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Interpretations of the Tutte and characteristic polynomials of matroids

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Abstract
We study interpretations of the Tutte and characteristic polynomials of matroids. If M is a matroid with rank function r whose ground set E is given with a linear ordering $<$, then $X \subseteq E$ is called $(M, <)$-compatible if $X \cap C \neq \{\min(C)\}$ for each circuit C of M. We show that the Tutte polynomial of M equals $\sum x^{r(M/X)} y^{r^*(M|X)}$ where X runs through the subsets of E such that X and $E \backslash X$ are $(M^*, <)$- and $(M, <)$-compatible, respectively. Similarly, the characteristic polynomial of M equals $\sum (-1)^{|X|}(k-1)^{r(M/X)}$ where X runs either through $(M^*, <)$-compatible subsets of E, or through the independent sets of M such that X and $E \backslash X$ are $(M^*, <)$- and $(M, <)$-compatible, respectively.

Keywords Tutte polynomial · Characteristic polynomial · Matroid · $(M, <)$-compatible set

Mathematics Subject Classification 05B35 · 05C31

1 Introduction

Let M be a matroid on a finite set E with rank function r. The *Tutte polynomial* of M is (see [4])

$$T(M; x, y) = \sum_{Z \subseteq E} (x-1)^{r(M)-r(Z)} (y-1)^{r^*(M)-r^*(E \backslash Z)}. \tag{1}$$

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This invariant was introduced in [22] and encodes many properties of graphs and matroids. Applications in combinatorics, knot theory, statistical physics, and coding theory are surveyed in [3–7,11,24,25]. Evaluating the Tutte polynomial at a point or finding its coefficients is in general a $\sharp P$-hard problem [2,14], even for planar graphs [23], and evaluations are in general difficult even to approximate [10]. $T(M; x, y)$ is fully characterized by the following rules

$$\begin{array}{ll} T(M; x, y) = 1 & \text{if } E = \emptyset, \\ T(M; x, y) = xT(M-e; x, y) & \text{if e is an isthmus of M}, \\ T(M; x, y) = yT(M-e; x, y) & \text{if e is a loop of M}, \\ T(M; x, y) = T(M/e; x, y) + T(M-e; x, y) & \text{otherwise.} \end{array} \tag{2}$$

The duality formula is

$$T(M; x, y) = T(M^*; y, x) \tag{3}$$

and the convolution formula (see [8,17,18]) is

$$T(M; x, y) = \sum_{Z \subseteq E} T(M/Z; x, 0)\, T(M|Z; 0, y), \tag{4}$$

where $M|Z = M - (E\backslash Z)$.

The *characteristic polynomial* of M is (see [4,26])

$$p(M; k) = \sum_{Z \subseteq E} (-1)^{|Z|} k^{r(M)-r(Z)} = (-1)^{r(M)} T(M; 1-k, 0). \tag{5}$$

This generalizes chromatic and flow polynomials of graphs (see [4,25,26]). Relations with some other combinatorial structures are studied in [1,26]. Polynomial $p(M; k)$ is fully characterized by the following rules

$$\begin{array}{ll} p(M; k) = 1 & \text{if } E = \emptyset, \\ p(M; k) = (k-1)p(M-e; k) & \text{if e is an isthmus of M}, \\ p(M; k) = p(M-e; k) - p(M/e; k) & \text{otherwise.} \end{array} \tag{6}$$

2 Interpretations

We recall some basic properties of matroids. If $X \subseteq E$, we shall usually write $r(M-X)$, $r(M/X)$, $r^*(M-X)$, $r^*(M/X)$ for $r_{M-X}(E\backslash X)$, $r_{M/X}(E\backslash X)$, $r_{(M-X)^*}(E\backslash X)$, $r_{(M/X)^*}(E\backslash X)$, respectively. For any $X, Y \subseteq E$ (see cf. [21]),

$$r_{M^*}(X) = |X| + r_M(E\backslash X) - r_M(X), \tag{7}$$

$$r_{M/Y}(X) = r_M(X \cup Y) - r_M(Y). \tag{8}$$

Denote by I_M the set of isthmuses of M. If $X \subseteq E$ and $e \in E$, $e \notin I_M, X$, then $r((M-e)/X) = r_{(M-e)}(E\backslash\{e\}) - r_{(M-e)}(X) = r_M(E\backslash\{e\}) - r_M(X) = r_M(E) - r_M(X) = r(M/X)$, i.e.,

$$r((M-e)/X) = r(M/X). \tag{9}$$

Let $\mathcal{C}(M)$ denote the family of circuits of M. By [21, Proposition 3.1.1],

$$\text{if } C \in \mathcal{C}(M/e), \text{ then either } C \in \mathcal{C}(M), \text{ or } C \cup \{e\} \in \mathcal{C}(M). \tag{10}$$

It is an easy exercise to prove that (see cf. Exercise 2 in [21, Section 3.1])

$$\begin{aligned} &\text{if } C \in \mathcal{C}(M), C \neq \{e\}, \text{ then either } e \in C \text{ and } C\backslash\{e\} \in \mathcal{C}(M/e), \\ &\quad \text{or } e \notin C \text{ and } C \text{ is a union of circuits from } \mathcal{C}(M/e). \end{aligned} \tag{11}$$

Let $<$ be a linear ordering of E. For each nonempty $X \subseteq E$, we denote by $\min(X)$ and $\max(X)$ the minimal and maximal element of X with respect to $<$, respectively.

We say that $X \subseteq E$ is $(M, <)$-*compatible* if for each $C \in \mathcal{C}(M)$, $C \cap X \neq \{\min(C)\}$. Clearly, no $(M, <)$-compatible set can contain a loop of M.

Denote by $\mathcal{E}(M, <)$ the family of all $(M^*, <)$-compatible subsets of E, by $\mathcal{D}(M, <) = \{X \in \mathcal{E}(M, <);\, E\backslash X \in \mathcal{E}(M^*, <)\}$, and by $\mathcal{P}(M, <)$ the set of couples (X, Y) such that $X, Y \subseteq E$, $X \cap Y = \emptyset$, X is $(M^*, <)$-compatible, Y is $(M, <)$-compatible, and X and Y are maximal with this property (i.e., for each $e' \in E\backslash(X \cup Y)$, $X \cup \{e'\}$ is not $(M^*, <)$-compatible and $Y \cup \{e'\}$ is not $(M, <)$-compatible). Clearly, $I_{M^*} \subseteq X$ and $I_M \subseteq Y$ for each $(X, Y) \in \mathcal{P}(M, <)$. Furthermore, $\{(X, E\backslash X);\, X \in \mathcal{D}(M, <)\} \subseteq \mathcal{P}(M, <)$, but the inclusion is an equality because in the following statement we prove that $\mathcal{D}(M, <)$ and $\mathcal{P}(M, <)$ have the same cardinality.

Theorem 1 *Let $<$ be a linear ordering of elements of a matroid M. Then $|\mathcal{D}(M, <)| = |\mathcal{P}(M, <)| = T(M; 1, 1)$ and*

$$T(M; x, y) = \sum_{X \in \mathcal{D}(M,<)} x^{r(M/X)} y^{r^*(M|X)}. \tag{12}$$

Proof In order to unify and simplify notation, denote by $\mathcal{P}_1(M, <) = \mathcal{P}(M, <)$, $\mathcal{P}_2(M, <) = \{(X, E\backslash X);\, X \in \mathcal{D}(M, <)\}$, and by $g(M, <; x, y)$ the right hand side of (12), i.e.,

$$g(M, <; x, y) = \sum_{(X,Y) \in \mathcal{P}_2(M,<)} x^{r(M/X)} y^{r^*(M-Y)}.$$

Clearly, $\mathcal{P}_2(M, <) \subseteq \mathcal{P}_1(M, <)$ and the equality occurs if and only if $|\mathcal{P}_1(M, <)| = |\mathcal{P}_2(M, <)|$. We use induction on $|E\backslash(I_{M^*} \cup I_M)|$ to prove that $T(M; x, y) = g(M, <; x, y)$ and $|\mathcal{P}_i(M, <)| = T(M; 1, 1)$ for $i = 1, 2$.

If $E = I_{M^*} \cup I_M$, then $\mathcal{P}_i(M, <) = \{(I_{M^*}, I_M)\}$, $|\mathcal{P}_i(M, <)| = 1 = T(M; 1, 1)$ $(i = 1, 2)$, and $T(M; x, y) = x^{|I_M|} y^{|I_{M^*}|} = g(M, <; x, y)$.

If $E \neq I_{M^*} \cup I_M$, choose $e = \max(E\backslash(I_{M^*} \cup I_M))$ and define

$$\mathcal{P}'_i(M, <) = \{(X, Y) \in \mathcal{P}_i(M, <);\, e \in Y\} \ \ (i = 1, 2),$$

$$g'(M, <; x, y) = \sum_{(X,Y)\in\mathcal{P}'_2(M,<)} x^{r(M/X)} y^{r^*(M-Y)}.$$

Suppose that $(X, Y) \in \mathcal{P}_1(M{-}e, <)$ and $e' \in E\backslash(X \cup Y \cup \{e\})$.

(i) Let $C \in \mathcal{C}(M^*)$. If $C\backslash\{e\} \in \mathcal{C}(M^*/e)$, then $e \notin X$, $\min(C) = \min(C\backslash\{e\})$, and $X \cap C = X \cap (C\backslash\{e\}) \neq \{\min(C\backslash\{e\})\}$. If $C = \bigcup_{i=1}^{n} C_i$, $C_i \in \mathcal{C}(M^*/e)$, then $X \cap C_i \neq \{\min(C_i)\}$, $i = 1, \ldots, n$, $\min(C) \in \{\min(C_i); i = 1, \ldots, n\}$, whence $X \cap C \neq \{\min(C)\}$. Thus by (11), $X \cap C \neq \{\min(C)\}$. Since this holds for each $C \in \mathcal{C}(M^*)$, X is $(M^*, <)$-compatible.

(ii) If $C \in \mathcal{C}(M)$ and $e \in C$ (resp. $e \notin C$), then $e \in (Y{\cup}\{e\}) \cap C \neq \{\min(C)\}$ because $e \neq \min(C)$ (resp. $(Y{\cup}\{e\}) \cap C = Y \cap C \neq \{\min(C)\}$), i.e., $Y \cup \{e\}$ is $(M, <)$-compatible.

(iii) If $X \cup \{e'\}$ is not $(M^*/e, <)$-compatible, there exists $C' \in \mathcal{C}(M^*/e)$ such that $C' \cap (X{\cup}\{e'\}) = \{\min(C')\}$, thus $e' = \min(C') \neq e$. If $C'{\cup}\{e\} \in \mathcal{C}(M^*)$ (resp. $C' \in \mathcal{C}(M^*)$) then $e \notin X$, $\min(C') = \min(C'{\cup}\{e\})$, and $(C'{\cup}\{e\}) \cap (X{\cup}\{e'\}) = C' \cap (X{\cup}\{e'\}) = \{\min(C'{\cup}\{e\})\}$ (resp. $C' \cap (X{\cup}\{e'\}) = \{\min(C')\}$), whence by (10), $X \cup \{e'\}$ is not $(M^*, <)$-compatible.

(iv) If $Y \cup \{e'\}$ is not $(M{-}e, <)$-compatible, there exists $C' \in \mathcal{C}(M{-}e)$ such that $C' \cap (Y{\cup}\{e, e'\}) = C' \cap (Y{\cup}\{e'\}) = \{\min(C')\}$, i.e., $Y \cup \{e, e'\}$ is not $(M, <)$-compatible.

By (i)–(iv), $(X, Y{\cup}\{e\})) \in \mathcal{P}'_1(M, <)$. From (i) and (ii), it also follows that if $(X, Y) \in \mathcal{P}_2(M{-}e, <)$ then $(X, Y{\cup}\{e\}) \in \mathcal{P}'_2(M, <)$.

Suppose that $(X, Y') \in \mathcal{P}'_1(M, <)$ and $e' \in E\backslash(X \cup Y')$.

(v) If $C \in \mathcal{C}(M^*/e)$ and $C \cup \{e\} \in \mathcal{C}(M^*)$ (resp. $C \in \mathcal{C}(M^*)$), then $\min(C) = \min(C{\cup}\{e\})$, and $X \cap (C{\cup}\{e\}) = X \cap C \neq \{\min(C{\cup}\{e\})\}$ (resp. $X \cap C \neq \{\min(C)\}$), whence by (10), X is $(M^*/e, <)$-compatible.

(vi) If $C \in \mathcal{C}(M{-}e)$, then $(Y'\backslash\{e\}) \cap C = Y' \cap C \neq \{\min(C)\}$, i.e., $Y'\backslash\{e\}$ is $(M{-}e, <)$-compatible.

(vii) If $X \cup \{e'\}$ is not $(M^*, <)$-compatible, there exists $C' \in \mathcal{C}(M^*)$ such that $C' \cap (X{\cup}\{e'\}) = \{\min(C')\}$, whence $e' = \min(C') \neq e$. If $C'\backslash\{e\} \in \mathcal{C}(M^*/e)$, then $(C'\backslash\{e\}) \cap (X{\cup}\{e'\}) = \{\min(C')\}$. If $C' = \bigcup_{i=1}^{n} C'_i$, $C'_i \in \mathcal{C}(M^*/e)$, there exists j such that $\min(C') \in C'_j$, whence $\min(C'_j) = \min(C')$ and $C'_j \cap (X{\cup}\{e'\}) = \{\min(C'_j)\}$. Thus by (11), $X \cup \{e'\}$ is not $(M^*/e, <)$-compatible.

(viii) If $Y' \cup \{e'\}$ is not $(M, <)$-compatible, there exists $C' \in \mathcal{C}(M)$ such that $C' \cap (Y'{\cup}\{e'\}) = \{\min(C')\}$, thus $e' = \min(C') \neq e$, $e \notin C'$ (because $e \in Y'$), and $C' \cap ((Y'\backslash\{e\}) \cup \{e'\}) = \{\min(C')\}$, i.e., $(Y'\backslash\{e\}) \cup \{e'\}$ is not $(M{-}e, <)$-compatible.

By (v)–(viii), $(X, Y'\backslash\{e\})) \in \mathcal{P}_1(M{-}e, <)$. From (v) and (vi), it also follows that if $(X, Y') \in \mathcal{P}'_2(M, <)$ then $(X, Y'\backslash\{e\}) \in \mathcal{P}_2(M{-}e, <)$.

Therefore $\mathcal{P}'_i(M, <) = \{(X, Y \cup \{e\}); (X, Y) \in \mathcal{P}_i(M{-}e, <)\}$ $(i = 1, 2)$. For any $(X, Y') \in \mathcal{P}'_2(M, <)$, we have $e \notin I_{M,X}$ whence by (9), $r((M{-}e)/X) = r(M/X)$ and clearly, $M - Y' = (M{-}e) - (Y'\backslash\{e\})$. Thus by induction hypothesis, $|\mathcal{P}'_i(M, <)| = |\mathcal{P}_i(M{-}e, <)| = T(M{-}e; 1, 1)$ $(i = 1, 2)$ and

$$T(M-e;x,y) = g(M-e,<;x,y) = \sum_{(X,Y)\in\mathcal{P}_2(M-e,<)} x^{r((M-e)/X)} y^{r^*((M-e)-Y)}$$
$$= \sum_{(X,Y')\in\mathcal{P}'_2(M,<)} x^{r(M/X)} y^{r^*(M-Y')} = g'(M,<;x,y).$$

The latter equation holds true also for M^*, whence by (3),

$$|\mathcal{P}'_i(M^*,<)| = |\mathcal{P}_i(M^*-e,<)| = T(M^*-e;1,1) = T(M/e;1,1) \;\; (i=1,2),$$
$$T(M/e;x,y) = T(M^*-e;y,x) = g'(M^*,<;y,x).$$

Since $\mathcal{P}_i(M^*,<) = \{(Y,X);\,(X,Y)\in\mathcal{P}_i(M,<)\}$ and for each $(X,Y)\in\mathcal{P}_i(M,<)$, $r(M/X) = r^*(M^*-X)$, $r^*(M-Y) = r(M^*/Y)$, and $e\in X\cup Y$, we have

$$|\mathcal{P}_i(M,<)| = |\mathcal{P}'_i(M,<)| + |\mathcal{P}'_i(M^*,<)| \;\; (i=1,2),$$
$$g(M,<;x,y) = g'(M,<;x,y) + g'(M^*,<;y,x).$$

Thus by (2), $|\mathcal{P}_i(M,<)| = T(M;1,1)$ $(i=1,2)$ and $T(M;x,y) = g(M,<;x,y)$. □

Denote by $\mathcal{S}(M,<) = \{X\in\mathcal{D}(M,<);\, r(X) = |X|\}$.

Theorem 2 *Let* $<$ *be a linear ordering of elements of a matroid* M*. Then* $|\mathcal{E}(M,<)| = T(M;1,2)$*,* $|\mathcal{S}(M,<)| = T(M;1,0)$*, and*

$$p(M;k) = \sum_{X\in\mathcal{E}(M,<)} (-1)^{|X|}(k-1)^{r(M/X)}$$
$$= \sum_{X\in\mathcal{S}(M,<)} (-1)^{|X|}(k-1)^{r(M)-|X|}. \tag{13}$$

Proof We use induction on $|E\backslash I_M|$ to prove the first equation from (13) and that $|\mathcal{E}(M,<)| = T(M;1,2)$. If $E = I_M$, then $\mathcal{E}(M,<) = \{\emptyset\}$, $|\mathcal{E}(M,<)| = 1 = T(M;1,2)$, $r(M) = |I_M|$, and $p(M;k) = (k-1)^{r(M)}$ as claimed.

If $E\neq I_M$, choose $e = \max(E\backslash I_M)$ and denote by $\mathcal{E}^+ = \{X\in\mathcal{E}(M,<);\; e\notin X\}$, $\mathcal{E}^- = \{X\in\mathcal{E}(M,<);\; e\in X\}$.

If $X\in\mathcal{E}(M-e,<)$, then by item (i) from the proof of Theorem 1, X is $(M^*,<)$-compatible, i.e., $\mathcal{E}(M-e,<)\subseteq\mathcal{E}^+$. Similarly, the reverse implication follows from item (v). Thus $\mathcal{E}(M-e,<) = \mathcal{E}^+$.

If $X\in\mathcal{E}(M/e,<)$, then for each $C\in\mathcal{C}(M^*)$ satisfying $e\in C$ (resp. $e\notin C$), we have $e\in(X\cup\{e\})\cap C\neq\{\min(C)\}$, because $e\neq\min(C)$ (resp. $(X\cup\{e\})\cap C = X\cap C\neq\{\min(C)\}$), whence $X\cup\{e\}$ is $(M^*,<)$-compatible. If $X'\in\mathcal{E}^-$, then for each $C\in\mathcal{C}(M^*-e)$, $(X'\backslash\{e\})\cap C = X'\cap C\neq\{\min(C)\}$, whence $X'\backslash\{e\}$ is $(M^*-e,<)$-compatible. Thus $\mathcal{E}(M/e,<) = \{X'\backslash\{e\};\, X'\in\mathcal{E}^-\}$.

By (6) and induction hypothesis,

$$p(M;k) = p(M-e;k) - p(M/e;k)$$

$$
\begin{aligned}
&= \sum_{X \in \mathcal{E}(M-e,<)} (-1)^{|X|}(k-1)^{r((M-e)/X)} \\
&\quad - \sum_{X \in \mathcal{E}(M/e,<)} (-1)^{|X|}(k-1)^{r((M/e)/X)} \\
&= \sum_{X \in \mathcal{E}^+} (-1)^{|X|}(k-1)^{r(M/X)} \\
&\quad - \sum_{X' \in \mathcal{E}^-} (-1)^{|X' \setminus \{e\}|}(k-1)^{r(M/X')} \\
&= \sum_{X \in \mathcal{E}(M,<)} (-1)^{|X|}(k-1)^{r(M/X)}
\end{aligned}
$$

because by (9), $r((M-e)/X) = r(M/X)$ for each $X \in \mathcal{E}(M-e,<) = \mathcal{E}^+$. Furthermore, $|\mathcal{E}(M,<)| = |\mathcal{E}^+| + |\mathcal{E}^-| = |\mathcal{E}(M-e,<)| + |\mathcal{E}(M/e,<)|$ and $M-e = M/e$ if e is a loop of M, thus by the last two rows of (2), $|\mathcal{E}(M,<)| = T(M;1,2)$.

We prove the second part of (13). By (7), $r^*(M|X) = |X| - r(X)$ whence $\mathcal{S}(M,<) = \{X \in \mathcal{D}(M,<);\ r^*(M|X) = 0\}$. Moreover by (8), $r(M/X) = r(M) - r(X) = r(M) - |X|$ for each $X \in \mathcal{S}(M,<)$. From (5) and (12), we have

$$
\begin{aligned}
p(M;k) &= (-1)^{r(M)}T(M;1-k,0) = (-1)^{r(M)} \sum_{X \in \mathcal{S}(M,<)} (1-k)^{r(M/X)} 0^{r^*(M|X)} \\
&= \sum_{X \in \mathcal{S}(M,<)} (-1)^{|X|}(k-1)^{r(M)-|X|}.
\end{aligned}
$$

Hence $p(M;0) = |\mathcal{S}(M,<)|(-1)^{r(M)}$ and by (5), $|\mathcal{S}(M,<)| = T(M;1,0)$. □

If $I_{M^*} \neq \emptyset$ then $\mathcal{S}(M,<) = \emptyset$ (because $I_{M^*} \subseteq X$ for each $X \in \mathcal{S}(M,<)$ and $r(I_{M^*}) = 0$) and by (13), $p(M;k) = 0$.

Clearly, $\mathcal{S}(M,<) \subseteq \mathcal{E}(M,<)$, whence by (13),

$$
0 = \sum_{X \in \mathcal{E}(M,<) \setminus \mathcal{S}(M,<)} (-1)^{|X|}(k-1)^{r(M/X)}. \tag{14}
$$

Corollary 1 *Let $<$ be a linear ordering of elements of a matroid M. Then*

$$
\begin{aligned}
T(M;x,y) &= \sum_{Z \subseteq E} (-1)^{r(M)+|Z|} \left(\sum_{X \in \mathcal{E}(M/Z,<)} (-1)^{|X|}(-x)^{r((M/Z)/X)} \right) \\
&\quad \left(\sum_{Y \in \mathcal{E}((M|Z)^*,<)} (-1)^{|Y|}(-y)^{r^*((M|Z)-Y)} \right) \\
&= \sum_{Z \subseteq E} \left(\sum_{X \in \mathcal{S}(M/Z,<)} x^{r(M/Z)-|X|} \right) \left(\sum_{Y \in \mathcal{S}((M|Z)^*,<)} y^{r^*(M|Z)-|Y|} \right).
\end{aligned} \tag{15}
$$

Proof By (3), (4), and (5),

$$\begin{aligned}T(M;x,y) &= \sum_{Z\subseteq E} T(M/Z;x,0)\,T((M|Z)^*;y,0)\\ &= \sum_{Z\subseteq E}(-1)^{r(M/Z)}p(M/Z;1-x)\,(-1)^{r^*(M|Z)}p((M|Z)^*;1-y).\end{aligned}$$

By (7) and (8), $r(M/Z)+r^*(M|Z) = r_{M/Z}(E\backslash Z)+r_{(M|Z)^*}(Z) = r_M(E)-r_M(Z)+|Z|+r_{M|Z}(\emptyset)-r_{M|Z}(Z) = r(M)+|Z|-2r(Z)+0$, whence

$$T(M;x,y) = \sum_{Z\subseteq E}(-1)^{r(M)+|Z|}p(M/Z;1-x)\,p((M|Z)^*;1-y)$$

and applying the first equation from (13) for $p(M/Z;1-x)$ and $p((M|Z)^*;1-y)$ we get the first part of (15). By the second equation from (13),

$$\begin{aligned}T(M;x,y) = {} & \sum_{Z\subseteq E}(-1)^{r(M)+|Z|}\left(\sum_{X\in\mathcal{S}(M/Z,<)}(-1)^{|X|}(-x)^{r(M/Z)-|X|}\right)\\ & \left(\sum_{Y\in\mathcal{S}((M|Z)^*,<)}(-1)^{|Y|}(-y)^{r^*(M|Z)-|Y|}\right)\\ = {} & \sum_{Z\subseteq E}(-1)^{r(M)+|Z|}\left(\sum_{X\in\mathcal{S}(M/Z,<)}(-1)^{r(M/Z)}x^{r(M/Z)-|X|}\right)\\ & \left(\sum_{Y\in\mathcal{S}((M|Z)^*,<)}(-1)^{r^*(M|Z)}y^{r^*(M|Z)-|Y|}\right)\end{aligned}$$

and the second part of (15) follows from the fact that $r(M)+|Z|+r(M/Z)+r^*(M|Z) = r(M)+|Z|+r(M)-r(Z)+|Z|+r(\emptyset)-r(Z)$ is even. □

For example, suppose that $M = U_{1,n}$, $M^* = U_{n-1,n}$, and $E = \{e_1,\ldots,e_n\}$ has a linear ordering $<$ such that $e_1 < \cdots < e_n$. Denote by 2^E the powerset of E, $S_i = \{e_1,\ldots,e_i\}$, and $S_i^* = E\backslash S_i$ ($i = 0,\ldots,n$, $S_n^* = S_0 = \emptyset$). We can check that

$$\begin{aligned}&\mathcal{E}(M,<) = 2^E\backslash\{S_1\}, && \mathcal{E}(M^*,<) = \{S_i^*;\, i=0,\ldots,n\},\\ &\mathcal{D}(M,<) = \{S_i;\, i=0,2,\ldots,n\}, && \mathcal{D}(M^*,<) = \{S_i^*;\, i=0,2,\ldots,n\},\\ &\mathcal{S}(M,<) = \{S_0\} = \{\emptyset\}, && \mathcal{S}(M^*,<) = \{S_i^*;\, i=2,\ldots,n\},\end{aligned}$$
$$\mathcal{P}(M,<) = \{(S_i,S_i^*);\, i=0,2,\ldots,n\},$$

and by (12) and (13),

$$\begin{aligned} P(M;k) &= k-1,\\ P(M^*;k) &= \textstyle\sum_{i=1}^{n-1}(-1)^{n-1-i}(k-1)^i,\\ T(M;x,y) &= x+\textstyle\sum_{i=1}^{n-1} y^i. \end{aligned}$$

In accordance with Theorems 1, 2, and (3),

$$\begin{array}{ll} |\mathcal{E}(M,<)| = 2^n-1 = T(M;1,2), & |\mathcal{E}(M^*,<)| = n+1 = T(M;2,1),\\ |\mathcal{S}(M,<)| = 1 = T(M;1,0), & |\mathcal{S}(M^*,<)| = n-1 = T(M;0,1),\\ |\mathcal{E}(M,<)\backslash\mathcal{S}(M,<)| = 2^n-2, & |\mathcal{E}(M^*,<)\backslash\mathcal{S}(M^*,<)| = 2,\\ \multicolumn{2}{l}{|\mathcal{D}(M,<)| = |\mathcal{D}(M^*,<)| = |\mathcal{P}(M,<)| = |\mathcal{P}(M^*,<)| = n = T(M;1,1),} \end{array}$$

and in accordance with (14),

$$\begin{aligned} 0 &= \sum_{X\in\mathcal{E}(M,<)\backslash\mathcal{S}(M,<)} (-1)^{|X|}(k-1)^{r(M/X)}\\ &= \sum_{Y\in\mathcal{E}(M^*,<)\backslash\mathcal{S}(M^*,<)} (-1)^{|Y|}(k-1)^{r^*(M-Y)}. \end{aligned}$$

It would be interesting to study relations between $\mathcal{E}(M,<)$, $\mathcal{D}(M,<)$, $\mathcal{P}(M,<)$, and $\mathcal{S}(M,<)$ and other structures, that have the same cardinalities or are known to give expressions to compute the Tutte polynomial or the characteristic polynomial. Obvious examples are the bases, independent sets, internal and external activities (see [5,22]), broken circuits (sets of the form $C\backslash\{\min(C)\}$, $C \in \mathcal{C}(M)$), and the partitioning of 2^E into $T(M;1,1)$ intervals of type $[B\backslash\mathrm{Int(B)}, \mathrm{B}\cup\mathrm{Ext(B)}]$ (where B ranges over all bases of M and $\mathrm{Int}(B)$ and $\mathrm{Ext}(B)$ denote the usual sets of active elements with respect to B, see [9,12,19,20]). Finally, notice a classical interpretation of $p(M;0) = T(M;1,0)$ introduced in [13] in terms of totally cyclic and acyclic orientations of graphs satisfying additional conditions. These orientations represent certain equivalence classes of totally cyclic and acyclic orientations of graphs (see [15,16]).

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