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# Bisymplectic Grassmannians of planes

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**Abstract**

The bisymplectic Grassmannian $\mathrm{I}_2\mathrm{Gr}(k, V)$ parametrizes $k$-dimensional subspaces of a vector space $V$ which are isotropic with respect to two general skew-symmetric forms; it is a Fano projective variety which admits an action of a torus with a finite number of fixed points. In this work, we study its equivariant cohomology with complex coefficients when $k = 2$; the central result of the paper is an equivariant Chevalley formula for the multiplication of the hyperplane class by any Schubert class. Moreover, we study in detail the case of $\mathrm{I}_2\mathrm{Gr}(2, \mathbb{C}^6)$, which is a quasi-homogeneous variety, we analyse its deformations, and we give a presentation of its cohomology.

## 1 Introduction

In complex algebraic geometry, classical Grassmannians are a special kind of homogeneous spaces for classical groups. They have been studied thoroughly for more than a century from different point of views: their geometry is governed by a rich combinatorial description, which manifests itself in many classical results about their cohomology. Moreover, the homogeneity condition has been very useful to investigate further properties of these varieties, such as their equivariant and quantum cohomology (see for instance [4,7,11,17]). Among classical Grassmannians, symplectic (respectively orthogonal) ones parametrize subspaces of a given vector space which are isotropic with respect to a non-degenerate symplectic (resp. orthogonal) form.

Even for varieties which admit an action of a *sufficiently big* algebraic group, when the homogeneity hypothesis is dropped less is known: some efforts have led to the notion of GKM varieties (for the action of tori with a finite number of zero- and one-dimensional orbits on complete varieties, they are defined in [8]) and some results have been obtained for specific examples (for instance, see [9,15,16]). In this paper, we present a work on a particular class of varieties, called bisymplectic Grassmannians, which are not homogeneous but admit an action of a *big* torus.

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In general, one can define multisymplectic (respectively multiorthogonal) Grassmannians as the varieties parametrizing subspaces of a given vector space which are isotropic with respect to a fixed number of general symplectic (resp. orthogonal) forms. As an example, consider the unique Fano threefold of degree 22, which is usually denoted by $\mathcal{V}_{22}$, and that appears in Iskovskikh's classification (see [10]); Mukai showed that it can be seen as a trisymplectic Grassmannian $\mathrm{I}_3\mathrm{Gr}(3,7)$ of 3-dimensional subspaces of $\mathbb{C}^7$.

Of course, in general, asking the isotropy condition with respect to many symplectic forms implies that the corresponding Grassmannian is no longer homogeneous. However, in the case of bisymplectic Grassmannians (two symplectic forms, denoted by $\mathrm{I}_2\mathrm{Gr}(k,2n)$) and of orthosymplectic Grassmannians (one symplectic and one orthogonal form), one can prove that it is still possible to define an action of a torus $T$ with a finite number of fixed points. Moreover, for extremal values of $k$, the bisymplectic Grassmannian is actually a homogeneous variety: $\mathrm{I}_2\mathrm{Gr}(1,2n) \cong \mathbf{P}^{2n-1}$ and $\mathrm{I}_2\mathrm{Gr}(n,2n) \cong (\mathbf{P}^1)^n$ (for the second isomorphism, which is a priori quite surprising, see [12]). Therefore, even though $\mathrm{I}_2\mathrm{Gr}(k,2n)$ is not homogeneous when $k \neq 1, n$ (consequence of the fact that it has non-trivial deformations, see Theorem 2.7), one may still expect it to behave quite similarly to homogeneous spaces.

However, this non-homogeneity implies that some difficulties appear when trying to study the $T$-equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k,2n)$. The finiteness of the number of fixed points and $T$-invariant curves implies that GKM theory applies and allows to compute the equivariant cohomology combinatorially in terms of fixed point data, but the combinatorics may still be irksome to work out. In this paper, we show how to determine the equivariant cohomology with complex coefficients of bisymplectic Grassmannians of planes, i.e. for $\mathrm{I}_2\mathrm{Gr}(2,2n)$ (when $k=2$). This variety has a simple geometric construction: it can be seen as the intersection of two hypersurfaces in $\mathrm{Gr}(2,2n)$. Even so, the determination of its equivariant cohomology is an interesting problem for different reasons: on the one hand, as already remarked, we can apply some equivariant tools in a rather simple non-homogeneous situation; on the other hand, we believe that the proofs of the results we state here for $\mathrm{I}_2\mathrm{Gr}(2,2n)$ can be adequately generalized in the case of bisymplectic Grassmannians $\mathrm{I}_2\mathrm{Gr}(k,2n)$ with $k \neq 2$. We intend to analyse this more general situation in the future.

The main results we obtain for $\mathrm{I}_2\mathrm{Gr}(2,2n)$ concern its equivariant cohomology. Firstly, we show that the classes of an additive basis of the cohomology can be uniquely determined by a finite number of relations coming from $T$-equivariant curves (Theorem 3.12); these classes correspond to the *Schubert* subvarieties that appear in the Bialynicki-Birula decomposition. Then, we find an equivariant Chevalley formula for the multiplication of any class with the hyperplane class (Theorem 3.24), from which one can recover the corresponding formula for the classical cohomology. As a result, one can compute the classes of Schubert varieties inductively (Corollary 3.25).

As an application, we give an explicit presentation of the cohomology of $\mathrm{I}_2\mathrm{Gr}(2,6)$. This bisymplectic Grassmannian is particularly interesting because it is quasi-homogeneous: it admits an action of $\mathrm{SL}(2)^3$ with a dense affine orbit. Moreover, even though it has no smooth deformations, we are able to describe all its singular flat deformations (Proposition 3.27).

The structure of the paper is as follows. In the first part, we recall general results about bisymplectic Grassmannians, whose detailed proofs can be found in [3]. We also recall some facts about symplectic Grassmannian, as they are useful to understand our situation better. In the central part of the paper, we deal with bisymplectic Grassmannians of planes $\mathrm{I}_2\mathrm{Gr}(2, 2n)$; after recalling some basic properties of the equivariant cohomology, we prove the two main results of the paper: the uniqueness for equivariant Schubert classes in Theorem 3.12 and the equivariant Chevalley formula in Theorem 3.24. Finally, we study in detail the quasi-homogeneous variety $\mathrm{I}_2\mathrm{Gr}(2, 6)$, we determine its orbit structure and its flat deformations, and we give a presentation of its classical cohomology.

## 2 Bisymplectic Grassmannians

In this section, we recall some basic definitions and facts about bisymplectic Grassmannians. The content of what follows can be found in [3, Chapter 4]; therefore, we will omit the proofs. Introducing the notations for general bisymplectic Grassmannians is useful for two reasons. On the one hand, it constitutes the natural framework in which to study the Grassmannians of planes, which can be seen as a particular example. On the other hand, it allows to compare what can be done in our particular example with the general situation; indeed, we believe that the ideas developed in this paper can be used fruitfully to obtain analogous results for general bisymplectic Grassmannians, which we intend to do in the future.

Let us consider the Grassmannian $\mathrm{Gr}(k, 2n)$ of $k$-dimensional subspaces inside a vector space of dimension $2n$. From now on, if not otherwise stated, we will assume that $2 \leq k \leq n$. By fixing a skew-symmetric form $\omega$ over $\mathbb{C}^{2n}$, one can consider the subvariety $\mathrm{IGr}(k, 2n)$ inside $\mathrm{Gr}(k, 2n)$ of isotropic subspaces with respect to $\omega$. If $\omega$ is non-degenerate, $\mathrm{IGr}(k, 2n)$ is smooth, and it is a rational homogeneous variety for the natural action of $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n)$. Denoting by $\mathcal{U}$ the tautological bundle over the Grassmannian, the variety $\mathrm{IGr}(k, 2n)$ can be seen as the zero locus of a general section of $\wedge^2\mathcal{U}^*$ over $\mathrm{Gr}(k, 2n)$; indeed notice that, by the Borel–Weil theorem, $\mathrm{H}^0(\mathrm{Gr}(k, 2n), \wedge^2\mathcal{U}^*) \cong \wedge^2(\mathbb{C}^{2n})^*$. We will refer to $\mathrm{IGr}(k, 2n)$ as the isotropic (or symplectic) Grassmannian.

Let us now fix two skew-symmetric forms $\omega_1, \omega_2$ over $\mathbb{C}^{2n}$.

**Definition 2.1** The bisymplectic Grassmannian is the subvariety $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ inside $\mathrm{Gr}(k, 2n)$ of subspaces isotropic with respect to $\omega_1$ and $\omega_2$. Equivalently, the points in $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ are isotropic with respect to the pencil $\langle\omega_1, \omega_2\rangle$.

***Remark 2.2*** As we will see later, there is not only one isomorphism class of bisymplectic Grassmannians. Indeed, the definition depends on the choice of a pencil $\langle\omega_1, \omega_2\rangle$. However, we will still refer to *the* bisymplectic Grassmannian in the following.

Of course, $\mathrm{I}_2\mathrm{Gr}(k, 2n) \subset \mathrm{IGr}(k, 2n)_i$, where $\mathrm{IGr}(k, 2n)_i$ is the symplectic Grassmannian with respect to $\omega_i$, $i = 1, 2$. The fact that $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ is not empty is a consequence of the fact that $\mathrm{I}_2\mathrm{Gr}(n, 2n) \neq \emptyset$ (see Example 2.3). Moreover, $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ can be seen as the zero locus of a section of $(\wedge^2\mathcal{U}^*)^{\oplus 2}$ over $\mathrm{Gr}(k, 2n)$; by

Bertini theorem, if the two forms $\omega_1$ and $\omega_2$ are in general position, the bisymplectic Grassmannian is smooth. Moreover, in this case, its dimension is $2k(n-k)+k$ and, by the adjunction formula, its canonical bundle is

$$K_{\mathrm{I}_2\mathrm{Gr}(k,2n)} = \mathcal{O}(-2n+2k-2);$$

therefore, $\mathrm{I}_2\mathrm{Gr}(k,2n)$ is a Fano variety. In the next sections, we will study under which conditions the bisymplectic Grassmannians are smooth (i.e. for what kind of pencils). Before doing so, let us deal with the case $k=n$.

***Example 2.3*** ($k=n$) Recall that the square root of the determinant of a $2n \times 2n$-skew-symmetric matrix is a polynomial of degree $n$ in the entries of the matrix, which is called Pfaffian. In [12, Theorem 3.1], Kuznetsov refines Bertini theorem proving that the variety $\mathrm{I}_2\mathrm{Gr}(n,2n)$ is smooth if and only if the pencil $\langle\omega_1,\omega_2\rangle$ intersects the Pfaffian divisor $D \subset \mathbf{P}(\wedge^2(\mathbb{C}^{2n})^*)$ (i.e. the locus of matrices where the Pfaffian is zero) in $n$ distinct points; in this case, the two forms are simultaneously block diagonalizable (with blocks of size $2\times 2$), and there exists an isomorphism

$$\mathrm{I}_2\mathrm{Gr}(n,2n) \cong (\mathbf{P}^1)^n. \tag{1}$$

Therefore, the automorphism group of $\mathrm{I}_2\mathrm{Gr}(n,2n)$ is $(\mathrm{PGL}(2))^n \times \mathfrak{S}_n$ (where $\mathfrak{S}_n$ is the group of permutations of $n$ elements). Surprisingly enough, from the isomorphism one realizes that $\mathrm{I}_2\mathrm{Gr}(n,2n)$ has no small deformations.

***Example 2.4*** ($k=2$) The bisymplectic Grassmannian of planes is $\mathrm{I}_2\mathrm{Gr}(2,2n)$. It is just the intersection of two hyperplane sections in $\mathrm{Gr}(2,2n)$. However, we will see that it shares with $\mathrm{I}_2\mathrm{Gr}(k,2n)$ for $k\neq 1$ and $k\leq n$ some useful properties, among them the existence of an action of an $n$-dimensional torus.

From now on, the zero locus of a section $s$ of a vector bundle over a variety will be denoted by $\mathscr{Z}(s)$. Moreover, let us denote by $V=\mathbb{C}^{2n}$.

As a consequence of Example 2.3, one may wonder whether all bisymplectic Grassmannians admit no small deformations. Moreover, we are interested in their automorphism group, because we will use the fact that a torus acts on bisymplectic Grassmannians with a finite number of fixed points (and therefore localization techniques apply). In the following, we recall two results from [3] where this automorphism group is explicitly computed (modulo a finite group). In order to do so, we need to state a result on the normal form of a pencil of skew-symmetric forms which defines a smooth bisymplectic Grassmannian. Let $D\subset \mathbf{P}(\wedge^2 V^*)$ be the Pfaffian divisor of degree $n$.

**Proposition 2.5** ([3, Proposition 4.1.6]) *Let $\Omega=\langle\omega_1,\omega_2\rangle\subset\mathbf{P}(\wedge^2(V^*))$ be a pencil of skew-symmetric forms such that $\mathscr{Z}(\Omega)\subset \mathrm{Gr}(k,V)$ has the expected dimension. $\mathscr{Z}(\Omega)$ is smooth if and only if $\Omega\cap D = p_1,\dots,p_n$, where the $p_i$'s are n distinct points such that:*

1. $\dim(\mathrm{Ker}(p_i)) = 2$ *for* $1\leq i\leq n$;
2. $V=\mathrm{Ker}(p_1)\oplus\cdots\oplus\mathrm{Ker}(p_n)$.

The proof of this result is essentially identical to the one used in [12, Theorem 3.1]; the result stated in [3] is weaker, but following its proof one can check that the statement made in Proposition 2.5 is correct. From now on, if not otherwise stated, we will assume that $\Omega$ is such that the corresponding bisymplectic Grassmannian is smooth of the expected dimension; by Bertini theorem, this is ensured by choosing $\Omega$ generic. We will denote by $K_i = \mathrm{Ker}(p_i)$.

***Remark 2.6*** The proof of the previous proposition (see [3]) actually shows that if $\mathscr{X}(\Omega)$ is smooth, then all the forms in $\Omega$ are simultaneously block diagonalizable. Moreover, as any non-degenerate form is conjugate to the standard one, one can suppose that $\Omega$ is generated by $\omega_1$ and $\omega_2$ with:

$$\omega_1 = \sum_{i=1}^{n} x_i \wedge x_{-i},$$
$$\omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i},$$

where $\langle x_i, x_{-i} \rangle = (K_i)^*$ for $1 \leq i \leq n$, and the $\lambda_i$'s are all distinct.

**Theorem 2.7** ([3, Theorem 4.1.12]) *Let* $1 < k < n$ *and* $X = \mathrm{I}_2\mathrm{Gr}(k, 2n)$. *The following isomorphisms hold:*

$$\mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(2)^n,$$
$$\mathrm{H}^1(X, T_X) \cong \mathbb{C}^{n-3}.$$

***Remark 2.8*** The fact that $\mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(2)^n$ should not be surprising; indeed, by Proposition 2.5 we know that the forms in $\Omega$ can be simultaneously block diagonalized, the blocks being the 2-dimensional subspaces $K_i$. A consequence of this is the fact that, for $1 \leq i \leq n$, the group $\mathrm{PGL}(K_i) \subset PGL(\mathbf{P}(\wedge^2 V^*))$ fixes the pencil $\Omega$. Therefore, it is contained in the automorphism group of $\mathscr{X}(\Omega) = X$. The fact that these are the only automorphisms of $X$ modulo a finite group is a consequence of the previous theorem. To state it more intrinsically, we can write:

$$T_{\mathrm{Aut}(X)} \cong \mathrm{H}^0(X, T_X) \cong \mathfrak{sl}(K_1) \oplus \cdots \oplus \mathfrak{sl}(K_n).$$

Moreover, this observation implies that an $n$-dimensional torus acts on $X$, which we will use later on.

***Remark 2.9*** What the second isomorphism of the theorem tells us is that the moduli stack $\mathcal{M}_{\mathrm{bisym}(k,n)}$ of bisymplectic Grassmannians should have dimension $n-3$. This is the same as the dimension of the moduli stack $\mathcal{M}_n$ of $n$ points inside $\mathbf{P}^1$. It is straightforward to see that there is a dominant rational morphism

$$\pi : \mathrm{Gr}(2, \wedge^2 V^*)/\mathrm{PGL}(V) \dashrightarrow \mathcal{M}_{\mathrm{bisym}(k,n)},$$

where $\mathrm{Gr}(2, \wedge^2 V^*)/\mathrm{PGL}(V)$ is the GIT quotient. This quotient has dimension $n-3$, i.e. it is not of the expected dimension. Indeed, on the open subset inside $\mathrm{Gr}(2, \wedge^2 V^*)$ of diagonalizable pencils, each point $\Omega$ is fixed by a copy of $\mathrm{SL}(2)^n \subset \mathrm{PGL}(V)$. Moreover, one can prove that there is a birational morphism

$$\mathrm{Gr}(2, \wedge^2 V^*)/\mathrm{PGL}(V) \dashleftarrow\dashrightarrow \mathcal{M}_n.$$

However, in order to have a birational model of $\mathcal{M}_{\mathrm{bisym}(k,n)}$, one should understand the degree of $\pi$; as it will not be needed in the following, we leave this question open for further work.

***Remark 2.10*** When $n = 3$ (and $k = 2$), the variety $X$ has no small deformations. Moreover, by Proposition 2.5, if $\mathscr{Z}(\Omega)$ is smooth (by Bertini theorem, for this it is sufficient to choose $\Omega$ generic), $\Omega$ intersects the Pfaffian divisor $D$ in three points $p_1, p_2, p_3$; by changing coordinates if necessary, we can suppose that $p_1 = [x_1 \wedge x_{-1} + x_2 \wedge x_{-2}]$ and $p_2 = [x_2 \wedge x_{-2} + x_3 \wedge x_{-3}]$, where $(x_{\pm 1}, x_{\pm 2}, x_{\pm 3})$ is a basis of $V^*$. Thus, there is only one smooth isomorphism class of $\mathrm{I}_2\mathrm{Gr}(2, 6)$ when, modulo a coordinate change, $\Omega = \langle p_1, p_2 \rangle$. In Sect. 3.4, we will study more in detail this variety, its flat deformations and its (equivariant) cohomology.

### 2.1 The torus action on $\mathrm{I_2Gr}(k, V)$

The variety $\mathrm{I}_2\mathrm{Gr}(k, V)$ is smooth and projective (according to the assumptions made in the previous section) and admits an action of a torus with a finite number of fixed points. We summarize here the first consequences of the existence of this action. This will allow us to introduce some useful notation.

Let $\mathrm{I}_2\mathrm{Gr}(k, V)$ be defined by the forms $\omega_1$ and $\omega_2$ described in Remark 2.6, and let $\mathrm{IGr}(k, V)$ be the symplectic Grassmannian defined by $\omega_1$ which contains $\mathrm{I}_2\mathrm{Gr}(k, V)$. Moreover let $T \cong (\mathbb{C}^*)^n$ be the maximal torus inside $\mathrm{Sp}(V)$ which is contained inside $\mathrm{SL}(2)^n \subset \mathrm{Aut}(\mathrm{I}_2\mathrm{Gr}(k, V))$. For simplicity, we will assume from now on that $T$ is the diagonal torus $\mathrm{diag}(t_n, \ldots, t_1, t_1^{-1}, \ldots, t_n^{-1}) \subset \mathrm{Sp}(V)$. It acts on $\mathrm{IGr}(k, V)$ with a finite number of fixed points, and as a consequence the induced action on $\mathrm{I}_2\mathrm{Gr}(k, V)$ has a finite fixed locus as well. The surprising fact is that the two fixed loci are the same:

**Proposition 2.11** ([3, Proposition 4.2.1]) *There are $2^k\binom{n}{k}$ fixed points for the action of $T$ on $\mathrm{IGr}(k, V)$ and on $\mathrm{I}_2\mathrm{Gr}(k, V)$. They are parametrized by subsets $I \subset \{\pm 1, \ldots, \pm n\}$ with k elements such that $I \cap (-I) = \emptyset$.*

If $V = \langle v_n \ldots, v_1, v_{-1}, \ldots, v_{-n} \rangle$, with $K_i = \langle v_i, v_{-i} \rangle$, then the fixed point corresponding to a subset $I = (i_1, \ldots, i_k)$ is given by $p_I = [v_I] = [v_{i_1} \wedge \cdots \wedge v_{i_k}]$.

**Definition 2.12** We will say that a subset $I \subset \{\pm 1, \ldots, \pm n\}$ of size $k$ is admissible if $I \cap (-I) = \emptyset$.

Therefore, by the Bialynicki-Birula decomposition (see [1]), by fixing a generic one-dimensional torus $\tau \subset T$ with the same fixed locus as $T$, we can associate with each fixed point $p_I$, where $I \subset \{\pm 1, \dots, \pm n\}$ is admissible, a *Schubert* variety $\sigma_I \subset \mathrm{I_2Gr}(k, V)$, which is the closure of a *Schubert* cell isomorphic to an affine space (the terminology is borrowed from the homogeneous situation). The Schubert cell is defined as the set of points which accumulate towards $p_I$ under the action of $\tau$. The condition that $\tau$ needs to satisfy in order to give the decomposition is that it acts with a finite number of fixed points. This is true for a general $\tau \subset T$. For instance, let us fix a particular one-dimensional torus with which we will work later on:

$$\tau = \mathrm{diag}(t^n, \dots, t, t^{-1}, \dots, t^{-n}) \subset T. \tag{2}$$

Then, by explicitly computing the action of $\tau$ on the Plücker coordinates of the Grassmannian (see e.g. [3, Lemma 4.2.3]), one can show that:

**Lemma 2.13** *The one-dimensional torus $\tau$ acts with a finite number of fixed points over* $\mathrm{IGr}(k, V)$ *and over* $\mathrm{I_2Gr}(k, V)$.

***Remark 2.14*** The symplectic Grassmannian $\mathrm{IGr}(k, V)$ is a homogeneous variety under the action of $\mathrm{Sp}(V)$, and as such it has a natural Bruhat decomposition in orbits under the action of a Borel subgroup of $\mathrm{Sp}(V)$. It turns out that the Bruhat decomposition and the Bialynicki-Birula one are the same if the chosen Borel subgroup and the one-dimensional torus are compatible (see [2][Book $II$, example 4.2]). We will denote by $\sigma'_I$ the Schubert varieties of $\mathrm{IGr}(k, V)$, and by $B$ a Borel subgroup of $\mathrm{Sp}(V)$.

The identification of the two decompositions implies that if a fixed point $p_J$ belongs to a Schubert variety $\sigma'_I$, then actually $\sigma'_J = \overline{B.p_J} \subset \overline{B.p_I} = \sigma'_I$. This fact is crucial when trying to compute the equivariant cohomology of $\mathrm{IGr}(k, V)$, as we will see. However, this property will not hold in the bisymplectic case, and it is one of the main reasons why computing the equivariant cohomology for $\mathrm{I_2Gr}(k, V)$ becomes more difficult.

As $p_I$ is fixed, the torus $T$ acts on the tangent space $T_I := T_{\mathrm{I_2Gr}(k,V), p_I}$. Let $\Xi(T)$ denote the character group of $T$, and let $\epsilon_i \in \Xi(T)$ be the character of $T$ given by $\mathrm{diag}(t_n, \dots, t_1, t_1^{-1}, \dots, t_n^{-1}) \mapsto t_i$. If $i < 0$, we denote by $\epsilon_i$ the character $-\epsilon_{-i}$.

**Lemma 2.15** ([3, Lemma 4.2.6]) *The weights of the action of $T$ on $T_I$ are*

$$\begin{aligned} &-2\epsilon_i \text{ for } i \in I \text{ and} \\ &\epsilon_i - \epsilon_j \text{ for } i \notin I \cup (-I), \quad j \in I. \end{aligned}$$

The weights of the action of $\tau$ are easily deduced from Lemma 2.15; indeed, under the identification $\Xi(\tau) \cong \mathbb{Z}$, it is sufficient to notice that $\epsilon_i \mapsto i$ under the morphism $j^* : \Xi(T) \to \Xi(\tau)$ induced by the natural inclusion $j : \tau \to T$. The Schubert variety $\sigma_I$ is smooth at $p_I$ and the tangent space $T_{\sigma_I, p_I}$ is the $\tau$-invariant subspace of $T_I$ whose weights with respect to $\tau$ are negative.

**Definition 2.16** From now on, we will say that $\xi \in \Xi(T)$ is $\tau$-positive (and we will denote it by $\xi > 0$) if $j^*(\xi) > 0$, and $\tau$-negative if $j^*(\xi) < 0$.

Therefore, given a certain subset $I$, it is possible to compute the codimension of $\sigma_I$. It is sufficient to count the number of $\tau$-positive weights for the action of $\tau$ on $T_I$. In order to do so, one starts with the weights in Lemma 2.15, apply the morphism $j^*$ (sending $e_i$ to $i$) and obtain:

$$\operatorname{codim}(\sigma_I) = \#\{(i, j) \text{ s.t. } i \notin I \cup (-I)\ ,\ j \in I, \text{ and } j > i\} + \#\{j \in I \text{ s.t. } j < 0\}.$$

$\mathrm{I}_2\mathrm{Gr}(k, V)$ is smooth and projective, thus filtrable via the Schubert cells; this implies that the cohomology of $\mathrm{I}_2\mathrm{Gr}(k, V)$ as a $\mathbb{Z}$-module is freely generated by the classes $\overline{\sigma_I}$ for $I$ admissible. The odd Betti numbers are therefore all equal to zero. Let $\{b^i_{k,n}\}_i$ be the even Betti numbers of $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ (where $i$ is the codimension). We will denote by $S_{k,n}$ the sequence of integers:

$$S_{k,n} = (b^0_{k,n}, \dots, b^{\dim(\mathrm{I}_2\mathrm{Gr}(k,2n))}_{k,n}, 0, \dots, 0, \dots).$$

Of course, the decomposition of $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ in Schubert cells whose closures are the $\sigma_I$'s implies that $b^i_{k,n}$ is equal to the number of subsets $I$ such that $\operatorname{codim}(\sigma_I) = i$. We will denote by $[h]$ the shift on the right by $h$. For instance, $S_{k,n}[1] = (0, b^0_{k,n}, \dots, b^{\dim(\mathrm{I}_2\mathrm{Gr}(k,2n))}_{k,n}, 0, \dots, 0, \dots)$.

**Theorem 2.17** ([3, Theorem 4.3.4]) *The following recursive formula holds for the Betti numbers of* $\mathrm{I}_2\mathrm{Gr}(k, 2(n+1))$*:*

$$S_{k,n+1} = S_{k,n}[k] + S_{k-1,n} + S_{k-1,n}[1 + 2(n + 1 - k)]. \tag{3}$$

***Example 2.18*** We give here a list of examples of Betti numbers of bisymplectic Grassmannians for small $k, n$:

$S_{2,3} = (1, 1, 2, 4, 2, 1, 1, 0, \dots)$;
$S_{3,4} = (1, 1, 2, 6, 6, 6, 6, 2, 1, 1, 0, \dots)$;
$S_{2,4} = (1, 1, 2, 2, 3, 6, 3, 2, 2, 1, 1, 0, \dots)$.

***Remark 2.19*** The case of the Grassmannian of planes is particularly easy because $\mathrm{I}_2\mathrm{Gr}(2, 2n)$ is a codimension 2 complete intersection inside $\mathrm{Gr}(2, 2n)$; all its Betti numbers except the middle term can be derived from those of $\mathrm{Gr}(2, 2n)$ (or of $\mathrm{IGr}(2, 2n)$) by applying Lefschetz hyperplane theorem. Moreover, as $\chi(\mathrm{IGr}(2, 2n)) = \chi(\mathrm{I}_2\mathrm{Gr}(2, 2n))$ (because the number of fixed points is the same for the two varieties), the middle term is the sum of the two middle Betti numbers of $\mathrm{IGr}(2, 2n)$.

***Remark 2.20*** By using the recursive formula, it is possible to prove that for any $1 < k < n$ we have: $b^0_{k,n} = 1$, $b^1_{k,n} = 1$, $b^2_{k,n} = 2$. In particular,

$$\operatorname{Pic}(\mathrm{I}_2\mathrm{Gr}(k, V)) \cong \mathbb{Z}.$$

Furthermore, let $H = (n, n-1, \dots, n-k+2, n-k)$ be the subset corresponding to the codimension 1 Schubert variety (both for $\mathrm{IGr}(k, V)$ and for $\mathrm{I}_2\mathrm{Gr}(k, V)$). Then $\sigma_H$ is a hyperplane section of $\mathcal{O}(1)$ inside $\mathrm{I}_2\mathrm{Gr}(k, V)$, and it is a line. Indeed, it is the restriction to $\mathrm{I}_2\mathrm{Gr}(k, V)$ of the hyperplane section $\sigma'_H \subset \mathrm{IGr}(k, V)$.

#### 2.1.1 $T$-equivariant curves

In order to compute the $T$-equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k, V)$ using GKM theory, one needs to understand which are the $T$-invariant curves, and what are the inclusions of fixed points in Schubert varieties $p_J \in \sigma_I$. Recall that $T$-invariant curves are rational curves whose intersection with the fixed locus has cardinality 2; these two fixed points will be denoted by $p_0$ and $p_\infty$.

**Lemma 2.21** ([3, Lemma 4.2.8 and Lemma 4.3.14]) *There is only a finite number of $T$-invariant curves inside* $\mathrm{IGr}(k, V)$. *They are of two types:*

*type $\alpha$: curves with $p_0 = p_I$ and $p_\infty = p_J$, where $\#(I \cap J) = k - 1$;*
*type $\beta$: curves with $p_0 = p_I$ and $p_\infty = p_J$, where $\#(I \cap J) = k - 2$, $I - J = \{a_1, a_2\}$, $J - I = \{-a_2, -a_1\}$.*

*Among these, the $T$-invariants curves which are also contained inside* $\mathrm{I}_2\mathrm{Gr}(k, V)$ *are those of type $\alpha$.*

***Remark 2.22*** From the proof of Lemma [3, Lemma 4.2.8], it is straightforward to see that the curves of type $\alpha$ are lines inside $\mathbf{P}(\wedge^k V^*)$, while the curves of type $\beta$ are conics inside a $\mathbf{P}^2 \subset \mathbf{P}(\wedge^k V^*)$.

**Definition 2.23** Let $I = \{a_k \geq \cdots \geq a_1\}$ and $J = \{b_k \geq \cdots \geq b_1\}$. If $a_i \geq b_i$ for $1 \leq i \leq k$, then we will say that $I$ is greater or equal than $J$, and we will denote this by $I \geq J$.

We will say that $C = C_1 \ldots C_m$ is a chain of $T$-equivariant curves from $p_I$ to $p_J$ if $C_i(\infty) = C_{i+1}(0)$ for any $1 \leq i \leq m$, and $C_1(0) = p_I$, $C_m(\infty) = p_J$.

**Lemma 2.24** ([3, Lemma 4.2.10]) *For two admissible subsets $I$ and $J$, the fact that $I \geq J$ is equivalent to $p_J \in \sigma'_I \subset$* $\mathrm{IGr}(k, V)$ *and to the fact that there is a chain of $T$-invariant curves inside* $\mathrm{IGr}(k, V)$ *from $p_I$ to $p_J$.*

The second statement in Lemma 2.24 is a consequence of the proof of [3, Lemma 4.2.10].

The results of this section show that $\mathrm{I}_2\mathrm{Gr}(k, V)$ is a projective $T$-skeletal variety, i.e. $T$ acts with a finite number of fixed points and invariant curves. Hence, results from [8] can be applied to describe the $T$-equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k, V)$ in terms of fixed point data (see 3.4).

## 3 Equivariant cohomology of bisymplectic Grassmannians of planes

In this section we study the $T$-equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(2, V)$. We begin by recalling some basic facts about equivariant cohomology. A reference for this subject is [6]; the general results we will cite can be found in [8] or [5]. Then we will analyse the case of the symplectic Grassmannian, in order to compare it with the behaviour of the bisymplectic one. The main result of this section will be a Chevalley formula for Schubert classes in $\mathrm{I}_2\mathrm{Gr}(2, V)$, which a priori determines inductively all the equivariant classes $\overline{\sigma_I}$ for $I$ admissible.

Let $X$ be a smooth projective variety on which a torus $T \cong (\mathbb{C}^*)^n$ acts with finitely many fixed points $X^T = \{p_1, \dots, p_r\}$. Denote by $\Xi(T) \cong \mathbb{Z}^n$ the character group of $T$. Moreover, let $\tau \in T$ be a general 1-dimensional torus such that its fixed locus is equal to $X^T$; then the Bialynicki-Birula decomposition for $\tau$ provides varieties $\sigma_i$ for all $1 \le i \le r$. They form a basis for the integral homology of $X$ and by Poincaré duality also a basis for the ordinary cohomology $\mathrm{H}^*(X, \mathbb{Z})$.

The equivariant cohomology ring $\mathrm{H}^*_T(X)$ with complex coefficients is an algebra over the polynomial ring $\mathrm{H}^*_T(\mathrm{pt}) \cong \mathbb{C}[\Xi(T)] = \mathrm{Sym}((\Xi(T)) \otimes_{\mathbb{Z}} \mathbb{C})$ via the push-forward map of the natural inclusion of a point pt inside $X$. An additive basis for this algebra is given by the (equivariant) classes $\overline{\sigma_i}$ for $1 \le i \le r$.

Denote by $\mathrm{H}^*(X) := \mathrm{H}^*(X, \mathbb{C})$. The pullback map $i^* : \mathrm{H}^*_T(X) \to \mathrm{H}^*_T(X^T)$ of the natural inclusion $i : X^T \to X$ is injective (by [8, Theorem 1.2.2]); therefore,

$$\mathrm{H}^*_T(X) = \Xi(T) \otimes_{\mathbb{Z}} \mathrm{H}^*(X) \cong \Xi(T) \otimes_{\mathbb{Z}} \bigoplus_{p_i} \mathbb{C}\overline{\sigma_i}$$

can be seen as a subring of

$$\mathrm{H}^*_T(X^T) \cong \Xi(T) \otimes_{\mathbb{Z}} \mathrm{H}^*(X^T) \cong \Xi(T) \otimes_{\mathbb{Z}} \bigoplus_{p_i} \mathbb{C}p_i \cong \mathbb{C}[\Xi(T)]^{\oplus r}.$$

Via this inclusion, we will denote by $f_{\sigma_i} \in \mathbb{C}[\Xi(T)]^{\oplus r}$ the pullback of the class $\overline{\sigma_i} \in \mathrm{H}^*_T(X)$, and by $f_{\sigma_i}(p_j) = (i \circ i_j)^* \overline{\sigma_i}$, where $i_j : p_j \to X^T$ is the natural inclusion. The polynomial $f_{\sigma_i}(p_j)$ is usually referred to as the localization of the class $\overline{\sigma_i}$ at the point $p_j$. Clearly, if $\epsilon_1, \dots, \epsilon_n$ is a $\mathbb{Z}$-basis of $\Xi(T)$, then $f_{\sigma_i}(p_j) \in \mathrm{H}^*_T(p_j)$ is a polynomial in $\epsilon_1, \dots, \epsilon_n$. Therefore, in order to understand the equivariant cohomology of $X$, we need to find the polynomials $f_{\sigma_i}(p_j)$. The following results hold:

**Theorem 3.1** *The polynomials $f_{\sigma_i}(p_j)$ satisfy the following properties:*

1. *$f_{\sigma_i}(p_j)$ is a homogeneous polynomial of degree* $\mathrm{codim}(\sigma_i)$*;*
2. *$f_{\sigma_i}(p_j) = 0$ if $p_j \notin \sigma_i$;*
3. *$f_{\sigma_i}(p_j)$ is the product of the $T$-characters of the normal bundle $N_{\sigma_i/X, p_j}$ whenever $\sigma_i$ is smooth at $p_j$;*
4. *If there exists a $T$-equivariant curve between $p_j$ and $p_k$ whose character is $\chi$, then $\chi$ divides $f_{\sigma_i}(p_j) - f_{\sigma_i}(p_k)$ for $1 \le i \le r$.*

**Theorem 3.2** ([8, Theorem 1.2.2]) *If there is only a finite number of $T$-invariant curves inside $X$, then the equivariant cohomology $\mathrm{H}^*_T(X)$ is the subalgebra of $\mathbb{C}[\Xi(T)]^{\oplus r}$ consisting of elements $f = (f_1, \dots, f_r)$ satisfying the last condition in Theorem 3.1, i.e.:*

$$\begin{gathered}\textit{if there exists a } T\textit{-equivariant curve between } p_j \textit{ and } p_k \\ \textit{whose character is } \chi, \textit{ then } \chi \textit{ divides } f_j - f_k.\end{gathered} \tag{4}$$

Moreover, from the equivariant cohomology, it is possible to recover the ordinary cohomology $\mathrm{H}^*(X)$:

**Theorem 3.3** *The classical cohomology* $\mathrm{H}^*(X)$ *can be recovered from the equivariant cohomology* $\mathrm{H}_T^*(X)$ *as*

$$\mathrm{H}^*(X) \cong \mathrm{H}_T^*(X)/(\epsilon_1, \cdots, \epsilon_n).$$

Therefore, the finiteness of the number of $T$-invariant curves inside $\mathrm{I}_2\mathrm{Gr}(k, V)$ (Lemma 2.21) implies that GKM theory can be applied:

**Theorem 3.4** *The relations in* (4) *are enough to determine the equivariant cohomology of* $\mathrm{I}_2\mathrm{Gr}(k, V)$.

***Remark 3.5*** Theorem 3.4 just states that Theorem 3.2 applies to bisymplectic Grassmannians.

One should be careful: being able to determine the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(k, V)$ does not imply that we are able to identify the equivariant classes $\overline{\sigma_I}$ in general.

### 3.1 Warm-up: the homogeneous case

In contrast to the bisymplectic case, in the homogeneous case we can identify the equivariant classes $f_{\sigma_I'}$. The following proposition is a well known result, whose argument goes back at least to [11, Section 2.1]:

**Proposition 3.6** ([3, Proposition 4.2.16]) *Let* $X = G/P$ *be a homogeneous rational variety under the action of a simple group* $G$. *Then the maximal torus* $T$ *inside a Borel subgroup* $B \subset G$ *acts with a finite number of fixed points on* $X$. *Moreover, if there is only a finite number of* $T$*-equivariant curves, then the equivariant classes of Schubert varieties inside* $\mathrm{H}_T^*(X)$ *are determined by the relations* 1, 2, 3 *in Theorem* 3.1.

The crucial property in order to prove this proposition is the one underlined by Remark 2.14. Later on we will prove the analogous result for $\mathrm{I}_2\mathrm{Gr}(2, V)$ by adapting the proof of the previous proposition.

We recall in the following that an equivariant Chevalley formula is known for $\mathrm{IGr}(k, V)$. This formula permits to compute inductively the polynomials $f_{\sigma_I'}(p_J)$. The inductive method proceeds as follows. Let us fix a Schubert variety $\sigma_I'$. Then

$$\text{if } p_J \notin \sigma_I', \text{ then } f_{\sigma_I'}(p_J) = 0.$$

Moreover, $f_{\sigma_I'}(p_I)$ is just the product of the (positive) $\tau$-weights of $T_I$ (because $\sigma_I'$ is smooth at $p_I$). Notice that these two assertions are general, and will hold for the bisymplectic Grassmannian as well.

Finally, the function $f_{\sigma'_I}(\cdot)(f_{\sigma'_H}(\cdot)-f_{\sigma'_H}(p_I))$ has support over the points $p_J \in \sigma'_I$, $J \neq I$ (recall that $\sigma'_H$ is the codimension 1 Schubert variety by using the notation of Remark 2.20). By applying Lemma 2.24, we obtain:

$$f_{\sigma'_I}(\cdot)(f_{\sigma'_H}(\cdot) - f_{\sigma'_H}(p_I)) = \sum_{J\in I_{-1}} a_{I,J} f_{\sigma'_J}(\cdot), \tag{5}$$

where $I_{-1} = \{J \text{ s.t. } I \geq J, \text{ and } \operatorname{codim}(\sigma'_J) = \operatorname{codim}(\sigma'_I)+1\}$. The condition on the codimension is a consequence of the fact that $\deg(f_{\sigma'_J}) = \operatorname{codim}(\sigma'_J)$. The coefficient $a_{I,J}$ turns out to be equal to 1 if there is a $\alpha$-curve between $p_I$ and $p_J$, and it is equal to 2 if there is a $\beta$-curve between $p_I$ and $p_J$. Knowing the coefficients $a_{I,J}$, one can determine inductively $f_{\sigma'_I}$ from the $f_{\sigma'_J}$'s in Eq. (5).

***Remark 3.7*** Equation (5) is just localization (and rearrangement) of the equivariant Chevalley formula

$$\sigma'_I\sigma'_H = f_{\sigma'_H}(p_I)\sigma'_I + \sum_{J\in I_{-1}} a_{I,J}\sigma'_J; \tag{6}$$

the validity of this formula is a direct consequence of the fact that Schubert classes form a basis over $\mathrm{H}^*_T(\mathrm{pt})$ and that $\sigma'_I\sigma'_H - f_{\sigma'_H}(p_I)\sigma'_I$ is supported on classes $\sigma'_J$ such that $J \in I_{-1}$ (by localization at $p_I$).

### 3.2 Schubert classes are determined

In this section we prove that the equivariant Schubert classes for $\mathrm{I}_2\mathrm{Gr}(2, V)$ are completely determined by the relations 1, 2, 3 in Theorem 3.1, i.e. the analogous of Proposition 3.6. In order to do so, we will need to understand (some) inclusions of fixed points inside Schubert varieties. In the end we will prove an equivariant Lefschetz hyperplane theorem, relating the equivariant cohomology of the symplectic Grassmannian to that of the bisymplectic one.

***Remark 3.8*** From now on, we will denote by $f_I(J) = f_{\sigma_I}(p_J)$.

The problem of determining the inclusions of fixed points in the case of the bisymplectic Grassmannian is more difficult to deal with. In order to understand this problem, notice that $\mathrm{I}_2\mathrm{Gr}(k, V) \subset \mathrm{IGr}(k, V)$ implies that if $p_J \in \sigma_I \subset \sigma'_I$, then $I \geq J$. Moreover $\geq$ is a partial order relation on the admissible subsets of $\{\pm 1, \ldots, \pm n\}$ of cardinality $k$. We define the relation $\geq_\in$ on admissible subsets: $I \geq_\in J$ if and only if there exist admissible subsets $J = J_1, J_2, \ldots, J_u = I$ such that $p_{J_i} \in \sigma_{J_{i+1}}$ for $i = 1, \ldots, u-1$. This relation is by construction reflexive and transitive. Moreover, it is antisymmetric because if $I \neq J$, $I \geq_\in J$ and $J \geq_\in I$, then $J \geq I \geq J$ and $I \neq J$, which is a contradiction by the definition of $\geq$. As a result, $\geq_\in$ is a partial order relation on admissible subsets of $\{\pm 1, \ldots, \pm n\}$, and as a consequence we get the following:

**Lemma 3.9** *There exist polynomials $a_{I,J} \in \mathbb{C}[\epsilon_1, \ldots, \epsilon_n]$ of degree* $\mathrm{codim}(\sigma_J) - \mathrm{codim}(\sigma_I) - 1$ *such that*

$$f_I(\cdot)(f_H(\cdot) - f_H(I)) = \sum_{J \in I_{\geq_\epsilon -1}} a_{I,J} f_J(\cdot), \tag{7}$$

*where* $I_{\geq_\epsilon -1} = \{J \text{ s.t. } I \geq_\epsilon J, \text{ and } \mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I) + 1\}$.

***Proof*** We already know that

$$f_I(\cdot)(f_H(\cdot) - f_H(I)) = \sum_J a_{I,J} f_J(\cdot)$$

for some polynomials $a_{I,J}$ because the classes of Schubert varieties generate the equivariant cohomology over $\Xi(T)$. We want to prove that $a_{I,J} = 0$ if $J \notin I_{\geq_\epsilon -1}$. Indeed, let $L \notin I_{\geq_\epsilon -1}$ be a subset which is maximal for the partial order relation $\geq_\epsilon$ such that $a_{I,L} \neq 0$. Then, by evaluating the previous equation at $p_L$, we obtain

$$0 = a_{I,L} f_L(L).$$

But $f_L(L)$ is the product of the weights of the normal bundle of $\sigma_L$ at $p_L$, and we have $f_L(L) \neq 0$. This gives a contradiction. The assertion on the degree of $a_{I,J}$ is a consequence of the fact that $f_L(\cdot)$ is a homogeneous polynomial of degree $\mathrm{codim}(\sigma_L)$ for any admissible $L$. ◻

In the next section, we will use this lemma to obtain an equivariant Chevalley formula for the multiplication of Schubert varieties with $f_H$. Now just notice that in general we are looking for coefficients $a_{I,J}$'s which are not constants, but actual polynomials; determining even one of them may need the use of a lot of relations. This problem comes from the fact that, as we are in the non-homogeneous case, the fact that $p_J \in \sigma_I$ does not necessarily imply that $\sigma_J \subset \sigma_I$, or, more concretely, that $\mathrm{codim}(\sigma_J) > \mathrm{codim}(\sigma_I)$. Hence we get that the coefficients $a_{I,J}$ may very well not be constant. However, for the Grassmannians of planes, this problem can be controlled, as shown in Lemma 3.11; before proving the lemma, let us point out that some of the inclusions which hold in $\mathrm{IGr}(2, V)$ do not hold in $\mathrm{I_2Gr}(2, V)$:

**Lemma 3.10** *Let $I = (i, -i+1)$. If $J = (i-1, -i)$ or $J = (i-1, -i-1)$ or $J = (i-2, -i)$, then $p_J \notin \sigma_I$.*

***Proof*** Let us fix some notation. We denote by $q_I$ the Plücker coordinates on the Grassmannian $\mathrm{Gr}(2, V)$. Then $\mathrm{Gr}(2, V) \subset \mathbf{P}(\wedge^2 V)$ is defined by the quadratic equations

$$q_{(a,b)}q_{(c,d)} - q_{(a,c)}q_{(b,d)} + q_{(b,c)}q_{(a,d)} = 0 \text{ for } a, b, c, d \in \{\pm 1, \ldots, \pm n\}, \tag{8}$$

which are also known as the Plücker relations. Moreover the two equations defining the bisymplectic Grassmannian (and coming from $\omega_1$ and $\omega_2$) are:

$$\sum_{i=1}^{n} q_{(i,-i)} = 0 \text{ and } \sum_{i=1}^{n} \lambda_i q_{(i,-i)} = 0. \tag{9}$$

Finally, the Schubert variety $\sigma_I$ is just the intersection of $\mathrm{I}_2\mathrm{Gr}(2, V)$ with the Schubert variety (associated with $p_I \in \mathbf{P}(\wedge^2 V)$) of the action of $\tau$ on $\mathbf{P}(\wedge^2 V)$; being this action linear, $\sigma_I$ is defined set theoretically inside $\mathrm{I}_2\mathrm{Gr}(2, V)$ by the linear relations

$$q_J = 0 \text{ for } I \not\geq J,$$

while in a neighbourhood of $p_I$ we can suppose that $q_I \neq 0$. The relations defining the Schubert variety $\sigma_I$ and those coming from $\omega_1$ and $\omega_2$ imply that

$$q_{(i,-i)} = q_{(i-1,-i+1)} = 0.$$

By using the Plücker equations with $a = i, b = -i+1, c = i-1, d = -i-1$ (respectively $a = i, b = -i+1, c = i-1, d = -i$, $a = i, b = -i+1, c = i-2, d = -i$), one gets that $\sigma_I$ is contained in the locus where $q_{(i-1,-i-1)} = 0$ (resp. $q_{(i-1,-i)} = 0$, $q_{(i-2,-i)} = 0$), which does not contain $p_J$ with $J = (i-1, -i-1)$ (resp. $J = (i-1, -i)$, $J = (i-2, -i)$). □

**Lemma 3.11** *Suppose that* $p_J \in \sigma_I$ *and* $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I)$. *Then* $\mathrm{codim}(\sigma_J) = \mathrm{codim}(\sigma_I) = 2n-3$, $I = (i, -i+1)$ *and* $J = (i, -i-1)$.

***Proof*** We will prove the lemma by comparison with the symplectic Grassmannian. The weights of the action of $T$ on $T_{\mathrm{IGr}(2,V),p_I}$ are

$$-2\epsilon_i \text{ for } i \in I\ , \epsilon_i - \epsilon_j \text{ for } i \notin I \cup (-I), \quad j \in I \text{ and}$$
$$-\epsilon_{i_1} - \epsilon_{i_2} \text{ for } i_1 > i_2 \in I.$$

Let $I = (i_1 > i_2)$. If $i_1 + i_2 > 0$, then the codimension of $\sigma'_I$ inside $\mathrm{IGr}(2, V)$ is the same as that of $\sigma_I$ inside $\mathrm{I}_2\mathrm{Gr}(2, V)$, and it is $\leq 2n-3$; if $i_1 + i_2 < 0$, the codimension of $\sigma'_I$ inside $\mathrm{IGr}(2, V)$ is equal to $\mathrm{codim}(\sigma_I) + 1 \geq 2n-2$. Moreover $p_J \in \sigma_I$ implies that $p_J \in \sigma'_I$ and $\mathrm{codim}(\sigma'_J) > \mathrm{codim}(\sigma'_I)$.

Therefore, if $p_J \in \sigma_I$ and $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I)$, then the only possibility is that $\mathrm{codim}(\sigma_J) = \mathrm{codim}(\sigma_I) = 2n-3$. As a consequence, $I$ must be of the form $I = (i, -i+1)$ for a certain $2 \leq i \leq n$, and this forces either $J = (i-1, -i)$ or $J = (i, -i-1)$. The first case is excluded by Lemma 3.10. □

Now we are ready to prove the analogous of Proposition 3.6:

**Theorem 3.12** *The equivariant classes $f_I$ of Schubert varieties inside the cohomology group* $\mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(2, V))$ *are determined by the relations* 1, 2, 3 *in Theorem* 3.1. *More explicitly, suppose that* $g \in \mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(2, V))$ *is such that* $g(p_J)$ *is homogeneous of degree* $\mathrm{codim}(\sigma_I)$ *for any admissible* $J$, $g(p_J) = 0$ *if* $p_J \notin \sigma_I$ *and* $g(p_J)$ *is the product of the* $T$*-characters of the normal bundle* $N_{\sigma_I/X,p_J}$ *whenever* $\sigma_I$ *is smooth at* $p_J$*; then* $g = f_I$.

***Remark 3.13*** Notice that relation 4 in Theorem 3.1 is always satisfied by any element inside $\mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(2, V))$.

***Proof*** The polynomials $f_I(p_J)$ of the equivariant class of a Schubert variety $\sigma_I$ satisfy the relations in Theorem 3.1. Moreover, by the finiteness of the number of $T$-invariant curves, we have that if two $T$-invariant curves with characters $\chi_1$, $\chi_2$ meet $p_I$, then $\chi_1$ and $\chi_2$ must be prime to each other.

Let us deal first with a Schubert variety $\sigma_I$, where $I$ is not of the form $I = (i, -i+1)$. This hypothesis implies by Lemma 3.11 that if $p_J \in \sigma_I$, then $\mathrm{codim}(\sigma_J) > \mathrm{codim}(\sigma_I)$. Let us consider an element

$$g = (g_1 \ldots, g_r) \in \mathrm{H}^*_T(X) \subset \mathbb{C}[\Xi(T)]^{\oplus r}$$

satisfying the conditions in the statement of the theorem (where we denoted by $g_J = g(p_J)$). Then $f_I - g$ is zero over all points $p_J$ such that $\mathrm{codim}(\sigma_J) \leq \mathrm{codim}(\sigma_I)$. We want to prove that $f_I - g = 0$. Let us suppose that $f_I - g \neq 0$. Then we can find a point $p_h \in \sigma_I$ such that $(f_I - g)(p_h) \neq 0$ and $\mathrm{codim}(\sigma_h)$ is minimal. Condition (4) and the finiteness of the number of $T$-invariant curves implies that $(f_I - g)(p_h)$ must be divisible by $f_h(p_h)$ (because the weights of the normal space at any fixed point $p_L$ are exactly those of the $T$-equivariant curves linking the point to points $p_M$ with $M \geq L$); but

$$\deg((f_I - g)(p_h)) = \mathrm{codim}(\sigma_I) < \mathrm{codim}(\sigma_h) = \deg(f_h(p_h)),$$

which gives a contradiction.

The previous argument must be adapted when $I = (i, -i+1)$. When this is the case, $\sigma_I$ can contain at most two points $p_h$ such that $\mathrm{codim}(\sigma_h) = \mathrm{codim}(\sigma_I)$, namely $h = (i, -i-1)$ and $h = (i-2, -i+1)$. Suppose for example that $h = (i, -i-1)$. In this case, by Lemma 3.10 $p_{(i-1,-i-1)} \notin \sigma_I$, if $(f_{\sigma_I} - g)(p_h) \neq 0$ it must be divisible by $f_{\sigma_h}(p_h)(\epsilon_i - \epsilon_{i-1})$, whose degree is greater than the codimension of $\sigma_I$. Similarly when $h = (i-2, -i+1)$ because $p_{(i-2,-i)} \notin \sigma_I$. □

### 3.2.1 An equivariant Lefschetz Hyperplane Theorem

Let $i : \mathrm{I}_2\mathrm{Gr}(k, V)^T \to \mathrm{I}_2\mathrm{Gr}(k, V)$ be the inclusion of the fixed points, and $j : \mathrm{I}_2\mathrm{Gr}(k, V) \to \mathrm{IGr}(k, V)$ the natural inclusion. As $(i \circ j)^* : \mathrm{H}^*_T(\mathrm{IGr}(k, V)) \to \mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(k, V)^T)$ is an inclusion (because $\mathrm{I}_2\mathrm{Gr}(k, V)^T = \mathrm{IGr}(k, V)^T$), we get that $j^*$ should be injective as well. We will denote by $f_{\sigma'_I}(J)$ the pullback of the equivariant classes of Schubert subvarieties of $\mathrm{IGr}(k, V)$ inside $\mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(k, V)^T)$. Moreover, let

$$f_I f_J = \sum_L N^L_{I,J} f_L$$

be the multiplication rule inside $\mathrm{H}^*_T(\mathrm{I}_2\mathrm{Gr}(k, V))$, and

$$f_{\sigma_I} f_{\sigma_J} = \sum_L M^L_{I,J} f_{\sigma_L}$$

the multiplication rule inside $\mathrm{H}^*_T(\mathrm{IGr}(k, V))$, where $I, J, L$ are admissible subsets and $N^L_{I,J}$, $M^L_{I,J}$ are polynomials of the right degree.

The Lefschetz hyperplane theorem says that the restriction of the cohomology of an ambient variety $X$ to an hypersurface $Y$ is an isomorphism in codimension $< \dim_{\mathbb{C}}(X)$. The following proposition is an equivariant version of this classical result for bisymplectic Grassmannians of planes:

**Theorem 3.14** (Equivariant Lefschetz) *Let $I$ be an admissible subset of* $\mathrm{I}_2\mathrm{Gr}(2, V)$ *such that* $\mathrm{codim}(\sigma_I) < 2n-3$. *Then $j^*\sigma'_I = \sigma_I$ and, as a consequence,*

$$f_I = f_{\sigma'_I}.$$

*Moreover, let $J, L$ be two admissible subsets as well such that* $\mathrm{codim}(\sigma_J) < 2n-3$ *and* $\mathrm{codim}(\sigma_L) < 2n-3$. *Then*

$$M^L_{I,J} = N^L_{I,J}.$$

***Proof*** Let us consider $f_{\sigma'_I}$. By Lemma 2.21 $f_{\sigma'_I}$ satisfies the relations 1, 2, 3 in Theorem 3.1 which are also satisfied by $f_I$, and all the relations in Theorem 3.2. Therefore, by Theorem 3.4 and Theorem 3.12 we get that $f_{\sigma'_I} = f_I$. The second statement follows at once. □

***Remark 3.15*** For what concerns the other classes, the problem becomes more involved. Indeed, if $\mathrm{codim}(\sigma_I) \geq 2n-3$, the class $f_{\sigma'_I}$ does not satisfy all the relations 1, 2, 3 in Theorem 3.1. For instance, if $\mathrm{codim}(\sigma_I) > 2n-3$, by the proof of Lemma 3.11 we know that $\mathrm{codim}(\sigma_I) = \mathrm{codim}(\sigma'_I) - 1$, and therefore relation 1 is not satisfied. Finding a formula which expresses all the classes $f_{\sigma'_I}$ in terms of the classes $f_I$ may help understanding better the equivariant cohomology of $\mathrm{I}_2\mathrm{Gr}(2, V)$. Indeed, one could try to derive an equivariant Pieri formula for multiplication of any Schubert class by a *special* Schubert class, as it is done in [14] for the symplectic (as well as the ordinary and the orthogonal) Grassmannians.

### 3.3 A Chevalley formula

The following lemma will be useful in the sequel:

**Lemma 3.16** *The Schubert variety $\sigma_H$, where $H = \{n, n-1, \ldots, n-k+3, n-k+2, n-k\}$, corresponding to the unique generator of* $\mathrm{Pic}(\mathrm{I}_2\mathrm{Gr}(k, V))$ *is represented in equivariant cohomology by the degree* 1 *polynomials*

$$f_H(I) = \sum_{i \in I} -\epsilon_i + \sum_{i=1}^{k} \epsilon_{n-i+1}.$$

***Proof*** We already know that $f_H$ in the equivariant cohomology is uniquely determined by the fact that $f_H(\{n, \ldots, n-k+2, n-k+1\}) = 0$ and $f_H(H) = -\epsilon_{n-k} + \epsilon_{n-k+1}$; these conditions, together with condition (4), are satisfied by the formula in the statement. □

***Remark 3.17*** This lemma follows also from Theorem 3.14 because localizations of $\sigma'_H$ and $\sigma_H$ coincide.

The next result we want to present is the computation of an equivariant Chevalley formula for bisymplectic Grassmannians of planes, i.e. of the coefficients $a_{I,J}$ appearing in Eq. (7) for $k = 2$. Having these coefficients will permit to compute all the equivariant classes of Schubert varieties, starting from that of maximal codimension down to the one of codimension 0. We will divide the proof into different lemmas, which deal with different situations. The most difficult part will be understanding the behaviour of classes of middle codimension, because in this case we have coefficients $a_{I,J}$ which are of degree one, and not just constants (Lemma 3.11). At the end of the proof, we have summarized the Chevalley formula in Theorem 3.24.

The first lemma deals with Schubert varieties for which the Chevalley formula is the same as that of symplectic Grassmannians:

**Lemma 3.18** *Let $I$, $J$ be admissible subsets such that one of the two following conditions are satisfied:* $\operatorname{codim}(\sigma_J) < 2n-3$ *or* $\operatorname{codim}(\sigma_I) > 2n-3$. *If* $\#(I \cap J) = 1$ *and* $\operatorname{codim}(\sigma_I) = \operatorname{codim}(\sigma_J) - 1$ *then* $a_{I,J} = 1$*, otherwise* $a_{I,J} = 0$.

***Remark 3.19*** When $\operatorname{codim}(\sigma_J) < 2n-3$, the statement of the lemma can be derived directly from Theorem 3.14 and the computation of $a_{I,J}$ for $\mathrm{IGr}(k, V)$ (see Sect. 3.1). However below we give another more concrete proof of the lemma.

***Proof*** By hypothesis, we have that $\operatorname{codim}(\sigma_I) - \operatorname{codim}(\sigma_J) = \operatorname{codim}(\sigma'_I) - \operatorname{codim}(\sigma'_J)$. Therefore, $p_J \in \sigma_I$ and $\operatorname{codim}(\sigma_I) = \operatorname{codim}(\sigma_J) - 1$ implies that $\#(I \cap J) = 1$. Moreover notice that $\deg(a_{I,J}) = 0$ in this case, so that the $a_{I,J}$ are always constant. Let us suppose that $I = (i_1, i_2)$ and $J = (j_1, j_2)$ are admissible subsets and that $i_1 = j_1$. (The case $i_2 = j_2$ is treated similarly.) By Eq. (7) and by the fact that there exists a $T$-invariant curve between $p_I$ and $p_J$ of weight $\epsilon_{i_2} - \epsilon_{j_2}$, we have the two following relations:

$$f_I(J)(\epsilon_{i_2} - \epsilon_{j_2}) = a_{I,J} f_J(J),$$
$$f_I(I) - f_I(J) \text{ is divisible by } (\epsilon_{i_2} - \epsilon_{j_2}).$$

Putting these two relations together implies that

$$f_I(I) - a_{I,J} \frac{f_J(J)}{\epsilon_{i_2} - \epsilon_{j_2}} \equiv 0 \bmod(\epsilon_{i_2} - \epsilon_{j_2}). \tag{10}$$

However, by condition (3) of Theorem 3.1 we know that $f_I(I)$ and $\tilde{f}_J(J) := f_J(J)/(\epsilon_{i_2} - \epsilon_{j_2})$ are not divisible by $(\epsilon_{i_2} - \epsilon_{j_2})$; thus, this third relation determines $a_{I,J}$. Indeed, if we impose the equality $\epsilon_{i_2} = \epsilon_{j_2}$ (i.e. modulo $(\epsilon_{i_2} - \epsilon_{j_2})$), it is easy to check (by using condition (3) of Theorem 3.1) that $f_I(I)$ is equal to $\tilde{f}_J(J)$; therefore, the only possibility in order to satisfy Eq. (10) is that $a_{I,J} = 1$ because this ensures that the LHS is equal to zero. In the other cases when $\operatorname{codim}(\sigma_I) = \operatorname{codim}(\sigma_J) - 1$, the coefficient $a_{I,J} = 0$ by applying Eq. (7) to $J$ because $p_J \notin p_I$. □

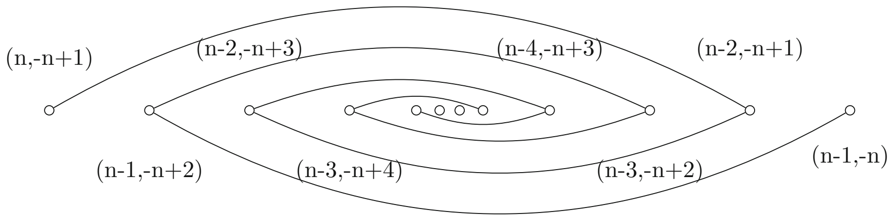



**Fig. 1** Inclusions of fixed points and $T$-invariant curves inside $\mathrm{I}_2\mathrm{Gr}(2, V)$ in codimension $= 2n - 3$

The following lemmas deal with the interesting part of the cohomology of $\mathrm{I}_2\mathrm{Gr}(2, V)$. In Fig. 1 we reported the inclusions of fixed points whose Schubert varieties have the same codimension (situation described in Lemmas 3.11 and 3.10). We deal first of all with these inclusions, i.e. with polynomials $a_{I,J}$ of degree 1:

**Lemma 3.20** *If* $I = (i, -i+1)$ *and* $J = (i, -i-1)$ *or* $J = (i-2, -i+1)$ *with* $i > 0$*, then* $a_{I,J} = \epsilon_{i-1} - \epsilon_i$.

***Proof*** Let us suppose $J = (i, -i-1)$. By Lemma 3.10, we know that $p_{(i-1,-i-1)} \notin \sigma_I$; therefore, $f_I((i-1, -i-1)) = 0$ and the existence of a $T$-invariant curve between $p_J$ and $p_{(i-1,-i-1)}$ gives that

$$f_I(J) \text{ is divisible by } (\epsilon_{i-1} - \epsilon_i).$$

As by Eq. (7)

$$f_I(J)(\epsilon_{i+1} - \epsilon_{i-1}) = a_{I,J} f_J(J),$$

and as by Theorem 3.1 $f_J(J)$ is not divisible by $(\epsilon_{i-1} - \epsilon_i)$, we get that the coefficient $a_{I,J}$ is of the form $a_{I,J} = a(\epsilon_{i-1} - \epsilon_i)$, for a certain constant $a$. Arguing as in the proof of Lemma 3.18, we have that $a = 1$ by the existence of a $T$-invariant curve between $p_I$ and $p_J$, which gives the relation determining $a$:

$$f_I(I) - a\frac{\epsilon_{i-1} - \epsilon_i}{\epsilon_{i+1} - \epsilon_{i-1}} f_J(J) \equiv 0 \bmod (\epsilon_{i+1} - \epsilon_{i-1}).$$

If $J = (i-2, -i+1)$, the proof is exactly the same, provided that we replace $p_{(i-1,-i-1)} \notin \sigma_I$ by the fact that $p_{(i-2,-i)} \notin \sigma_I$. □

The following facts can be verified easily: if $\mathrm{codim}(\sigma_I) = 2n-2$, then either $I = (i, -i+2)$ with $i > 0$ or $I = (2, 1)$. By symmetry, if $\mathrm{codim}(\sigma_I) = 2n-4$, then $I = (i-2, -i)$ with $i > 0$ or $I = (-1, -2)$. Finally, if $\mathrm{codim}(\sigma_I) = 2n-3$, then either $I = (i, -i+1)$ or $I = (i-1, -i)$ with $i > 0$.

Figure 2 represents the inclusions of fixed points which are relevant for the following proposition:

**Lemma 3.21** *Let* $I = (i-1, -i)$*, with* $i > 0$*. The only nonzero coefficients* $a_{I,J}$ *are:*

$$a_{I,(i-2,-i)} = a_{I,(i-1,-i-1)} = 1.$$

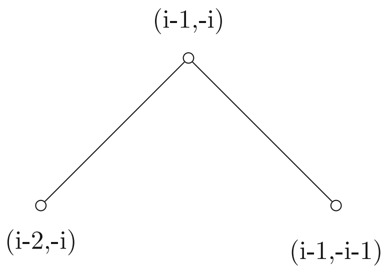



**Fig. 2** Inclusions of fixed points inside $\sigma_I$ with $I = (i-1, -i)$, $i > 0$

***Proof*** The proof of this result follows the same lines of the proof of Lemma 3.18. The reason why this happens is that in this case as well $\operatorname{codim}(\sigma_I) - \operatorname{codim}(\sigma_J) = \operatorname{codim}(\sigma'_I) - \operatorname{codim}(\sigma'_J)$. □

Figure 3 represents the inclusions of fixed points which are relevant for the following proposition:

**Lemma 3.22** *Let $I = (i, -i+2)$, with $i > 0$. The only nonzero coefficients $a_{I,J}$ are:*

$$a_{I,(i,-i+1)} = a_{I,(i-1,-i+2)} = a_{I,(i-3,-i+2)} = a_{I,(i,-i-1)} = 1,$$
$$a_{I,(i-2,-i+1)} = a_{I,(i-1,-i)} = 2.$$

***Proof*** The coefficients $a_{I,(i,-i+1)}$ and $a_{I,(i-1,-i+2)}$ are computed as it is done in the proof of Lemma 3.18. Let us deal with the remaining coefficients for points $p_J \in \sigma_I$ such that $\operatorname{codim}(\sigma_I) = \operatorname{codim}(\sigma_J) - 1$:

- Let $J = (i-3, -i+2)$. By Eq. (7),

$$f_I(J)(\epsilon_i - \epsilon_{i-3}) = f_{(i-1,-i+2)}(J) + a_{I,J} f_J(J).$$

  Moreover by Lemma 3.20 we know that

$$f_{(i-1,-i+2)}(J) = \frac{\epsilon_{i-2} - \epsilon_{i-1}}{\epsilon_{i-1} - \epsilon_{i-3}} f_J(J).$$

  Therefore, the existence of a $T$-equivariant curve between $p_I$ and $p_J$ of weight $(\epsilon_i - \epsilon_{i-3})$ gives the relation

$$f_I(I) - \frac{\epsilon_{i-2} - \epsilon_{i-1} + a_{I,J}(\epsilon_{i-1} - \epsilon_{i-3})}{(\epsilon_i - \epsilon_{i-3})(\epsilon_{i-1} - \epsilon_{i-3})} f_J(J) \equiv 0 \bmod (\epsilon_i - \epsilon_{i-3}).$$

  By condition (3) of Theorem 3.1, the two terms of the LHS are not divisible by $(\epsilon_i - \epsilon_{i-3})$. As $f_I(I)$ is divisible by $(\epsilon_{i-2} - \epsilon_{i-3})$ (by Lemma 2.15 and condition (3) of Theorem 3.1), the only possibility is that the second term of the LHS is divisible either by $(\epsilon_{i-2} - \epsilon_{i-3})$ or by $(\epsilon_{i-2} - \epsilon_i)$; this forces $a_{I,J} = 1$.

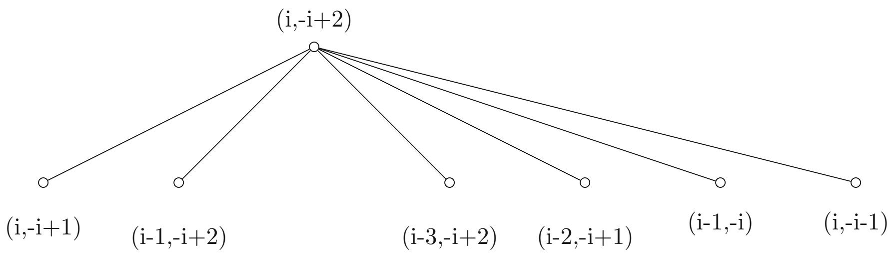



**Fig. 3** Inclusions of fixed points inside $\sigma_I$ with $I = (i, -i+2), i > 0$

- Let $J = (i, -i-1)$. The argument is similar to the previous one; the last relation becomes

$$f_I(I) - \frac{\epsilon_{i-1} - \epsilon_i + a_{I,J}(\epsilon_{i+1} - \epsilon_{i-1})}{(\epsilon_{i+1} - \epsilon_{i-2})(\epsilon_{i+1} - \epsilon_{i-1})} f_J(J) \equiv 0 \operatorname{mod}(\epsilon_{i+1} - \epsilon_{i-2}).$$

  As $f_I(I)$ is divisible by $(\epsilon_{i+1} - \epsilon_i)$, we get that $a_{I,J} = 1$.
- Let $J = (i-2, -i+1)$. Lemma 3.20 gives

$$f_{(i,-i+1)}(J) = (\epsilon_{i-1} - \epsilon_i) f_J(J).$$

  Using this relation and Eq. (7), we obtain

$$f_I(J)(\epsilon_i - 2\epsilon_{i-2} + \epsilon_{i-1}) = \frac{\epsilon_{i-1} - \epsilon_i + a_{I,J}(\epsilon_i - \epsilon_{i-2})}{\epsilon_i - \epsilon_{i-2}} f_J(J),$$

  which implies that $a_{I,J} = 2$.
- Let $J = (i-1, -i)$. The argument is similar to the previous one; Lemma 3.20 and Eq. (7) give the relation

$$f_I(J)(2\epsilon_i - \epsilon_{i-2} - \epsilon_{i-1}) = \frac{\epsilon_{i-2} - \epsilon_{i-1} + a_{I,J}(\epsilon_i - \epsilon_{i-2})}{\epsilon_i - \epsilon_{i-2}} f_J(J),$$

  which implies that $a_{I,J} = 2$.

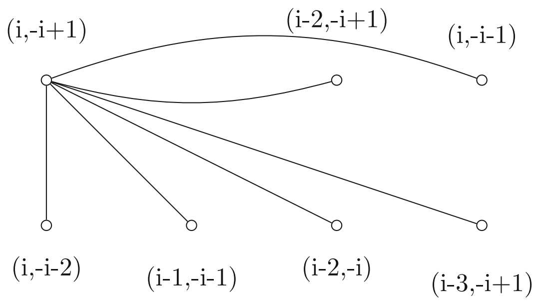
□

Figure 4 represents the inclusions of fixed points which are relevant for the following proposition:

**Lemma 3.23** *Let $I = (i, -i+1)$, with $i > 0$. The only nonzero constant coefficients $a_{I,J}$ are:*

$$a_{I,(i-1,-i-1)} = a_{I,(i-2,-i)} = 1.$$

**Fig. 4** Inclusions of fixed points inside $\sigma_I$ with $I = (i, -i+i)$, $i > 0$




***Proof*** The proof uses the same arguments of the proof of Lemma 3.22; therefore, we will be more concise. We need to deal with the coefficients for points $p_J \in \sigma_I$ such that $\operatorname{codim}(\sigma_I) = \operatorname{codim}(\sigma_J) - 1$:

- Let $J = (i-1, -i-1)$. Lemma 3.20 and Eq. (7) give the relation

$$f_I(J)(\epsilon_i - 2\epsilon_{i-2} + \epsilon_{i-1}) = \frac{\epsilon_{i-2} - \epsilon_{i-1} + a_{I,J}(\epsilon_{i-1} - \epsilon_{i-2})}{\epsilon_{i-1} - \epsilon_{i-2}} f_J(J),$$

  which implies that $a_{I,J} = 1$ because by Lemma 3.10 $f_I(J) = 0$.
- Let $J = (i-2, -1)$. Lemma 3.20 and Eq. (7) give the relation

$$f_I(J)(2\epsilon_i - \epsilon_{i-2} - \epsilon_{i-1}) = \frac{\epsilon_{i-1} - \epsilon_i + a_{I,J}(\epsilon_i - \epsilon_{i-1})}{\epsilon_i - \epsilon_{i-1}} f_J(J),$$

  which implies that $a_{I,J} = 1$ because by Lemma 3.10 $f_I(J) = 0$.
- Let $J = (i, -i-2)$. By using Lemma 3.20 and Eq. (7) repeatedly, and the existence of a $T$-equivariant curve between $p_I$ and $p_J$, we obtain the relation

$$f_I(I) - \frac{\epsilon_{i-1} - \epsilon_i + a_{I,J}(\epsilon_{i+2} - \epsilon_{i+1})}{(\epsilon_{i+2} - \epsilon_{i-1})(\epsilon_{i+2} - \epsilon_{i+1})} f_J(J) \equiv 0 \operatorname{mod}(\epsilon_{i+2} - \epsilon_{i-1}).$$

  As $f_I(I)$ is divisible by $(\epsilon_{i+2} - \epsilon_i)$, we get that $a_{I,J} = 0$.
- Let $J = (i-3, -i+1)$. By using Lemma 3.20 and Eq. (7) repeatedly, and the existence of a $T$-equivariant curve between $p_I$ and $p_J$, we obtain the relation

$$f_I(I) - \frac{\epsilon_{i-1} - \epsilon_i + a_{I,J}(\epsilon_{i-2} - \epsilon_{i-3})}{(\epsilon_i - \epsilon_{i-3})(\epsilon_{i-2} - \epsilon_{i-3})} f_J(J) \equiv 0 \operatorname{mod}(\epsilon_i - \epsilon_{i-3}).$$

  As $f_I(I)$ is divisible by $(\epsilon_{i-1} - \epsilon_{i-3})$, we get that $a_{I,J} = 0$.

□

Putting all the lemmas together, we have proved:

**Theorem 3.24** (Chevalley formula) *The coefficients $a_{I,J}$ for $I = (i_1, i_2)$, $J = (j_1, j_2)$ two admissible subsets in the Chevalley formula (7) for the bisymplectic Grassmannian* *of planes* $\mathrm{I}_2\mathrm{Gr}(2, V)$ *are given by the following rules (the integer i is always supposed to be* $> 0$*):*

*(1)* $a_{(i,-i+1),(i,-i-1)} = a_{(i,-i+1),(i-2,-i+1)} = \epsilon_{i-1} - \epsilon_i$;
*(2)* $a_{(i,-i+2),(i-3,-i+2)} = a_{(i,-i+2),(i,-i-1)} = a_{(i,-i+1),(i-1,-i-1)} = a_{(i,-i+1),(i-2,-i)} = 1$;
*(3)* $a_{(i,-i+2),(i-2,-i+1)} = a_{(i,-i+2),(i-1,-i)} = 2$;
*(4)* *if* $I \geq J$, $\#(I \cap J) = 1$, $\mathrm{codim}(\sigma_I) = \mathrm{codim}(\sigma_J) - 1$ *and* $a_{(I,J)}$ *does not appear in the cases* 1), 2), 3) *above, then* $a_{I,J} = 1$;
*(5)* *in all remaining cases* $a_{I,J} = 0$.

Thus, we obtain:

**Corollary 3.25** *Equation* (7) *and Theorem* 3.24 *determine inductively the equivariant classes of all the Schubert varieties inside* $\mathrm{I}_2\mathrm{Gr}(2, V)$.

***Remark 3.26*** ($\mathrm{I}_2\mathrm{Gr}(2, 8)$) Let us point out that the constant coefficients $a_{I,J}$ computed in Theorem 3.24 give the Chevalley formula for the classical cohomology (by Theorem 3.3) and therefore allow to compute the degrees of Shubert varieties. In Fig. 5 we reported the degrees of Schubert varieties inside $\mathrm{I}_2\mathrm{Gr}(2, 8)$ (the case of $\mathrm{I}_2\mathrm{Gr}(2, 6)$ will be dealt with in the next section). As it was expected classically, we find that the degree of $\mathrm{I}_2\mathrm{Gr}(2, 8)$ is equal to $\deg(\mathrm{Gr}(2, 8)) = 132$, and this is an evidence of the fact that our formula is correct.

### 3.4 A quasi-homogeneous example

As an application of the previous general results, in this section we study in detail the smallest non-trivial bisymplectic Grassmannian of planes, i.e. $\mathrm{I}_2\mathrm{Gr}(2, V)$ with $V \cong \mathbb{C}^6$. This variety is interesting not only because computations are still feasible by hand, but because it is a quasi-homogeneous variety, i.e. it admits an action of a group with a dense orbit. Moreover, it has no small deformations, and it admits only a finite number of flat deformations. In the following we study its decomposition in orbits and its flat deformations. Then, we will give a presentation of its (classical) cohomology ring.

The variety $\mathrm{I}_2\mathrm{Gr}(2, V)$ with $V \cong \mathbb{C}^6$ admits an action of

$$G = \mathrm{SL}(2)^3 \cong \mathrm{SL}(K_1) \times \mathrm{SL}(K_2) \times \mathrm{SL}(K_3),$$

where the 2-dimensional planes $K_1, K_2, K_3$ have been defined in Sect. 2. We will denote a vector inside $K_i$ by the subscript $i$ (e.g. $v_i$, $v_i'$, etc.). The list of $G$-orbits inside $\mathrm{I}_2\mathrm{Gr}(2, V)$ with their representatives is the following one:

- A representative of the dense orbit is $[P] = (v_1 + v_2 + v_3) \wedge (v_1' + v_2' + v_3')$. This orbit is isomorphic to the quotient $\mathrm{SL}(2)^3/\mathrm{SL}(2)$, where the quotient factor $\mathrm{SL}(2)$ is the image of the diagonal morphism $\mathrm{SL}(2) \to SL(2)^3$. Being the quotient of two reductive groups, the dense orbit is an affine variety. Indeed, as the Plücker coordinate $q_{(1,-1)}([P]) \neq 0$, all the points $[Q]$ of the orbit satisfy $q_{(1,-1)}([Q]) \neq$

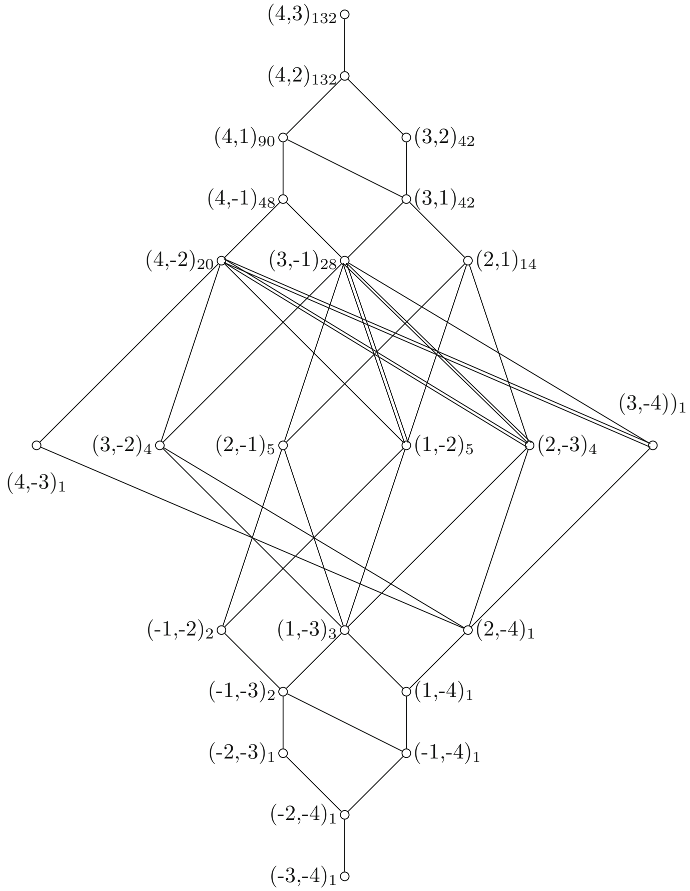



**Fig. 5** Degree of Schubert varieties (notation $I_{\deg(\sigma_I)}$) inside $\mathrm{I}_2\mathrm{Gr}(2, 8)$; the number of edges between vertex $I$ and vertex $J$ gives the coefficient $a_{I,J}$ in classical cohomology

0. Therefore, the orbit is contained inside the affine variety $\{q_{(1,-1)} \neq 0\} \subset \mathrm{I}_2\mathrm{Gr}(2, V)$; in fact the dense orbit is equal to $\{q_{(1,-1)} \neq 0\}$ (or equivalently $q_{(2,-2)} \neq 0$ or $q_{(3,-3)} \neq 0$).

- There is one orbit with representatives of type $(v_i + v_j) \wedge (v_j + v_k)$ (or, which is the same, $(v_i + 2v_j + v_k) \wedge (v_i + v_j)$). Let $\mathcal{U}_i$ be the tautological bundle over $\mathbf{P}(K_i)$. Then this orbit is isomorphic to the total space of

$$(\mathbf{P}(\mathcal{U}_i \oplus \mathcal{U}_j) \setminus (\mathbf{P}(\mathcal{U}_i) \cup \mathbf{P}(\mathcal{U}_j))) \times (\mathbf{P}(\mathcal{U}_k \oplus \mathcal{U}_j) \setminus (\mathbf{P}(\mathcal{U}_k) \cup \mathbf{P}(\mathcal{U}_j)))$$
$$\text{over } \mathbf{P}(K_1) \times \mathbf{P}(K_2) \times \mathbf{P}(K_3).$$

Its closure is the irreducible divisor that compactifies the dense orbit.

- There are three orbits with representatives of type $v_i \wedge (v_j + v_k)$, each one isomorphic to

$$\mathbf{P}(K_i) \times (\mathbf{P}(K_j \oplus K_k) \setminus (\mathbf{P}(K_j) \cup \mathbf{P}(K_k))).$$

- There are three minimal orbits with representatives of type $v_i \wedge v_j$, each one isomorphic to

$$\mathbf{P}(K_i) \times \mathbf{P}(K_j).$$

#### 3.4.1 The Hilbert scheme of $\mathrm{I}_2\mathrm{Gr}(2, 6)$

We have already seen that $\mathrm{I}_2\mathrm{Gr}(2, 6)$ has no small deformations (Theorem 2.7), and that there is only one smooth isomorphism class (see Remark 2.10). This is related to the fact that if $V \cong \mathbb{C}^6$, then $(\wedge^2 V^*) \otimes \mathbb{C}^2$ is a prehomogeneous space for the action of $\mathrm{SL}(V) \times \mathrm{SL}(2) \times \mathbb{C}^*$ (see [13]). This implies that there are just a finite number of orbits and therefore that all pencils $\Omega$ in a dense subset of $\mathbf{P}(\wedge^2 V^*)$ are conjugated under the action of $\mathrm{PGL}(V)$. As a consequence, there are only finitely many isomorphism classes of varieties of the form $\mathscr{Z}(\Omega)$. In the following we intend to describe these varieties.

We will consider $\mathrm{I}_2\mathrm{Gr}(2, V)$ with $V \cong \mathbb{C}^6$ as a subvariety $\mathrm{I}_2\mathrm{Gr}(2, V) \subset \mathrm{Gr}(2, V) \subset \mathbf{P}(\wedge^2 V^*)$, and we will denote by $p(t)$ the Hilbert polynomial

$$p(t) = \chi(\mathrm{I}_2\mathrm{Gr}(2, V), \mathcal{O}(t)) = \mathrm{H}^0(\mathrm{I}_2\mathrm{Gr}(2, V), \mathcal{O}(t)) \text{ for } t >> 0.$$

**Proposition 3.27** *There are* 11 *flat deformations (included the smooth one) of* $\mathrm{I}_2\mathrm{Gr}(2, V)$ *inside* $\mathrm{Gr}(2, V)$. *They correspond to the orbits of* $\mathrm{SL}(V)$ *inside* $\mathrm{Gr}(2, \wedge^2 V^*)$, *which can be identified as a smooth component of the Hilbert scheme of* $\mathrm{I}_2\mathrm{Gr}(2, V) \subset \mathrm{Gr}(2, V)$.

***Proof*** Let us consider a pencil $\Omega$. In order to have that $\mathscr{Z}(\Omega)$ is a flat deformation of the (smooth) bisymplectic Grassmannian, we only need to verify that it has the expected codimension (equal to 6). Indeed, in that case, we can compute $p(t) = \chi(\mathrm{I}_2\mathrm{Gr}(2, V), \mathcal{O}(t))$ by using the Koszul complex as

$$p(t) = \chi(Gr(2, V), \mathcal{O}(t)) - \chi(\mathrm{Gr}(2, V), 2\mathcal{O}(t-1)) + \chi(\mathrm{Gr}(2, V), \mathcal{O}(t-2)),$$

obtaining that the Hilbert polynomial does not depend on the particular choice of $\Omega$.

By [13][Case $E_7, \alpha_3$], there are 15 orbits of $\mathrm{SL}(V) \times \mathrm{SL}(2) \times \mathbb{C}^*$ inside $(\wedge^2 V^*) \otimes \mathbb{C}^2$. Four of them are generated by one form; therefore, the corresponding zero locus $\mathscr{Z}(\Omega)$ has dimension $\geq 7$ and cannot be a flat deformation of the (smooth) bisymplectic Grassmannian. The orbits of actual pencils $\Omega$ have been reported in Fig. 6. Among them:

- the pencils inside $O_0, O_1, O_2, O_{5_I}, O_6$ contain a non-degenerate form; therefore, $\mathscr{Z}(\Omega)$ is a hypersurface in the irreducible variety $\mathrm{IGr}(2, V)$ and has dimension equal to 6;

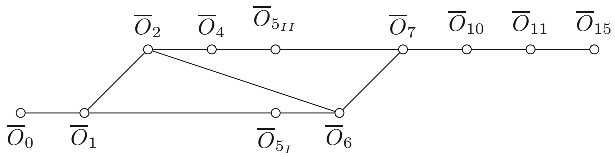



**Fig. 6** Orbit closures of non-degenerate pencils of 2-forms with respective codimensions as labels

- the pencils inside $O_7$, $O_{10}$, $O_{11}$, $O_{15}$ contain a form of type $x_1 \wedge x_{-1}$, whose zero locus defines a (irreducible) Schubert variety inside $\mathrm{Gr}(2, V)$. Therefore, $\mathscr{Z}(\Omega)$ is again 6-dimensional;
- the pencils inside $O_4$, $O_{5_{II}}$ contain a form of type $x_1 \wedge x_{-1} + x_2 \wedge x_{-2}$, which is singular only at one point and irreducible as well. Therefore, once more $\mathscr{Z}(\Omega)$ is 6-dimensional.

We have thus shown that the family $\{(\mathscr{Z}(\Omega), \Omega) \subset \mathrm{Gr}(2, V) \times \mathrm{Gr}(2, \wedge^2 V^*)\}$ is flat over $\mathrm{Gr}(2, \wedge^2 V^*)$, and this gives a morphism $\psi$ from $\mathrm{Gr}(2, \wedge^2 V^*)$ to the Hilbert scheme of $\mathrm{I}_2\mathrm{Gr}(2, V) \subset \mathrm{Gr}(2, V)$. Moreover, this Hilbert scheme has tangent space at $\psi(\Omega) = \mathscr{Z}(\Omega)$ equal to

$$\mathrm{H}^0(\mathscr{Z}(\Omega), \mathcal{N}_{\mathscr{Z}(\Omega),\mathrm{Gr}(2,V)}) = \mathrm{H}^0(\mathscr{Z}(\Omega), 2\mathcal{O}(1)) \cong T_{\mathrm{Gr}(2,\wedge^2 V^*),\Omega},$$

and the differential of the morphism $\psi$ is an isomorphism at each point (notice that the chain of isomorphisms does not depend on the fact that $\mathscr{Z}(\Omega)$ is smooth). We get that $\psi$ is étale; moreover, it is injective because $\Omega$ can be recovered as the codimension two linear space inside $\wedge^2 V$ generated by the linear system $|\mathcal{O}(1)|$ over $\mathscr{Z}(\Omega)$. Therefore, $\mathrm{Gr}(2, \wedge^2 V^*)$ is exactly one irreducible component of the Hilbert scheme. □

### 3.4.2 Presentation of the cohomology for $\mathrm{I}_2\mathrm{Gr}(2, 6)$

In this last section, we compute explicitly the (equivariant) cohomology of $\mathrm{I}_2\mathrm{Gr}(2, V)$ for $V \cong \mathbb{C}^6$. We give a presentation of the cohomology ring, and we discuss some related questions, such as the existence of a certain symmetry or of a self-dual basis. We begin with an application of the Chevalley formula for bisymplectic Grassmannians of planes:

**Proposition 3.28** *The coefficients $a_{I,J}$ that appear in Eq.* (7) *for* $\mathrm{I}_2\mathrm{Gr}(2, 6)$ *are uniquely determined by the relations in Theorem* 3.1*. They are reported in Fig.* 7.

This proposition is a restatement of Theorems 3.12 and 3.24. Thus, by Corollary 3.25, we know that it is possible to determine inductively the equivariant classes of all the Schubert varieties inside $\mathrm{I}_2\mathrm{Gr}(2, 6)$.

***Remark 3.29*** The constant coefficients $a_{I,J}$ determine the multiplication of a Schubert variety with the hyperplane section in the ordinary cohomology, i.e. a Pieri type formula for $\mathrm{I}_2\mathrm{Gr}(2, 6)$. In particular, our computations are coherent with the fact that the degree of $\mathrm{I}_2\mathrm{Gr}(2, 6)$ is 14, as we know because it is the degree of $\mathrm{Gr}(2, 6)$.

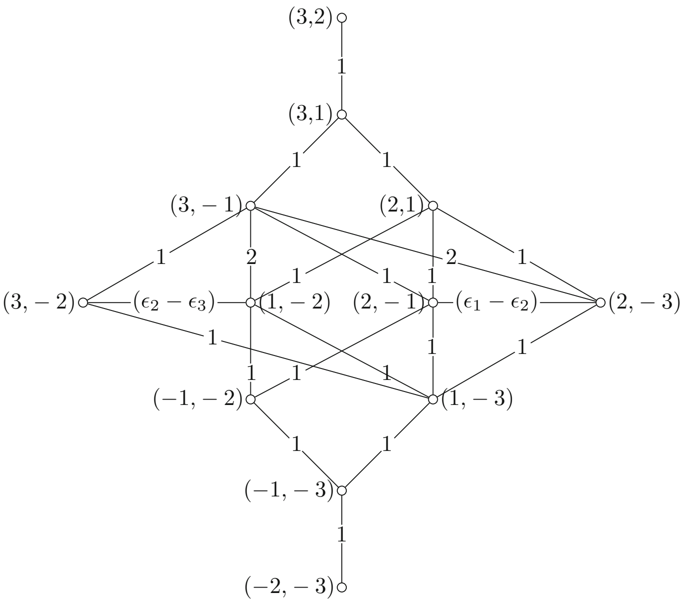
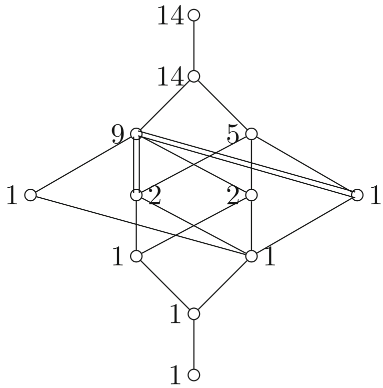



**Fig. 7** Coefficients $a_{I,J}$ in $\mathrm{I}_2\mathrm{Gr}(2, 6)$

**Fig. 8** Degrees of Schubert varieties inside $\mathrm{I}_2\mathrm{Gr}(2, 6)$




From the equivariant cohomology, one can recover the classical cohomology of $\mathrm{I}_2\mathrm{Gr}(2, 6)$ (Theorem 3.3). We will use the following notations:

$$\sigma_1 := \sigma_{3,1} \; , \sigma_2 := \sigma_{2,1} \; , \sigma_3 := \sigma_{3,-2} \; , \sigma_3' := \sigma_{2,-3},$$

with

$$\deg(\sigma_1) = 14 \; , \; \deg(\sigma_2) = 5 \; , \; \deg(\sigma_3) = 1 \; , \; \deg(\sigma_3') = 1.$$

**Theorem 3.30** *A presentation of the cohomology of the bisymplectic Grassmannian* $\mathrm{I_2Gr}(2,6)$ *is given by:*

$$\mathrm{H}^*(\mathrm{I_2Gr}(2,6),\mathbb{Z}) \cong \mathbb{Z}[\sigma_1,\sigma_2,\sigma_3,\sigma_3']/I,$$

*where $I$ is the ideal generated by the following elements:*

$$\begin{array}{ccc} 2\sigma_1^4-2\sigma_1^2\sigma_2-3\sigma_1\sigma_3' & , & \sigma_2\sigma_3' & , \\ \sigma_1\sigma_3-\sigma_1\sigma_3' & , & \sigma_3\sigma_3'-\sigma_1^3\sigma_3' & , \\ \sigma_2^2-\sigma_1^4+2\sigma_1^2\sigma_2+2\sigma_1\sigma_3' & , & \sigma_3^2 & , \\ \sigma_1^5-14\sigma_1^2\sigma_3' & , & \sigma_3'^2 & , \\ \sigma_2\sigma_3 & , & \sigma_1^4\sigma_3' & . \end{array}$$

***Proof*** First, we prove that $\sigma_1,\sigma_2,\sigma_3,\sigma_3'$ generate the cohomology by showing that they generate all the Schubert classes $\sigma_I$. This is a consequence of the following formulas, which can be derived directly from Fig. 8:

$$\begin{aligned}
\sigma_{(3,-1)} &= \sigma_1^2-\sigma_2,\\
\sigma_{(2,-1)} &= 3\sigma_1\sigma_2-\sigma_1^3+\sigma_3,\\
\sigma_{(1,-2)} &= \sigma_1^3-2\sigma_1\sigma_2-\sigma_3-\sigma_3',\\
\sigma_{(-1,-2)} &= \sigma_1^4-2\sigma_1^2\sigma_2-3\sigma_1\sigma_3',\\
\sigma_{(1,-3)} &= \sigma_1\sigma_3',\\
\sigma_{(-1,-3)} &= \sigma_1^2\sigma_3',\\
\sigma_{(-2,-3)} &= \sigma_1^3\sigma_3'.
\end{aligned}$$

The relations generating $I$ involving the product of $\sigma_1$ with other classes can be derived from Fig. 8 too. For the remaining relations, they can be derived from the following identities, which hold in the equivariant cohomology, and can be verified by computing explicitly the classes $\sigma_I$:

$$\begin{aligned}
\sigma_2^2 &= \sigma_2(\epsilon_3-\epsilon_1)(\epsilon_3-\epsilon_2)+\sigma_{(1,-2)}(\epsilon_3-\epsilon_1)+\sigma_{(2,-1)}(\epsilon_3-\epsilon_2)+\\
&\quad +\sigma_3'(\epsilon_3-\epsilon_2)+\sigma_1\sigma_{(1,-2)},\\
\sigma_2\sigma_3 &= (\epsilon_2+\epsilon_3)(\sigma_{(1,-2)}(\epsilon_2-\epsilon_3)+\sigma_{(1,-3)}),\\
\sigma_2\sigma_3' &= 2\epsilon_3(\sigma_3'(\epsilon_3-\epsilon_2)+\sigma_{(1,-3)}),\\
\sigma_3\sigma_3' &= \sigma_{(-2,-3)},\\
\sigma_3^2 &= 2\epsilon_2(\sigma_3(\epsilon_1+\epsilon_2)(\epsilon_2-\epsilon_1)+\sigma_{(1,-2)}(\epsilon_1+\epsilon_3)(\epsilon_3-\epsilon_2)+\\
&\quad -\sigma_{(-1,-2)}(\epsilon_3-\epsilon_2)-\sigma_{(1,-3)}(\epsilon_1+\epsilon_3)+\sigma_{(-1,-3)}),\\
\sigma_3'^2 &= 2\epsilon_3(\sigma_3'(\epsilon_3-\epsilon_1)(\epsilon_3+\epsilon_1)+\sigma_{(1,-3)}(\epsilon_1+\epsilon_2)-\sigma_{(-1,-3)}).
\end{aligned}$$

We have verified that these are all the relations inside $I$ by showing that they generate all products involving $\sigma_1,\sigma_2,\sigma_3,\sigma_3'$. □

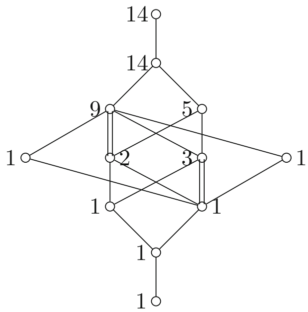



**Fig. 9** Degree of classes in a self-dual basis; the codimension 3 classes are, from left to right: $\sigma_{(3,-2)}, \sigma_{(1,-2)}, \sigma_x, \sigma_{(2,-3)}$

***Remark 3.31*** The basis given by the Schubert classes inside $\mathrm{I}_2\mathrm{Gr}(2,6)$ is not self-dual with respect to the intersection product. For instance, the nonzero products of codimension 3 Schubert classes are as follows:

$$\sigma_{(3,-2)}\sigma_{(2,-3)} = 1,$$
$$\sigma_{(1,-2)}\sigma_{(2,-1)} = 1,$$
$$\sigma_{(3,-2)}\sigma_{(2,-1)} = -1.$$

A self-dual basis in codimension 3 would be given by $\sigma_{(3,-2)}, \sigma_{(2,-3)}, \sigma_{(1,-2)}, \sigma_x = \sigma_{(2,-1)} + \sigma_{(2,-3)}$. In this basis, the degree diagram is the one shown in Fig. 9. Notice that the diagram is symmetric with respect to a central reflection; this is a consequence of the fact that the additive basis chosen is self-dual.

***Remark 3.32*** The group of permutations $\mathfrak{S}_n$ acts on the cohomology of the bisymplectic Grassmannians, even though it does not act on the varieties themselves; the action is a consequence of a monodromy phenomenon.

Let $X$ be a bisymplectic Grassmannian $\mathrm{I}_2\mathrm{Gr}(k, 2n)$ defined by the forms

$$\omega_1 = \sum_{i=1}^{n} x_i \wedge x_{-i} \text{ and } \omega_2 = \sum_{i=1}^{n} \lambda_i x_i \wedge x_{-i}.$$

Let $\eta$ be an element of the group of permutations $\mathfrak{S}_n$. There exists a curve $\gamma$ inside the space of pencils of bisymplectic forms that goes from $\Omega = \langle \omega_1, \omega_2 \rangle$ to $\eta.\Omega = \langle \omega_1, \eta.\omega_2 \rangle$, where

$$\eta.\omega_2 = \sum_{i=1}^{n} \lambda_{\eta(i)} x_i \wedge x_{-i}.$$

Following the curve, one obtains a continuous deformation $\gamma$ such that $\gamma(0) = X = \gamma(1)$, and which sends a Schubert variety $\sigma_I$ to $\eta.\sigma_I$, where the action on $\sigma_I$ is induced by the one of $\mathfrak{S}_n$ on the pencils. As the cohomology is locally constant, the action on Schubert varieties induces an action in cohomology. In the following we show concretely what it means in the case when $k = 2, n = 3$.

As the irreducible representations of $\mathfrak{S}_3$ given by Schubert classes with codimension different from 3 are only 1-dimensional, we will focus on codimension 3 Schubert varieties. They admit the following explicit description:

$$
\begin{aligned}
\alpha_2 &:= \sigma_{(3,-2)} = v_{-2} \wedge \mathbf{P}(\langle v_{\pm 3}, v_{\pm 1}\rangle),\\
\beta_1 &:= \sigma_{(1,-2)} = \{x \in \mathbf{P}(\langle v_{-2}, v_{-3}\rangle) \wedge \mathbf{P}(\langle v_{\pm 1}, v_{-2}, v_{-3}\rangle) \text{ s.t. } x \neq 0\},\\
\beta_2 &:= \sigma_{(2,-1)} = \{x \in \mathbf{P}(\langle v_{-1}, v_{-3}\rangle) \wedge \mathbf{P}(\langle v_{\pm 2}, v_{-1}, v_{-3}\rangle) \text{ s.t. } x \neq 0\},\\
\alpha_3 &:= \sigma_{(2,-3)} = v_{-3} \wedge \mathbf{P}(\langle v_{\pm 2}, v_{\pm 1}\rangle).
\end{aligned}
$$

Moreover, inside the cohomology of $\mathrm{I}_2\mathrm{Gr}(2,6)$ there are two more remarkable varieties:

$$
\begin{aligned}
\alpha_1 &:= v_{-1} \wedge \mathbf{P}(\langle v_{\pm 3}, v_{\pm 2}\rangle),\\
\beta_3 &:= \{x \in \mathbf{P}(\langle v_{-1}, v_{-2}\rangle) \wedge \mathbf{P}(\langle v_{\pm 3}, v_{-1}, v_{-2}\rangle) \text{ s.t. } x \neq 0\}.
\end{aligned}
$$

Actually, there are also varieties $\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \beta_{-1}, \beta_{-2}, \beta_{-3}$, but one can prove that in cohomology $\alpha_i = \alpha_{-i}$ and $\beta_i = \beta_{-i}$ for $i = 1, 2, 3$. The action of $\mathfrak{S}_3$ on the $\alpha_i$'s and the $\beta_i$'s is the expected one. By using the products of the codimension 3 Schubert varieties and the symmetries given by $\mathfrak{S}_3$, one can prove that

$$
\begin{aligned}
\alpha_1 - \alpha_2 &= \beta_1 - \beta_2,\\
\alpha_2 - \alpha_3 &= \beta_2 - \beta_3.
\end{aligned}
$$

To summarize, the action of $\mathfrak{S}_3$ on $\mathrm{H}^i(\mathrm{I}_2\mathrm{Gr}(2,6), \mathbb{Z})$ is trivial if $i \neq 6$ because all representations are completely reducible and $\mathfrak{S}_3$ acts trivially on $\sigma_H^i$; moreover $\mathrm{H}^6(\mathrm{I}_2\mathrm{Gr}(2,6), \mathbb{Z})$ decomposes in the sum of two trivial representations generated by the classes of $\sigma_H^3 = \alpha_2 + 3\beta_1 + 2\beta_2 + 3\alpha_3$ and $\sigma_{(2,1)}\sigma_H = \beta_1 + \beta_2 + \alpha_3$, and one natural 2-dimensional representation given by the action on $\langle \alpha_1, \alpha_2, \alpha_3\rangle$, with $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

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