ) biclosed (resp., biconvex) sets (of Ψ^+). From the definitions, one easily sees that $\mathcal{B}_{c_\Phi}(\Psi^+) \subseteq \mathcal{B}_{c_2}(\Psi^+)$.

Most of the rest of this section is devoted to a proof of the following fact.

Theorem 6.3 *There is a finite rank Coxeter system (W, S) such that one has $\bigcup_{\Phi} \mathcal{B}_{c_{\Phi}}(\Psi^+) \subsetneq \mathcal{B}_{c_2}(\Psi^+)$ where the union is over all realized root systems Φ of (W, S) .*

6.4. For the proof of the theorem, we shall require a description of the canonical simple roots of the root subsystem consisting of roots which are orthogonal to a fixed simple root α , in case the NGCM associated to a realized root system Φ is symmetric. This description may be obtained for general (even infinite rank) (W, S) by comparison of results from [4,5] as follows.

Define a groupoid (i.e., a small category in which all morphisms are invertible) G as follows. The set of objects of G is Π . For $\alpha, \beta \in \Pi$, a morphism $\alpha \rightarrow \beta$ is a triple (β, w, α) where $w \in W$ satisfies $w(\alpha) = \beta$. Composition of morphisms in G is determined by the formula $(\gamma, v, \beta)(\beta, w, \alpha) = (\gamma, vw, \alpha)$.

For any distinct $\alpha, \beta \in \Pi$ with $m_{\alpha,\beta} := m(s_{\alpha}s_{\beta}) < \infty$, let $v(\beta, \alpha) := w_{\alpha,\beta}s_{\alpha}$ where $w_{\alpha,\beta}$ is the longest element of the finite standard parabolic subgroup $\langle s_{\alpha}, s_{\beta} \rangle$. Then one has a morphism $v'(\beta, \alpha) = (\gamma, v(\beta, \alpha), \alpha) \in G$ where $\gamma := v(\beta, \alpha)\alpha \in \{\alpha, \beta\}$. The following observations, especially (1), are used frequently below:

- (1) If $m := m_{\alpha,\beta}$ is even, $v(\beta, \alpha) = (s_{\beta}s_{\alpha})^{m/2-1}s_{\beta}$ is a reflection in a root orthogonal to α , and $\gamma = \alpha$
- (2) If m is odd, then $v(\beta, \alpha) = (s_{\alpha}s_{\beta})^{(m-1)/2}$ is of even length and $\gamma = \beta$.
- (3) In either case, $v'(\beta, \alpha)^{-1} = v'(\gamma', \gamma)$ where $\{\gamma, \gamma'\} = \{\alpha, \beta\}$.

It is easily seen (and a special case of well-known results of Howlett and Deodhar) that the groupoid G is generated by the morphisms $v'(\beta, \alpha)$ for distinct $\alpha, \beta \in \Pi$ with $m_{\alpha,\beta}$ finite. In [5], where relations for G in terms of these generators are given, the generators $v'(\beta, \alpha) : \alpha \rightarrow \beta$ with $m_{\beta,\alpha}$ odd are called *movers* and the other generators, of the form $v'(\beta, \alpha) : \alpha \rightarrow \alpha$ with $m_{\alpha,\beta}$ even, are called *shakers*, though “fixers” would be more apt in this rank one case. Let M be the subgroupoid of G generated by all the movers.

For $\alpha \in \Pi$, let $G_{\alpha} := \text{Hom}_G(\alpha, \alpha)$. The map $(\alpha, w, \alpha) \mapsto w$ induces an isomorphism of G_{α} with the stabilizer $\{w \in W | w\alpha = \alpha\}$ of α in W . We regard this as an identification below. Let $M_{\alpha} := \text{Hom}_M(\alpha, \alpha)$, which is a subgroup of G_{α} . Also, let $S''_{\alpha} := \{v'(\beta, \alpha) | \beta \in \Pi, m_{\alpha,\beta} \text{ is even}\}$ denote the set of all shakers with α as domain (and codomain). Set $T'_{\alpha} := \{g^{-1}sg | \beta \in \Pi, g \in \text{Hom}_G(\alpha, \beta), s \in S''_{\beta}\}$, $W'_{\alpha} := \langle T'_{\alpha} \rangle \subseteq G_{\alpha}$ and $S'_{\alpha} := \{g^{-1}sg | \beta \in \Pi, g \in \text{Hom}_M(\alpha, \beta), s \in S''_{\beta}\} \subseteq T'_{\alpha}$.

For $\gamma \in \Phi^+$ define $\gamma^{\perp} = \{\rho \in \Phi | (\rho, \gamma^{\vee}) = (\gamma, \rho^{\vee}) = 0\}$. Also let $T_{\alpha} := \{s_{\beta} | \beta \in \Phi \cap \alpha^{\perp}\} \subseteq T$, $W_{\alpha} = \langle T_{\alpha} \rangle \subseteq W$ and $S_{\alpha} := \chi(W_{\alpha})$. The point of the following result is to show that two natural sets of Coxeter group generators for the reflection subgroup W_{α} generated by all roots orthogonal to a fixed simple root α coincide.

Proposition 6.5

- (a) $G_{\alpha} = M_{\alpha} \rtimes W'_{\alpha}$ (a semidirect product of groups with W'_{α} normal in G_{α}).
- (b) M_{α} is a free group (isomorphic to the fundamental group based at α of the “odd Coxeter graph”, on vertex set Π , of (W, S)).
- (c) $(W'_{\alpha}, S'_{\alpha})$ is a Coxeter system with reflections T'_{α} and the natural (conjugation) action of M_{α} on W'_{α} from (a) fixes the set S'_{α} and hence T'_{α} .

(d) One has $W'_\alpha = W_\alpha$, $T'_\alpha = T_\alpha$ and $S'_\alpha = S_\alpha$.

Proof With the above identification of G_α with the centralizer of α in W , it follows easily from (a) above that $S''_\alpha \subseteq S'_\alpha \subseteq T'_\alpha \subseteq T_\alpha$ and hence $W'_\alpha \subseteq W_\alpha$. From [4], one has $G_\alpha = M_\alpha \ltimes W_\alpha$ and (b), while from [5], (as the rank one case of results in any rank) one has $G_\alpha = M_\alpha \ltimes W'_\alpha$ and (c). This implies $W_\alpha = W'_\alpha$.

In the proof of (c) in [5], a realized root system Υ for W'_α in the real vector space α^\perp is constructed, whose roots are certain positive scalar multiples of projections of certain roots from Φ on α^\perp and whose set of reflections is T'_α . Tracing through the construction of the simple roots Δ for Υ (we omit details), it follows using (1) above that the simple root corresponding to the simple reflection $g^{-1}sg \in S'_\alpha$, where $\beta \in \Pi$, $g \in \text{Hom}_M(\alpha, \beta)$ and $s \in S''_\alpha$ is $g^{-1}(\delta) \in \Phi^+$ where $\delta \in \Phi^+$ with $s_\delta = s$.

It follows that Υ is (in this special rank one situation, though not in general) a root subsystem of Ψ . Further, Δ , as a basis of simple roots for this subsystem contained in the positive roots Φ^+ , corresponds to $\chi(W')$ by [11]. It follows that $S'_\alpha = S_\alpha$ and hence $T'_\alpha = T_\alpha$ since they are both the sets of reflections of $(W_\alpha, S_\alpha) = (W'_\alpha, S'_\alpha)$. This completes the proof. \square

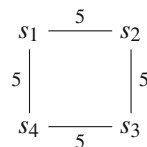
6.6. We remark that in general, the listed elements $g^{-1}sg$ in the set defining S'_α are not necessarily distinct. However, in our application below, (W, S) has no finite standard parabolic subgroups of rank three. This implies that, in the terminology of [5], the above-mentioned presentation of the groupoid G has no (R2) relations, i.e., only involves (R1) relations (as in (c)). This assures further that the elements $g^{-1}sg$ in the set defining S'_α are pairwise distinct and that (W_α, S_α) has no non-trivial braid relations, i.e., is a universal Coxeter system.

Theorem 6.3 is equivalent to the statement below.

Proposition 6.7 *There is a finite rank Coxeter system (W, S) with a c_2 -biclosed subset Ξ of Ψ^+ such that, for every realized root system Φ of (W, S) , with canonical surjection $\pi : \Phi \rightarrow \Psi$, $\pi^{-1}(\Xi)$ is not a cone $_\Phi$ -biclosed subset of Φ^+ .*

Proof To prove the theorem, we take for (W, S) a finite rank Coxeter system, mentioned to the first author by Bob Howlett, in which the reflection subgroup generated by reflections in roots orthogonal to some (in fact, any) fixed root α is an infinite rank universal Coxeter group. We then prove the assertion of the theorem in the case of the standard root system Φ . Finally, we deduce the assertion of the theorem for arbitrary Φ by a homotopy argument similar to that in the proof of Theorem 5.2.

Consider the rank four Coxeter system (W, S) with $S = \{s_i | i = 1, 2, 3, 4\}$ and Coxeter graph



and its standard root system Φ with simple roots $\{\alpha_i | i = 1, 2, 3, 4\}$.

To take advantage of the symmetry, take the indices modulo 4 i.e., set $s_{i+4n} := s_i$ and $\alpha_{i+4n} := \alpha_i$ for any integers $1 \leq i \leq 4$ and n .

All the roots are W -conjugate, so we consider for definiteness the simple root $\alpha = \alpha_1$ and the corresponding reflection group W_α as defined above. Let $x_i := s_i s_{i+1} s_i s_{i+1}$ for $i \in \mathbb{Z}$ and $w := x_4 x_3 x_2 x_1$ in W . From Proposition 6.5, one sees that $M_\alpha = \langle w \rangle \cong \mathbb{Z}$ and that the canonical simple system $\Delta \subseteq \Phi^+$ for W_α is

$$\Delta = \bigcup_{n \in \mathbb{Z}} w^n \Delta', \quad \Delta' := \{\beta_1 := \alpha_3, \beta_2 := x_4 \alpha_2, \beta_3 := x_1^{-1} \alpha_4, \beta_4 := x_4 x_3 \alpha_1\}.$$

In fact, M_α acts freely on Δ with Δ' as a set of (four) orbit representatives which we now describe explicitly for subsequent use.

Let $\tau := 2 \cos \frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$ be the golden ratio, so $\tau^2 = \tau + 1$. We have $s_{\alpha_i} \alpha_{i \pm 1} = \alpha_{i \pm 1} + \tau \alpha_i$, $s_{\alpha_i}(\alpha_{i+2}) = \alpha_{i+2}$ and of course $s_{\alpha_i}(\alpha_i) = -\alpha_i$. From this, one computes $\beta_1 = \alpha_3$, $\beta_2 = (1 + 2\tau)\alpha_1 + \alpha_2 + (1 + 2\tau)\alpha_4$, $\beta_3 = (1 + 2\tau)\alpha_1 + (1 + 2\tau)\alpha_2 + \alpha_4$, $\beta_4 = (6\tau + 4)\alpha_1 + (2\tau + 1)\alpha_3 + (8\tau + 4)\alpha_4$.

Maintain the notation above. We give an example of a $c_{2, \Phi}$ -biclosed subset Ξ' of Φ^+ which is not cone_Φ -biclosed. Note that $\beta_3 + \beta_4 = (2\tau + 1)(\beta_1 + \beta_2)$. Also, $s_{\beta_1}(\beta_2) = \beta_2 + (4\tau + 2)\beta_1$ so $(2\tau + 1)s_{\beta_1}(\beta_2) = \beta_3 + \beta_4 + (2\tau + 1)(4\tau + 1)\beta_1$. This shows that $s_{\beta_1}(\beta_2) \in \text{cone}(\{\beta_1, \beta_3, \beta_4\})$.

Now let $\Xi' := \Gamma \cup \Gamma'$ where $\Gamma = \{\gamma \in \Phi^+ | (\gamma, \alpha^\vee) < 0\}$ and Γ' is any $c_{2, \Phi_{W_\alpha}}$ -biclosed subset of $\Phi_{W_\alpha}^+$. We claim that $\Xi' \in \mathcal{B}_{c_{2, \Phi}}(\Phi^+)$. To see this, let $\Gamma'' := \{\gamma \in \Phi^+ | (\gamma, \alpha^\vee) > 0\}$ and consider $\delta, \beta, \gamma \in \Phi^+$ such that $\gamma = b\beta + d\delta$ for some $b, d > 0$. We have to show that if δ and β are in Ξ' (resp., $\Phi^+ \setminus \Xi'$) then γ is in Ξ' (resp., $\Phi^+ \setminus \Xi'$). But this holds since if either δ or β is in Γ (resp., Γ''), then so is γ since $\Phi_{W_\alpha} = \{\epsilon \in \Phi^+ | (\epsilon, \alpha^\vee) = 0\}$, while if both δ and β are in Γ' (resp., $\Phi_{W_\alpha}^+ \setminus \Gamma'$) then so is γ since Γ' is $c_{2, \Phi_{W_\alpha}}$ -biclosed in $\Phi_{W_\alpha}^+$.

Take for Γ' , using [12, Proposition 2.3], the positive root system of the standard parabolic subgroup $\langle s_{\beta_1}, s_{\beta_3}, s_{\beta_4} \rangle$ of W_α . Then $\Xi' \in \mathcal{B}_{c_{2, \Phi}}(\Phi^+)$. However, Ξ' is not cone_Φ -biclosed since from above, $s_{\beta_1}(\beta_2) \in \text{cone}(\Xi') \cap (\Phi^+ \setminus \Xi')$. Note for later use that this argument implies that $\Xi' \cap \Phi_{W_\alpha}$ is not $\text{cone}_{\Phi_{W_\alpha}}$ -biclosed in $\Phi_{W_\alpha}^+$.

Let Ψ be the abstract root system of (W, S) . We transfer the cone_Φ -biclosed subset Ξ' of Φ to a subset Ξ of Ψ via the canonical bijection $\pi: \Phi \rightarrow \Psi$, setting $\Xi = \pi(\Xi') = \{(s_\alpha, 1) | \alpha \in \Xi'\} \subseteq \Psi$. Also, let $U := W_\alpha$.

Now we change notation to allow, more generally, Φ to be any realized root system of (W, S) and $\pi: \Phi \rightarrow \Psi$ to denote the canonical W -equivariant surjection. (Similarly to the case of $\tilde{A}_n, n \geq 2$, such realized root systems are parameterized up to rescaling and isomorphism by elements of $\mathbb{R}_{>0}$). Let $\delta_i = (s_i, 1)$ for $i = 1, \dots, 4$ be the simple roots of Ψ . The oriented matroid structure on Φ induces by transfer via π a structure of reduced oriented matroid on Ψ , with closure operator $c_\Phi = c$ given by $c(\Gamma) = \pi(\text{cone}_\Phi(\pi^{-1}(\Gamma)))$ for any $\Gamma \subseteq \Psi$. As the notation suggests, c_Φ could possibly depend on the choice of Φ , though we do not know if it actually does (and it does not matter for our purposes). We shall show however that

- (*) Ψ_U is c_Φ -closed and the restriction c_U of c_Φ to a closure operator on Ψ_U is independent of the choice of Φ .

This will imply that $\Xi'' := \pi^{-1}(\Xi)$ is not cone_Φ -biclosed in Φ^+ . For suppose to the contrary that Ξ'' is cone_Φ -biclosed in Φ^+ . Then $\Xi'' \cap \Phi_U$ would be cone_{Φ_U} biclosed in Φ_U^+ . The corresponding subset $\pi(\Xi'' \cap \Phi_U) = \Xi \cap \Psi_U$ would then be c_Φ -biclosed (i.e., c_U -biclosed) in Ψ_U . But then it would be c_Φ -biclosed in the case Φ is the standard root system of (W, S) (by $(*)$) and we have seen already that it is not.

We now turn to the proof of the key point $(*)$. Denote the simple root of Φ corresponding to s_i as α_i as before, for $i = 1, \dots, 4$. We have

$$\begin{aligned} \Phi_U &= \pi^{-1}(\Psi_U) = \{ \beta \in \Phi \mid s_\beta(\alpha_1) \in \mathbb{R}_{>0}\alpha_1 \} = \{ \beta \in \Phi \mid s_\beta(\alpha_1) = \alpha_1 \} \\ &= \{ \beta \in \Phi \mid (\alpha_1, \beta^\vee) = 0 \} = \{ \beta \in \Phi \mid (\beta, \alpha_1^\vee) = 0 \}. \end{aligned}$$

So Φ_U is clearly a flat of $(\Phi, *, \text{cone}_\Phi)$, which implies $\Psi_U = \pi(\Phi_U)$ is a flat of the corresponding reduced oriented matroid $(\Psi, *, c)$. In particular, Φ_U is c -closed.

Next we observe that, since U is an infinite rank reflection subgroup, it is not of rank two or less and its root system Φ_U must therefore span a subspace of dimension three or greater. Since that subspace does not contain α_1 , it follows that its dimension is exactly three and that the simple roots of Φ must be linearly independent.

Hence Φ is completely determined, as a subset of the real vector space with $\Pi = \{ \alpha_i \mid i = 1, 2, 3, 4 \}$ as basis, by its NGCM. Suppose A and B are two NGCMs for (W, S) . Choose by Lemma 5.1 a continuous map $\theta: [0, 1] \rightarrow \mathbb{R}^{4 \times 4}$ with NGCMs for values, with $\theta(0) = A$ and $\theta(1) = B$. We consider four roots $\gamma_i = w_i(\delta_{j_i})$ for $i = 1, 2, 3, 4$ such that $w_1 = 1, j_1 = 1$ (so $\gamma_1 = \delta_1$) and $\gamma_2, \gamma_3, \gamma_4 \in \Psi_U$. We have corresponding roots $(w_i(\alpha_{j_i}))_t$ in the root system $\Phi(t)$ with NGCM $\theta(t)$ and simple roots α_i for $i = 1, \dots, 4$. We claim that the sign of $a_t := \det((w_1(\alpha_{j_1}))_t, \dots, w_4(\alpha_{j_4}))_t) \in \mathbb{R}$ is independent of $t \in [0, 1]$.

Observe that $(w_i(\alpha_{j_i}))_t \in \Phi(t)_U$ if and only if $i = 2, 3, 4$. Since $\Phi(t)_U = \{ \beta \in \Phi(t) \mid (\beta, \alpha_1^\vee) = 0 \}$ is the intersection of $\Phi(t)$ with a subspace of dimension three, one has $a_t = 0$ if and only if two of the roots $(w_i(\alpha_{j_i}))_t$ for $i = 2, 3, 4$ span the same ray (i.e., two abstract roots $(w_i(\delta_{j_i}))$ for $i = 1, \dots, 4$ coincide or differ only by sign) or these three roots span a plane (i.e., the corresponding abstract roots $(w_i(\delta_{j_i}))$ for $i = 2, 3, 4$ lie in a maximal dihedral root subsystem of Ψ). Since the formulations in terms of Ψ are independent of t , it follows that if $a_t = 0$ for one $t \in [0, 1]$, then $a_t = 0$ for all $t \in [0, 1]$. Since the map $t \mapsto a_t$ is continuous, the sign $\text{sgn}(a_t)$ is independent of t . Since $\gamma_2, \gamma_3, \gamma_4 \in \Psi_U$ are arbitrary, it follows that the map $(w_2(\delta_{j_2}), \dots, w_4(\delta_{j_4})) \mapsto \text{sgn}(\det((w_1(\alpha_{j_1}))_t, \dots, w_4(\alpha_{j_4}))_t)$ defines a chirotope for Ψ_U corresponding to the restriction of $c_{\Phi(t)}$ to a closure operator on Ψ_U and that it is independent of t as claimed. In particular, any two NGCMs A, B for (W, S) give the same oriented matroid $(\Psi_U, *, c_U)$ as asserted. \square

6.8 Oriented matroid root systems

One may regard Theorem 6.3 showing that natural combinatorially defined notions of c_2 -biclosed sets and initial sections of reflection orders do not correspond well to the geometry of realized root systems.

Define a (reduced) oriented matroid root system for a (possibly infinite rank Coxeter system (W, S) to be a reduced oriented matroid $(\Psi, -, d)$ on the abstract root system Ψ of (W, S) such that

- (i) d is W -equivariant (i.e., W acts on Ψ by automorphisms of this oriented matroid),
- (ii) Ψ^+ is d -closed,
- (iii) for each maximal dihedral reflection subgroup W' of W , the corresponding root subsystem $\Psi_{W'} := (W' \cap T) \times \{\pm 1\}$ of Ψ is d -closed,
- (iv) for W' as in (iii), the restriction of the oriented matroid structure on Ψ to an oriented matroid structure on $\Psi_{W'}$ is equal to the reduced oriented matroid structure on $\Psi_{W'}$ from any realized root system $\Phi_{W'}$ of $(W', \chi(W'))$.

We recall that in (iv), $\Psi_{W'}$ is independent of the choice of $\Phi_{W'}$ (in fact, it may be described combinatorially in terms of $(W', \chi(W'))$).

It is easily seen that any $(\Psi, -, c_\Phi)$ is an oriented matroid root system in this sense; the first author will give basic properties of oriented matroid root systems in general elsewhere. It is natural to ask whether one may have, say, $\mathcal{B}_{c_2}(\Psi^+) = \mathcal{B}_d(\Psi^+)$ for some such d or $\mathcal{B}_{c_2}(\Psi^+) = \bigcup_d \mathcal{B}_d(\Psi^+)$ where the union is over all oriented matroid root systems $(\Psi, -, d)$. An even more basic open question is whether there exist any non-realizable d (i.e., d as above which are not of the form c_Φ for any realizable root system Φ). One may conjecture on the basis of Theorem 5.2 that there is a unique oriented matroid root system for (W, S) of rank three. However, we conjecture that there are finite rank Coxeter systems for which such non-realizable d exist (possibly even in great abundance). The question of whether they exist for irreducible affine Weyl groups is an interesting and accessible one.

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