Journal of Algebraic Combinatorics (2021) 53:157–175
https://doi.org/10.1007/s10801-019-00924-4

On a conjecture about an analogue of Tokuyama's theorem for G_2

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Received: 19 May 2018 / Accepted: 29 October 2019 / Published online: 28 November 2019

Abstract
We prove the conjecture of Friedlander et al. (J Algebr Comb 41:1089, 2015) about sums over Littelmann patterns for the root system of type G_2, which is an analogue of Tokuyama's theorem Tokuyama (J Math Soc Jpn 40(4):671–685, 1988) for root systems of type A_r. We use elementary means to show that the conjecture is implied by a finite set of polynomial identities.

Keywords Character formulas · Whittaker functions · Littelmann patterns

Mathematics Subject Classification 17B10 · 11F68 · 20C15 · 05E15

1 Introduction

For integer $r \geq 2$, let $\mathfrak{sl}_r(\mathbb{C})$ be the Lie algebra of $r \times r$ complex matrices with trace zero equipped with the Lie bracket $[X, Y] = XY - YX$. Let θ be a dominant weight for $\mathfrak{sl}_r(\mathbb{C})$, and let $V(\theta)$ be the irreducible representation of highest weight θ. A basis for $V(\theta)$ can be given in terms of a set of Gel'fand–Cetlin patterns $GC(\theta)$; this is a set of ordered tuples of integers satisfying certain inequalities depending on θ and is closely connected to the canonical bases of [6].

Now let $\chi(\theta)$ be the character of $V(\theta)$, and let ρ be the sum of the fundamental weights of $\mathfrak{sl}_r(\mathbb{C})$. Then Tokuyama's theorem gives a formula for $\chi(\theta)$ in the form

$$\sum_{P \in GC(\theta+\rho)} H(P) = D(\mathbf{x}, q; A_{r-1})\chi(\theta) \tag{1}$$

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where $D(\mathbf{x}, q; A_{r-1})$, defined in Definition 2, is a q-deformation of the usual Weyl denominator, and where each summand $H(P)$ is a polynomial in q^{-1} computed using statistics taken from the pattern P. Therefore, Tokuyama's theorem connects two fundamental objects in the representation theory of $\mathfrak{sl}_r(\mathbb{C})$.

We note that since a semisimple Lie algebra over an algebraically closed field is determined by its root system, Tokuyama's theorem can be expressed solely in terms of the root system A_{r-1} of $\mathfrak{sl}_r(\mathbb{C})$.

A major question is how Eq. (1) generalizes to other root systems. Now, the polynomial $D(\mathbf{x}, q; A_{r-1})$ has a natural generalization $D(\mathbf{x}, q; R)$ to arbitrary root systems R, as does $\chi(\theta)$ by construction, so the right side of Eq. (1) is easily generalized. A series of papers mentioned below seek combinatorial structures that generalize the left side of (1) for various root systems R. This paper proves a conjecture of Friedlander et al. [5], called Theorem 2 in this paper, for such a generalization to G_2 using an elementary bijective proof.

Generalizing Tokuyama's theorem is a major question in part because the right side of Eq. (1) is the same formula that provides a closed-form evaluation of certain functions called *matrix coefficients*. These matrix coefficients arise from the principal series representation of an algebraic group over a non-Archimedean field. More specifically, such a matrix coefficient is defined by a certain integral of a canonical function on the algebraic group, where that function is viewed as a vector in the principal series representation of the algebraic group. Shintani [10] and Casselman–Shalika [2] evaluate the integral for one such matrix coefficient, known the spherical-Whittaker function, to be the right side of (1). Generalizations of Tokuyama's theorem thus would provide insight into evaluating integrals over algebraic groups and the structure of algebraic groups in general.

In addition, Tokuyama's theorem has played an important part in the development of Weyl group multiple Dirichlet series. These are Dirichlet series in multiple variables and are defined in terms of a given root system. A Weyl group multiple Dirichlet series possesses a set of functional equations isomorphic to the Weyl group of that root system. These series are also conjectured to be the Fourier–Whittaker coefficients of certain Eisenstein series on metaplectic groups. This relationship is described in more detail in [1].

We now mention generalizations of Tokuyama's theorem to other root systems. Tokuyama's theorem has been generalized to type C by Hamel and King [7] in which they use symplectic shifted tableaux in place of Gel'fand–Cetlin patterns; and to type B by Friedberg and Zhang [4] in which they use metaplectic double covers in their proof. A conjectural generalization to type D was given by Chinta and Gunnells [3]. For arbitrary root systems, McNamara [9] shows how to compute p-adic Whittaker functions as sums over crystal graphs. For type A, these sums may be interpreted as sums over Gel'fand–Cetlin patterns mentioned above, but have not been computed explicitly for other root systems. The conjecture of [5] (Theorem 2 here) is the first conjecture about an analogue of Tokuyama's theorem for an exceptional root system.

We now describe our proof strategy and compare it to the work mentioned above. In [11], Tokuyama proves his original theorem using the Pieri rule applied to Schur polynomials. Our proof of Theorem 2 is different from the proof in [11] and other proofs for the analogues to other root systems in that the proof in this paper is purely algebraic, using elementary combinatorics to establish a bijection. Specifically, we express both sides of Theorem 2 as polynomials in four indeterminates whose coefficients are rational functions. We then show that the coefficients are equal. We demonstrate these techniques for the root system A_2 in Sect. 3.

2 Background material for Littelmann patterns and root systems

The authors of [5] replace the Gel'fand–Cetlin patterns of (1) with a set of inequalities called *Littelmann patterns* [8]. Littelmann patterns can be defined for any root system R and yield the Gel'fand–Cetlin patterns when $R = A_r$. We now briefly describe these Littelmann patterns following the description in [5].

Let $\mathfrak{g}$ be a simple complex Lie algebra with root system R. Let w_L be a reduced expression for the longest element in the Weyl group of R. Let θ be a dominant weight, and let V_θ the irreducible representation of $\mathfrak{g}$ with lowest weight $-\theta$. As noted in [5], for some root systems, such as G_2, V_θ coincides with the representation with highest weight θ, but differs for other root systems, such as A_r. Ultimately what matters for us is the normalized definition of χ_θ in Definition 1 and the Littelmann inequalities discussed next.

Given the above data, Littelmann describes how to construct a rational polyhedral cone $C_\theta \in \mathbb{R}^N$, where N is the number of positive roots in R. He also constructs a bijection between the lattice points of C_θ and the vertices of the crystal graph $B(\theta)$. This $B(\theta)$ is a finite directed graph constructed from representation-theoretic properties of V_θ. The Littelmann patterns π are then N-tuples of non-negative integers that index the lattice points in C_θ. The pattern π satisfies certain inequalities which we call Littelmann inequalities below. When $R = A_r$ and $w_L = s1(s_2 s_1)(s_3 s_2 s_1)...(s_r ... s_1)$, the Littelmann patterns are equivalent to Gel'fand–Cetlin patterns. No knowledge of crystal graphs is necessary to understand the proof in this paper; we need to consider only the Littelmann inequalities themselves.

We review the concepts related to root systems necessary to state Theorem 2. Let R be an irreducible root system of rank r in a vector space V with inner product $\langle \cdot, \cdot \rangle$. Let R_+ denote a choice of positive roots in R with simple roots $\alpha_1, \ldots, \alpha_r$. Let $\varpi_1, \ldots, \varpi_r \in V$ denote the fundamental weights defined by

$$2\frac{(\varpi_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

Let $\Lambda_W \subset V$ denote the weight lattice, and $\Lambda \subset V$ the root lattice, and $\mathbb{C}[\Lambda]$ the associated ring of Laurent polynomials. Identifying x^{α_i} with x_i, we express $\mathbb{C}[\Lambda]$ as $\mathbb{C}[x_1^{\pm}, \ldots, x_r^{\pm}]$. For any linear combination θ of the α_i

$$\theta = \sum_{i=1}^{r} c_i \alpha_i$$

for some rational numbers c_i, we write

$$\mathbf{x}^\theta = \prod_{i=1}^{r} x_i^{c_i}.$$

Let ρ denote the half-sum of positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha = \sum_{i=1}^{r} \varpi_i.$$

For a dominant weight $\theta \in \Lambda_W$, the weight $\theta + \rho$ is equal to

$$\theta + \rho = \sum_{i=1}^{r} \ell_i \varpi_i$$

for some positive integers ℓ_i. Let W be the Weyl group of R generated by the simple reflection s_i. For $w \in W$, let length(w) be the length of a reduced expression for w and let $\mathrm{sgn}(w)$ be

$$\mathrm{sgn}(w) = (-1)^{\mathrm{length}(w)}.$$

Now let w_L denote the long element in W. We define χ_θ to be the character of V_θ normalized to have constant term 1:

Definition 1

$$\chi_\theta = \frac{\mathrm{sgn}(w_L)}{\mathbf{x}^{w_L(\theta+\rho)})} \frac{\sum_{w \in W} \mathrm{sgn}(w) \mathbf{x}^{w(\theta+\rho)}}{\prod_{\alpha>0}(1 - x^\alpha)}.$$

We next define the notation for factored expressions alluded to in the introduction.

Definition 2 Let q is an indeterminate. For $\alpha \in R$, denote

$$D(\mathbf{x}, q; \alpha) = \left(1 - \frac{\mathbf{x}^\alpha}{q}\right)$$

and

$$T(\mathbf{x}, q; \alpha) = \frac{\left(1 - \frac{\mathbf{x}^\alpha)}{q}\right)}{(1 - \mathbf{x}^\alpha)}.$$

We also denote

$$D(\mathbf{x}, q; R) = \prod_{\alpha \in R_+} D(\mathbf{x}, q; \alpha)$$

$$T(\mathbf{x}, q; R) = \prod_{\alpha \in R_+} T(\mathbf{x}, q; \alpha).$$

3 Demonstration for A_2

We demonstrate our proof techniques for Tokuyama's theorem for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ with root system $R = A_2$. The result we prove is

Theorem 1

$$\sum_{GC(\ell_1,\ell_2)} h(a_{11})h(a_{12})h(a_{21})x^{a_{11}+a_{12}}y^{a_{21}} = D(\boldsymbol{x}, q; A_2)\chi_\theta$$

where the terms are defined below. Note that χ_θ denotes the character normalized to have constant term 1 given in Eq. (2).

For A_2, the positive simple roots are

$$\alpha_1 = (\sqrt{2}, 0)$$
$$\alpha_2 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right).$$

The fundamental weights ϖ_1 and ϖ_2

$$\varpi_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$$
$$\varpi_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2.$$

The simple reflections s_1 and s_2 act linearly on R according to $s_i(\alpha_i) = -\alpha_i$ and

$$s_1(\alpha_2) = s_2(\alpha_1) = \alpha_1 + \alpha_2.$$

Recall $\theta \in \Lambda_W$ denotes a dominant weight and

$$\theta + \rho = \ell_1\varpi_1 + \ell_2\varpi_2$$

where ℓ_1 and ℓ_2 are positive integers. With $x = x_1$ and $y = x_2$, the normalized character χ_θ becomes

$$\chi_\theta = \frac{1 - x^{\ell_2} - y^{\ell_1} - (xy)^{\ell_1+\ell_2} + x^{\ell_1+\ell_2}y^{\ell_1} + x^{\ell_2}y^{\ell_1+\ell_2}}{(1-x)(1-y)(1-xy)}. \tag{2}$$

Therefore

$$D(\mathbf{x}, q; A_2)\chi_\theta = T(\mathbf{x}, q; A_2)(1-x^{\ell_2}-y^{\ell_1}-(xy)^{\ell_1+\ell_2}+x^{\ell_1+\ell_2}y^{\ell_1}+x^{\ell_2}y^{\ell_1+\ell_2}) \tag{3}$$

Table 1 Terms for $P_{W;A_2}$

Multi-degree	Coefficient
((0,0),(0,0))	$T(\mathbf{x}, q; A_2)$
((0,0),(1,0))	$-T(\mathbf{x}, q; A_2)$
((0,1),(0,0))	$-T(\mathbf{x}, q; A_2)$
((1,1),(1,1))	$-T(\mathbf{x}, q; A_2)$
((1,1),(1,0))	$T(\mathbf{x}, q; A_2)$
((0,1),(1,1))	$T(\mathbf{x}, q; A_2)$

We consider (3) as a polynomial $P_{W;A_2}$ in the four indeterminates x^{ℓ_1}, y^{ℓ_1}, x^{ℓ_2} and y^{ℓ_2}. We say that the term

$$(x^{m_1}y^{n_1})^{\ell_1}(x^{m_2}y^{n_2})^{\ell_2}$$

has multi-degree $((m_1, n_1), (m_2, n_2))$. Therefore the polynomial $P_{W;A_2}$ has six terms with the multi-degrees and coefficients listed in Table 1.

We define the set $GC(\ell_1, \ell_2)$ of Gel'fand–Cetlin patterns π for A_2. Consider the set of triples of integers (A_{11}, A_{12}, A_{22}) such that

1. $0 \leq A_{11} \leq \ell_1$
2. $\ell_1 \leq A_{12} \leq \ell_1 + \ell_2$
3. $A_{11} \leq A_{21} \leq A_{12}$

Such a triple is called a Gel'fand–Cetlin pattern. These patterns are identified with the Littelmann patterns for A_2 after the following change of variables:

$$A_{11} = a_{11},\ A_{12} = \ell_1 + a_{12},\ A_{21} = a_{21}.$$

The above inequalities then become

1. $0 \leq a_{11} \leq \ell_1$
2. $0 \leq a_{12} \leq \ell_2$
3. $a_{11} \leq a_{21} \leq \ell_1 + a_{12}$.

These are the Littelmann inequalities for the weight $\theta + \rho = \ell_1\varpi_1 + \ell_2\varpi_2$ and reduced word $s_1s_2s_1$. Define $GC(\ell_1, \ell_2)$ to be the set of 3-tuples of integers $\pi = (a_{11}, a_{12}, a_{21})$ that satisfy the Littelmann inequalities. We denote the lower bounds of these inequalities by L_{ij} and the upper bounds by U_{ij}:

$$L_{11} = 0, L_{12} = 0, L_{21} = a_{11}$$

and

$$U_{11} = \ell_1, U_{12} = \ell_2, U_{21} = \ell_1 + a_{12}.$$

Following the terminology of [5], we say that an entry a_{ij} of π is "circled", denoted by a_{ij}°, if a_{ij} attains the lower bound in its inequality; i.e., a_{ij} is circled if $a_{ij} = L_{ij}$. We say that a_{ij} is "boxed", denoted by $\underline{a_{ij}}$, if a_{ij} attains its upper bound in its inequality; i.e., a_{ij} is boxed if $a_{ij} = U_{ij}$.

We then define $h(a_{ij}) \in \mathbb{Z}[q^{-1}]$ by

Definition 3

$$h(a_{ij}) = \begin{cases} 1, & \text{if } a_{ij} \text{ is circled and not boxed} \\ 1 - q^{-1}, & \text{if } a_{ij} \text{ is neither circled nor boxed} \\ -q^{-1}, & \text{if } a_{ij} \text{ is boxed and not circled} \\ 0, & \text{if } a_{ij} \text{ is both boxed and circled.} \end{cases}$$

Now we prove Theorem 1. We note that this proof can be generalized to $A_r, r \geq 1$ to provide an elementary proof of Tokuyama's theorem which we give in another paper.

Proof We compute the sum

$$\sum_{\pi \in GC(\ell_1, \ell_2)} h(a_{11})h(a_{12})h(a_{21})x^{a_{11}+a_{12}}y^{a_{21}}.$$

We first perform the sum over a_{21} and then over a_{11} and a_{12}. By Lemma 1 in Sect. 5,

$$\sum_{a_{21}=a_{11}}^{\ell_1+a_{12}} h(a_{21})y^{a_{21}} = \frac{(1-\frac{y}{q})}{1-y}(y^{a_{11}} - y^{\ell_1+a_{12}}) \tag{4}$$

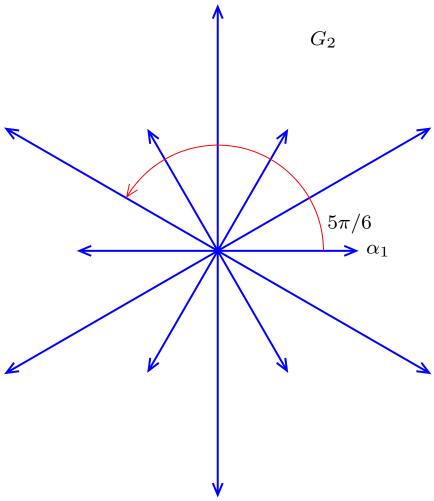
We remember the factor $\frac{1-\frac{y}{q}}{1-y} = T(\mathbf{x}, q; \alpha_2)$ and take the $y^{a_{11}}$ term in (4). We sum this term over a_{11} and a_{12} again using Lemma 1 in Sect. 5 to get

$$\begin{aligned} \sum_{a_{12}=0}^{\ell_2} \sum_{a_{11}=0}^{\ell_1} (xy)^{a_{11}} x^{a_{12}} &= \left(\sum_{a_{11}=0}^{\ell_1} (xy)^{a_{11}} \right) \left(\sum_{a_{12}=0}^{\ell_2} x^{a_{12}} \right) \\ &= \left(\frac{\left(1 - \frac{xy}{q}\right)}{1-xy} (1-(xy)^{\ell_1}) \right) \left(\frac{\left(1-\frac{x}{q}\right)}{1-x}(1-x^{\ell_2}) \right). \end{aligned} \tag{5}$$

Similarly we take the $y^{\ell_1+a_{12}}$ term in (4) and sum over a_{11} and a_{12} to get

$$\begin{aligned} -y^{\ell_1} \sum_{a_{12}=0}^{\ell_2} \sum_{a_{11}=0}^{\ell_1} x^{a_{11}}(xy)^{a_{12}} &= -y^{\ell_1} \left(\sum_{a_{11}=0}^{\ell_1} x^{a_{11}} \right) \left(\sum_{a_{12}=0}^{\ell_2} (xy)^{a_{12}} \right) \\ &= -y^{\ell_1} \left(\frac{\left(1-\frac{x}{q}\right)}{1-x}(1-x^{\ell_1}) \right) \left(\frac{\left(1-\frac{xy}{q}\right)}{1-xy}(1-(xy)^{\ell_2}) \right). \end{aligned} \tag{6}$$

Fig. 1 The root system G_2



Adding (5) and (6) and multiplying by $T(\mathrm{x}, q; \alpha_2)$ gives

$$\begin{aligned} & T(\mathbf{x}, q; A_2)\left((1-(xy)^{\ell_1})(1-x^{\ell_2}) - y^{\ell_1}(1-x^{\ell_1})(1-(xy)^{\ell_2})\right) \\ &\quad = T(\mathbf{x}, q; A_2)(1-x^{\ell_2}+x^{\ell_2}(xy)^{\ell_1} - y^{\ell_1} + y^{\ell_1}(xy)^{\ell_2}) - (xy)^{\ell_2}(xy)^{\ell_1}). \end{aligned}$$

This proves Tokuyama's theorem for $R = A_2$. □

4 Statement of conjecture

We define the terms necessary to state Theorem 2.

Let $\mathfrak{g}$ be the simple complex Lie algebra of type with root system $R = G_2$ (see Fig. 1). The simple roots of R are

$$\alpha_1 = (\sqrt{2}, 0)$$
$$\alpha_2 = \left(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\right).$$

The fundamental weights are therefore

$$\varpi_1 = 2\alpha_1 + \alpha_2$$
$$\varpi_2 = 3\alpha_1 + 2\alpha_2.$$

The Weyl group W has order 12 and is generated by the simple reflections s_1 and s_2, which act on R via $s_i(\alpha_i) = -\alpha_i$ and

$$s_1(\alpha_2) = 3\alpha_1 + \alpha_2$$

$$s_2(\alpha_1) = \alpha_1 + \alpha_2.$$

Letting $x_1 = x$ and $x_2 = y$, we have

$$D(\mathbf{x}, q; G_2) = \left(1 - \frac{x}{q}\right)\left(1 - \frac{y}{q}\right)\left(1 - \frac{xy}{q}\right)\left(1 - \frac{x^2y}{q}\right)\left(1 - \frac{x^3y}{q}\right)\left(1 - \frac{x^3y^2}{q}\right).$$

Let θ be a dominant weight. Then $\theta + \rho = \ell_1\varpi_1 + \ell_2\varpi_2$. Recall that the set $B(\theta + \rho)$ is the crystal graph whose vertex set is identified with the set of Littelmann patterns; we thus refer to $B(\theta + \rho)$ as the set of Littelmann patterns which are 6-tuples $\pi = (a, b, c, d, e, f)$ such that the entries a, b, c, d, e, f are non-negative integers that satisfy the following Littelmann inequalities.
Littelmann Inequalities:

1. $0 \leq f \leq \ell_2 + a - 2b + c - 2d + e$
2. $b \leq a \leq \ell_1 + 3b - 2c + 3d - 2e$
3. $\frac{c}{2} \leq b \leq \ell_2 + c - 2d + e$
4. $2d \leq c \leq \ell_1 + 3d - 2e$
5. $e \leq d \leq \ell_2 + e$
6. $0 \leq e \leq \ell_1$

These are the inequalities that come from choice of long word $s_2s_1s_2s_1s_2s_1$. It is straightforward to check that inequalities $i + 1$ through 6 imply that the lower bound of inequality i is less than or equal to the upper bound of inequality i. We refer to inequalities 1 through 6 as the inequalities for the entries f, a, b, c, d and e, in that order. As defined for the root system A_2 above, we say that an entry u of π is "circled", denoted by u°, if u attains the lower bound in its inequality; e.g., f is circled if $f = 0$, a is circled if $a = b$, b is circled if $b = \frac{c}{2}$, etc. Likewise, we say that u is "boxed", denoted by $\underline{u}$, if u attains its upper bound in its inequality; e.g., f is boxed if $f = \ell_2 + a - 2b + c - 2d + e$, a is boxed if $a = \ell_1 + 3b - 2c + 3d - 2e$, b is boxed if $b = \ell_2 + c - 2d + e$, etc. Then for any Littelmann pattern π, the authors of [5] define $\hat{H}(\pi)$ to be a certain polynomial in $\mathbb{Z}[q^{-1}]$. We express this $\hat{H}(\pi)$ as a sum of two terms

$$\hat{H}(\pi) = H_{\text{std}}(\pi) + H_{\text{adj}}(\pi)$$

where $H_{\text{std}}(\pi)$ encompasses the "standard" terms that come from Tokuyama's original definition; and $H_{\text{adj}}(\pi)$ encompasses the special cases, or "adjusted" terms. That is,

$$H_{\text{std}}(\pi) = h(a)h(b)h(c)h(d)h(e)h(f) \tag{7}$$

where $h(u)$ is defined by Definition 3. The $H_{\text{adj}}(\pi)$ is defined below via Eqs. (8) and (9).

Their conjecture is then

Theorem 2

$$\sum_{\pi \in B(\theta+\rho)} \hat{H}(\pi)\, x^{a+c+e} y^{b+d+f} = D(\boldsymbol{x}, q;\, R_+)\chi_\theta$$

We note that to define $\hat{H}(\pi)$, the authors of [5] use a definition depending on whether π is generic or one of twenty special cases. Considering $H_{\mathrm{adj}}(\pi)$ on its own allows us to consolidate the special cases, to simplify their characterization, and also to simplify the values of $H_{\mathrm{adj}}(\pi)$. We also see that the after expressing

$$\sum_{\pi \in B(\theta+\rho)} H_{\mathrm{adj}}(\pi)\, x^{a+c+e} y^{b+d+f}$$

as a polynomial, the coefficients have a factored form similar to that of $T(\mathbf{x}, q;\, R_+)$.

We now define $H_{\mathrm{adj}}(\pi)$ by consolidating below the twenty special cases used by [5]. The adjusted contribution $H_{\mathrm{adj}}(\pi)$ is defined in general to be 0 unless $\pi = (a, b, c, d, e, f)$ satisfies certain conditions. The first condition is that π has what in [5] is called a "bad middle", which means $b = d + 1$ and $c = 2d + 1$. Therefore the Littelmann inequalities for π with a bad middle become

"Bad Middle" Littelmann Inequalities:

1. $0 \leq f \leq \ell_2 + a - 2d + e - 1$
2. $d + 1 \leq a \leq \ell_1 + 2d - 2e + 1$
3. $d + .5 \leq d + 1 \leq \ell_2 + e + 1$
4. $2d \leq 2d + 1 \leq \ell_1 + 3d - 2e$
5. $e \leq d \leq \ell_2 + e$
6. $0 \leq e \leq \ell_1$

Thus such π are determined by the values of a, d, e and f. The definitions for circling and boxing the entries of π still hold. Let $\pi'' = (a, d, e)$, and we set

$$H_{\mathrm{adj}}(\pi) = H_{\mathrm{adj}}(\pi'')h(f). \tag{8}$$

By calculating $\hat{H}(\pi)$ and $H_{\mathrm{std}}(\pi)$ in each of the twenty cases of [5], we can determine $H_{\mathrm{adj}}(\pi'')$. We see that the values of $H_{\mathrm{adj}}(\pi'')$ become more concise than those given in [5] and that the twenty cases are consolidated to the following definition.

$$H_{\mathrm{adj}}(\pi'') = \begin{cases} \frac{(1-q^{-1})}{q}, \text{ if } \pi'' = (a^\circ, d^\circ, e^\circ,) \\ -\frac{(1-q^{-1})}{q^2}, \text{ if } \pi'' = (\underline{a}, d \text{ or } d^\circ, e^\circ) \\ \frac{(1-q^{-1})}{q^3}, \text{ if } \pi'' = (a = 2d+1-e, \underline{d}, e \text{ or } e^\circ) \\ \frac{(1-q^{-1})^2}{q} \text{ if } \pi'' = (a^\circ, d^\circ, e), (a, d^\circ, e^\circ), (a^\circ, d, e^\circ), (a, d^\circ, e), (a^\circ, d, e) \\ -\frac{(1-q^{-1})^2}{q^2}, \text{ if } \pi'' = (a, \underline{d}, e \text{ or } e^\circ,) \text{ such that } a \neq 2d+1-e \\ \frac{(1-q^{-1})^3}{q}, \text{ if } \pi'' = (a, d, e) \\ \frac{(1-q^{-1})^3}{q}, \text{ if } \pi'' = (a, d, e^\circ) \text{ such that } a \neq 2d+1-e \\ \frac{(1-q^{-1})((1-q^{-1})^2+q^{-1})}{q}, \text{ if } \pi'' = (a = 2d+1-e, d, e^\circ). \end{cases} \tag{9}$$

This means, for example, that if $\pi = (a, b, c, d, e, f) = (1, 1, 1, 0, 0, 0)$, then π has a bad middle with a, d and e circled (because we assume $\ell_1, \ell_2 > 0$), so

$$H_{\text{adj}}((1, 1, 1, 0, 0, 0)) = \frac{(1 - q^{-1})}{q}.$$

We see that the definition of $H_{\text{adj}}(\pi)$ depends only on the circling and boxing of a, d and e and whether $a = 2d + 1 - e$.

5 Proof of conjecture

Now we can prove Theorem 2.

The strategy of the proof is to interpret

$$\sum_{\pi \in B(\theta+\rho)} \hat{H}(\pi)\, x^{a+c+e} y^{b+d+f} \tag{10}$$

as a rational function in x, y and q^{-1}. This rational function depends on the numbers ℓ_1 and ℓ_2, which appear only as exponents of x and y in the numerator of the rational function. We therefore interpret this rational function as a polynomial, say P_H, in the four indeterminates

$$x^{\ell_1}, y^{\ell_1}, x^{\ell_2}, y^{\ell_2} \tag{11}$$

whose coefficients we prove will be of the form

$$\frac{p_1(x, y, q^{-1})}{p_2(x, y)} \tag{12}$$

where p_1 and p_2 are polynomials. Now the right side of Theorem 2 is also a polynomial, say P_W, in the four indeterminates (11) with coefficients of the form (12). Therefore Theorem 2 can be proved by equating the coefficients of the two polynomials P_H and P_W. Recall

$$P_W = D(\mathbf{x}, q; G_2)\chi_\theta$$

where χ_θ is the normalized character defined in Definition 1. The polynomial P_W has 12 terms, as there are 12 elements in the Weyl group W, and the coefficients are of the form

$$\operatorname{sgn}(w) T(\mathbf{x}, q; G_2).$$

We denote the multi-degree of the term

$$(x^{m_1} y^{n_1})^{\ell_1} (x^{m_2} y^{n_2})^{\ell_2}$$

Table 2 Terms for P_W

Multi-degree	Coefficient	Multi-degree	Coefficient
((0,0),(0,0))	$T(\mathbf{x}, q; G_2)$	((3,1),(6,3))	$T(\mathbf{x}, q; G_2)$
((1,0),(0,0))	$-T(\mathbf{x}, q; G_2)$	((4,2),(6,3))	$-T(\mathbf{x}, q; G_2)$
((1,1),(0,1))	$T(\mathbf{x}, q; G_2)$	((1,1),(3,3))	$-T(\mathbf{x}, q; G_2)$
((0,0),(0,1))	$-T(\mathbf{x}, q; G_2)$	((3,2),(3,3))	$T(\mathbf{x}, q; G_2)$
((1,0),(3,0))	$T(\mathbf{x}, q; G_2)$	((3,2),(6,4))	$-T(\mathbf{x}, q; G_2)$
((3,1),(3,1))	$-T(\mathbf{x}, q; G_2)$	((4,2),(6,4))	$T(\mathbf{x}, q; G_2)$

by

$$((m_1, n_1), (m_2, n_2)).$$

The twelve multi-degrees of P_W and the coefficients are given in Table 2.

We show how to express (10) as a polynomial in the indeterminates (11). As $\hat{H}(\pi) = H_{\mathrm{std}}(\pi) + H_{\mathrm{adj}}(\pi)$, we compute separately the two sums

$$\sum_{\pi \in B(\theta+\rho)} H_{\mathrm{std}}(\pi)\, x^{a+c+e} y^{b+d+f} \tag{13}$$

and

$$\sum_{\pi \in B(\theta+\rho)} H_{\mathrm{adj}}(\pi)\, x^{a+c+e} y^{b+d+f}. \tag{14}$$

We first compute (13). We sum over the six entries in the order f, a, b, c, d, e. We write (13) as

$$\sum_e h(e)x^e\Big(\sum_d h(d)y^d\Big(\sum_c h(c)x^c\Big(\sum_b h(b)y^b\Big(\sum_a h(a)x^a\Big(\sum_f h(f)y^f\Big)\Big)\Big)\Big)\Big) \tag{15}$$

where each entry ranges over integers satisfying its respective Littelmann inequality. Thus f ranges from 0 to $\ell_2+a-2b+c-2d+e$, a ranges from b to $\ell_1+3b-2c+3d-2e$, b ranges from $\frac{c}{2}$ to $\ell_2+c-2d+e$, etc.

Note that, depending on the values of a, b, c, d, and e, the bounds in the Littelmann inequalities may or may not be integers. In particular, the lower bound for the entry b is $\frac{c}{2}$. This means that if c is odd, then b, which we require to be an integer, cannot attain the lower bound in its Littelmann equality, and furthermore when $b = \lceil \frac{c}{2} \rceil$ it is not circled. Therefore, in the following lemmas we evaluate sums over Littelmann patterns considering whether or not the bounds may be attained.

Lemma 1 *Let u be an entry of π and let U and L be the upper and lower bounds, respectively, of the Littelmann inequality for u. Recall the expression $h(u)$ from*

Definition 3. If U and L are integers and $U \geq L$, then

$$\sum_{L \leq u \leq U} h(u) X^u = \frac{(1-q^{-1}X)}{1-X}(X^L - X^U). \tag{16}$$

Proof Since U and L are integers, u attains the values of U and L. First consider $U > L$. Directly applying the definition of $h(u)$ yields

$$\sum_{L \leq u \leq U} h(u) X^u = X^L + (1-q^{-1})\frac{X^{L+1} - X^U}{1-X} - q^{-1}X^U$$

which simplifies to

$$\frac{(1-q^{-1}X)}{1-X}(X^L - X^U). \tag{17}$$

If $U = L$, then in the left side of (16) u attains only one value which is both boxed and circled. Thus $h(u) = 0$ and the sum is zero. But if $U = L$, then (17) is also zero. Therefore the lemma is true in the case $U = L$ as well. □

Lemma 1 suffices to evaluate the sums for the entries f and a in the nested sum (15). Lemma 1 also shows that, in order to evaluate the sum for b, we must consider sums of the form

$$\sum_{c/2 \leq b \leq U} h(b) X^b$$

where $X = x^m y^n$ for some integers m and n. As discussed above, this sum will depend on the parity of c. We therefore make use of characteristic function $\boldsymbol{1}_{0\text{mod}2}(n)$, where

$$\boldsymbol{1}_{0\text{mod}2}(n) = \begin{cases} 1, & \text{if } n \equiv 0 \mod 2 \\ 0, & \text{if } n \equiv 1 \mod 2. \end{cases}$$

The next lemma evaluates the sum for b.

Lemma 2 *Let $U = \ell_2 + c - 2d + e$ be the upper bound in the Littelmann inequality for the entry b. If $U \geq \frac{c}{2}$, then*

$$\begin{aligned} \sum_{\frac{c}{2} \leq b \leq U} h(b) X^b = &\left(X^{\frac{c}{2}} + \frac{X^{\frac{c}{2}+1}(1-q^{-1})}{1-X} \right) \boldsymbol{1}_{0\text{mod}2}(c) \\ &+ \left(\frac{X^{\frac{c+1}{2}}(1-q^{-1})}{1-X} \right) \boldsymbol{1}_{0\text{mod}2}(c+1) \\ &- \frac{X^U(1-q^{-1})}{1-X} - q^{-1}X^U. \end{aligned}$$

Proof If c is even, then u attains the lower bound in its Littelmann inequality. If $U > \frac{c}{2}$, then directly applying the definition of $h(u)$ as in Lemma 1 yields

$$\sum_{\frac{c}{2} \leq b \leq U} h(b) X^b = X^{\frac{c}{2}} + \frac{X^{\frac{c}{2}+1}(1-q^{-1})}{1-X} - \frac{X^U(1-q^{-1})}{1-X} - q^{-1}X^U. \tag{18}$$

Equation (18) is also true if $U = \frac{c}{2}$, because both sides are 0 as in Lemma 1.

If c is odd, then the smallest value b attains is $\lceil \frac{c}{2} \rceil = \frac{c+1}{2}$; at this value b is not circled. Therefore in this case for $U > \frac{c+1}{2}$

$$\sum_{\frac{c}{2} \leq b \leq U} h(b) X^b = \sum_{b=\frac{c+1}{2}}^{U} h(b) X^b = \left(\frac{X^{\frac{c+1}{2}}(1-q^{-1})}{1-X} \right) - \frac{X^U(1-q^{-1})}{1-X} - q^{-1}X^U. \tag{19}$$

Equation (19) is also true when $U = \frac{c+1}{2}$, as the sum then consists of only one term for which the entry u is boxed.

Combining (18) and (19) using characteristic functions proves the lemma. □

From Lemma 2, we see that the sum over c in (15) is a sum of sums of the form

$$\frac{p_1(x, y, q^{-1})}{p_2(x, y)} \sum_{L \leq c \leq U} h(c) Y^c X^{\frac{C_1 + C_2 c}{2}} \mathbf{1}_{0 \bmod 2}(C_1 + C_2 c) \tag{20}$$

where X and Y each are monomials in x and y; p_1 and p_2 are polynomials with integer coefficients; C_1 and C_2 are integers; and $U = \ell_1 + 3d - 2e$ and $L = 2d$ are integers as well. If C_2 is even, this sum may be evaluated by Lemma 1. If C_2 is odd, we use Lemma 3.

Lemma 3 *Let U and L be the upper and lower bounds, respectively, of the Littelmann inequality for the entry u. Let C_1 and C_2 be integers with C_2 odd. If $U \geq L$, then*

$$\begin{aligned}
&\sum_{L \leq u \leq U} h(u) Y^u X^{(C_1 + C_2 u)/2} \mathbf{1}_{0 \bmod 2}(C_1 + C_2 u) \\
&= \left(Y^L X^{(C_1 + C_2 L)/2} + \frac{(1-q^{-1}) Y^{L+2} X^{(C_1 + C_2(L+2))/2}}{1 - Y^2 X^{C_2}} \right) \mathbf{1}_{0 \bmod 2}(C_1 + C_2 L) \\
&\quad + \frac{(1-q^{-1}) Y^{L+1} X^{(C_1 + C_2(L+1))/2}}{1 - Y^2 X^{C_2}} \mathbf{1}_{0 \bmod 2}(C_1 + C_2(L+1)) \\
&\quad - \left(\frac{(1-q^{-1}) Y^U X^{(C_1 + C_2 U)/2}}{1 - Y^2 X^{C_2}} + q^{-1} Y^U X^{(C_1 + C_2 U)/2} \right) \mathbf{1}_{0 \bmod 2}(C_1 + C_2 U) \\
&\quad - \frac{(1-q^{-1}) Y^{U+1} X^{(C_1 + C_2(U+1))/2}}{1 - Y^2 X^{C_2}} \mathbf{1}_{0 \bmod 2}(C_1 + C_2(U+1)).
\end{aligned}$$

Proof The proof is very similar to that of Lemma 2. Since C_2 is odd, the entries that are neither boxed nor circled contribute a geometric sum with common ratio $Y^2X^{C_2}$. Then the parities of U and L determine the highest and lowest values, respectively, of u that can contribute a nonzero term to the sum. The four terms of characteristic functions in the lemma thus come from these cases. □

Lemma 3 allows us to evaluate the sum for c in (15) and shows that the sum for d is a sum of sums of the form

$$\sum_{L\le d\le U} h(c)Y^d X^{\frac{C_1+C_2 d}{2}} \mathbf{1}_{0\bmod 2}(C_1+C_2 d)$$

where X and Y each are some monomials in x and y; where C_1 and C_2 are integers; and where $U = \ell_2 + e$ and $L = e$ are integers as well. But this sum for d can be evaluated using Lemmas 1 or 3, depending on whether C_2 is even or odd, respectively. Likewise these lemmas can also evaluate the sum for e.

Lemma 3 motivates the following definition.

Definition 4 A function $f : \mathbb{Z}^2 \to \mathbb{Q}(x, y)[q^{-1}]$ is called *exponential-congruent* if

$$f(\ell_1, \ell_2) = AX^{\ell_1}Y^{\ell_2}Z^{(C_1\ell_1+C_2\ell_2+C_3)/2}\mathbf{1}_{0\bmod 2}(C_1\ell_1 + C_2\ell_2 + C_3)$$

where $A \in \mathbb{Q}(x, y)[q^{-1}]$; X, Y and Z are monomials in x and y; and C_1, C_2 and C_3 are integers. If C_1 and C_2 are both even, then such a function f is called *exponential*.

Note that we abuse notation in the following because even though we interpret P_H and P_W as polynomials in the indeterminates (11), we write $P_H(\ell_1, \ell_2)$ and $P_W(\ell_1, \ell_2)$.

Theorem 3 *Let $B(\theta+\rho)$ be the set of Littelmann patterns π for $\theta+\rho = \ell_1\varpi_1+\ell_2\varpi_2$, where ℓ_1 and ℓ_2 are positive integers. Let $\hat{H}(\pi) = H_{\mathrm{std}}(\pi)+H_{\mathrm{adj}}(\pi)$ be the coefficient defined previously. Then*

$$\sum_{\pi\in B(\theta+\rho)} \hat{H}(\pi)\, x^{a+c+e} y^{b+d+f} = P_H(\ell_1, \ell_2)$$

where $P_H(\ell_1, \ell_2)$ is a finite sum of exponential-congruent functions.

Proof Recall from Sect. 4 that

$$\hat{H}(\pi) = H_{\mathrm{std}}(\pi) + H_{\mathrm{adj}}(\pi)$$

where $H_{\mathrm{std}}(\pi)$ is defined by Eq. (7) and $H_{\mathrm{adj}}(\pi)$ by (8) and (9). Let $f_{\mathrm{std}}(\ell_1, \ell_2)$ denote

$$f_{\mathrm{std}}(\ell_1, \ell_2) = \sum_{\pi\in B(\theta+\rho)} H_{\mathrm{std}}(\pi)\, x^{a+c+e} y^{b+d+f}.$$

The function $f_{\mathrm{std}}(\ell_1,\ell_2)$ is expressed as the nested sum (15), which Lemmas 1, 2, and 3 evaluate as a finite sum of exponential-congruent functions. Now let $f_{\mathrm{adj}}(\ell_1,\ell_2)$ denote

$$f_{\mathrm{adj}}(\ell_1,\ell_2) = \sum_{\pi\in B(\theta+\rho)} H_{\mathrm{adj}}(\pi)\, x^{a+c+e} y^{b+d+f}. \tag{21}$$

This is a sum over Littelmann patterns with bad middle which are described by the "bad middle" Littelmann inequalities in Sect. 4. This sum can also be expressed as a nested sum over the entries f, a, d and e whose upper and lower bounds are always integer. Therefore Lemma 1 suffices to evaluate the sum for the entry f. For each of the cases in (9), geometric sums can evaluate the sums for entries a, d and e. This shows that $f_{\mathrm{adj}}(\ell_1,\ell_2)$ is finite sum of exponential functions.

Thus

$$f_{\mathrm{std}}(\ell_1,\ell_2) + f_{\mathrm{adj}}(\ell_1,\ell_2) = P_H(\ell_1,\ell_2)$$

is a finite sum of exponential-congruent functions. □

Let $P_W(\ell_2,\ell_2)$ denote the right side of Theorem 2. Theorem 3 shows that Theorem 2 is equivalent to the statement

$$P_H(\ell_1,\ell_2) = P_W(\ell_1,\ell_2)$$

for all integer $\ell_1,\ell_2 > 0$. The function $P_W(\ell_1,\ell_2)$ is a sum of twelve exponential functions, while $P_H(\ell_1,\ell_2)$ is a finite sum of congruent-exponential functions. Now *a priori* an arbitrary finite sum of exponential-congruent functions may not reduce to a finite sum of exponential functions. However, in Theorem 4 we verify that this is the case for $P_H(\ell_1,\ell_2)$ and that it is equivalent to $P_W(\ell_1,\ell_2)$, proving the theorem.

Theorem 4 *Let $P_H(\ell_1,\ell_2)$ and $P_W(\ell_1,\ell_2)$ be as defined above. For any positive integers ℓ_1 and ℓ_2, then*

$$P_H(\ell_1,\ell_2) = P_W(\ell_1,\ell_2).$$

Proof The strategy of the proof is to calculate the coefficients of the exponential terms on either side of the theorem statement and verify that they are equal. Lemmas 1, 2, and 3 allow for automated calculation of $P_H(\ell_1,\ell_2)$ by a computer.

Recall that

$$P_H(\ell_1,\ell_2) = f_{\mathrm{std}}(\ell_1,\ell_2) + f_{\mathrm{adj}}(\ell_1,\ell_2)$$

in the notation of the proof of Theorem 3. We calculate that f_{std} is a sum of 544 terms that are exponential-congruent functions and that f_{adj} is a sum of 106 terms that are exponential functions. We now specify parities for ℓ_1 and ℓ_2 so that P_H, when restricted to such ℓ_1,ℓ_2, becomes a finite sum of *exponential* functions. That is, suppose

$$\frac{p_1(x,y,q^{-1})}{p_2(x,y)}(x^{m_1}y^{n_1})^{\ell_1}(x^{m_2}y^{n_2})^{\ell_2}(x^{m_3}y^{n_3})^{\frac{C_1\ell_1+C_2\ell_2+C_3}{2}}\mathbf{1}_{0\bmod 2}(C_1\ell_1+C_2\ell_2+C_3) \tag{22}$$

is a term in $P_H(\ell_1,\ell_2)$. For a choice of $\epsilon_1,\epsilon_2\in\{0,1\}$, we set

$$\ell_1=2L_1+\epsilon_1,\ \ell_2=2L_2+\epsilon_2$$

where L_1 and L_2 range over the set of positive integers. The term (22) then becomes

$$\begin{aligned}&\frac{p_1(x,y,q^{-1})}{p_2(x,y)}(x^{m_1}y^{n_1})^{\epsilon_1}(x^{m_2}y^{n_2})^{\epsilon_2}(x^{m_3}y^{n_3})^{\frac{C_1\epsilon_1+C_2\epsilon_2+C_3}{2}}\\&\times(x^{2m_1+C_1m_3}y^{2n_1+C_1n_3})^{L_1}(x^{2m_2+C_2m_3}y^{2n_2+C_2n_3})^{L_2}\end{aligned} \tag{23}$$

if

$$C_1\epsilon_1+C_2\epsilon_2+C_3\equiv 0 \mod 2$$

and 0 otherwise. If the term (23) is not 0, then we say it has multi-degree

$$((2m_1+C_1m_3,2n_2+C_1n_3),(2m_2+C_2m_3,2n_2+C_2n_3))$$

and coefficient

$$\frac{p_1(x,y,q^{-1})}{p_2(x,y)}(x^{m_1}y^{n_1})^{\epsilon_1}(x^{m_2}y^{n_2})^{\epsilon_2}(x^{m_3}y^{n_3})^{\frac{C_1\epsilon_1+C_2\epsilon_2+C_3}{2}}.$$

We then consider the set of multi-degrees that occur in these sums. We choose $\epsilon_1=\epsilon_2=0$. There are 33 distinct multi-degrees that occur from the standard terms, and 14 distinct multi-degrees that come from the adjusted terms. The union of these sets contains 35 distinct multi-degrees. When we combine like terms for the standard terms, there are 18 multi-degrees with nonzero coefficients, and when we combine like terms for the adjusted terms, there are 10 multi-degrees with nonzero coefficients. We present the multi-degrees and coefficients with $\epsilon_1=\epsilon_2=0$ for the standard terms in Table 3 and for the adjusted terms in Table 4. Note that because we are considering $\ell_i=2L_i+\epsilon_i$ for $i=1$ and 2, a multi-degree in Table 2 must be multiplied by 2 to obtain the corresponding multi-degree in Tables 3 and 4.

To express these coefficients, we define

$$T_1(\mathbf{x})=\frac{(1-q^{-1})(1-q^{-1}x)(1-q^{-1}y)(1-q^{-1}x^3y^2)}{(1-x)(1-y)(1-x^4y^2)(1-x^3y^2)}$$

$$T_2(\mathbf{x})=\frac{(1-q^{-1})(1-q^{-1}y)(1-q^{-1}xy)(1-q^{-1}x^3y)}{(1-y)(1-xy)(1-x^4y^2)(1-x^3y)}$$

Table 3 Standard terms for P_H with $\epsilon_1 = \epsilon_2 = 0$

Multi-degree	Coefficient	Multi-degree	Coefficient
((0,0),(0,0))	$T(\mathbf{x}) - q^{-1}x^2yT_1(\mathbf{x})$	((8,4),(8,6))	$-q^{-1}x^2yT_2(\mathbf{x})$
((2,0),(0,0))	$-T(\mathbf{x}) + q^{-1}x^2yT_1(\mathbf{x})$	((2,0),(6,0))	$T(\mathbf{x})$
((0,0),(0,2))	$-T(\mathbf{x}) + q^{-1}x^2yT_2(\mathbf{x})$	((6,2),(6,2))	$-T(\mathbf{x})$
((2,2),(0,2))	$T(\mathbf{x}) - q^{-1}x^2yT_2(\mathbf{x})$	((6,2),(12,6))	$T(\mathbf{x})$
((2,0),(8,4))	$-q^{-1}x^2yT_1(\mathbf{x})$	((8,4),(12,6))	$-T(\mathbf{x})$
((2,2),(8,6))	$q^{-1}x^2yT_2(\mathbf{x})$	((2,2),(6,6))	$-T(\mathbf{x})$
((0,0),(6,4))	$q^{-1}x^3yT_3(\mathbf{x})$	((6,4),(6,6))	$T(\mathbf{x})$
((8,4),(6,4))	$-q^{-1}x^3yT_3(\mathbf{x})$	((6,4),(12,8))	$-T(\mathbf{x})$
((8,4),(8,4))	$q^{-1}x^2yT_1(\mathbf{x})$	((8,4),(12,8))	$T(\mathbf{x})$

Table 4 Adjusted terms for P_H with $\epsilon_1 = \epsilon_2 = 0$

Multi-degree	Coefficient	Multi-degree	Coefficient
((0,0),(0,0))	$q^{-1}x^2yT_1(\mathbf{x})$	((2,2),(8,6))	$-q^{-1}x^2yT_2(\mathbf{x})$
((2,0),(0,0))	$-q^{-1}x^2yT_1(\mathbf{x})$	((0,0),(6,4))	$-q^{-1}x^3yT_3(\mathbf{x})$
((0,0),(0,2))	$-q^{-1}x^2yT_2(\mathbf{x})$	((8,4),(6,4)	$q^{-1}x^3yT_3(\mathbf{x})$
((2,2),(0,2))	$q^{-1}x^2yT_2(\mathbf{x})$	((8,4),(8,4))	$-q^{-1}x^2yT_1(\mathbf{x})$
((2,0),(8,4))	$q^{-1}x^2yT_1(\mathbf{x})$	((8,4),(8,6))	$q^{-1}x^2yT_2(\mathbf{x})$

$$T_3(\mathbf{x}) = \frac{(1-q^{-1})^2(1-q^{-1}y)(1-q^{-1}x^4y^2)}{(1-x)(1-xy)(1+x^2y)(1-x^3y)(1-x^3y^2)}$$

and abbreviate $T(\mathbf{x}) = T(\mathbf{x}, q; G_2)$ defined above.

Now we add the coefficients for the standard terms and adjusted terms for each multi-degree and see that they add up to

$$\operatorname{sgn}(w)T(\mathbf{x})$$

which gives us P_W. We do the same procedure the three other combinations of ϵ_1 and ϵ_2 and obtain similar results; the coefficient in Tables 3 and 4 for multi-degree $((m_1, n_1), (m_2, n_2))$ gets multiplied by $(x^{m_1}y^{n_1})^{\frac{\epsilon_1}{2}}(x^{m_2}y^{n_2})^{\frac{\epsilon_2}{2}}$. This proves the theorem. $\square$

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