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Quasi-quantum groups obtained from tensor braided Hopf algebras

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Abstract
Let H be a quasi-Hopf algebra, ${}_H^H\mathcal{M}_H^H$ the category of two-sided two-cosided Hopf modules over H and ${}_H^H\mathcal{YD}$ the category of left Yetter–Drinfeld modules over H. We show that ${}_H^H\mathcal{M}_H^H$ admits a braided monoidal structure for which the strong monoidal equivalence ${}_H^H\mathcal{M}_H^H \cong {}_H^H\mathcal{YD}$ established by the structure theorem for quasi-Hopf bimodules becomes braided monoidal. Using this braided monoidal equivalence, we prove that Hopf algebras within ${}_H^H\mathcal{M}_H^H$ can be characterized as quasi-Hopf algebras with a projection or as biproduct quasi-Hopf algebras in the sense of Bulacu and Nauwelaerts (J Pure Appl Algebra 174:1–42, 2002) . A particular class of such (braided, quasi-) Hopf algebras is obtained from a tensor product Hopf algebra type construction. Our arguments rely on general categorical facts.

Keywords Braided category · Biproduct · Projection · Braided tensor Hopf algebra · Quantum shuffle quasi-Hopf algebra

Mathematics Subject Classification 16T05 · 18D10

1 Introduction

The so-called quantum shuffle Hopf algebras are cotensor Hopf algebras of a Hopf bimodule M over a Hopf algebra H. Their importance resides on the fact that all quantized enveloping algebras associated with finite-dimensional simple Lie algebras or with affine Kac–Moody Lie algebras are of this type; see [25]. As the cotensor defines a monoidal structure on the category of Hopf H-bimodules isomorphic to the one determined by the tensor product over H, it follows that quantum shuffle algebras

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can be as well introduced as tensor Hopf algebras within the braided category of Hopf H-bimodules (also known under the name of two-sided two-cosided Hopf modules).

The structure of a Hopf algebra H with a projection $\pi : B \to H$ is due to Radford [24]. Up to an isomorphism, B is a biproduct Hopf algebra $A \times H$ between a left H-module algebra and left H-comodule coalgebra A and H, satisfying appropriate compatibility relations. Majid [19] observed that all these conditions are equivalent to the fact that A is a Hopf algebra within ${}_H^H\mathcal{Y}D$, the braided monoidal category of left Yetter–Drinfeld modules over H. A second characterization of Hopf algebras with a projection is due to Bespalov and Drabant [1], where Hopf algebras with a projection are identified with Hopf algebras within ${}_H^H\mathcal{M}_H^H$, the braided monoidal category of two-sided two-cosided Hopf modules over H introduced by Woronowicz in [28]. The connection with the Hopf algebras in ${}_H^H\mathcal{Y}D$ becomes clear now, since ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{Y}D$ are braided monoidally equivalent. The latest result was proved by Schauenburg in [26]; see also [25]. We should mention that in all this theory a key role is played by the structure theorem for two-sided two-cosided Hopf modules. Furthermore, by moving backwards, these equivalences associate to any vector space (viewed in a canonical way as Yetter–Drinfeld module) a two-sided two-cosided Hopf module, and then a quantum shuffle Hopf algebra.

The purpose of this note is to construct quasi-quantum shuffle groups, i.e., tensor Hopf algebras within categories of quasi-Hopf bimodules. This is possible because many of the above-mentioned results have already been generalized to the quasi-Hopf case. For instance, a structure theorem for quasi-Hopf (bi)comodule algebras was given in [9,23]. It is not possible to prove a similar structure theorem for quasi-Hopf module coalgebras, since H is not, in general, a module coalgebra over itself. Instead, it is more natural to try describing the bimodule coalgebras C over a quasi-Hopf algebra H, as H is a bimodule coalgebra over itself in a canonical way. We did this in [2, Theorem 5.6] where we proved that, up to an isomorphism, C is a smash product coalgebra between a coalgebra in ${}_H^H\mathcal{Y}D$ and H. Note that all the mentioned structure theorems actually characterize the (co)algebras within some monoidal categories of quasi-Hopf (bi)modules. Furthermore, the involved structures are a smash product algebra and a smash product coalgebra, as they were defined in [2,7]; they are required to define a Hopf like object, and this leads naturally to the biproduct quasi-Hopf algebra construction from [5], as well to the structure of a quasi-Hopf algebra with a projection and its relation to the Hopf algebras in ${}_H^H\mathcal{Y}D$. Although a quasi-Hopf algebra cannot be regarded as a braided Hopf algebra, we were able to adapt the categorical techniques used in [1] to the setting provided by quasi-Hopf algebras. Otherwise stated, we could produce structure theorems for the bialgebras and Hopf algebras in ${}_H^H\mathcal{M}_H^H$, similar to the ones in [1]. The choice of the category ${}_H^H\mathcal{M}_H^H$ rather than ${}_H\mathcal{M}_H^H$ is imposed by the fact that the former is braided, while the latter is not, and so we can consider bialgebras and Hopf algebras only within ${}_H^H\mathcal{M}_H^H$. Finally, ${}_H^H\mathcal{M}_H^H$ with $\otimes_H$ is strict monoidal and although is isomorphic to the monoidal structure given by the cotensor product, the latter is not strict; this led us to work with tensor braided Hopf algebras instead of cotensor ones.

The paper is organized as follows. In Sect. 2, we briefly recall the definition of a quasi-Hopf algebra, the language of braided monoidal categories and braided monoidally equivalences, and the monoidally equivalence between ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{YD}$. Using a general categorical result, in Sect. 3 we uncover in a canonical way a braiding on ${}_H^H\mathcal{M}_H^H$ for which the strong monoidally equivalence ${}_H^H\mathcal{M}_H^H \cong {}_H^H\mathcal{YD}$ from [2,27] becomes a braided monoidal equivalence. In Sect. 4, we characterize the Hopf algebras B in ${}_H^H\mathcal{M}_H^H$ as quasi-Hopf algebras with a projection and show that, up to an isomorphism, such a B is nothing but a biproduct quasi-Hopf algebra in the sense of [5]. We should stress that our techniques allow us to show in a more elegant and less computational way that the biproduct is indeed a quasi-Hopf algebra. In addition, we get almost for free the converse of the construction in [5]: if the smash product algebra $A\#H$ of an algebra A in ${}_H^H\mathcal{YD}$ and H, and the smash product coalgebra $A \rtimes H$ between the coalgebra A in ${}_H^H\mathcal{YD}$ and H afford a quasi-Hopf algebra structure on $A \otimes H$, then A is a Hopf algebra in ${}_H^H\mathcal{YD}$. Inspired by the work of Nichols [22], in Sect. 5 we associate to any object $M \in {}_H^H\mathcal{M}_H^H$ a braided Hopf algebra $T_H(M)$ within ${}_H^H\mathcal{M}_H^H$, the so-called tensor Hopf algebra of M over H. Furthermore, we describe the quasi-Hopf algebra structure of $T_H(M)$ and show that it is isomorphic to the biproduct quasi-Hopf algebra of $T(V)$ and H, where V is a certain set of coinvariants of M and $T(V)$ is the tensor Hopf algebra of V built within the braided monoidal category of left H-Yetter–Drinfeld modules. Actually, the construction of $T(V)$ within ${}_H^H\mathcal{YD}$ makes sense for any $V \in {}_H^H\mathcal{YD}$. This fact is fully exploited in Sect. 6 where a concrete class of quasi-Hopf algebras with a projection is constructed out of a vector space, a cyclic group of order n and a primitive root of unity of degree n^2 in k, $n \geq 2$.

2 Preliminaries

2.1 Quasi-bialgebras and quasi-Hopf algebras

We work over a field k. All algebras, linear spaces, etc., will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [10], a quasi-bialgebra is a quadruple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are algebra homomorphisms satisfying the identities

$$(\mathrm{Id}_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \mathrm{Id}_H)(\Delta(h))\Phi^{-1}, \tag{2.1}$$

$$(\mathrm{Id}_H \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes \mathrm{Id}_H)(\Delta(h)) = h, \tag{2.2}$$

for all $h \in H$, where Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\mathrm{Id}_H \otimes \Delta \otimes \mathrm{Id}_H)(\Phi)(\Phi \otimes 1)$$
$$= (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi), \tag{2.3}$$

$$(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id}_H)(\Phi) = 1 \otimes 1. \tag{2.4}$$

The map Δ is called the coproduct or the comultiplication, ε is the counit, and Φ is the reassociator. As for Hopf algebras, we denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes \mathrm{Id}_H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2,$$
$$(\mathrm{Id}_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by lower case letters, namely

$$\Phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3 = \cdots$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = y^1 \otimes y^2 \otimes y^3 = z^1 \otimes z^2 \otimes z^3 = \cdots$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{2.5}$$
$$X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2\beta S(x^3) = 1. \tag{2.6}$$

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld [10] in the sense that we do not require the antipode to be bijective. In the case where H is finite-dimensional or quasi-triangular, bijectivity of the antipode follows from the other axioms, see [3,6], so the two definitions are equivalent. Anyway, the bijectivity of the antipode S will be implicitly understood in the case when S^{-1}, the inverse of S, appears is formulas or computations.

It is well known that the antipode of a Hopf algebra is an anti-morphism of coalgebras. For a quasi-Hopf algebra H, there exists an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, called the Drinfeld twist or the gauge transformation, such that $\varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1$ and

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\mathrm{cop}}(h)), \tag{2.7}$$

for all $h \in H$, where $\Delta^{\mathrm{cop}}(h) = h_2 \otimes h_1$. f can be described explicitly: first we define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(x^1X^2)\alpha x^2X^3_1 \otimes S(X^1)\alpha x^3X^3_2 \overset{(2.3,2.5)}{=} S(X^2x^1_2)\alpha X^3x^2 \otimes S(X^1x^1_1)\alpha x^3, \tag{2.8}$$
$$\delta = X^1_1x^1\beta S(X^3) \otimes X^1_2x^2\beta S(X^2x^3) \overset{(2.3,2.5)}{=} x^1\beta S(x^3_2X^3) \otimes x^2X^1\beta S(x^3_1X^2). \tag{2.9}$$

With this notation f and f^{-1} are given by the formulas

$$f = (S \otimes S)(\Delta^{\mathrm{op}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \tag{2.10}$$
$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\mathrm{cop}}(x^3)). \tag{2.11}$$

Moreover, f satisfies the following relations:

$$f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta. \tag{2.12}$$

We will need the appropriate generalization of the formula $h_1 \otimes h_2 S(h_3) = h \otimes 1$ in classical Hopf algebra theory. Following [13,14], we define

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \tag{2.13}$$
$$q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2. \tag{2.14}$$

For all $h \in H$, we then have

$$\Delta(h_1) p_R (1 \otimes S(h_2)) = p_R (h \otimes 1), \tag{2.15}$$
$$(1 \otimes S^{-1}(h_2)) q_R \Delta(h_1) = (h \otimes 1) q_R, \tag{2.16}$$

and the following relations hold:

$$\Delta(q^1) p_R (1 \otimes S(q^2)) = 1 \otimes 1, \tag{2.17}$$
$$q_1^1 x^1 \otimes q_2^1 x^2 \otimes q^2 x^3 = X^1 \otimes q^1 X_1^2 \otimes S^{-1}(X^3) q^2 X_2^2, \tag{2.18}$$
$$X^1 p_1^1 \otimes X^2 p_2^1 \otimes X^3 p^2 = x^1 \otimes x_1^2 p^1 \otimes x_2^2 p^2 S(x^3). \tag{2.19}$$

2.2 Braided monoidal equivalences

For the definition of a (co)algebra (resp. bialgebra, Hopf algebra) in a monoidal (resp. braided monoidal) category $\mathcal{C}$ and related topics, we refer to [11,16,21]. Usually, for a monoidal category $\mathcal{C}$, we denote by $\otimes$ the tensor product, by $\underline{1}$ the unit object, and by a, l, r the associativity constraint and the left and right unit constraints, respectively.

A strong monoidal functor between two monoidal categories $\mathcal{C}$, $\mathcal{C}'$ is a triple $(F, \varphi_2, \varphi_0)$, where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $\varphi_0 : \underline{1} \to F(\underline{1}')$ is an isomorphism, and $\varphi_{2,U,V} : F(U) \otimes' F(V) \to F(U \otimes V)$ is a family of natural isomorphisms in $\mathcal{C}'$. φ_0 and φ_2 have to satisfy certain properties, see for example [16, XI.4].

When $(\mathcal{C}, c)$ and $(\mathcal{C}', c')$ are (pre)braided monoidal categories, a (pre)braided functor $F : (\mathcal{C}, c) \to (\mathcal{C}', c')$ is a strong monoidal functor $(F, \varphi_2, \varphi_0) : \mathcal{C} \to \mathcal{C}'$ compatible with the (pre)braidings c and c', in the sense that, for any objects $X, Y \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{\varphi_{2,X,Y}} & F(X \otimes Y) \\ {\scriptstyle c'_{F(X),F(Y)}} \downarrow & & \downarrow {\scriptstyle F(c_{X,Y})} \\ F(Y) \otimes' F(X) & \xrightarrow[\varphi_{2,Y,X}]{} & F(Y \otimes X) \end{array} \tag{2.20}$$

commutes.

Finally, for the definition of a natural tensor isomorphism ω between two strong monoidal functors $(F, \varphi_2^F, \varphi_0^F), (G, \varphi_2^G, \varphi_0^G) : \mathcal{C} \to \mathcal{C}'$ we refer to [16, Definition XI.4.1]. Note that, according to our terminology, in loc. cit. a tensor functor is nothing but a strong monoidal functor. This is why, for consistency, we will call ω as above a natural strong monoidal isomorphism.

We say that F is a strong monoidal equivalence if there exists a strong monoidal functor $G : \mathcal{C}' \to \mathcal{C}$ such that FG is naturally strongly monoidally isomorphic to $\mathrm{Id}_{\mathcal{C}'}$ and GF is naturally strongly monoidally isomorphic to $\mathrm{Id}_{\mathcal{C}}$. If a functor $F : \mathcal{C} \to \mathcal{C}'$ defines a strong monoidal equivalence between $\mathcal{C}$ and $\mathcal{C}'$ we say that the categories $\mathcal{C}$ and $\mathcal{C}'$ are strongly monoidally equivalent.

If a functor $F : \mathcal{C} \to \mathcal{C}'$ defines a strong monoidal equivalence between two (pre)braided categories $\mathcal{C}$ and $\mathcal{C}'$ we say that the categories $\mathcal{C}$ and $\mathcal{C}'$ are (pre)braided monoidally equivalent, provided that F is a (pre)braided functor, too.

2.3 A strong monoidal equivalence

Let H be a quasi-bialgebra. Then, the category of H-bimodules ${}_H\mathcal{M}_H$ is monoidal, since it can be identified with the category of left modules over the quasi-Hopf algebra $H^{\mathrm{op}} \otimes H$, where H^{op} is the opposite quasi-bialgebra associated to H. Explicitly, ${}_H\mathcal{M}_H$ is monoidal with the following structure. The associativity constraints $a'_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ are given by

$$a'_{M,N,P}((m \otimes n) \otimes p) = X^1 \cdot m \cdot x^1 \otimes (X^2 \cdot n \cdot x^2 \otimes X^3 \cdot p \cdot x^3). \qquad (2.21)$$

The unit object is k viewed as an H-bimodule via the counit ε of H, and the left and right unit constraints are given by the natural isomorphisms $k \otimes M \cong M \cong M \otimes k$.

A (co)algebra in ${}_H\mathcal{M}_H$ is called an H-bimodule (co)algebra.

With its regular comultiplication and counit, H is a coalgebra in ${}_H\mathcal{M}_H$. Then, we can define ${}_H^H\mathcal{M}_H^H$ as being the category of H-bicomodules in ${}_H\mathcal{M}_H$. For the explicit definition of an object M in ${}_H^H\mathcal{M}_H^H$, we refer to [2,8,27]. Roughly speaking, we have a left H-coaction on M, denoted by $\lambda_M : M \ni m \mapsto m_{\{-1\}} \otimes m_{\{0\}} \in H \otimes M$, and at the same time a right H-coaction on M, denoted by $\rho_M : M \ni m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H$, which are counital and coassociative up to conjugation by the reassociator Φ of H and, moreover, compatible each other, and also with the H-bimodule structure of M, respectively.

${}_H^H\mathcal{M}_H^H$ is monoidal in such a way that the forgetful functor $\mathcal{U} : {}_H^H\mathcal{M}_H^H \to ({}_H\mathcal{M}_H, \otimes_H, H)$ is strong monoidal. If $M, N \in {}_H^H\mathcal{M}_H^H$ then the left and right coactions of H on $M \otimes_H M$ are defined by those of M and N, and the multiplication of H.

It was proved by Schauenburg in [27] that ${}_H^H\mathcal{M}_H^H$ is monoidally equivalent to the left center of the monoidal category ${}_H\mathcal{M}$. The latter is denoted by ${}_H^H\mathcal{Y}D$ and called the category of left Yetter–Drinfeld modules over H. Its objects were described for the first time by Majid in [20]. They are left H-modules M on which H coacts from the left such that $\varepsilon(m_{[-1]})m_{[0]} = m$ and

$$\begin{aligned} &X^1 m_{[-1]} \otimes (X^2 \cdot m_{[0]})_{[-1]} X^3 \otimes (X^2 \cdot m_{[0]})_{[0]} \\ &\quad = X^1 (Y^1 \cdot m)_{[-1]_1} Y^2 \otimes X^2 (Y^1 \cdot m)_{[-1]_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{[0]}, \end{aligned} \qquad (2.22)$$

for all $m \in M$. Here, and in what follows, we denote by $\lambda_M : M \to H \otimes M$, $\lambda_M(m) = m_{[-1]} \otimes m_{[0]}$ the left H-coaction on M. It is compatible with the left H-module structure on M, in the sense that, for all $h \in H$ and $m \in M$,

$$h_1 m_{[-1]} \otimes h_2 \cdot m_{[0]} = (h_1 \cdot m)_{[-1]} h_2 \otimes (h_1 \cdot m)_{[0]}. \tag{2.23}$$

The monoidal structure on ${}^H_H\mathcal{YD}$ is such that the forgetful functor ${}^H_H\mathcal{YD} \to {}_H\mathcal{M}$ is strong monoidal. The coaction on the tensor product $M \otimes N$ of two Yetter–Drinfeld modules M and N is given by

$$\begin{aligned}\lambda_{M\otimes N}(m \otimes n) = X^1(x^1Y^1 \cdot m)_{[-1]}x^2(Y^2 \cdot n)_{[-1]}Y^3 \\ \otimes X^2 \cdot (x^1Y^1 \cdot m)_{[0]} \otimes X^3x^3 \cdot (Y^2 \cdot n)_{[0]},\end{aligned} \tag{2.24}$$

for all $m \in M$ and $n \in N$.

The strongly monoidally equivalence between ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$ is produced by the following functors, see [2].

Proposition 2.1 *Consider the functors $\mathcal{F} : {}^H_H\mathcal{YD} \to {}^H_H\mathcal{M}^H_H$ and $\mathcal{G} : {}^H_H\mathcal{M}^H_H \to {}^H_H\mathcal{YD}$ defined as follows:*

– For $M \in {}^H_H\mathcal{YD}$, we have $\mathcal{F}(M) = M \otimes H \in {}^H_H\mathcal{M}^H_H$ with the structure given by

$$h \cdot (m \otimes h') \cdot h'' = h_1 \cdot m \otimes h_2h'h'', \tag{2.25}$$

$$\lambda_{M\otimes H}(m \otimes h) = X^1 \cdot (x^1 \cdot m)_{[-1]} \cdot x^2h_1 \otimes \left(X^2 \cdot (x^1 \cdot m)_{[0]} \otimes X^3x^3h_2\right), \tag{2.26}$$

$$\rho_{M\otimes H}(m \otimes h) = (x^1 \cdot m \otimes x^2h_1) \otimes x^3h_2, \tag{2.27}$$

for all $h, h', h'' \in H$ and $m \in M$. If $f : M \to N$ is a morphisms in ${}^H_H\mathcal{YD}$ then $\mathcal{F}(f) = f \otimes \mathrm{Id}_H$.

– If $M \in {}^H_H\mathcal{M}^H_H$ then

$$\mathcal{G}(M) = M^{\overline{co(H)}} := \{m \in M \mid \rho_M(m) = x^1 \cdot m \cdot S(x^3_2X^3)f^1 \otimes x^2X^1\beta S(x^3_1X^2)f^2\},$$

the set of alternative coinvariants of M, which belongs to ${}^H_H\mathcal{YD}$ via the structure defined by

$$h \triangleright m = h_1 \cdot m \cdot S(h_2), \tag{2.28}$$

$$\lambda_{M^{\overline{co(H)}}}(m) = X^1Y^1_1m_{\{-1\}}g^1S(Z^2Y^2_2)\alpha Z^3Y^3 \otimes X^2Y^1_2 \cdot m_{\{0\}} \cdot g^2S(X^3Z^1Y^2_1), \tag{2.29}$$

for all $h \in H$ and $m \in M^{\overline{co(H)}}$, where $f^{-1} = g^1 \otimes g^2$ is the inverse of the Drinfeld's twist f. On morphisms, we have that $\mathcal{G}(f) = f|_{M^{\overline{co(H)}}}$, a well-defined morphism in ${}^H_H\mathcal{YD}$, for any morphism $f : M \to N$ in ${}^H_H\mathcal{M}^H_H$.

Then, $\mathcal{F}$ and $\mathcal{G}$ are inverse strong monoidal equivalence functors.

According to [2], the strong monoidal structure on $\mathcal{F}$ is given, for all $M, N \in {}^H_H\mathcal{YD}$, $m \in M$, $h, h' \in H$ and $n \in N$, by

$$\varphi_{2,M,N}((m \otimes h) \otimes_H (n \otimes h')) = (x^1 \cdot m \otimes x^2h_1 \cdot n) \otimes x^3h_2h', \tag{2.30}$$

and the morphism $\varphi_0 = \mathrm{Id}_H : H \to F(k) = k \otimes H \cong H$.

The strong monoidal structure on $\mathcal{G}$ is determined by

$$\overline{\phi}_{2,M,N}(m \otimes n) = q^1 x^1_1 \cdot m \cdot S(q^2 x^1_2) x^2 \otimes_H n \cdot S(x^3), \tag{2.31}$$

for all $M, N \in {}^H_H\mathcal{M}^H_H$, $m \in M^{\overline{co(H)}}$ and $n \in N^{\overline{co(H)}}$, and $\overline{\phi}_0 : k \to \mathcal{G}(H) = k\beta$ defined by $\overline{\phi}_0(\kappa) = \kappa\beta$, for all $\kappa \in k$, respectively. Using arguments similar to the ones in the proof of [2, Corollary 3.2], a straightforward computation ensures us that

$$\overline{\phi}^{-1}_{2,M,N}(m \otimes_H n) = \overline{E}_M(m_{(0)}) \otimes \overline{E}_N(m_{(1)} \cdot n), \tag{2.32}$$

for all $M, N \in {}^H_H\mathcal{M}^H_H$ and $m \otimes_H n \in (M \otimes_H N)^{\overline{\mathrm{co}(H)}}$, where $\overline{E}_M : M \to M^{\overline{co(H)}}$ determined by $\overline{E}_M(m) = m_{(0)} \cdot \beta S(m_{(1)})$, for all $m \in M$, is the projection defined in [8].

Furthermore, for all $M \in {}^H_H\mathcal{M}^H_H$,

$$\overline{\nu}_M : M^{\overline{co(H)}} \otimes H \ni m \otimes h \mapsto X^1 \cdot m \cdot S(X^2)\alpha X^3 h \in M, \tag{2.33}$$

is an isomorphism in ${}^H_H\mathcal{M}^H_H$ with inverse $\overline{\nu}^{-1}_M : M \ni m \mapsto \overline{E}(m_{(0)}) \otimes m_{(1)} \in M^{\overline{co(H)}} \otimes H$. The family of all morphisms $\overline{\nu}_M$ define a natural strong monoidal isomorphism $\overline{\nu}$ between $\mathcal{FG}$ and $\mathrm{Id}_{{}^H_H\mathcal{M}^H_H}$. Likewise, for all $M \in {}^H_H\mathcal{YD}$,

$$\zeta_M : (M \otimes H)^{\overline{co(H)}} \ni m \otimes h \mapsto \varepsilon(h)m \in M \tag{2.34}$$

is an isomorphism in ${}^H_H\mathcal{YD}$ and defines a natural strong monoidal isomorphism ζ between $\mathcal{GF}$ and $\mathrm{Id}_{{}^H_H\mathcal{YD}}$.

3 A braided monoidal equivalence

In Hopf algebra theory, it is well known that ${}^H_H\mathcal{M}^H_H$ is braided monoidally equivalent to ${}^H_H\mathcal{YD}$. A remarkable braiding on ${}^H_H\mathcal{M}^H_H$ was introduced by Woronowicz [28], and the fact that with respect to this braiding ${}^H_H\mathcal{M}^H_H$ and ${}^H_H\mathcal{YD}$ are braided monoidally equivalent was proved by Schauenburg in [26, Theorem 5.7].

The aim of this section is to generalize the two results above to the quasi-Hopf setting. To this end, we start with a lemma of independent interest.

The results below are stated without proofs in [11, Remark 2.4.10] and [15, Example 2.4]. For the sake of completeness and also for further use, we outline them in what follows.

Lemma 3.1 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two monoidal categories $(\mathcal{C}, \otimes, a, \underline{1}, l, r)$ and $(\mathcal{D}, \square, \mathbf{a}, \underline{I}, \lambda, \rho)$.*

(i) *F defines a strong monoidal equivalence if and only if F is strong monoidal and an equivalence of categories.*

(ii) *If F is as in (i) and $\mathcal{C}$ is, moreover, braided then there exists a unique braiding on $\mathcal{D}$ that turns F into a braided monoidal functor. Consequently, a functor defines a braided monoidal equivalence if and only if it is braided and an equivalence of categories.*

Proof If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories then by the proof of [18, IV.4 Theorem 1] there exist a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mu : \mathrm{Id}_{\mathcal{D}} \to FG$ and $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that, for all $X \in \mathcal{C}$,

$$F(\nu_X) = \mu^{-1}_{F(X)}. \tag{3.1}$$

(i) The direct implication is immediate. For the converse, we only indicate the unique strong monoidal structure $(\varphi_2^G, \varphi_0^G)$ of G that turns μ and ν into monoidal transformations. To this end, we denote by $(\varphi_2^F := (\varphi^F_{2,X,Y})_{X,Y\in\mathcal{C}}, \varphi_0^F)$ the strong monoidal structure of F, and by $\widetilde{\varphi}_2^F$, $\widetilde{\varphi}_0^F$ the inverse morphisms of φ_2^F, respectively, φ_0^F. Then, $\varphi_0^G = G(\widetilde{\varphi}_0^F)\nu^{-1}_{\underline{1}}$ and

$$\varphi^G_{2,U,V} = G((\mu_U^{-1}\Box\mu_V^{-1})\widetilde{\varphi}^F_{2,G(U),G(V)})\nu^{-1}_{G(U)\otimes G(V)}, \ \forall\, U,\ V \in \mathcal{D}. \tag{3.2}$$

(ii) Any braiding c for $\mathcal{C}$ defines a braiding d on $\mathcal{D}$ as follows. For any objects U, V of $\mathcal{D}$ take $d_{U,V}$ to be the following composition:

$$\begin{array}{ccccc} U\Box V & \xrightarrow{\mu_U\Box\mu_V} & FG(U)\Box FG(V) & \xrightarrow{\varphi_{2,G(U),G(V)}} & F(G(U)\otimes G(V)) \\ {\scriptstyle d_{U,V}}\downarrow & & & & \downarrow{\scriptstyle F(c_{G(U),G(V)})} \\ V\Box U & \xleftarrow[\mu_V^{-1}\Box\mu_U^{-1}]{} & FG(V)\Box FG(U) & \xleftarrow[\varphi^{-1}_{2,G(V),G(U)}]{} & F(G(V)\otimes G(U)) \end{array} \tag{3.3}$$

Then, $(\mathcal{D}, d = (d_{U,V})_{U,V\in\mathcal{D}})$ is a braided category and $F : (\mathcal{C}, c) \to (\mathcal{D}, d)$ becomes a braided monoidal functor. □

We specialize the above result to the strong monoidal equivalence in Proposition 2.1. Note that, according to [20] the category ${}^H_H\mathcal{Y}D$ is braided via the braiding given by $c = (c_{M,N})_{M,N\in{}^H_H\mathcal{Y}D}$, where, for all $m \in M$ and $n \in N$,

$$c_{M,N}(m \otimes n) = m_{[-1]} \cdot n \otimes m_{[0]}. \tag{3.4}$$

From now on, throughout the paper H is a quasi-Hopf algebra with bijective antipode. If $M \in {}^H_H\mathcal{M}^H_H$ then by $E_M : M \to M$ we denote the projection on the space of coinvariants of M, defined by $E_M(m) = X^1 \cdot m_{(0)} \cdot \beta S(X^2 m_{(1)})\alpha X^3 = q^1 \cdot \overline{E}_M(m) \cdot S(q^2)$, for all $m \in M$. Here, $M \ni m \mapsto \rho_M(m) := m_{(0)} \otimes m_{(1)} \in M \otimes H$ denotes the right coaction of H on M and q_R is the element in (2.14).

Record from [14] the following properties of E_M:

$$h \cdot E_M(m) = E_M(h_1 \cdot m) \cdot h_2\ , \tag{3.5}$$

$$E_M(m\cdot h)=\varepsilon(h)E_M(m)\ ,\ \ E_M(h\cdot E_M(m))=E_M(h\cdot m)\ , \tag{3.6}$$
$$E_M^2=E_M,\ E_M(m_{(0)})\cdot m_{(1)}=m\ ,$$
$$E_M(E_M(m)_{(0)})\otimes E_M(m)_{(1)}$$
$$=E_M(m)\otimes 1, \tag{3.7}$$
$$\rho(E_M(m))=E_M(x^1\cdot m)\cdot x^2\otimes x^3, \tag{3.8}$$

for all $m\in M$ and $h\in H$. Also, recall from [8] that $M^{\overline{\mathrm{co(H)}}}$ in invariant under the left adjoint action of H, that is $h\triangleright\overline{E}_M(m)=\overline{E}_M(h\cdot m)$, for all $h\in H$ and $m\in M$, where $\triangleright$ is defined by (2.28). Furthermore, the image of $\overline{E}_M$ is $M^{\overline{\mathrm{co(H)}}}$.

Another property of $\overline{E}_M$ is the following.

Lemma 3.2 *Let H be a quasi-Hopf algebra and M a two-sided two-cosided Hopf module over H. Then, for all $m\in M$, we have that*

$$m_{\{-1\}}\otimes\overline{E}_M(m_{\{0\}})=X^1Y_1^1\overline{E}_M(m_{(0)})_{\{-1\}}g^1S(q^2Y_2^2)Y^3m_{(1)}$$
$$\otimes X^2Y_2^1\cdot\overline{E}_M(m_{(0)})_{\{0\}}\cdot g^2S(X^3q^1Y_1^2). \tag{3.9}$$

Proof For all $m\in M\in{}^H_H\mathcal{M}^H_H$, we have

$$X^1Y_1^1\overline{E}_M(m_{(0)})_{\{-1\}}g^1S(q^2Y_2^2)Y^3m_{(1)}\otimes X^2Y_2^1\cdot\overline{E}_M(m_{(0)})_{\{0\}}\cdot g^2S(X^3q^1Y_1^2)$$
$$\overset{(2.7)}{=}X^1Y_1^1m_{(0,0)_{\{-1\}}}\beta_1g^1S(q^2Y_2^2m_{(0,1)_2})Y^3m_{(1)}$$
$$\otimes X^2Y_2^1\cdot m_{(0,0)_{\{0\}}}\cdot\beta_2g^2S(X^3q^1Y_1^2m_{(0,1)_1})$$
$$\overset{(2.12)}{=}X^1m_{(0)_{\{-1\}}}Y_1^1\delta^1S(q^2m_{(1)_{(1,2)}}Y_2^2)m_{(1)_2}Y^3$$
$$\otimes X^2\cdot m_{(0)_{\{0\}}}\cdot Y_2^1\delta^2S(X^3q^1m_{(1)_{(1,1)}}Y_1^2)$$
$$\overset{(2.16)}{=}m_{\{-1\}}X^1Y_1^1\delta^1S(q^2Y_2^2)Y^3\otimes m_{\{0\}_{(0)}}\cdot X^2Y_2^1\delta^2S(m_{\{0\}_{(1)}}X^3q^1Y_1^2)$$
$$\overset{(2.18)}{=}m_{\{-1\}}X^1q^1_{(1,1)}x_1^1\delta^1S(q^2x^3)\otimes m_{\{0\}_{(0)}}\cdot X^2q^1_{(1,2)}x_2^1\delta^2S(m_{\{0\}_{(1)}}X^3q_2^1x^2)$$
$$\overset{(2.9),(2.1)}{=}m_{\{-1\}}q_1^1\beta S(q^2)\otimes m_{\{0\}_{(0)}}\cdot q^1_{(2,1)}\beta S(m_{\{0\}_{(1)}}q^1_{(2,2)})$$
$$\overset{(2.5),(2.6)}{=}m_{\{-1\}}\otimes m_{\{0\}_{(0)}}\cdot\beta S(m_{\{0\}_{(1)}})$$
$$=m_{\{-1\}}\otimes\overline{E}_M(m_{\{0\}}),$$

as needed. □

One can provide now a braiding for ${}^H_H\mathcal{M}^H_H$.

Theorem 3.3 *If H is a quasi-Hopf algebra then ${}^H_H\mathcal{M}^H_H$ is a braided monoidal category with the braiding defined by*

$$d_{M,N}:M\otimes_H N\ni m\otimes_H n\mapsto E_N(m_{\{-1\}}\cdot n_{(0)})\otimes_H m_{\{0\}}\cdot n_{(1)}\in N\otimes_H M, \tag{3.10}$$

for all $M,N\in{}^H_H\mathcal{M}^H_H$.

Furthermore, if we consider ${}^H_H\mathcal{M}^H_H$ braided with the braiding d, then ${}^H_H\mathcal{M}^H_H$ is braided monoidally equivalent to ${}^H_H\mathcal{YD}$, where the braiding on ${}^H_H\mathcal{YD}$ is c as in (3.4).

Proof Let ${}^H_H\mathcal{YD} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} {}^H_H\mathcal{M}^H_H$ be the inverse strong monoidal equivalence functors defined in Proposition 2.1. We have that $\overline{\nu}$ defined by (2.33) is a natural strong monoidal isomorphism between $\mathcal{FG}$ and $\mathrm{Id}_{{}^H_H\mathcal{M}^H_H}$, while ζ given by (2.34) is a natural strong monoidal isomorphism between $\mathcal{GF}$ and $\mathrm{Id}_{{}^H_H\mathcal{YD}}$, respectively.

We prove now that $(\mathcal{F}, \overline{\nu}, \zeta)$ obeys the condition in (3.1), i.e., that $\overline{\nu}_{\mathcal{F}(M)} = \mathcal{F}(\zeta_M) : \overline{(M \otimes H)^{\mathrm{co(H)}}} \otimes H \to M \otimes H$, for all $M \in {}^H_H\mathcal{YD}$.

To this end, by [8, Remark 2.4] we have $\overline{(M \otimes H)^{\mathrm{co(H)}}} = \{p^1 \cdot m \otimes p^2 \mid m \in M\}$, where $p_R = p^1 \otimes p^2$ is the element defined in (2.13). Thus, $\zeta_M(p^1 \cdot m \otimes p^2) = m$, for all $m \in M$, and therefore $\mathcal{F}(\zeta_M)((p^1 \cdot m \otimes p^2) \otimes h) = m \otimes h$, for all $m \in M$ and $h \in H$.

If q_R is the element in (2.14) we then compute that

$$\begin{aligned}
&\overline{\nu}_{\mathcal{F}(M)}((p^1 \cdot m \otimes p^2) \otimes h) \\
&\overset{(2.33),(2.14)}{=} q^1 \cdot (p^1 \cdot m \otimes p^2) \cdot S(q^2)h \\
&\overset{(2.25)}{=} q^1_1 p^1 \cdot m \otimes q^1_2 p^2 S(q^2)h \overset{(2.17)}{=} m \otimes h,
\end{aligned}$$

for all $m \in M$ and $h \in H$. We conclude that $\overline{\nu}_{\mathcal{F}(M)} = \mathcal{F}(\zeta_M)$, as stated.

It follows by Lemma 3.1 that the braiding c for ${}^H_H\mathcal{YD}$ transports along $\mathcal{F}$ to a braiding d on ${}^H_H\mathcal{M}^H_H$ such that $\mathcal{F}$ becomes a braided monoidal equivalence. It only remains to show that d is as in (3.10). Using (3.3), we see that

$$\begin{aligned}
d_{M,N} &= \Big(m \otimes_H n \xrightarrow{\overline{\nu}_M^{-1} \otimes_H \overline{\nu}_N^{-1}} (\overline{E}_M(m_{(0)}) \otimes m_{(1)}) \otimes_H (\overline{E}_N(n_{(0)}) \otimes n_{(1)}) \\
&\xrightarrow{\varphi_{2,\mathcal{G}(M),\mathcal{G}(N)}} (x^1 \triangleright \overline{E}_M(m_{(0)}) \otimes x^2 m_{(1)_1} \triangleright \overline{E}_N(n_{(0)})) \otimes x^3 m_{(1)_2} n_{(1)} \\
&= (\overline{E}_M(x^1 \cdot m_{(0)}) \otimes x^2 m_{(1)_1} \triangleright \overline{E}_N(n_{(0)})) \otimes x^3 m_{(1)_2} n_{(1)} \\
&\overset{(3.6)}{=} (\overline{E}_M(m_{(0,0)}) \otimes m_{(0,1)} \triangleright \overline{E}_N(n_{(0)})) \otimes m_{(1)} n_{(1)} \\
&\xrightarrow{c_{\mathcal{G}(M),\mathcal{G}(N)} \otimes \mathrm{Id}_H} (\overline{E}_M(m_{(0,0)})_{[-1]} m_{(0,1)} \triangleright \overline{E}_N(n_{(0)}) \otimes \overline{E}_M(m_{(0,0)})_{[0]}) \otimes m_{(1)} n_{(1)} \\
&\overset{(2.29)}{=} (X^1 Y^1_1 \overline{E}_M(m_{(0,0)})_{\{-1\}} g^1 S(q^2 Y^2_2) Y^3 m_{(0,1)} \triangleright \overline{E}_N(n_{(0)}) \\
&\quad \otimes X^2 Y^1_2 \cdot \overline{E}_M(m_{(0,0)})_{\{0\}} \cdot g^2 S(X^3 q^1 Y^2_1)) \otimes m_{(1)} n_{(1)} \\
&\overset{(3.9)}{=} (m_{(0)_{\{-1\}}} \triangleright \overline{E}_N(n_{(0)}) \otimes \overline{E}_N(m_{(0)_{\{0\}}})) \otimes m_{(1)} n_{(1)} \\
&\xrightarrow{\varphi^{-1}_{2,\mathcal{G}(N),\mathcal{G}(M)}} (X^1 m_{(0)_{\{-1\}}} \triangleright \overline{E}_N(n_{(0)}) \otimes 1_H) \otimes_H (X^2 \triangleright \overline{E}_N(m_{(0)_{\{0\}}}) \otimes X^3 m_{(1)} n_{(1)}) \\
&\overset{(3.6)}{=} (m_{\{-1\}} \triangleright \overline{E}_N(n_{(0)}) \otimes 1_H) \otimes_H (\overline{E}_M(m_{\{0\}_{(0)}}) \otimes m_{\{0\}_{(1)}} n_{(1)}) \\
&\xrightarrow{\overline{\nu}_N \otimes \overline{\nu}_M} q^1 \cdot \overline{E}_N(m_{\{-1\}} \cdot n_{(0)}) \cdot S(q^2) \otimes_H Q^1 \cdot \overline{E}_M(m_{\{0\}_{(0)}}) \cdot S(Q^2) m_{\{0\}_{(1)}} n_{(1)} \\
&\overset{(2.14)}{=} X^1 m_{\{-1\}_1} \cdot \overline{E}_N(n_{(0)}) \cdot S(X^2 m_{\{-1\}_2})\alpha \\
&\quad \otimes_H X^3 Q^1 \cdot \overline{E}_M(m_{\{0\}_{(0)}}) \cdot S(Q^2) m_{\{0\}_{(1)}} n_{(1)} \\
&\overset{(2.16)}{=} X^1 m_{\{-1\}_1} \cdot \overline{E}_N(n_{(0)}) \cdot S(X^2 m_{\{-1\}_2})\alpha
\end{aligned}$$

$$
\begin{aligned}
&\otimes_H Q^1 \cdot \overline{E}_M(X^3_1 \cdot m_{\{0\}_{(0)}}) \cdot S(Q^2)X^3_2 m_{\{0\}_{(1)}} n_{(1)} \\
&\overset{(3.6)}{=} m_{\{-1\}}X^1 \cdot \overline{E}_N(n_{(0)}) \cdot S(m_{\{0,-1\}}X^2)\alpha \otimes_H E_M(m_{\{0,0\}_{(0)}}) \cdot m_{\{0,0\}_{(1)}} X^3 n_{(1)} \\
&\overset{(3.7)}{=} m_{\{-1\}}X^1 \cdot \overline{E}_N(n_{(0)}) \cdot S(m_{\{0,-1\}}X^2)\alpha \otimes_H m_{\{0,0\}} \cdot X^3 n_{(1)} \\
&\overset{(2.14)}{=} q^1 \cdot \overline{E}_N(m_{\{-1\}} \cdot n_{(0)}) \cdot S(q^2) \otimes_H m_{\{0\}} \cdot n_{(1)} \\
&= E_N(m_{\{-1\}} \cdot n_{(0)}) \otimes_H m_{\{0\}} \cdot n_{(1)} \Big),
\end{aligned}
$$

for all $m \in M$ and $n \in N$, as desired. □

4 Hopf algebras within ${}^H_H\mathcal{M}^H_H$

The aim of this section is to characterize the bialgebras and the Hopf algebras in ${}^H_H\mathcal{M}^H_H$. Inspired by some categorical results of Bespalov and Drabant [1], we show that giving a Hopf algebra in ${}^H_H\mathcal{M}^H_H$ is equivalent to giving a quasi-Hopf algebra projection for H as in [5]. Consequently, we obtain almost for free that quasi-Hopf algebra projections are characterized by the biproduct quasi-Hopf algebras constructed in [5], and therefore by Hopf algebras in ${}^H_H\mathcal{YD}$, too.

A quasi-bialgebra map between two quasi-bialgebras H and A is an algebra map $i : H \to A$ which intertwines the quasi-coalgebras structures, respects the counits and satisfies $(i \otimes i \otimes i)(\Phi_H) = \Phi_A$. If H, A are quasi-Hopf algebras then i is a quasi-Hopf algebra map if, in addition, $i(\alpha_H) = \alpha_A$, $i(\beta_H) = \beta_A$ and $S_A \circ i = i \circ S_H$.

For a quasi-Hopf algebra H denote by $H - \underline{\text{qBialgProj}}$ (resp. $H - \underline{\text{qHopfProj}}$) the category whose objects are triples (A, i, π) consisting of a quasi-bialgebra (resp. quasi-Hopf algebra) A and two quasi-bialgebra (resp. quasi-Hopf algebra) morphisms $H \underset{\pi}{\overset{i}{\rightleftarrows}} A$ such that $\pi i = \mathrm{Id}_H$. A morphism between (A, i, π) and (A', i', π') in $H - \underline{\text{qBialgProj}}$ (resp. $H - \underline{\text{qHopfProj}}$) is a quasi-bialgebra (resp. quasi-Hopf algebra) morphism $\tau : A \to A'$ such that $\tau i = i'$ and $\pi' \tau = \pi$. In what follows, the objects of $H - \underline{\text{qBialgProj}}$ (resp. $H - \underline{\text{qHopfProj}}$) will be called quasi-bialgebra (resp. quasi-Hopf algebra) projections for H.

We also denote by $\mathrm{Bialg}({}^H_H\mathcal{M}^H_H)$ (resp. $\mathrm{Hopf}({}^H_H\mathcal{M}^H_H)$) the category of bialgebras (resp. Hopf algebras) and bialgebra morphisms within ${}^H_H\mathcal{M}^H_H$.

As expected, we next prove that the categories $\mathrm{Bialg}({}^H_H\mathcal{M}^H_H)$ and $H - \underline{\text{qBialgProj}}$ (resp. $\mathrm{Hopf}({}^H_H\mathcal{M}^H_H)$ and $H - \underline{\text{qHopfProj}}$) are isomorphic. We first need some lemmas.

Lemma 4.1 *Take $M, N \in {}^H_H\mathcal{M}^H_H$, and the elements $m, m' \in M$ and $n, n' \in N$. Then,*

$$m \otimes_H n = m' \otimes_H n' \quad \Leftrightarrow \quad E(m_{(0)}) \otimes m_{(1)} \cdot n = E(m'_{(0)}) \otimes m'_{(1)} \cdot n'. \tag{4.1}$$

Proof From [14], we have that $\nu_M^{-1} : M \ni m \mapsto E_M(m_{(0)}) \otimes m_{(1)} \in M^{\mathrm{co(H)}} \otimes H$ is an isomorphism in ${}^H_H\mathcal{M}^H_H$, for all $M \in {}^H_H\mathcal{M}^H_H$. Here, $M^{\mathrm{co(H)}}$ is the image of E_M, a left H-module via the structure given by $h \neg E_M(m) = E_M(h \cdot m)$, for all $h \in H$ and $m \in M$. For a k-space U and $V \in {}_H\mathcal{M}$ denote by $\Upsilon_{U,V} : (U \otimes H) \otimes_H V \to U \otimes V$

the canonical isomorphism. We then have that $m \otimes_H n = m' \otimes_H n'$ if and only if

$$\begin{aligned}
&\Upsilon_{M^{\mathrm{co(H)}}, N^{\mathrm{co(H)}} \otimes H}(\nu_M^{-1} \otimes_H \nu_N^{-1})(m \otimes_H n) \\
&\quad = \Upsilon_{M^{\mathrm{co(H)}}, N^{\mathrm{co(H)}} \otimes H}(\nu_M^{-1} \otimes_H \nu_N^{-1})(m' \otimes_H n') \\
&\Leftrightarrow E_M(m_{(0)}) \otimes E_N(m_{(1)_1} \cdot n_{(0)}) \otimes m_{(1)_2} n_{(1)} \\
&\quad = E_M(m'_{(0)}) \otimes E_N(m'_{(1)_1} \cdot n'_{(0)}) \otimes m'_{(1)_2} n'_{(1)}.
\end{aligned}$$

Thus, if $m \otimes_H n = m' \otimes_H n'$ then

$$\begin{aligned}
&E_M(m_{(0)}) \otimes E_N(m_{(1)_1} \cdot n_{(0)}) \cdot m_{(1)_2} n_{(1)} = E_M(m'_{(0)}) \otimes E_N(m'_{(1)_1} \cdot n'_{(0)}) \cdot m'_{(1)_2} n'_{(1)} \\
&\overset{(3.5)}{\Leftrightarrow} E_M(m_{(0)}) \otimes m_{(1)} \cdot E_N(n_{(0)}) \cdot n_{(1)} = E_M(m'_{(0)}) \otimes m'_{(1)} \cdot E_N(n'_{(0)}) \cdot n'_{(1)} \\
&\overset{(3.7)}{\Leftrightarrow} E_M(m_{(0)}) \otimes m_{(1)} \cdot n = E_M(m'_{(0)}) \otimes m'_{(1)} \cdot n'.
\end{aligned}$$

The converse follows easily from (3.7), and we are done. □

Now we can construct the functor that gives the desired categorical isomorphism. For the definition of an H-bicomodule algebra $(\mathcal{A}, \lambda, \rho, \Phi_\lambda, \Phi_r, \Phi_{\lambda,r})$, we refer to [13].

Proposition 4.2 *Let H be a quasi-Hopf algebra. Then, there is a functor*

$$\mathcal{V} : \mathrm{Bialg}({}_H^H\mathcal{M}_H^H) \to H - \underline{\mathrm{qBialgProj}}.$$

On objects, $\mathcal{V}$ sends a bialgebra $(B, \underline{m}_B, i : H \to B, \underline{\Delta}_B, \pi : B \to H)$ in ${}_H^H\mathcal{M}_H^H$ to the triple (B, i, π), where B is considered as a quasi-bialgebra via $m_B := \underline{m}_B q_{B,B}$, $q_{B,B} : B \otimes B \to B \otimes_H B$ being the canonical surjection, $1_B = i(1_H)$,

$$\Delta_B(b) = b_{\underline{1}_{(0)}} \cdot b_{\underline{2}_{\{-1\}}} \otimes b_{\underline{1}_{(1)}} \cdot b_{\underline{2}_{\{0\}}}, \quad \varepsilon_B = \varepsilon\pi : B \to k, \tag{4.2}$$

and $\Phi_B = (i \otimes i \otimes i)(\Phi)$. $\mathcal{V}$ acts as identity on morphisms.

Proof We must check that $(B, m_B, 1_B, \Delta_B, \varepsilon_B, \Phi_B)$ is indeed a quasi-bialgebra and, moreover, that i, π become quasi-bialgebra morphisms.

By [2, Lemma 4.9], $(B, \underline{m}_B, i)$ is an algebra in ${}_H^H\mathcal{M}_H^H$ if and only if $(B, m_B, 1_B)$ is a k-algebra and at the same time an H-bicomodule algebra via the original left and right H-coactions and reassociators $\Phi_\lambda = X^1 \otimes X^2 \otimes i(X^3)$, $\Phi_\rho = i(X^1) \otimes X^2 \otimes X^3$ and $\Phi_{\lambda,\rho} = X^1 \otimes i(X^2) \otimes X^3$, such that, for all $h \in H$,

$$\lambda(i(h)) = h_1 \otimes i(h_2) \;\text{ and }\; \rho(i(h)) = i(h_1) \otimes h_2. \tag{4.3}$$

Otherwise stated, i is an H-bicomodule algebra morphism. Furthermore, the H-bimodule structure on B is nothing but the one induced by the restriction of scalars functor defined by i.

Likewise, by [2, Theorem 5.3], we have that $(B, \Delta_B, \varepsilon_B = \varepsilon\pi)$ is a coalgebra in ${}_H\overline{\mathcal{M}}_H := ({}_H\mathcal{M}_H, \otimes, k, a', l', r')$, i.e., an H-bimodule coalgebra, and $\pi : B \to H$ is a coalgebra morphism in ${}_H\overline{\mathcal{M}}_H$. If we denote $\Delta_B(b) = b_1 \otimes b_2$ we then have

$$i(X^1)b_{(1,1)}i(x^1) \otimes i(X^2)b_{(1,2)}i(x^2) \otimes i(X^3)b_2 i(x^3) = b_1 \otimes b_{(2,1)} \otimes b_{(2,2)}, \tag{4.4}$$

for all $b \in B$, $\varepsilon\pi = \varepsilon_B$ and $\Delta(\pi(b)) = \pi(b_1) \otimes \pi(b_2)$, for all $b \in B$.

The left and right H-coactions on B can be recovered from Δ_B and π as

$$\lambda(b) = \pi(b_1) \otimes b_2 \text{ and } \rho(b) = b_1 \otimes \pi(b_2),\ \forall\, b \in B. \tag{4.5}$$

Since i is the unit and π is the counit of the bialgebra B within ${}^H_H\mathcal{M}^H_H$ it follows that $\pi i = \mathrm{Id}_H$, and therefore, π is surjective. Furthermore, $\pi : B \to H$ is an algebra morphism in ${}^H_H\mathcal{M}^H_H$, so π is a k-algebra morphism as well. As we have seen, π intertwines the comultiplications Δ_B and Δ of B and H, too. If we define $\Phi_B := (i \otimes i \otimes i)(\Phi)$, it is clear that $(\pi \otimes \pi \otimes \pi)(\Phi_B) = \Phi$.

Combining (4.3) and (4.5) we get

$$\lambda(i(h)) = \pi(i(h)_1) \otimes i(h)_2 = h_1 \otimes i(h_2),\ \forall\, h \in H,$$

and therefore $\pi(i(h)_1) \otimes i(h)_2 = \pi(i(h_1)) \otimes i(h_2)$, for all $h \in H$. As π is surjective, we obtain that i intertwines the comultiplications Δ and Δ_B of H and B, and so $\Delta_B(1_B) = \Delta_B(i(1_H)) = i(1_H) \otimes i(1_H) = 1_B \otimes 1_B$. It is also an algebra morphism such that $(i \otimes i \otimes i)(\Phi) = \Phi_B$ and $\varepsilon_B i = \varepsilon$.

The most difficult part is to show that Δ_B is multiplicative, that is

$$\Delta_B(bb') = (b_{\underline{1}_{(0)}} \cdot b_{\underline{2}_{\{-1\}}})(b'_{\underline{1}_{(0)}} \cdot b'_{\underline{2}_{\{-1\}}}) \otimes (b_{\underline{1}_{(1)}} \cdot b_{\underline{2}_{\{0\}}})(b'_{\underline{1}_{(1)}} \cdot b'_{\underline{2}_{\{0\}}}), \tag{4.6}$$

for all $b, b' \in B$. Toward this end, observe first that by (3.10) and (4.1) we have that $\underline{\Delta}_B$ is multiplicative in ${}^H_H\mathcal{M}^H_H$ if and only if

$$\begin{aligned} &E((bb')_{\underline{1}_{(0)}}) \otimes (bb')_{\underline{1}_{(1)}} \cdot (bb')_{\underline{2}} \\ &\quad = E(b_{\underline{1}_{(0)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})_{(0)}) \otimes b_{\underline{1}_{(1)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})_{(1)} \cdot (b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}}) b'_{\underline{2}}, \end{aligned} \tag{4.7}$$

for all $b, b' \in B$, where, for simplicity, from now on we denote E_B by E. This allows us to compute that

$$\begin{aligned} \Delta_B(bb') &= (bb')_{\underline{1}_{(0)}} \cdot (bb')_{\underline{2}_{\{-1\}}} \otimes (bb')_{\underline{1}_{(1)}} \cdot (bb')_{\underline{2}_{\{0\}}} \\ &\overset{(3.7)}{=} E((bb')_{\underline{1}_{(0,0)}}) \cdot (bb')_{\underline{1}_{(0,1)}} (bb')_{\underline{2}_{\{-1\}}} \otimes (bb')_{\underline{1}_{(1)}} \cdot (bb')_{\underline{2}_{\{0\}}} \\ &\overset{(3.6)}{=} x^1 \neg E((bb')_{\underline{1}_{(0)}}) \cdot x^2((bb')_{\underline{1}_{(1)}} \cdot (bb')_{\underline{2}})_{\{-1\}} \otimes x^3 \cdot ((bb')_{\underline{1}_{(1)}} \cdot (bb')_{\underline{2}})_{\{0\}} \\ &\overset{(4.7)}{=} E(x^1 \cdot b_{\underline{1}_{(0)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})_{(0)}) \cdot x^2 b_{\underline{1}_{(1)_1}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})_{(1)_1} ((b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}}) b'_{\underline{2}})_{\{-1\}} \\ &\quad \otimes x^3 b_{\underline{1}_{(1)_2}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})_{(1)_2} \cdot ((b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}}) b'_{\underline{2}})_{\{0\}} \end{aligned}$$

$$
\begin{aligned}
&\overset{(3.6)}{=} E(b_{\underline{1}_{(0,0)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})(0,0)) \cdot b_{\underline{1}_{(0,1)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})(0,1)(b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}})\{-1\} b'_{\underline{2}_{\{-1\}}} \\
&\quad \otimes b_{\underline{1}_{(1)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})(1) \cdot (b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}})\{0\} b'_{\underline{2}_{\{0\}}} \\
&\overset{(3.7)}{=} b_{\underline{1}_{(0)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})(0) \cdot b_{\underline{2}_{\{0,-1\}}} b'_{\underline{1}_{(1)_1}} b'_{\underline{2}_{\{-1\}}} \\
&\quad \otimes b_{\underline{1}_{(1)}} E(b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}})(1) \cdot (b_{\underline{2}_{\{0,0\}}} \cdot b'_{\underline{1}_{(1)_2}}) b'_{\underline{2}_{\{0\}}} \\
&\overset{(3.8)}{=} b_{\underline{1}_{(0)}} E(x^1 b_{\underline{2}_{\{-1\}}} \cdot b'_{\underline{1}_{(0)}}) \cdot x^2 b_{\underline{2}_{\{0,-1\}}} b'_{\underline{1}_{(1)_1}} b'_{\underline{2}_{\{-1\}}} \otimes (b_{\underline{1}_{(1)}} x^3 \cdot b_{\underline{2}_{\{0,0\}}} \cdot b'_{\underline{1}_{(1)_2}}) b'_{\underline{2}_{\{0\}}} \\
&\overset{(3.6)}{=} b_{\underline{1}_{(0)}} E(b_{\underline{2}_{\{-1\}_1}} \cdot b'_{\underline{1}_{(0,0)}}) \cdot b_{\underline{2}_{\{-1\}_2}} b'_{\underline{1}_{(0,1)}} b'_{\underline{2}_{\{-1\}}} \otimes b_{\underline{1}_{(1)}} \cdot (b_{\underline{2}_{\{0\}}} \cdot b'_{\underline{1}_{(1)}}) b'_{\underline{2}_{\{0\}}} \\
&\overset{(3.5)}{=} (b_{\underline{1}_{(0)}} \cdot b_{\underline{2}_{\{-1\}}})(E(b'_{\underline{1}_{(0,0)}}) \cdot b'_{\underline{1}_{(0,1)}} b'_{\underline{2}_{\{-1\}}}) \otimes (b_{\underline{1}_{(1)}} \cdot b_{\underline{2}_{\{0\}}})(b'_{\underline{1}_{(1)}} \cdot b'_{\underline{2}_{\{0\}}}) \\
&\overset{(3.7)}{=} (b_{\underline{1}_{(0)}} \cdot b_{\underline{2}_{\{-1\}}})(b'_{\underline{1}_{(0)}} \cdot b'_{\underline{2}_{\{-1\}}}) \otimes (b_{\underline{1}_{(1)}} \cdot b_{\underline{2}_{\{0\}}})(b'_{\underline{1}_{(1)}} \cdot b'_{\underline{2}_{\{0\}}}),
\end{aligned}
$$

for all $b, b' \in B$, as needed. The remaining details are left to the reader. □

We can construct an inverse for $\mathcal{V}$ as follows.

Proposition 4.3 *Let H be a quasi-Hopf algebra and (B, i, π) a quasi-bialgebra projection for it. Then, B is a bialgebra in ${}_H^H\mathcal{M}_H^H$ with the structure given, for all $h, h' \in H$ and $b, b' \in B$, by*

$$h \cdot b \cdot h' = i(h) b i(h'); \tag{4.8}$$

$$\lambda : B \ni b \mapsto \pi(b_1) \otimes b_2 \in H \otimes B, \quad \rho : B \ni b \mapsto b_1 \otimes \pi(b_2) \in B \otimes H; \tag{4.9}$$

$$\underline{m}_B(b \otimes_H b') = bb', \quad i : H \to B; \tag{4.10}$$

$$\underline{\Delta}_B(b) = E(b_1) \otimes_H b_2 \ \text{ and } \ \underline{\varepsilon}_B = \pi. \tag{4.11}$$

In this way we have a well-defined functor $\mathcal{T} : H - \underline{\mathrm{qBialgProj}} \to \mathrm{Bialg}({}_H^H\mathcal{M}_H^H)$. $\mathcal{T}$ acts as identity on morphisms.

Proof It is easy to see that B is an object in ${}_H^H\mathcal{M}_H^H$ with the structure as in (4.8) and (4.9). Since $(b \cdot h)b' = b(h \cdot b')$, for all $b, b' \in B$ and $h \in H$, it follows that $\underline{m}_B : B \otimes_H B \to B$ given by $\underline{m}_B(b \otimes_H b') = bb'$, for all $b, b' \in B$, is well defined. By [2, Lemma 4.9] we deduce that $(B, \underline{m}_B, i)$ is an algebra in ${}_H^H\mathcal{M}_H^H$, since

$$\lambda(i(h)) = \pi(i(h)_1) \otimes i(h)_2 = h_1 \otimes i(h_2) \ \text{ and}$$
$$\rho(i(h)) = i(h)_1 \otimes \pi(i(h)_2) = i(h_1) \otimes h_2,$$

for all $h \in H$, i.e., i is an H-bicomodule morphism, where the H-bicomodule structure of B is $(B, \lambda, \rho, \Phi_\lambda = X^1 \otimes X^2 \otimes i(X^3), \Phi_\rho = i(X^1) \otimes X^2 \otimes X^3, \Phi_{\lambda,\rho} = X^1 \otimes i(X^2) \otimes X^3)$. We should point out that all these facts follow because $i : H \to B$ is a quasi-bialgebra morphism.

[2, Theorem 5.3] guarantees that B is a coalgebra in ${}_H^H\mathcal{M}_H^H$ with the structure in (4.11). Thus, it only remains to show that $\underline{\Delta}_B$ is an algebra morphism, where the algebra structure on $B \otimes_H B$ is the tensor product algebra one, modulo the braiding in (3.10). We compute

$$
\begin{aligned}
\underline{\Delta}_B i(h) &= E(i(h)_1) \otimes_H i(h)_2 \\
&= q^1 \cdot i(h_1)_{(0)} \cdot \beta S(q^2 i(h_1)_{(1)}) \otimes i(h_2) \\
&= i(q^1 h_{(1,1)} \beta S(q^2 h_{(1,2)}) h_2) \otimes_H 1_H \\
&\overset{(2.16)}{=} i(hq^1 \beta S(q^2)) \otimes_H 1_H \\
&\overset{(2.6)}{=} i(h) \otimes_H 1_H,
\end{aligned}
$$

and this shows that, up to the identification given by the unit constraints of the monoidal category $({}_H\mathcal{M}_H, \otimes_H, H)$, $\underline{\Delta}_B i = i \otimes_H i$.

Due to (4.7) and (4.11), that $\underline{\Delta}_B$ is multiplicative is equivalent to

$$
\begin{aligned}
&\underline{\Delta}_B(bb') \\
&\quad = E(b_1)E(b_{2_{\{-1\}}} \cdot E(b'_1)_{(0)}) \otimes_H (b_{2_{\{0\}}} \cdot E(b'_1)_{(1)})b'_2 \\
&\quad \overset{(3.8)}{=} E(b_1)E(\pi(b_{(2,1)}) \cdot E(x^1 \cdot b'_1) \cdot x^2) \otimes_H (b_{(2,2)} \cdot x^3)b'_2 \\
&\quad \overset{(3.6)}{=} E(X^1 \cdot b_{(1,1)})E(X^2\pi(b_{(1,2)}) \cdot b'_1) \cdot X^3 \otimes_H b_2 b'_2,
\end{aligned}
$$

for all $b, b' \in B$. Since, for all $b, b' \in B$, we have that

$$
\begin{aligned}
&E(X^1 \cdot b_1)E(X^2\pi(b_2) \cdot b') \cdot X^3 \\
&\quad = i(q^1 X^1_1) b_{(1,1)} i(\beta S(q^2 X^1_2 \pi(b_{(1,2)}) Q^1 X^2_1 \pi(b_{(2,1)})) b'_1 i(\beta S(Q^2 X^2_2 \pi(b_{(2,2)}) \pi(b'_2)) X^3) \\
&\quad \overset{(2.18)}{=} i(q^1 Q^1_{(1,1)})(x^1 \cdot b_1)_1 i(\beta S(q^2 Q^1_{(1,2)} \pi((x^1 \cdot b_1)_2)) Q^1_2 \pi(x^2 \cdot b_{(2,1)})) \\
&\quad\quad b'_1 i(\beta S(Q^2 \pi(x^3 \cdot b_{(2,2)}) \pi(b'_2)) \\
&\quad \overset{(2.16)}{=} i(Q^1 q^1)(b_1)_{(1,1)} i(\beta S(q^2 \pi((b_1)_{(1,2)})) \pi((b_1)_2)) b'_1 i(\beta S(Q^2 \pi(b_2) \pi(b'_2))) \\
&\quad = i(Q^1) b_1 i(q^1 \beta S(q^2)) b'_1 i(\beta S(Q^2 \pi(b_2 b'_2))) \\
&\quad \overset{(2.6)}{=} E(bb'),
\end{aligned}
$$

it follows that $\underline{\Delta}_B$ is multiplicative if and only if

$$
E((bb')_1) \otimes_H (bb')_2 = E(b_1 b'_1) \otimes_H b_2 b'_2, \ \forall\, b,\ b' \in B.
$$

The latter equivalence is immediate since Δ_B is multiplicative. This ends the proof. □

At this point, we can prove one of the main results of this paper.

Theorem 4.4 *Let H be a quasi-Hopf algebra. Then, the functors*

$$
\mathrm{Bialg}({}_H^H\mathcal{M}_H^H) \underset{\mathcal{T}}{\overset{\mathcal{V}}{\rightleftarrows}} H - \underline{\mathrm{qBialgProj}}
$$

define a category isomorphism.

They also produce a category isomorphism between $\mathrm{Hopf}({}_H^H\mathcal{M}_H^H)$ *and* $H -$ $\underline{\mathrm{qHopfProj}}$.

Proof One can check directly that $\mathcal{V}$ and $\mathcal{T}$ are inverse to each other; see also [2, Lemma 4.9 & Corollary 5.4].

Take $(B, i, \pi) \in H - \underline{\text{qHopfProj}}$, and denote by S_B the antipode of B. We claim that $\mathcal{T}((B, i, \pi)) = B$ is a Hopf algebra in ${}_H^H\mathcal{M}_H^H$ with antipode determined by

$$\underline{S}(b) = q^1\pi(b_{(1,1)})\beta \cdot S_B(q^2 \cdot b_{(1,2)}) \cdot \pi(b_2),\ \forall\, b \in B.$$

Indeed, a technical but straightforward computation ensures that $\underline{S}$ is a morphism in ${}_H^H\mathcal{M}_H^H$. Then, one can check that

$$\underline{S}(E(b)) = q^1\pi(b_1)\beta \cdot S_B(q^2 \cdot b_2),\ \forall\, b \in B,$$

and this fact allows us to compute that

$$\begin{aligned} \underline{S}(b_{\underline{1}})b_{\underline{2}} &= \underline{S}(E(b_1))b_2 \\ &= i(\pi(X^1 \cdot b_{(1,1)})\beta)S_B(X^2 \cdot b_{(1,2)})i(\alpha)(X^3 \cdot b_2) \\ &= i\pi(b)i(X^1\beta S(X^2)\alpha X^3) = i\pi(b), \end{aligned}$$

for all $b \in B$, as required. Similarly, one can easily see that

$$i(S(\pi(b_1))\alpha)\underline{S}(b_2) = \varepsilon_B(b)i(\alpha),\ \forall\, b \in B,$$

and from here we get that

$$\begin{aligned} b_{\underline{1}}\underline{S}(b_{\underline{2}}) &= E(b_1)\underline{S}_B(b_2) \\ &= i(\pi(X^1 \cdot b_{(1,1)})\beta S(\pi(X^2 \cdot b_{(1,2)}))\alpha)\underline{S}(X^3 \cdot b_2) \\ &= i(\pi(b_1)X^1\beta S(\pi(b_{(2,1)})X^2)\alpha)\underline{S}(b_{(2,2)})i(X^3) \\ &= \varepsilon_B(b_2)i(\pi(b_1)i(X^1\beta S(X^2)\alpha X^3) = i\pi(b), \end{aligned}$$

for all $b \in B$. Hence, our claim is proved.

In a similar manner, we can prove that if $\underline{S}$ is antipode for the bialgebra B in ${}_H^H\mathcal{M}_H^H$ then the quasi-bialgebra $\mathcal{V}(B)$ is actually a quasi-Hopf algebra with antipode determined by

$$S_B(b) = S(b_{(0)_{\{-1\}}}p^1)\alpha \cdot \underline{S}(b_{(0)_{\{0\}}}) \cdot p^2S(b_{(1)}),\ \forall\, b \in B, \tag{4.12}$$

and distinguished elements $\alpha_B = i(\alpha)$ and $\beta_B = i(\beta)$. We leave the verification of the remaining details to the reader. □

We end this paper by presenting a second characterization for the bialgebras (resp. Hopf algebras) in ${}_H^H\mathcal{M}_H^H$.

By Theorem 3.3, the categories ${}_H^H\mathcal{M}_H^H$ and ${}_H^H\mathcal{YD}$ are braided monoidally equivalent. Therefore, bialgebras (resp. Hopf algebras) in ${}_H^H\mathcal{M}_H^H$ are in a one to one correspondence to bialgebra (resp. Hopf algebra) structures in ${}_H^H\mathcal{YD}$. More precisely, if B is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{M}_H^H$ then $A := \overline{B^{\mathrm{co(H)}}}$ is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{YD}$. The inverse of this correspondence associates to any bialgebra (resp. Hopf algebra) A in ${}_H^H\mathcal{YD}$ the bialgebra (resp. Hopf algebra) $\mathcal{F}(A) = A \otimes H$ in ${}_H^H\mathcal{M}_H^H$. Thus, B and $A \otimes H$ are isomorphic as braided bialgebras (resp. Hopf algebras) in ${}_H^H\mathcal{M}_H^H$. Consequently, $\mathcal{V}(B)$ and $\mathcal{V}(A \otimes H)$ are isomorphic as objects in $H -$ qBialgProj (resp. H-qHopfProj).

Firstly, $A \otimes H$ is an object in ${}_H^H\mathcal{M}_H^H$ with the structure as in (2.25)–(2.27). By [2, Theorem 4.11], as an algebra $\mathcal{V}(A \otimes H) = A\#H$, the smash product algebra of A and H from [7]. The multiplication of $A\#H$ is given by

$$(a\#h)(a'\#h') = (x^1 \cdot a)(x^2 h_1 \cdot a')\#x^3 h_2 h',$$

for all $a, a' \in A$ and $h, h' \in H$, and its unit is $1_A \otimes 1_H$. This contributes to the structure of $\mathcal{V}(A \otimes H)$ with $j : H \ni h \mapsto 1_A \otimes h \in A\#H$, so far an H-bicomodule algebra morphism, provided that A is an algebra in ${}_H^H\mathcal{YD}$ (see [2, Proposition 4.10] for more details).

Secondly, by [2, Theorem 5.6], as a coalgebra $\mathcal{V}(A \otimes H) = A \ltimes H$, the smash product coalgebra of A and H. More exactly, the comultiplication is defined by

$$\begin{aligned}\Delta(a \ltimes h) = (&y^1 X^1 \cdot a_1 \ltimes y^2 Y^1 (x^1 X^2 \cdot a_2)_{[-1]} x^2 X_1^3 h_1) \\ &\otimes (y_1^3 Y^2 \cdot (x^1 X^2 \cdot a_2)_{[0]} \ltimes y_2^3 Y^3 x^3 X_2^3 h_2), \qquad (4.13)\end{aligned}$$

and the counit is $\varepsilon(a \otimes h) = \varepsilon_A(a)\varepsilon(h)$, for all $a \in A$ and $h \in H$. This contributes to the structure of $\mathcal{V}(A \otimes H)$ with $p : A \ltimes H \ni a \ltimes h \mapsto \varepsilon_A(a)h \in H$, so far an H-bimodule coalgebra morphism, provided that A is a coalgebra in ${}_H^H\mathcal{YD}$. As before, $a \mapsto a_{[-1]} \otimes a_{[0]}$ is the left coaction of H on A, $\Delta_A(a) = a_1 \otimes a_2$ is the comultiplication of A in ${}_H^H\mathcal{YD}$ and ε_A is its counit.

Summing up, $\mathcal{V}(\mathcal{F}(A)) = (A \times H, j, p)$, the biproduct quasi-bialgebra (resp. quasi-Hopf algebra) constructed in [5], provided that A is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{YD}$. Note that, in [5] we gave the coalgebra structure of $A \times H$ by adapting the one in the Hopf algebra case, and that by hard computations we showed that $A \times H$ is a quasi-bialgebra (resp. quasi-Hopf algebra), provided that A is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{YD}$. Now we have a more conceptual and less computational proof, and at the same time a converse for the cited result in [5].

Corollary 4.5 *Let H be a quasi-Hopf algebra, and B an object of ${}_H^H\mathcal{YD}$ which is at the same time an algebra and a coalgebra in ${}_H^H\mathcal{YD}$. Then, the smash product algebra and the smash product coalgebra afford a quasi-bialgebra (resp. quasi-Hopf algebra) structure on $A \otimes H$ if and only if A is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{YD}$. If this is the case, then $A \times H$ is a bialgebra (resp. Hopf algebra) in ${}_H^H\mathcal{M}_H^H$.*

Proof Everything follows from the above comments, and the fact that $\mathcal{F} : {}_H^H\mathcal{YD} \to {}_H^H\mathcal{M}_H^H$ is a braided monoidal equivalence, and that $\mathcal{T}$, $\mathcal{V}$ are inverse isomorphism functors.

Remark that, the antipode s of the quasi-Hopf algebra $A \times H$ can be obtained from the antipode S_A of A in ${}_H^H\mathcal{YD}$ and the antipode S of H as follows. The antipode $\underline{S}$ of

$\mathcal{F}(A)$ in ${}_H^H\mathcal{M}_H^H$ is $\mathcal{F}(S_A) = S_A \otimes \mathrm{Id}_H$, and so we have that

$$\begin{aligned}
&s(a \times h)\\
&\overset{(4.12)}{=} S((a \times h)_{(0)_{\{-1\}}} p^1)\alpha \cdot \underline{S}((a \times h)_{(0)_{\{0\}}}) \cdot p^2 S((a \times h)_{(1)})\\
&\overset{(2.27)}{=} S((x^1 \cdot a \times x^2 h_1)_{\{-1\}} p^1)\alpha \cdot \underline{S}((x^1 \cdot a \times x^2 h_1)_{\{0\}}) \cdot p^2 S(x^3 h_2)\\
&\overset{(2.25),(2.26)}{=} S(X^1(y^1 x^1 \cdot a)_{[-1]} y^2 x_1^2 h_{(1,1)} p^1)\alpha\\
&\quad \cdot \underline{S}(X^2 \cdot (y^1 x^1 \cdot a)_{[0]} \times X^3 y^3 x_2^2 h_{(1,2)} p^2 S(x^3 h_2))\\
&\overset{(2.15),(2.19)}{=} S(X^1 (p_1^1 \cdot a)_{[-1]} p_2^1 h)\alpha \cdot \underline{S}(X^2 \cdot (p_1^1 \cdot a)_{[0]} \times X^3 p^2)\\
&\overset{(2.23)}{=} S(X^1 p_1^1 a_{[-1]} h)\alpha \cdot (S_A(X^2 p_2^1 \cdot a_{[0]}) \times X^3 p^2)\\
&\overset{(2.25)}{=} (1_A \times S(X^1 p_1^1 a_{[-1]} h)\alpha)(X^2 p_2^1 \cdot S_A(a_{[0]}) \times X^3 p^2),
\end{aligned}$$

for all $a \in A$ and $h \in H$. Clearly, the distinguished elements that together with s define the antipode for $A \times H$ are $j(\alpha) = 1_A \times \alpha$ and $j(\beta) = 1_A \times \beta$. In this way, we gave an alternative proof for [5, Lemma 3.3]. In the computation above, we wrote $a \times h$ in place of $a \otimes h$ in order to distinguish the quasi-bialgebra structure on $A \otimes H$ given by the biproduct construction. □

Collecting the results proved in this section, we get the following.

Theorem 4.6 *Let H be a quasi-Hopf algebra. Then, there is a one-to-one correspondence between:*

- *bialgebras (resp. Hopf algebras) in ${}_H^H\mathcal{M}_H^H$;*
- *quasi-bialgebra (resp. quasi-Hopf algebra) projections for H;*
- *bialgebras (resp. Hopf algebras) in ${}_H^H\mathcal{YD}$;*
- *biproduct quasi-bialgebra (resp. quasi-Hopf algebra) structures for H.*

We end this paper by applying Theorem 4.6 to a class of braided Hopf algebras in ${}_H^H\mathcal{M}_H^H$ obtained from a tensor Hopf algebra type construction.

5 Tensor Hopf algebras within ${}_H^H\mathcal{M}_H^H$

Let H be a quasi-Hopf algebra and M an object of ${}_H^H\mathcal{M}_H^H$. We show that the tensor algebra $T_H(M)$ associated to M within $({}_H^H\mathcal{M}_H^H, \otimes_H, H)$ admits a braided Hopf algebra structure in $({}_H^H\mathcal{M}_H^H, \otimes_H, H)$ or, equivalently, a quasi-Hopf algebra structure with a projection.

Recall that the tensor algebra $T_H(M)$ of M within ${}_H^H\mathcal{M}_H^H$ is $T_H(M) = H \oplus \bigoplus_{n\geq 1} M^{\otimes_H n}$, where $M^{\otimes_H 1} := M$ and $M^{\otimes_H n} := M^{\otimes_H n-1} \otimes_H M$, for all $n \geq 2$. For $l < n$, we denote by $m^{\otimes_H l+1,n}$ the element $m^{l+1} \otimes_H \cdots \otimes_H m^n \in M^{\otimes_H n-l}$; when $l = 0$ and $n \geq 1$ we will write $m^{\otimes_H n}$ instead of $m^{\otimes_H 1,n}$.

The product $*$ on $T_H(M)$ is given by concatenation over H, i.e.,

$$h * h' = h \otimes_H h' \equiv hh',$$
$$h * m^{\otimes_H l} = h \otimes_H m^{\otimes_H l} \equiv hm^1 \otimes_H m^2 \otimes_H \cdots \otimes_H m^l,$$
$$m^{\otimes_H l} * h = m^{\otimes_H l} \otimes_H h \equiv m^1 \otimes_H \cdots \otimes_H m^l h,$$
$$m^{\otimes_H l} * m^{\otimes_H l+1,n} = m^{\otimes_H n},$$

for all $h, h' \in H$, $l \geq 1$, $n \geq 2$ and $m^1, \ldots, m^n \in M$. The unit of $T_H(M)$ is given by the unit 1 of H. As the monoidal category $({}_H^H\mathcal{M}_H^H, \otimes_H, H)$ is strict, in the writing of an element of $M^{\otimes_H n}$ we do not have to pay attention to parenthesis.

Using the monoidal structure on ${}_H^H\mathcal{M}_H^H$ given by $\otimes_H$, we find that $T_H(M)$ is an object in ${}_H^H\mathcal{M}_H^H$ via the structure induced by those of H and M, as follows: the H-bimodule structure of $T_H(M)$ is given by the above product $*$, while the H-bicomodule structure is defined, for all $h \in H$, $l \geq 1$ and $m^1, \ldots, m^l \in M$, by

$$\lambda(h) = \rho(h) = \Delta(h) = h_1 \otimes h_2 \in (H \otimes T_H(M)) \cap (T_H(M) \otimes H),$$
$$\lambda(m^{\otimes_H l}) = m^1_{\{-1\}} \cdots m^l_{\{-1\}} \otimes m^1_{\{0\}} \otimes_H \cdots \otimes_H m^l_{\{0\}},$$
$$\rho(m^{\otimes_H l}) = m^1_{(0)} \otimes_H \cdots \otimes_H m^l_{(0)} \otimes m^1_{(1)} \cdots m^l_{(1)}.$$

5.1 A braided Hopf algebra structure on $T_H(M)$

Denote by $i : H \to T_H(M)$ and $j : M \to T_H(M)$ the canonical embedding maps. It can be easily checked that i is an H-bicomodule algebra map, provided that $T_H(M)$ is considered as an H-bicomodule algebra via $(*, 1, \lambda, \rho)$ as above and reassociators $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = X^1 \underline{\otimes} X^2 \underline{\otimes} X^3$, where, in general, by $\underline{\otimes}$ we denote the tensor product between $T_H(M)$ and itself within the category of k-vector spaces. Otherwise stated, $(T_H(M), *, i)$ is an algebra in $({}_H^H\mathcal{M}_H^H, \otimes_H, H)$ and $i : H \to T_H(M)$ is an algebra morphism in ${}_H^H\mathcal{M}_H^H$. Finally, it is immediate that j is a morphism in ${}_H^H\mathcal{M}_H^H$.

Similar to the Hopf case [22], the tensor algebra $T_H(M)$ in ${}_H^H\mathcal{M}_H^H$ is uniquely determined by the following universal property.

Proposition 5.1 *Let H be a quasi-Hopf algebra and $M \in {}_H^H\mathcal{M}_H^H$. Then for any algebra morphism $u : A \to A'$ in ${}_H^H\mathcal{M}_H^H$ and any morphism $\zeta : M \to A'$ in ${}_H^H\mathcal{M}_H^H$, there exists a unique morphism $\overline{\zeta} : T_H(M) \to A'$ of algebras in ${}_H^H\mathcal{M}_H^H$ such that $\overline{\zeta} j = \zeta$.*

Proof It is similar to the one given for [22, Proposition 1.4.1]. □

The above universal property allows to define a Hopf algebra structure on $T_H(M)$ as follows. To avoid any possible confusion, by $\overline{\otimes}$ we denote the tensor product between $T_H(M)$ and itself within the strict braided monoidal category $({}_H^H\mathcal{M}_H^H, \otimes_H, H)$.

Proposition 5.2 *If H is a quasi-Hopf algebra and $M \in {}_H^H\mathcal{M}_H^H$ then there exist algebra morphisms $\underline{\Delta} : T_H(M) \to T_H(M) \overline{\otimes} T_H(M)$ and $\underline{\varepsilon} : T_H(M) \to H$ in ${}_H^H\mathcal{M}_H^H$, uniquely determined by*

$$\underline{\Delta}(h) = h \overline{\otimes} 1 = 1 \overline{\otimes} h \ \text{ and } \ \underline{\Delta}(m) = 1 \overline{\otimes} m + m \overline{\otimes} 1, \quad \text{resp. } \ \underline{\varepsilon}(h) = h \ \text{ and } \ \underline{\varepsilon}(m) = 0,$$

for all $h \in H$ and $m \in M$. Furthermore, $(*, 1, \underline{\Delta}, \underline{\varepsilon})$ provides a bialgebra structure on $T_H(M)$ within ${}_H^H\mathcal{M}_H^H$.

Proof To define $\underline{\Delta}$, we apply Proposition 5.1 for $A = H$, $A' = T_H(M) \overline{\otimes} T_H(M)$, $\zeta : M \ni m \mapsto m\overline{\otimes}1 + 1\overline{\otimes}m \in A'$ and $u : A \to A'$ the unit morphism of A', where A' has the tensor product algebra structure of $T_H(M)$ and itself, within ${}_H^H\mathcal{M}_H^H$. Thus, u is an algebra morphism in ${}_H^H\mathcal{M}_H^H$ and is given by $u(h) = h\overline{\otimes}1 = 1\overline{\otimes}h$, for all $h \in H$. Keeping in mind the monoidal structure of ${}_H^H\mathcal{M}_H^H$, one can easily check that ζ is a morphism in ${}_H^H\mathcal{M}_H^H$. Therefore, there is a unique algebra morphism $\underline{\Delta} : T_H(M) \to T_H(M) \overline{\otimes} T_H(M)$ in ${}_H^H\mathcal{M}_H^H$ such that $\underline{\Delta} j = \zeta$. Equivalently, $\underline{\Delta}$ is the algebra morphism in ${}_H^H\mathcal{M}_H^H$ completely determined by

$$\begin{aligned}\underline{\Delta}(h) &= \underline{\Delta}i(h) = u(h) = h\overline{\otimes}1 = 1\overline{\otimes}h\ ,\ \forall\, h \in H\\ \underline{\Delta}(m) &= \underline{\Delta}j(m) = \zeta(m) = m\overline{\otimes}1 + 1\overline{\otimes}m\ ,\ \forall\, m \in M.\end{aligned}$$

Since $\underline{\Delta}$ is an algebra morphism, inductively, we can uncover how it acts on an arbitrary element of $T_H(M)$. For instance, $\underline{\Delta}(h \otimes_H m) = h\underline{\Delta}(m) = hm\overline{\otimes}1 + 1\overline{\otimes}hm = \underline{\Delta}(hm)$, and

$$\begin{aligned}\underline{\Delta}(m^{\otimes_H 2}) &= (m^1\overline{\otimes}1 + 1\overline{\otimes}m^1)(m^2\overline{\otimes}1 + 1\overline{\otimes}m^2)\\ &= m^1 \otimes_H m^2\overline{\otimes}1 + m^1\overline{\otimes}m^2 + 1\overline{\otimes}m^1 \otimes_H m^2 + d_{T_H(M),T_H(M)}(m^1\overline{\otimes}m^2),\end{aligned}$$

for all $h \in H$ and $m, m^1, m^2 \in M$, where, as before, d is the braiding on ${}_H^H\mathcal{M}_H^H$ as in (3.10). And so on.

To define $\underline{\varepsilon}$, we proceed in a similar manner. This time we apply Proposition 5.1 to $A = A' = H$, $u = \mathrm{Id}_H$ and $\zeta : M \ni m \mapsto 0 \in A$, the null morphism. This gives an algebra morphism $\underline{\varepsilon} : T_H(M) \to H$, completely determined by $\underline{\varepsilon}(h) = \underline{\varepsilon}i(h) = u(h) = h$, for all $h \in H$, and $\underline{\varepsilon}(m) = \underline{\varepsilon}j(m) = \zeta(m) = 0$, for all $m \in M$. Consequently, for any nonzero natural number n we have

$$\underline{\varepsilon}(h \otimes_H m^{\otimes_H n}) = 0\ ,\ \forall\, h \in H \text{ and } m^1, \ldots, m^n \in M.$$

So it remains to prove that $(T_H(M), \underline{\Delta}, \underline{\varepsilon})$ is a coalgebra in ${}_H^H\mathcal{M}_H^H$. To show that $\underline{\Delta}$ is coassociative we apply again Proposition 5.1, this time to the following datum: $A = H$, A' equals the tensor product algebra $T_H(M) \overline{\otimes} T_H(M) \overline{\otimes} T_H(M)$ in ${}_H^H\mathcal{M}_H^H$, u equals the unit morphism of A' and

$$\zeta : M \ni m \mapsto m\overline{\otimes}1\overline{\otimes}1 + 1\overline{\otimes}m\overline{\otimes}1 + 1\overline{\otimes}1\overline{\otimes}m \in A',$$

a morphism in ${}_H^H\mathcal{M}_H^H$. We have, for all $m \in M$, that

$$\begin{aligned}(\underline{\Delta} \overline{\otimes} \mathrm{Id}_{T_H(M)})\underline{\Delta}(m) &= \underline{\Delta}(m)\overline{\otimes}1 + \underline{\Delta}(1)\overline{\otimes}m\\ &= m\overline{\otimes}1\overline{\otimes}1 + 1\overline{\otimes}m\overline{\otimes}1 + 1\overline{\otimes}1\overline{\otimes}m\\ &= m\overline{\otimes}\underline{\Delta}(1) + 1\overline{\otimes}\underline{\Delta}(m)\\ &= (\mathrm{Id}_{T_H(M)} \overline{\otimes} \underline{\Delta})\underline{\Delta}(m),\end{aligned}$$

so $(\underline{\Delta}\ \overline{\otimes}\ \mathrm{Id}_{T_H(M)})\underline{\Delta} j = (\mathrm{Id}_{T_H(M)}\ \overline{\otimes}\ \underline{\Delta})\underline{\Delta} j = \zeta$. As $(\underline{\Delta}\ \overline{\otimes}\ \mathrm{Id}_{T_H(M)})\underline{\Delta}$ and $(\mathrm{Id}_{T_H(M)}\ \overline{\otimes}\ \underline{\Delta})\underline{\Delta}$ are algebras morphisms in ${}_H^H\mathcal{M}_H^H$, it follows from the universal property of $T_H(M)$ that $(\underline{\Delta}\ \overline{\otimes}\ \mathrm{Id}_{T_H(M)})\underline{\Delta} = (\mathrm{Id}_{T_H(M)}\ \overline{\otimes}\ \underline{\Delta})\underline{\Delta}$, as desired.

Up to the identifications given by the left and right unit constraints of ${}_H^H\mathcal{M}_H^H$, we compute that $(\underline{\varepsilon} \otimes_H \mathrm{Id}_{T_H(M)})\underline{\Delta} j = j = (\mathrm{Id}_{T_H(M)} \otimes_H \underline{\varepsilon})\underline{\Delta} j$. $(\underline{\varepsilon} \otimes_H \mathrm{Id}_{T_H(M)})\underline{\Delta}$ and $(\mathrm{Id}_{T_H(M)} \otimes_H \underline{\varepsilon})\underline{\Delta}$ are algebra morphisms in ${}_H^H\mathcal{M}_H^H$, implying $(\underline{\varepsilon} \otimes_H \mathrm{Id}_{T_H(M)})\underline{\Delta} = (\mathrm{Id}_{T_H(M)} \otimes_H \underline{\varepsilon})\underline{\Delta} = \mathrm{Id}_{T_H(M)}$, as required. □

Next, we construct the antipode of $T_H(M)$. It is well known that, in general, the antipode $\underline{S}$ of a braided Hopf algebra B is an anti-morphism of the algebra B. Otherwise stated, $\underline{S}$ is an algebra morphism from B to B^{op}, where B^{op} is the opposite algebra associated to B. Coming back to our setting, $T_H(M)^{\mathrm{op}}$ equals $T_H(M)$ as object in ${}_H^H\mathcal{M}_H^H$, and is the algebra in ${}_H^H\mathcal{M}_H^H$ having the same unit as $T_H(M)$ and multiplication $*_{\mathrm{op}}$ given by $*_{\mathrm{op}} = * \circ d_{T_H(M),T_H(M)}$. Thus, if $T_H(M)$ admits an antipode $\underline{S}$ then it will be completely determined by its restrictions to H and M, since, for all $z, w \in T_H(M)$,

$$\underline{S}(z \overline{\otimes} w) = * \circ d_{T_H(M),T_H(M)}(\underline{S}(z) \overline{\otimes} \underline{S}(w)) = * \circ (\underline{S} \overline{\otimes} \underline{S}) d_{T_H(M),T_H(M)}(z \overline{\otimes} w). \tag{5.1}$$

Theorem 5.3 *Let H be a quasi-Hopf algebra and M an object in ${}_H^H\mathcal{M}_H^H$. Then, the tensor product algebra $T_H(M)$ in ${}_H^H\mathcal{M}_H^H$ admits a Hopf algebra structure within ${}_H^H\mathcal{M}_H^H$.*

Proof We know that $T_H(M)$ is a braided bialgebra. As in the proof of Proposition 5.2, if we take $A = H$, $A' = T_H(M)^{\mathrm{op}}$, u the unit morphism of A' and $\zeta : M \ni m \mapsto -m \in A'$, then from the universal property of $T_H(M)$ we get an algebra morphism $\underline{S} : T_H(M) \to T_H(M)^{\mathrm{op}}$ in ${}_H^H\mathcal{M}_H^H$, uniquely determined by

$$\underline{S}(h) = h\ ,\ \forall\, h \in H \text{ and } \underline{S}(m) = -m\ ,\ \forall\, m \in M.$$

With the help of (5.1), we can see how $\underline{S}$ acts on an arbitrary element of $T_H(M)$. To this end, for a fixed natural number $n \geq 2$, let $\{s_l = (l, l+1) \mid 1 \leq l \leq n-1\}$ be the set of generators s_l of the symmetric group S_n permuting l and $l+1$. Also, for any $1 \leq l \leq n-1$, take $d_l = \mathrm{Id}_{M^{\otimes_H l-1}} \otimes_H d_{M,M} \otimes_H \mathrm{Id}_{M^{\otimes_H n-l-1}}$, an automorphism of $M^{\otimes_H n}$, for all $1 \leq l \leq n-1$. Finally, for $\sigma \in S_n$ define $T_\sigma := d_{l_1} \ldots d_{l_r}$, where $\sigma = s_{l_1} \ldots s_{l_r}$ is a reduced expression for σ (i.e., r is minimal among all such expressions of σ). Note that, according to [17, Theorem 4.12], T_σ is well defined.

Now, if $\sigma_0 \in S_n$ is given by $\sigma_0(l) = n - l + 1$, for all $1 \leq l \leq n$, then

$$\underline{S}(m^{\otimes_H n}) = (-1)^n T_{\sigma_0}(m^{\otimes_H n}), \tag{5.2}$$

for all $n \geq 2$ and $m^1, \ldots, m^n \in M$. Observe that $s_1(s_2 s_1) \ldots (s_{n-1} \ldots s_1)$ is a reduced expression for σ_0, since σ_0 is what is called the longest element of S_n (for more details see the comments made before [17, Lemma 4.13]). Thus, $T_{\sigma_0} = d_1(d_2 d_1) \cdots (d_{n-1} \cdots d_1)$.

We show that $\underline{S}$ is antipode for the bialgebra structure of $T_H(M)$, that is,

$$* (\underline{S} \overline{\otimes} \mathrm{Id}_{T_H(M)})\underline{\Delta}(z) = i\underline{\varepsilon}(z) = *(\mathrm{Id}_{T_H(M)} \overline{\otimes} \underline{S})\underline{\Delta}(z), \tag{5.3}$$

for all $z \in T_H(M)$. Toward this end, remark first that (5.3) is satisfied for any $z = h \in H$ and $z = m \in M$. Also, if we define $*^2 = *$ and, in general, $*^k = *(*^{k-1} \otimes_H \mathrm{Id}_{T_H(M)})$, for all $k \geq 3$, we have

$$\begin{aligned}
&*(\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)})\underline{\Delta} * \\
&\quad = *(\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)})(*\overline{\otimes}*)(\mathrm{Id}_{T_H(M)}\overline{\otimes}d_{T_H(M),T_H(M)}\overline{\otimes}\mathrm{Id}_{T_H(M)})(\underline{\Delta}\overline{\otimes}\underline{\Delta}) \\
&\quad = *^3(\underline{S}\overline{\otimes}\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)})(d_{T_H(M),T_H(M)}\overline{\otimes}*)(\mathrm{Id}_{T_H(M)}\overline{\otimes}d_{T_H(M),T_H(M)}\overline{\otimes}\mathrm{Id}_{T_H(M)})(\underline{\Delta}\overline{\otimes}\underline{\Delta}) \\
&\quad = *^4(\underline{S}\overline{\otimes}\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)^{\overline{\otimes}2}})(\mathrm{Id}_{T_H(M)}\overline{\otimes}\underline{\Delta}\overline{\otimes}\mathrm{Id}_{T_H(M)})(d_{T_H(M),T_H(M)}\overline{\otimes}\mathrm{Id}_{T_H(M)})(\mathrm{Id}_{T_H(M)}\overline{\otimes}\underline{\Delta}) \\
&\quad = *^3(\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)^{\overline{\otimes}2}})(\mathrm{Id}_{T_H(M)}\overline{\otimes}(*(\underline{S}\overline{\otimes}\mathrm{Id}_{T_H(M)})\underline{\Delta})\overline{\otimes}\mathrm{Id}_{T_H(M)}) \\
&\qquad\quad (d_{T_H(M),T_H(M)}\overline{\otimes}\mathrm{Id}_{T_H(M)})(\mathrm{Id}_{T_H(M)}\overline{\otimes}\underline{\Delta}).
\end{aligned}$$

We used that $\underline{\Delta}$ is an algebra morphism in the first equality, the fact that $\underline{S} : T_H(M) \to T_H(M)^{\mathrm{op}}$ is an algebra morphism in ${}_H^H\mathcal{M}_H^H$ in the second equality, the naturality of the braiding d in the third equality, and the associativity of $*$ in the last equality.

The above computation says that if the first equality in (5.3) is satisfied by two elements of $T_H(M)$ then it is also satisfied by their product in $T_H(M)$. As M generates $T_H(M)$ as an algebra, this implies that the first equality of (5.3) is satisfied by any $z \in T_H(M)$. In a similar manner, we can show the second equality in (5.3), so our proof is finished. □

5.2 A quasi-Hopf algebra structure on $T_H(M)$

It follows now from Theorem 4.6 that $T_H(M)$ admits also the structure of a quasi-Hopf algebra with a projection or, equivalently, it has the structure of a biproduct quasi-Hopf algebra. More exactly, we have the following.

Proposition 5.4 *Let H be a quasi-Hopf algebra and M an object of ${}_H^H\mathcal{M}_H^H$. Then, the tensor algebra $(T_H(M), *, 1)$ within ${}_H^H\mathcal{M}_H^H$ admits the structure of a quasi-Hopf algebra with a projection. Its comultiplication $\widetilde{\Delta}$ is given by $\widetilde{\Delta}(h) = \Delta(h)$, for all $h \in H$, and*

$$\widetilde{\Delta}(m) = \lambda_M(m) + \rho_M(m) = m_{\{-1\}}\underline{\otimes}m_{\{0\}} + m_{(0)}\underline{\otimes}m_{(1)} \in T_H(M)\underline{\otimes}T_H(M)\ ,$$

for all $m \in M$, extended to the whole $T_H(M)$ as an algebra morphism from $T_H(M)$ to $T_H(M)\underline{\otimes}T_H(M)$, while its counit is determined by $\widetilde{\varepsilon}(h) = \varepsilon(h)$, for all $h \in H$, and $\widetilde{\varepsilon}(m) = 0$, for all $m \in M$, extended this time to the whole $T_H(M)$ as an algebra morphism from $T_H(M)$ to k. The reassociator of $T_H(M)$ is $\widetilde{\Phi} = X^1\underline{\otimes}X^2\underline{\otimes}X^3$, where $\Phi = X^1 \otimes X^2 \otimes X^3$ is the reassociator of H. An antipode $(\widetilde{S}, \widetilde{\alpha}, \widetilde{\beta})$ for $T_H(M)$ can be obtained from an antipode (S, α, β) of H as follows: $\widetilde{\alpha} = i(\alpha) = \alpha$, $\widetilde{\beta} = i(\beta) = \beta$, $\widetilde{S}(h) = S(h)$, for all $h \in H$, and

$$\widetilde{S}(m) = -S(m_{(0)_{\{-1\}}}p^1)\alpha \cdot m_{(0)_{\{0\}}} \cdot p^2 S(m_{(1)}),$$

for all $m \in M$, extended to the whole $T_H(M)$ as an anti-morphism of k-algebras from $T_H(M)$ to itself, that is, for all $n \geq 1$ and $m^1, \ldots, m^n \in M$, we have

$$\begin{aligned}\widetilde{S}(m^{\otimes_H n}) = (-1)^n S(m^n_{(0)_{\{-1\}}} p^1)\alpha \cdot m^n_{(0)_{\{0\}}} \cdot p^2 S(m^n_{(1)}) \otimes_H \\ \otimes_H \cdots \otimes_H S(m^1_{(0)_{\{-1\}}} \mathbf{p}^1)\alpha \cdot m^1_{(0)_{\{0\}}} \cdot \mathbf{p}^2 S(m^1_{(1)})\end{aligned} \tag{5.4}$$

(each tensor component over H contains a different copy of $p_R = p^1 \otimes p^2 = \cdots = \mathbf{p}^1 \otimes \mathbf{p}^2$).

Finally, via this structure we have quasi-Hopf algebra morphisms $H \underset{\pi}{\overset{i}{\rightleftarrows}} T_H(M)$ such that $\pi i = \mathrm{Id}_H$, where $\pi = \underline{\varepsilon}$ is the counit of $T_H(M)$ in ${}_H^H\mathcal{M}_H^H$.

Proof The unital k-algebra structure on $T_H(M)$ is given by concatenation, i.e., by $*$, and 1, the unit of H. The fact that $(T_H(M), *, 1, \widetilde{\Delta}, \widetilde{\varepsilon}, \widetilde{\Phi}, \widetilde{S}, \alpha, \beta)$ is a quasi-Hopf algebra is an immediate consequence of Theorems 4.4 and 5.3. □

Remarks 5.5 (i) The formula in (5.4) can be also obtained from (5.2) and (4.12), we leave the verification of this fact to the reader.

(ii) The quasi-Hopf algebra structure of $T_H(M)$ can be deduced as well from the following universal property of $T_H(M)$: for any k-algebras A, A', any k-algebra morphisms $u : H \to A, v : A \to A'$ and any H-bimodule morphism $\zeta : M \to A'$ there exists a unique k-algebra morphism $\overline{\zeta} : T_H(M) \to A'$ which is H-bilinear and such that $\overline{\zeta} i = vu$ and $\overline{\zeta} j = \zeta$; here A, A' are considered H-bimodules via u vu, respectively.

Explicitly, $\overline{\zeta}(h) = vu(h)$, for all $h \in H$, and, for all $n \geq 1$ and $m^1, \ldots, m^n \in M$,

$$\overline{\zeta}(m^{\otimes_H n}) = \zeta(m^1) \cdots \zeta(m^n).$$

Now, $\widetilde{\Delta}, \widetilde{\varepsilon}$ and $\widetilde{S}$ are uniquely determined by the following data: $(A = H \otimes H, A' = T_H(M)\underline{\otimes} T_H(M), u = \Delta, v = i \otimes i, \zeta = \lambda_M + \rho_M)$, $(A = H, A' = k, u = \mathrm{Id}_H, v = \varepsilon, \zeta = 0)$ and $(A = H, A' = T_H(M)^{\mathrm{opp}}, u = \mathrm{Id}_H, v = iS, \zeta : M \ni m \mapsto -S(m_{(0)_{\{-1\}}} p^1)\alpha \cdot m_{(0)_{\{0\}}} \cdot p^2 S(m_{(1)}) \in T_H(M)^{\mathrm{opp}})$, respectively. Note that $H \otimes H$ and $T_H(M)\underline{\otimes} T_H(M)$ are viewed as H-bimodules via Δ, and $T_H(M)^{\mathrm{opp}}$ is the opposite k-algebra associated to $T_H(M)$, regarded as an H-bimodule via the actions $h *_{op} z *_{op} h' = S(h') * z * S(h)$, for all $h, h' \in H$ and $z \in T_H(M)$.

Next, we want to describe, in two equivalent ways, how $\widetilde{\Delta}$ acts on an element of $T_H(M)$. By the universal property of $\otimes_H$, if A is a k-algebra and an H-bimodule, and $f_1, f_2 : M \to A$ are H-bimodule morphisms, then we have a well-defined H-bimodule morphism $f_1 \cdot f_2 : M \otimes_H M \to A$ sending $m^1 \otimes_H m^2 \in M \otimes_H M$ to $f_1(m^1) f_2(m^2) \in A$. We use this simple observation in order to see how $\widetilde{\Delta}$ extends to the whole $T_H(M)$. Actually, we have that $\lambda_M + \rho_M : M \to T_H(M)\underline{\otimes} T_H(M)$ is an H-bimodule morphism and

$$\widetilde{\Delta}(m^{\otimes_H n}) = (\lambda_M + \rho_M)(m^1)\cdots(\lambda_M + \rho_M)(m^n),$$

for all $n \geq 1$ and $m^1, \ldots, m^n \in M$, where, once more, the product in the right hand side is made in the tensor product algebra $T_H(M)\underline{\otimes}T_H(M)$ built within the category of k-vector spaces, viewed as an H-bimodule via the monoidal structure of ${}_H\mathcal{M}_H$ given by $\otimes$. Equivalently,

$$\widetilde{\Delta}(m^{\otimes_H n}) = (\lambda_M + \rho_M)^n(m^1 \otimes_H \cdots \otimes_H m^n),$$

where, in general, by f^n we denote the product $\cdot$ of n copies of f, an H-bimodule morphism from M to a k-algebra that is an H-bimodule, too. Since $\cdot$ is not commutative, we get that $\widetilde{\Delta}$ restricted to $M^{\otimes_H n}$ is the sum of 2^n distinct terms, each of them having the form $f_1 \cdot \ldots \cdot f_n$ with $f_l \in \{\lambda_M, \rho_M\}$, for all $1 \leq l \leq n$. For instance,

$$\begin{aligned}\widetilde{\Delta}(m^{\otimes_H 2}) &= (\lambda_M^2 + \lambda_M \cdot \rho_M + \rho_M \cdot \lambda_M + \rho_M^2)(m^1 \otimes_H m^2)\\ &= m^1_{\{-1\}}m^2_{\{-1\}}\underline{\otimes}m^1_{\{0\}} \otimes_H m^2_{\{0\}} + m^1_{\{-1\}} \cdot m^2_{(0)}\underline{\otimes}m^1_{\{0\}} \cdot m^2_{(1)}\\ &\qquad + m^1_{(0)} \cdot m^2_{\{-1\}}\underline{\otimes}m^1_{(1)} \cdot m^2_{\{0\}} + m^1_{(0)} \otimes_H m^2_{(0)}\underline{\otimes}m^1_{(1)}m^2_{(1)},\end{aligned}$$

for all $m^1, m^2 \in M$.

A second description for $\widetilde{\Delta}$ can be derived from the following result. It is a generalization of [25, Lemma 7] to the quasi-Hopf setting.

Lemma 5.6 *For any $M \in {}^H_H\mathcal{M}^H_H$, we have $\lambda_M \cdot \rho_M = (\rho_M \cdot \lambda_M) \circ d_{M,M}$.*

Proof For $m^1, m^2 \in M$, we compute

$$\begin{aligned}&(\rho_M \cdot \lambda_M) \circ d_{M,M}(m^1 \otimes_H m^2)\\ &= \rho_M(E(m^1_{\{-1\}} \cdot m^2_{(0)}))\lambda_M(m^1_{\{0\}} \cdot m^2_{(1)})\\ &= (E(x^1 m^1_{\{-1\}} \cdot m^2_{(0)}) \cdot x^2 \underline{\otimes} x^3)(m^1_{\{0\}_{\{-1\}}} m^2_{(1)_1} \underline{\otimes} m^1_{\{0\}_{\{0\}}} \cdot m^2_{(1)_2})\\ &= E(x^1 m^1_{\{-1\}} \cdot m^2_{(0)}) \cdot x^2 m^1_{\{0\}_{\{-1\}}} m^2_{(1)_1} \underline{\otimes} x^3 \cdot m^1_{\{0\}_{\{0\}}} \cdot m^2_{(1)_2}\\ &= E(m^1_{\{-1\}_1} x^1 \cdot m^2_{(0)}) \cdot m^1_{\{-1\}_2} x^2 m^2_{(1)_1} \underline{\otimes} m^1_{\{0\}} \cdot x^3 m^2_{(1)_2}\\ &= m^1_{\{-1\}} \cdot E(m^2_{(0)_{(0)}}) \cdot m^2_{(0)_{(1)}} \underline{\otimes} m^1_{\{0\}} \cdot m^2_{(1)}\\ &= m^1_{\{-1\}} \cdot m^2_{(0)} \underline{\otimes} m^1_{\{0\}} \cdot m^2_{(1)} = (\lambda_M \cdot \rho_M)(m^1 \otimes_H m^2),\end{aligned}$$

as needed. □

For any $1 \leq k \leq n$, let $S_{k,n-k}$ be the set of $(k, n-k)$-shuffles, that is the set of permutations $\sigma \in S_n$ for which $\sigma(1) < \cdots \sigma(k)$ and $\sigma(k+1) < \cdots \sigma(n)$. It can be easily seen that giving an element $\sigma \in S_{k,n-k}$ is equivalent to giving a subset $X_k = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$: we can assume that the elements of X_k are arranged in ascending order, and thus, the one-to-one correspondence maps X_k to the permutation σ given by $\sigma(1) = i_1, \ldots, \sigma(k) = i_k$ and, for $j > k, \sigma(j)$ equals the j^{th}-element of the set $\{1, \ldots, n\}\backslash X_k$, ordered in ascending order. Consequently, $S_{k,n-k}$ has $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ elements.

Following [25], $\widetilde{B}_{k,n-k} := \sum_{\sigma^{-1}\in S_{k,n-k}} T_\sigma$, where T_σ is defined by a reduced expression of σ as in the proof of Theorem 5.3.

Corollary 5.7 *With the above notation, the comultiplication $\widetilde{\Delta}$ of the quasi-Hopf algebra $T_H(M)$ is given by $\widetilde{\Delta}(h) = \Delta(h)$, for all $h \in H$, and*

$$\widetilde{\Delta}(m^{\otimes_H n}) = \sum_{k=0}^{n} (\rho_M^k \cdot \lambda_M^{n-k}) \circ \widetilde{B}_{k,n-k}(m^{\otimes_H n}),$$

for all $n \geq 1$ and $m^1, \ldots, m^n \in M$.

Proof Exactly as in the proof of [25, Proposition 6], we can show that, for any two H-bimodule morphisms f_1, f_2 from M to a k-algebra A that is also an H-bimodule, we have

$$(f_1 + f_2)^n = \sum_{k=0}^{n} (f_1^k \cdot f_2^{n-k}) \circ \widetilde{B}_{k,n-k},$$

provided that $f_2 \cdot f_1 = (f_1 \cdot f_2) \circ d_{M,M}$. Our assertion follows now by taking in the above formula $A = T_H(M)\underline{\otimes} T_H(M)$, $f_1 = \rho_M$ and $f_2 = \lambda_M$. □

5.3 $T_H(M)$ as a biproduct quasi-Hopf algebra

For simplicity, for $M \in {}_H^H\mathcal{M}_H^H$ we denote $M^{\overline{\mathrm{co}(H)}}$ by V. Also, by $T(V)$ we denote the k-vector space $\bigoplus_{n\geq 0} T^{n)}(V)$, where $T^{0)}(V) = k$, $T^{\otimes 1)}(V) = V$, $T^{\otimes 2)}(V) = V \otimes V$ and $T^{\otimes n)} = V \otimes T^{\otimes n-1)}(V)$, for all $n \geq 3$. Since $V \in {}_H^H\mathcal{YD}$ with structure (2.28–2.29) and ${}_H^H\mathcal{YD}$ is monoidal, it follows that $T(V)$ is a left Yetter–Drinfeld module over H. As ${}_H^H\mathcal{YD}$ is not strict monoidal, the order of the parenthesis in the definition of $T(V)$ is essential for the structure of $T(V)$ in ${}_H^H\mathcal{YD}$; the notation $T^{\otimes n)}(V)$ suggests that we deal with the tensor product of n-copies of V in ${}_H^H\mathcal{YD}$ such that all the closing parentheses are placed on the right-handed side of the last term of $\otimes$, i.e., $T^{\otimes n)}(V) = V \otimes (V \otimes (\cdots \otimes (V \otimes V) \cdots))$, as objects in ${}_H^H\mathcal{YD}$. This also motivates to denote by $\overleftarrow{v^{l+1,n}} = v^{l+1} \otimes (v^{l+2} \otimes (\cdots \otimes (v^{n-1} \otimes v^n) \cdots))$ an element of $T^{\otimes n-l)}(V)$, for all $l < n$; in the case when $l = 0$ and $n \geq 1$, in place of $\overleftarrow{v^{1,n}}$ we simply write $\overleftarrow{v^{n}}$.

We next show that $T_H(M)^{\overline{\mathrm{co}(H)}}$ and $T(V)$ are isomorphic objects of ${}_H^H\mathcal{YD}$.

Lemma 5.8 *Let H be a quasi-Hopf algebra, $M \in {}_H^H\mathcal{M}_H^H$, $V = M^{\overline{\mathrm{co}(H)}}$ and $\overline{E} = \overline{E}_M$. Then, for any $n \geq 2$, $\overline{\phi}_n^{-1} : (M^{\otimes_H n})^{\overline{\mathrm{co}(H)}} \to V^{\otimes n)}$ given by*

$$\overline{\phi}_k^{-1}(m^{\otimes_H n}) = \overline{E}(m^1_{(0)}) \otimes m^1_{(1)} \cdot \Big(\overline{E}(m^2_{(0)}) \otimes m^2_{(1)} \cdot \Big(\cdots \otimes m^{n-2}_{(1)} \cdot \big(\overline{E}(m^{n-1}_{(0)}) \otimes m^{n-1}_{(1)} \triangleright \overline{E}(m^n)\big) \cdots\Big)\Big),$$

for all $m^{\otimes_H n} \in (M^{\otimes_H n})^{\overline{\mathrm{co}(H)}}$*, is an isomorphism of left Yetter–Drinfeld modules. Consequently, if we set* $\overline{\phi}_1^{-1} = \mathrm{Id}_V$ *and* $\overline{\phi}_0^{-1} : H^{\overline{\mathrm{co}(H)}} = k\beta \ni \kappa\beta \to \kappa \in k$ *then*

$$\overline{\phi}^{-1} := \bigoplus_{n\geq 0} \phi_k^{-1} : T_H(M)^{\overline{\mathrm{co}(H)}} = \bigoplus_{n\geq 0} (M^{\otimes_H n})^{\overline{\mathrm{co}(H)}} \to \bigoplus_{n\geq 0} T^{\otimes n)}(V) = T(V)$$

is an isomorphism in ${}^H_H\mathcal{YD}$.

Proof Observe that $\overline{\phi}_2^{-1} = \overline{\phi}_{2,M,M}^{-1}$ defined by (2.32), and this justifies our notation. By mathematical induction on $n \geq 2$, we show that

$$\overline{\phi}_n^{-1} = (\mathrm{Id}_V^{\otimes n-2)} \otimes \overline{\phi}_{2,M,M}^{-1}) \cdots (\mathrm{Id}_V \otimes \overline{\phi}_{2,M,M^{\otimes_H n-2}}^{-1}) \overline{\phi}_{2,M,M^{\otimes_H n-1}}^{-1}.$$

Actually, $\overline{\phi}_{n+1}^{-1} = (\mathrm{Id}_V \otimes \overline{\phi}_{n-1}^{-1}) \overline{\phi}_{2,M,M^{\otimes_H n}}^{-1}$, for all $n \geq 2$, and since

$$\begin{aligned}
&\overline{\phi}_{2,M,M^{\otimes_H n}}^{-1}(m^{\otimes_H n+1}) \\
&\quad = \overline{E}(m^1_{(0)}) \otimes m^1_{(1)_1} \cdot m^2_{(0)} \otimes_H \cdots \otimes_H m^n_{(0)} \otimes_H \overline{E}(m^{n+1}) \cdot S(m^1_{(1)_2} m^2_{(1)} \cdots m^n_{(1)}),
\end{aligned}$$

for all $m^{\otimes_H n+1} \in (M^{\otimes_H n+1})^{\overline{\mathrm{co}(H)}}$, it follows that

$$\begin{aligned}
&\overline{\phi}_{n+1}^{-1}(m^{\otimes_H n+1}) \\
&\quad = \overline{E}(m^1_{(0)}) \otimes \overline{\phi}_n^{-1}(m^1_{(1)_1} \cdot m^2_{(0)} \otimes_H \cdots \otimes_H \overline{E}(m^{n+1}) \cdot S(m^1_{(1)_2} m^2_{(1)} \cdots m^n_{(1)})) \\
&\quad = \overline{E}(m^1_{(0)}) \otimes \Big(\overline{E}(m^1_{(1)_{(1,1)}} \cdot m^2_{(0,0)}) \otimes m^1_{(1)_{(1,2)}} m^2_{(0,1)} \cdot \Big(\overline{E}(m^3_{(0,0)}) \otimes m^3_{(0,1)} \cdot \\
&\qquad\quad \cdot \Big(\cdots \big(\overline{E}(m^n_{(0,0)}) \otimes m^n_{(0,1)} \triangleright \overline{E}(\overline{E}(m^{n+1}) \cdot S(m^1_{(1)_2} m^2_{(1)} \cdots m^n_{(1)}))\big) \cdots \Big)\Big)\Big) \\
&\quad = \overline{E}(m^1_{(0)}) \otimes m^1_{(1)} \cdot \Big(\overline{E}(m^2_{(0)}) \otimes m^2_{(1)} \cdot \Big(\cdots \otimes m^{n-1}_{(1)} \\
&\qquad\quad \cdot \big(\overline{E}(m^n_{(0)}) \otimes m^n_{(1)} \triangleright \overline{E}(m^{n+1}) \big) \cdots \Big)\Big)
\end{aligned}$$

for all $m^{\otimes_H n+1} \in (M^{\otimes_H n+1})^{\overline{\mathrm{co}(H)}}$, as required. It is clear at this moment that $\overline{\phi}_n^{-1}$ is an isomorphism in ${}^H_H\mathcal{YD}$, for all $n \geq 2$. Note that its inverse, denoted by $\overline{\phi}_n$, is determined by

$$\overline{\phi}_n(\overleftarrow{v^n}) = q^1 x^1_1 \cdot v^1 \cdot S(q^2 x^1_2) x^2 \otimes_H \cdots \otimes_H \mathbf{q}^1 y^1_1 \cdot v^{n-1} \cdot S(\mathbf{q}^2 y^1_2) y^2 \otimes_H v^n \cdot S(x^3 \cdots y^3),$$

for all $\overleftarrow{v^n} \in T^{n)}(V)$, where each tensor component contains a distinct copy of $q_R = q^1 \otimes q^1 = \cdots = \mathbf{q}^1 \otimes \mathbf{q}^2$; also, $\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = \cdots = y^1 \otimes y^2 \otimes y^3$ appears in the definition of $\overline{\phi}_n$ for $(n-1)$-times. □

$T_H(M)^{\overline{\mathrm{co}(H)}}$ is a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$, and this induces a Hopf algebra structure on $T(V)$ within ${}^H_H\mathcal{YD}$ as follows.

The comultiplication Δ of H is not coassociative, and this forces to introduce the notation $\Delta_{2)} = \Delta$, $\Delta_{3)} = (\mathrm{Id}_H \otimes \Delta)\Delta$ and, in general, $\Delta_{n)} = (\mathrm{Id}_H \otimes \Delta_{n-1)})\Delta$, for all $n \geq 3$. If $h \in H$ and $n \geq 2$, we denote

$$\Delta_{n)}(h) := h_{\mathbf{1}} \otimes \cdots \otimes h_{\mathbf{n}} = h_1 \otimes h_{(2,1)} \otimes \cdots \otimes h_{(\underbrace{2,\ldots,2}_{n-2},1)} \otimes h_{(\underbrace{2,\ldots,2}_{n-2},2)}.$$

Proposition 5.9 *In the hypothesis of Lemma 5.8, we have that $T(V)$ is a Hopf algebra in ${}^H_H\mathcal{YD}$ via the following structure. The multiplication, denoted by $\odot$, is given by*

$$\begin{aligned}
v^1 \odot v^2 &= v^1 \otimes v^2, \quad v^1 \odot \overleftarrow{v^{2,n+1}} = \overleftarrow{v^{n+1}},\\
\overleftarrow{v^{n}} \odot v^{n+1} &= X^1 \triangleright v^1 \otimes (Y^1 X^2_{\mathbf{1}} \triangleright v^2 \otimes (\cdots\\
&\quad \otimes (Z^1 \cdots Y^2_{\mathbf{n-3}} X^2_{\mathbf{n-2}} \triangleright v^{n-1}\\
&\quad \otimes (Z^2 \cdots Y^2_{\mathbf{n-2}} X^2_{\mathbf{n-1}} \triangleright v^n \otimes Z^3 \cdots Y^3 X^3 \triangleright v^{n+1})) \cdots)),\\
\overleftarrow{v^{m}} \odot \overleftarrow{v^{m+1,m+n}} &= X^1 \triangleright v^1 \otimes (Y^1 X^2_{\mathbf{1}} \triangleright v^2 \otimes (\cdots \otimes (Z^1 \cdots Y^2_{\mathbf{m-3}} X^2_{\mathbf{m-2}} \triangleright v^{m-1}\\
&\quad \otimes (Z^2 \cdots Y^2_{\mathbf{m-2}} X^2_{\mathbf{m-1}} \triangleright v^m \otimes Z^3 \cdots Y^3 X^3 \cdot \overleftarrow{v^{m+1,m+n}})) \cdots)),
\end{aligned}$$

for all $m, n \geq 2$ and $v^1, \ldots, v^{m+n} \in V$, and the unit equals the unit of the field k.

The comultiplication $\underline{\underline{\Delta}}$ and the counit $\underline{\underline{\varepsilon}}$ are defined, for all $\kappa \in k$ and $v \in V$, by

$$\underline{\underline{\Delta}}(\kappa) = \kappa \underline{\otimes} 1 = 1 \underline{\otimes} \kappa \quad \textit{and} \quad \underline{\underline{\Delta}}(v) = v \underline{\otimes} 1 + 1 \underline{\otimes} v,$$

and, respectively, by $\underline{\underline{\varepsilon}}(\kappa) = \kappa$ and $\underline{\underline{\varepsilon}}(v) = 0$, extended to the whole $T(V)$ as algebra morphisms in ${}^H_H\mathcal{YD}$. As before $\underline{\otimes}$ stands for the tensor product over k between $T(V)$ and itself.

The antipode $\underline{\underline{S}}$ of $T(V)$ is determined by $\underline{\underline{S}}(\kappa) = \kappa$ and $\underline{\underline{S}}(v) = -v$, for all $\kappa \in k$ and $v \in V$, extended as an anti-morphism of algebras in ${}^H_H\mathcal{YD}$ between $T(V)$ and itself.

Proof We show that the structure in the statement is the unique Hopf algebra structure on $T(V)$ within ${}^H_H\mathcal{YD}$ that turns $\overline{\phi}^{-1} : T_H(M)^{\overline{\mathrm{co}(H)}} \to T(V)$ into a braided Hopf algebra isomorphism. In this sense, the multiplication $\odot$ is given by

$$\begin{aligned}
\odot : T(V) \underline{\otimes} T(V) &\xrightarrow{\overline{\phi} \underline{\otimes} \overline{\phi}} T_H(M)^{\overline{\mathrm{co}(H)}} \underline{\otimes} T_H(M)^{\overline{\mathrm{co}(H)}} \xrightarrow{\overline{\phi}_{2,T_H(M),T_H(M)}}\\
&(T_H(M) \overline{\otimes} T_H(M))^{\overline{\mathrm{co}(H)}} \xrightarrow{\mathcal{G}(*)} T_H(M)^{\overline{\mathrm{co}(H)}} \xrightarrow{\overline{\phi}^{-1}} T(V),
\end{aligned}$$

where $*$ is the multiplication on $T_H(M)$ and $\mathcal{G}$ is the functor defined in Proposition 2.1. We have $\overline{\phi}_1 = \mathrm{Id}_V$, and therefore $v^1 \odot v^2 = v^1 \otimes v^2$, for all $v^1, v^2 \in V$.

For a generic $v \in V$, let us denote $w \otimes x^3 = q^1x^1_1 \cdot v \cdot S(q^2x^1_2)x^2 \otimes x^3$. Then, since $\overline{E}(v_{(0)}) \otimes v_{(1)} = p^1 \triangleright v \otimes p^2$, for all $v \in V$, we have

$$\begin{aligned}\overline{E}(w_{(0)}) \otimes w_{(1)} \otimes x^3 &= (q^1x^1_1)_1 \triangleright \overline{E}(v_{(0)}) \otimes (q^1x^1_1)_2 v_{(1)} S(q^2x^1_2)x^2 \otimes x^3 \\ &= (q^1x^1_1)_1 p^1 \triangleright v \otimes (q^1x^1_1)_2 p^2 S(q^2x^1_2)x^2 \otimes x^3 \\ &= q^1_1 p^1 x^1 \triangleright v \otimes q^1_2 p^2 S(q^2)x^2 \otimes x^3 = x^1 \triangleright v \otimes x^2 \otimes x^3.\end{aligned}$$

This fact allows to compute

$$\begin{aligned}&\overleftarrow{v^m} \odot \overleftarrow{v^{m+1,m+n}} \\ &= \overline{\phi}^{-1}_{m+n} \mathcal{G}(*) \overline{\phi}_{2,T_H(M),T_H(M)} \big(q^1x^1_1 \cdot v^1 \cdot S(q^2x^1_2)x^2 \\ &\quad \otimes_H \cdots \otimes_H \mathbf{q}^1 y^1_1 \cdot v^{m-1} \cdot S(\mathbf{q}^2 y^1_2) y^2 \\ &\quad \otimes_H v^m \cdot S(x^3 \cdots y^3) \otimes Q^1 z^1_1 \cdot v^{m+1} \cdot S(Q^2 z^1_2) z^2 \otimes_H \cdots \\ &\quad \otimes_H \mathbf{Q}^1 t^1_1 \cdot v^{m+n-1} \cdot S(\mathbf{Q}^2 t^1_2) t^2 \otimes_H v^{m+n} \cdot S(z^3 \cdots t^3)\big) \\ &= \overline{\phi}^{-1}_{m+n} \big(\mathfrak{q}^1 u^1_1 \cdot w^1 \otimes_H w^2 \otimes_H \cdots \otimes_H w^{m-1} \otimes_H v^m \cdot S(\mathfrak{q}^2 u^1_2 x^3 \cdots y^3) u^2 \\ &\quad \otimes_H w^{m+1} \otimes_H \cdots \otimes_H w^{m+n-1} \otimes_H v^{m+n} \cdot S(u^3 z^3 \cdots t^3)\big) \\ &= \mathfrak{q}^1_1 u^1_{(1,1)} \triangleright \overline{E}(w^1_{(0)}) \otimes (\mathfrak{q}^1_2 u^1_{(1,2)} 2 w^1_{(1)} \cdot (\overline{E}(w^2_{(0)}) \otimes w^2_{(1)} \cdot (\cdots \otimes w^{m-1}_{(1)} \cdot (\overline{E}(v^m_{(0)}) \otimes \\ &\quad v^m_{(1)} S(\mathfrak{q}^2 u^1_2 x^3 \cdots y^3) u^2 \cdot (\overline{E}(w^{m+1}_{(0)}) \otimes w^{m+1}_{(1)} \cdot (\cdots \otimes w^{m+n-2}_{(1)} \cdot (\overline{E}(w^{m+n-1}_{(0)}) \\ &\quad \otimes w^{m+n-1}_{(1)} \triangleright \overline{E}(v^{m+n} \cdot S(u^3 z^3 \cdots t^3))) \cdots))) \cdots))) \\ &= \mathfrak{q}^1_1 x^1 \triangleright v^1 \otimes (\mathfrak{q}^1_2 x^2 \cdot (y^1 \triangleright v^2 \otimes y^2 \cdot (\cdots \otimes (z^1 \triangleright v^{m-1} \otimes z^2 \cdot (p^1 \triangleright v^m \\ &\quad \otimes p^2 S(\mathfrak{q}^2 x^3 y^3 \cdots z^3) \cdot (v^{m+1} \otimes (v^{m+2} \otimes (\cdots \otimes (v^{m+n-1} \otimes v^{m+n}) \cdots))))) \cdots))),\end{aligned}$$

for all $m, n \geq 2$ and $v^1, \ldots, v^{m+n} \in V$. Now, to get the formula claimed in the statement we must apply (2.19) and (2.1) until we are able to use (2.17). We illustrate this way of computation with few examples: for all $v^1, \ldots, v^5 \in V$ we have

$$\begin{aligned}&(v^1 \otimes v^2) \odot (v^3 \otimes v^4) \\ &\quad = q^1_1 x^1 \triangleright v^1 \otimes (q^1_2 x^2 \cdot (p^1 \triangleright v^2 \otimes p^2 S(q^2x^3) \cdot (v^3 \otimes v^4))) \\ &\quad = q^1_1 x^1 \triangleright v^1 \otimes (q^1_{(2,1)} x^2_1 p^1 \triangleright v^2 \otimes q^1_{(2,2)} x^2_2 p^2 S(q^2x^3) \cdot (v^3 \otimes v^4)) \\ &\overset{(2.19),(2.1)}{=} X^1 (q^1_1 p^1)_1 \triangleright v^1 \otimes (X^2 (q^1_1 p^1)_2 \triangleright v^2 \otimes X^3 q^1_2 p^2 S(q^2) \cdot (v^3 \otimes v^4)) \\ &\overset{(2.17)}{=} X^1 \triangleright v^1 \otimes (X^2 \triangleright v^2 \otimes X^3 \cdot (v^3 \otimes v^4)),\end{aligned}$$

and similarly

$$\begin{aligned}&\overleftarrow{v^3} \odot (v^4 \otimes v^5) \\ &\quad = q^1_1 x^1 \triangleright v^1 \otimes (q^1_2 x^2 \cdot (y^1 \triangleright v^2 \otimes y^2 \cdot (p^1 \triangleright v^3 \otimes p^2 S(q^2 x^3 y^3) \cdot (v^4 \otimes v^5)))) \\ &\overset{(2.19),(2.1)}{=} q^1_1 x^1 \triangleright v^1 \\ &\qquad \otimes (q^1_2 \cdot (Y^1 (x^2_1 p^1)_1 \triangleright v^2 \otimes (Y^2 (x^1_2 p^2)_2 \triangleright v^3 \otimes Y^3 x^2_2 p^2 S(q^2 x^3) \cdot (v^4 \otimes v^5))))\end{aligned}$$

$$\overset{(2.19),(2.1)}{=} X^1(q_1^1p^1)_{\mathbf{1}} \rhd (Y^1X_1^2(q_1^1p^1)_{\mathbf{2}} \rhd v^2 \otimes (Y^2X_2^2(q_1^1p^1)_{\mathbf{3}}$$
$$\otimes Y^3X^3q_2^1p^2S(q^2)\cdot(v^4\otimes v^5)))$$
$$\overset{(2.17)}{=} X^1 \rhd (Y^1X_1^2 \rhd v^2 \otimes (Y^2X_2^2 \otimes Y^3X^3\cdot(v^4\otimes v^5))).$$

In a manner similar to the one above we can show the remaining two relations related to the definition of $\odot$, we leave the verification of this fact to the reader.

As $\overline{\phi}_0(\kappa) = \kappa\beta$, for all $\kappa \in k$, we get $\kappa \odot \kappa' = \kappa\kappa'$ and $\kappa \odot \overleftarrow{v^n} = \overleftarrow{v^n} \odot \kappa = \kappa\overleftarrow{v^n}$, for all $\kappa, \kappa' \in k$ and $v^1, \ldots, v^n \in V$. In particular, we deduce that $\odot$ is unital with unit given by the unit of k. This completes the algebra structure of $T(V)$ in ${}_H^H\mathcal{YD}$.

The coalgebra structure $(\underline{\underline{\Delta}}, \underline{\underline{\varepsilon}})$ of $T(V)$ in ${}_H^H\mathcal{YD}$ is obtained from the one of $T_H(M)^{\overline{\mathrm{co}(H)}}$ as follows: $\underline{\underline{\varepsilon}} : T(V) \xrightarrow{\overline{\phi}} T_H(M)^{\overline{\mathrm{co}(H)}} \xrightarrow{\mathcal{G}(\underline{\varepsilon})} H^{\overline{\mathrm{co}(H)}} \xrightarrow{\overline{\phi}_0^{-1}} k$ and

$$\underline{\underline{\Delta}} : T(V) \xrightarrow{\overline{\phi}} T_H(M)^{\overline{\mathrm{co}(H)}} \xrightarrow{\mathcal{G}(\underline{\Delta})} (T_H(M)\overline{\otimes}T_H(M))^{\overline{\mathrm{co}(H)}}$$
$$\xrightarrow{\overline{\phi}_{2,T_H(M),T_H(M)}^{-1}} T_H(M)^{\overline{\mathrm{co}(H)}}\underline{\otimes}T_H(M)^{\overline{\mathrm{co}(H)}} \xrightarrow{\overline{\phi}^{-1}\underline{\otimes}\overline{\phi}^{-1}} T(V)\underline{\otimes}T(V).$$

Explicitly, $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\Delta}}$ are algebra morphisms in ${}_H^H\mathcal{YD}$, completely determined by $\underline{\underline{\varepsilon}}(\kappa) = \kappa$, $\underline{\underline{\varepsilon}}(v) = 0$, $\underline{\underline{\Delta}}(\kappa) = \kappa(\overline{\phi}^{-1}\underline{\otimes}\overline{\phi}^{-1})(\beta\underline{\otimes}\beta) = \kappa\underline{\otimes}1 = 1\underline{\otimes}\kappa$ and

$$\begin{aligned}\underline{\underline{\Delta}}(v) &= (\overline{\phi}^{-1}\underline{\otimes}\overline{\phi}^{-1})(\overline{E}(v_{(0)})\overline{\otimes}\overline{E}_H(v_{(1)}) + \overline{E}_H(1)\underline{\otimes}\overline{E}(v))\\ &= (\overline{\phi}^{-1}\underline{\otimes}\overline{\phi}^{-1})(v\underline{\otimes}\beta + \beta\underline{\otimes}v)\\ &= v\underline{\otimes}1 + 1\underline{\otimes}v,\end{aligned}$$

for all $\kappa \in k$ and $v \in V$. In general, for all $n \geq 2$ and $v^1, \ldots v^n \in V$ we have $\underline{\underline{\varepsilon}}(\overleftarrow{v^n}) = 0$ and

$$\underline{\underline{\Delta}}(\overleftarrow{v^n}) = \underline{\underline{\Delta}}(v^1)(\underline{\underline{\Delta}}(v^2)(\cdots(\underline{\underline{\Delta}}(v^{n-1})\underline{\underline{\Delta}}(v^n))\cdots)),$$

where in the right hand side the product is made in the tensor product algebra $T(V)\underline{\otimes}T(V)$ within ${}_H^H\mathcal{YD}$.

Finally, the formula for the antipode $\underline{\underline{S}}$ follows from the equality $\underline{\underline{S}} = \overline{\phi}^{-1}\mathcal{G}(\underline{S})\overline{\phi}$. Note only that it extends to the whole $T(V)$ as an anti-morphism of algebras in ${}_H^H\mathcal{YD}$; thus, the braiding c of ${}_H^H\mathcal{YD}$ plays an important role in this case. □

Remark 5.10 Any object V of ${}_H^H\mathcal{YD}$ is the set of right coinvariants of a certain $M \in {}_H^H\mathcal{M}_H^H$. Thus, the braided tensor Hopf algebra construction $T(V)$ makes sense for any $V \in {}_H^H\mathcal{YD}$: the structure is the one in Proposition 5.9, of course wth the left adjoint H-action $\rhd$ replaced by the given left H-action, say $\cdot$, on V.

By the above results, we get the following.

Theorem 5.11 *Let H be a quasi-Hopf algebra, $M \in {}_H^H\mathcal{M}_H^H$ and $V = M^{\overline{\mathrm{co}(H)}}$. Then, $T_H(M)$ is isomorphic to the biproduct quasi-Hopf algebra $T(V) \times H$.*

Proof We know that $\overline{\phi}^{-1} : T_H(M)^{\overline{\mathrm{co}(H)}} \to T(V)$ is an isomorphism of Hopf algebras in ${}_H^H\mathcal{Y}D$, and therefore $\overline{\phi}^{-1} \times \mathrm{Id}_H : T_H(M)^{\overline{\mathrm{co}(H)}} \times H \to T(V) \times H$ is an isomorphism of quasi-Hopf algebras. But $\overline{\nu}_{T_H(M)}^{-1} : T_H(M) \to T_H(M)^{\overline{\mathrm{co}(H)}} \times H$ is a quasi-Hopf algebra isomorphism as well, and from here we conclude that $\Gamma := (\overline{\phi}^{-1} \times \mathrm{Id}_H)\overline{\nu}_{T_H(M)}^{-1} : T_H(M) \to T(V) \times H$ is a quasi-Hopf algebra isomorphism. More precisely, $\Gamma(h) = 1 \times h$, $\Gamma(m) = \overline{E}(m_{(0)}) \times m_{(1)}$ and

$$\Gamma(m^{\otimes_H n}) = \overline{E}(m^1_{(0,0)}) \otimes (m^1_{(0,1)} \cdot (\overline{E}(m^2_{(0,0)}) \otimes m^2_{(0,1)} \cdot (\cdots \otimes m^{n-2}_{(0,1)} \cdot (\overline{E}(m^{n-1}_{(0,0)}) \otimes m^{n-1}_{(0,1)} \triangleright \overline{E}(m^n_{(0)})) \cdots))) \times m^1_{(1)} \cdots m^n_{(1)},$$

for all $h \in H$, $m \in M$, and $n \geq 2$ and $m^1, \ldots, m^n \in M$. The inverse of Γ is Γ^{-1} given by $\Gamma^{-1}(z \times h) = q^1 \cdot \overline{\phi}(z) \cdot S(q^2)h$, for all $z \in T(V)$ and $h \in H$. □

6 An example

Denote by C_n the cyclic group of order $n \geq 2$, assume that k contains a primitive root of unity $\mathfrak{q}$ of order n^2 and take $q := \mathfrak{q}^n$, a primitive root of unity in k of order n (in particular, $n \neq 0$ in k). If g is a generator of C_n then, for any $0 \leq j \leq n-1$,

$$1_j := \frac{1}{n} \sum_{i=0}^{n-1} q^{(n-j)i} g^i$$

is an idempotent of the group algebra $k[C_n]$. Furthermore, $g 1_j = q^j 1_j$, and so $g^l 1_j = q^{lj} 1_j$, for all $0 \leq l, j \leq n-1$. This implies $1_l 1_j = \delta_{l,j} 1_l$, for all $0 \leq l, j \leq n-1$, and $\sum_{j=0}^{n-1} 1_j = \mathbf{1}$, the identity element of C_n.

For a rational number r, denote by $[r]$ the integer part of r. According to [12, Lemma 3.4] or [4, Proposition 5.1], we know that

$$\Phi := \sum_{i,j,l=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]} 1_i \otimes 1_j \otimes 1_l \tag{6.1}$$

is a non-trivial normalized 3-cocycle on C_n (in the Harrison cohomology, we refer to [4] for more details). Thus, we can endow $k[C_n]$ with a quasi-Hopf algebra structure as follows: the algebra structure is that of the group algebra $k[C_n]$, the coalgebra structure is given by

$$\Delta(g^s) = g^s \otimes g^s \ \text{ and } \ \varepsilon(g^s) = 1,$$

for all $1 \leq s \leq n-1$, the reassociator is Φ as above, and the antipode is determined by $S(g^s) = g^{n-s}$, for all $0 \leq s \leq n-1$, and distinguished elements $\alpha = g^{-1}$ and $\beta = \mathbf{1}$. Otherwise stated, the fact that $k[C_n]$ is a commutative algebra allows to view the Hopf group algebra $k[C_n]$ as a quasi-Hopf algebra with reassociator Φ. We will denote this quasi-Hopf algebra structure on $k[C_n]$ by $k_\Phi[C_n]$.

Let now V be a k-vector space. We equip V with a left Yetter–Drinfeld module structure over $k_\Phi[C_n]$, and then, we construct a Hopf algebra $T(V)$ in the braided category ${}_{k_\Phi[C_n]}^{k_\Phi[C_n]}\mathcal{YD}$. Thus, $T(V) \times k_\Phi[C_n]$ is a quasi-Hopf algebra with projection, and our goal is to compute explicitly this quasi-Hopf algebra structure.

Lemma 6.1 *With the above notation, V is a left $k_\Phi[C_n]$-Yetter–Drinfeld module via the structure given, for all $v \in V$, by*

$$g^s \cdot v = q^s v\ , \quad \forall\, 0 \leq s \leq n-1 \ \textit{and}\ \lambda_V : V \ni v \mapsto K \otimes v \in k_\Phi[C_n] \otimes V,$$

where $K := \sum_{j=0}^{n-1} \mathfrak{q}^j 1_j = \sum_{j=0}^{n-1} q^{\frac{j}{n}} 1_j$.

Proof For any $0 \leq j, l \leq n-1$, we have

$$\begin{aligned}
\Delta(K)(1_j \otimes 1_l) &= \sum_{s=0}^{n-1} q^{\frac{s}{n}} \Delta(1_s)(1_j \otimes 1_l) \\
&= \frac{1}{n} \sum_{s,i=0}^{n-1} q^{(n-s)i+\frac{s}{n}} g^i 1_j \otimes g^i 1_l \\
&= \frac{1}{n} \sum_{s=0}^{n-1} q^{\frac{s}{n}} \left(\sum_{i=0}^{n-1} q^{(j+l-s)i} \right) 1_j \otimes 1_l = \begin{cases} q^{\frac{j+l}{n}} 1_j \otimes 1_l\ , & \text{if } j+l < n \\ q^{\frac{j+l-n}{n}} 1_j \otimes 1_l\ , & \text{if } j+l \geq n. \end{cases}
\end{aligned}$$

We use this equality together with $1_j \cdot v = \frac{1}{n} \sum_{i=0}^{n-1} q^{(n-j)i} g^i \cdot v = \frac{1}{n} \sum_{i=0}^{n-1} q^{(n-j+1)i} v$ $= \delta_{j,1} v$, for all $0 \leq j \leq n-1$ and $v \in V$, to compute that

$$\begin{aligned}
&X^1 (Y^1 \cdot v)_{[-1]_1} Y^2 \otimes X^2 (Y^1 \cdot v)_{[-1]_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot v)_{[0]} \\
&= \sum_{j,l=0}^{n-1} q^{\left[\frac{j+l}{n}\right]} X^1 K_1 1_j \otimes X^2 K_2 1_l \otimes X^3 \cdot v \\
&= \sum_{j+l<n} q^{\left[\frac{j+l}{n}\right] + \frac{j+l}{n}} X^1 1_j \otimes X^2 1_l \otimes X^3 \cdot v \\
&\quad + \sum_{j+l \geq n} q^{\left[\frac{j+l}{n}\right] + \frac{j+l-n}{n}} X^1 1_j \otimes X^2 1_l \otimes X^3 \cdot v
\end{aligned}$$

$$= \sum_{j+l<n} q^{\frac{j+l}{n}+j\left[\frac{l+1}{n}\right]} 1_j \otimes 1_l \otimes v + \sum_{j+l\geq n} q^{\frac{j+l}{n}+j\left[\frac{l+1}{n}\right]} 1_j \otimes 1_l \otimes v$$

$$= \sum_{j,l=0}^{n-1} q^{\frac{j+l}{n}+j\left[\frac{l+1}{n}\right]} 1_j \otimes 1_l \otimes v,$$

for all $v \in V$. Likewise, since $1_i K = q^{\frac{i}{n}} 1_i$, for all $0 \leq i \leq n-1$, we see that

$$\begin{aligned} X^1 v_{[-1]} \otimes (X^2 \cdot v_{[0]})_{[-1]} X^3 \otimes (X^2 \cdot v_{[0]})_{[0]} &= X^1 K \otimes (X^2 \cdot v)_{[-1]} X^3 \otimes (X^2 \cdot v)_{[0]} \\ &= \sum_{j,l=0}^{n-1} q^{j\left[\frac{l+1}{n}\right]} 1_j K \otimes v_{[-1]} 1_l \otimes v_{[0]} \\ &= \sum_{j,l=0}^{n-1} q^{j\left[\frac{l+1}{n}\right]+\frac{j}{n}} 1_j \otimes K 1_l \otimes v \\ &= \sum_{j,l=0}^{n-1} q^{j\left[\frac{l+1}{n}\right]+\frac{j+l}{n}} 1_j \otimes 1_l \otimes v, \end{aligned}$$

for all $v \in V$, and this shows (2.22). The Yetter–Drinfeld condition in (2.23) is satisfied by our structure since $k_\Phi[C_n]$ is a commutative algebra. Finally, as $\varepsilon(1_j) = \delta_{j,0}$, for all $0 \leq j \leq n-1$, we deduce that $\varepsilon(K) = 1$, and this finishes our proof. □

We start to describe the braided Hopf algebra structure of $T(V)$ by computing its algebra structure in ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$. For this, we need first a lemma.

Lemma 6.2 *For any $m \geq 2$, we have*

$$\begin{aligned} &(\mathrm{Id}_{k_\Phi[C_n]} \otimes \Delta_{m)} \otimes \mathrm{Id}_{k_\Phi[C_n]})\Phi \\ &= \sum_{i,j_1,\ldots,j_m,l=0}^{n-1} q^{i\left[\frac{j_1+\cdots+j_m+l}{n}\right]-i\left[\frac{j_1+\cdots+j_m}{n}\right]} 1_i \otimes 1_{j_1} \otimes \cdots \otimes 1_{j_m} \otimes 1_l. \end{aligned}$$

Proof We prove the formula by mathematical induction on $m \geq 2$. To this end, for any natural number p we denote by p' the remainder of the division of p by n, that is $p = \left[\frac{p}{n}\right] n + p'$. Consequently, $\left[\frac{p'+l}{n}\right] = \left[\frac{p+l}{n}\right] - \left[\frac{p}{n}\right]$, for any natural numbers p, l.

We have $g^s = g^s \mathbf{1} = \sum_{a=0}^{n-1} g^s 1_a = \sum_{a=0}^{n-1} q^{as} 1_a$, for all $0 \leq s \leq n-1$, and therefore

$$\begin{aligned} \sum_{j=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]} \Delta(1_j) &= \frac{1}{n} \sum_{j,s=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]+(n-j)s} g^s \otimes g^s \\ &= \frac{1}{n} \sum_{j=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]} \left(\sum_{a,b=0}^{n-1} \left(\sum_{s=0}^{n-1} q^{(a+b-j)s} \right) 1_a \otimes 1_b \right) \end{aligned}$$

$$= \sum_{j=0}^{n-1} \left(\sum_{a+b=j} q^{i\left[\frac{(a+b)'+l}{n}\right]} 1_a \otimes 1_b + \sum_{a+b=n+j} q^{i\left[\frac{(a+b)'+l}{n}\right]} 1_a \otimes 1_b \right)$$

$$= \sum_{j_1, j_2=0}^{n-1} q^{i\left[\frac{(j_1+j_2)'+l}{n}\right]} 1_{j_1} \otimes 1_{j_2}$$

$$= \sum_{j_1, j_2=0}^{n-1} q^{i\left[\frac{j_1+j_2+l}{n}\right]-i\left[\frac{j_1+j_2}{n}\right]} 1_{j_1} \otimes 1_{j_2}$$

for all $0 \leq i, l \leq n-1$, proving the equality in the statement for $m = 2$. Using the mathematical induction and a computation similar to the one above, we get that

$$(\mathrm{Id}_{k_\Phi[C_n]} \otimes \Delta_{m+1}) \otimes \mathrm{Id}_{k_\Phi[C_n]})\Phi$$
$$= \sum_{i, j_1, \ldots, j_m, l=0}^{n-1} q^{i\left[\frac{j_1+\cdots+j_m+l}{n}\right]-i\left[\frac{j_1+\cdots+j_m}{n}\right]} 1_i \otimes 1_{j_1} \otimes \cdots \otimes 1_{j_{m-1}} \otimes \Delta(1_{j_m}) \otimes 1_l$$
$$= \sum_{i, j_1, \ldots, j_{m-1}, l=0}^{n-1} 1_i \otimes 1_{j_1} \cdots \otimes 1_{j_{m-1}}$$
$$\otimes \left(\sum_{j_m=0}^{n-1} q^{i\left[\frac{j_1+\cdots+j_m+l}{n}\right]-i\left[\frac{j_1+\cdots+j_m}{n}\right]} \Delta(1_{j_m}) \right) \otimes 1_l$$
$$= \sum_{i, j_1, \ldots, j_{m+1}, l=0}^{n-1} q^{i\left[\frac{j_1+\cdots+j_{m-1}+(j_m+j_{m+1})'+l}{n}\right]-i\left[\frac{j_1+\cdots+j_{m-1}+(j_m+j_{m+1})'}{n}\right]} 1_i$$
$$\otimes 1_{j_1} \cdots \otimes 1_{j_{m+1}} \otimes 1_l$$
$$= \sum_{i, j_1, \ldots, j_{m+1}, l=0}^{n-1} q^{i\left[\frac{j_1+\cdots+j_{m+1}+l}{n}\right]-i\left[\frac{j_1+\cdots+j_{m+1}}{n}\right]} 1_i \otimes 1_{j_1} \cdots \otimes 1_{j_{m+1}} \otimes 1_l,$$

as needed. □

We can describe now the monoidal algebra structure of $T(V)$.

Proposition 6.3 *Let V be the left $k_\Phi[C_n]$-Yetter–Drinfeld module defined in Lemma 6.1. Then, $T(V)$ is a left $k_\Phi[C_n]$-Yetter–Drinfeld module via the structure given by*

$$g^s \cdot \kappa = \kappa, \quad g^s \cdot \overleftarrow{v^m} = q^{sm} \overleftarrow{v^m}, \quad \overleftarrow{v^m} \mapsto g^{\left[\frac{m}{n}\right]} K^m \otimes \overleftarrow{v^m} = K^{m+n\left[\frac{m}{n}\right]} \otimes \overleftarrow{v^m},$$

for all $0 \leq s \leq n-1$, $\kappa \in k$, $m \geq 1$ and $v^1, \ldots, v^m \in V$.

Furthermore, $T(V)$ is an algebra in ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$ via the multiplication $\odot$ determined by

$$v^1 \odot v^2 = v^1 \otimes v^2, \quad v^1 \odot \overleftarrow{v^{2,m+1}} = \overleftarrow{v^{m+1}}, \quad \overleftarrow{v^m} \odot v^{m+1} = q^{\left[\frac{m}{n}\right]}\overleftarrow{v^{m+1}},$$
$$\overleftarrow{v^m} \odot \overleftarrow{v^{m+1,m+p}} = q^{(m+p)\left[\frac{m-1+p'}{n}\right]-m\left[\frac{m}{n}\right]}\overleftarrow{v^{m+p}} = q^{m\left[\frac{m'+p'}{n}\right]+p\left[\frac{m+p'}{n}\right]}\overleftarrow{v^{m+p}},$$

for all $m, p \geq 2$ and $v^1, \ldots, v^{m+p} \in V$, and is unital with unit equals the unit of k.

Proof We specialize the structure in Proposition 5.9 for $H = k_\Phi[C_n]$ and V as in Lemma 6.1. As any element of C_n is grouplike, it follows from the monoidal structure on ${}_{k_\Phi[C_n]}\mathcal{M}$ that $T(V)$ is a left $k_\Phi[C_n]$-module via the action $\cdot$ defined above.

We prove the formula concerning the left $k_\Phi[C_n]$-coaction on $T(V)$ by mathematical induction on $m \geq 1$. Let us start by noting that, for all $m \geq 2$ and $1 \leq l \leq n-1$, we have

$$1_l \cdot \overleftarrow{v^m} = \delta_{l,m'}\overleftarrow{v^m}. \tag{6.2}$$

For $m = 1$, we recover the formula for the left $k_\Phi[C_n]$-coaction on V. If we assume that it is true for $m \geq 1$ and for any m elements of V, then it is also true for any $m+1$ elements $v^1, \cdots, v^{m+1} \in V$, since, by (2.24) and the fact that $1_j \cdot v = \delta_{j,1}v$, for all $0 \leq j \leq n-1$ and $v \in V$, we have

$$\begin{aligned}
\overleftarrow{v^{m+1}} &\mapsto \sum_{j,l=0}^{n-1} q^{\left[\frac{j+l}{n}\right]} X^1(x^1 \cdot v^1)_{[-1]}x^2(1_j \cdot \overleftarrow{v^{2,m+1}})_{[-1]}1_l \\
&\quad \otimes (X^2 \cdot (x^1 \cdot v^1)_{[0]} \otimes X^3x^3 \cdot (1_j \cdot \overleftarrow{v^{2,m+1}})_{[0]}) \\
&\overset{(6.2)}{=} \sum_{l=0}^{n-1} q^{\left[\frac{l+m'}{n}\right]} X^1(x^1 \cdot v^1)_{[-1]}x^2 g^{\left[\frac{m}{n}\right]}K^m 1_l \otimes (X^2 \cdot (x^1 \cdot v^1)_{[0]} \otimes X^3x^3 \cdot \overleftarrow{v^{2,m+1}}) \\
&\overset{(*_1)}{=} \sum_{l=0}^{n-1} g^{\left[\frac{m}{n}\right]}K^{m+1}X^1 1_l \otimes (X^2 \cdot v^1 \otimes X^3 \cdot \overleftarrow{v^{2,m+1}}) \\
&= \sum_{l=0}^{n-1} q^{l\left[\frac{1+m'}{n}\right]} g^{\left[\frac{m}{n}\right]}K^{m+1}1_l \otimes \overleftarrow{v^{m+1}} \\
&\overset{(*_2)}{=} g^{\left[\frac{m}{n}\right]+\left[\frac{m'+1}{n}\right]}K^{m+1} \otimes \overleftarrow{v^{m+1}} = g^{\left[\frac{m+1}{n}\right]}K^{m+1} \otimes \overleftarrow{v^{m+1}},
\end{aligned}$$

as required. In $(*_1)$, we used that $\Phi^{-1} = \sum_{i,j,l=0}^{n-1} q^{-i\left[\frac{j+l}{n}\right]}1_i \otimes 1_j \otimes 1_l$, and in $(*_2)$ the facts that $g^a 1_l = q^{la}1_l$, for all $a \in \mathbb{N}$ and $0 \leq l \leq n-1$, and $\sum_{l=0}^{n-1} 1_l = \mathbf{1}$. We have also that $K^a = \sum_{l=0}^{n-1} q^{\frac{al}{n}} 1_l$, for all $a \in \mathbb{N}$, and therefore $K^n = \sum_{l=0}^{n-1} q^l 1_l = g$. This implies $g^a K^b = K^{na+b}$, for all $a, b \in \mathbb{N}$, proving the second formula for the left $k_\Phi[C_n]$-coaction on $T(V)$ claimed in the statement.

Now, the first two relations defining $\odot$ follow directly from the definition of $\odot$, while the third one can be derived from $1_j \cdot v = \delta_{j,1}v$, for all $0 \leq j \leq n-1$ and $v \in V$, and the formula in Lemma 6.2 as follows:

$$
\begin{aligned}
\overleftarrow{v^{m}} \odot v^{m+1} &= X^1 \cdot v^1 \otimes (Y^1 X^2_{\mathbf{1}} \cdot v^2 \otimes (\cdots \otimes (Z^1 \cdots Y^2_{\mathbf{m-3}} X^2_{\mathbf{m-2}} \cdot v^{m-1} \\
&\quad \otimes (Z^2 \cdots Y^2_{\mathbf{m-2}} X^2_{\mathbf{m-1}} \cdot v^m \otimes Z^3 \cdots Y^3 X^3 \cdot v^{m+1})) \cdots)) \\
&= q^{\left[\frac{m}{n}\right]-\left[\frac{m-1}{n}\right]} v^1 \otimes (Y^1 \cdot v^2 \otimes (T^1 Y^2_{\mathbf{1}} \cdot v^3 \\
&\qquad \otimes (\cdots \otimes (Z^1 \cdots T^2_{\mathbf{m-4}} Y^2_{\mathbf{m-3}} \cdot v^{m-1} \\
&\quad \otimes (Z^2 \cdots T^2_{\mathbf{m-3}} Y^2_{\mathbf{m-2}} \cdot v^m \otimes Z^3 \cdots T^3 Y^3 \cdot v^{m+1})) \cdots)) \\
&= q^{\left[\frac{m}{n}\right]-\left[\frac{m-1}{n}\right]+\left[\frac{m-1}{n}\right]-\left[\frac{m-2}{n}\right]} v^1 \otimes (v^2 \otimes (T^1 \cdot v^3 \otimes (\cdots \\
&\quad \otimes (Z^1 \cdots T^2_{\mathbf{m-4}} \cdot v^{m-1} \otimes (Z^2 \cdots T^2_{\mathbf{m-3}} \cdot v^m \otimes Z^3 \cdots T^3 \cdot v^{m+1})) \cdots)) \\
&= q^{\left[\frac{m}{n}\right]-\left[\frac{m-1}{n}\right]+\left[\frac{m-1}{n}\right]-\left[\frac{m-2}{n}\right]+\cdots+\left[\frac{3}{n}\right]-\left[\frac{2}{n}\right]} \\
&\quad v^1 \otimes (v^2 \otimes (\cdots \otimes (v^{m-2} \otimes (Z^1 \cdot v^{m-1} \otimes (Z^2 \cdot v^m \otimes Z^3 \cdot v^{m+1}))) \cdots)) \\
&= q^{\left[\frac{m}{n}\right]} \overleftarrow{v^{m+1}}.
\end{aligned}
$$

The proof of the fourth relation involving $\odot$ is quite technical. Note that (6.2) implies

$$
\begin{aligned}
&\overleftarrow{v^{m}} \odot \overleftarrow{v^{m+1,m+p}} \\
&= X^1 \cdot v^1 \otimes (Y^1 X^2_{\mathbf{1}} \cdot v^2 \otimes (\cdots \otimes (Z^1 \cdots Y^2_{\mathbf{m-3}} X^2_{\mathbf{m-2}} \cdot v^{m-1} \\
&\quad \otimes (Z^2 \cdots Y^2_{\mathbf{m-2}} X^2_{\mathbf{m-1}} \cdot v^m \\
&\quad \otimes Z^3 \cdots Y^3 X^3 \cdot (v^{m+1} \otimes (v^{m+2} \otimes (\cdots \otimes (v^{m+p-1} \otimes v^{m+p}) \cdots))))) \cdots)) \\
&= \sum_{l_1,\ldots,l_{m-2}=0}^{n-1} q^{\left[\frac{m-1+l_1}{n}\right]-\left[\frac{m-1}{n}\right]+\left[\frac{m-2+l_2}{n}\right]-\left[\frac{m-2}{n}\right]+\cdots+\left[\frac{2+l_{m-2}}{n}\right]-\left[\frac{2}{n}\right]} \\
&\quad v^1 \otimes (v^2 \otimes (\cdots \otimes (v^{m-2} \otimes (Z^1 \cdot v^{m-1} \otimes (Z^2 \cdot v^m \\
&\quad \otimes Z^3 1_{l_{m-2}} \cdots 1_{l_1} \cdot \Big(v^{m+1} \\
&\quad \otimes (v^{m+2} \otimes (\cdots \otimes (v^{m+p-1} \otimes v^{m+p}) \cdots))\Big)))) \cdots)) \\
&= q^{\left[\frac{m-1+p'}{n}\right]-\left[\frac{m-1}{n}\right]+\left[\frac{m-2+p'}{n}\right]-\left[\frac{m-2}{n}\right]+\cdots+\left[\frac{2+p'}{n}\right]-\left[\frac{2}{n}\right]} v^1 \\
&\quad \otimes (v^2 \otimes (\cdots \otimes (v^{m-2} \otimes (Z^1 \cdot v^{m-1} \\
&\quad \otimes (Z^2 \cdot v^m \otimes Z^3 \cdot \Big(v^{m+1} \otimes (v^{m+2} \otimes (\cdots \otimes (v^{m+p-1} \otimes v^{m+p}) \cdots))\Big)))) \cdots)) \\
&= q^{\left[\frac{m-1+p'}{n}\right]-\left[\frac{m-1}{n}\right]+\left[\frac{m-2+p'}{n}\right]-\left[\frac{m-2}{n}\right]+\cdots+\left[\frac{1+p'}{n}\right]-\left[\frac{1}{n}\right]} \overleftarrow{v^{m+p}}.
\end{aligned}
$$

This leads to the first formula for the $\odot$ mentioned above, since

$$
\begin{aligned}
\left[\frac{1}{n}\right] + \cdots + \left[\frac{a}{n}\right] &= (1 + 2 + \cdots + \left[\frac{a}{n}\right] - 1)n + (a' + 1)\left[\frac{a}{n}\right] \\
&= (a+1)\left[\frac{a}{n}\right] - \left[\frac{a}{n}\right]\left(\left[\frac{a}{n}\right] + 1\right)\frac{n}{2},
\end{aligned}
$$

for any nonzero natural number a, and therefore

$$\begin{aligned}&\left[\frac{p'+1}{n}\right]+\cdots+\left[\frac{p'+m-1}{n}\right]\\&=\left[\frac{p+1}{n}\right]+\cdots+\left[\frac{p+m-1}{n}\right]-(m-1)\left[\frac{p}{n}\right]\\&=(p+m)\left[\frac{p+m-1}{n}\right]-\left[\frac{p+m-1}{n}\right]\left(\left[\frac{p+m-1}{n}\right]+1\right)\frac{n}{2}\\&\quad-(p+1)\left[\frac{p}{n}\right]+\left[\frac{p}{n}\right]\left(\left[\frac{p}{n}\right]+1\right)\frac{n}{2}-(m-1)\left[\frac{p}{n}\right]\end{aligned}$$

and

$$\begin{aligned}\left[\frac{1}{n}\right]+\cdots+\left[\frac{m-1}{n}\right]&=(m+1)\left[\frac{m}{n}\right]-\left[\frac{m}{n}\right]\left(\left[\frac{m}{n}\right]+1\right)\frac{n}{2}-\left[\frac{m}{n}\right]\\&=m\left[\frac{m}{n}\right]-\left[\frac{m}{n}\right]\left(\left[\frac{m}{n}\right]+1\right)\frac{n}{2}.\end{aligned}$$

The second formula involving the product $v^{\overleftarrow{m}}\odot v^{\overleftarrow{m+1,m+p}}$ follows from the first one and the fact that

$$a\left(\left[\frac{a}{n}\right]-\left[\frac{a-1}{n}\right]\right)\equiv 0 \bmod n\ ,\ \forall\, a\in\mathbb{N}. \tag{6.3}$$

So our proof is finished. □

Next, we complete the algebra structure on $T(V)$ up to a braided Hopf algebra one, making the coalgebra structure of it explicit in terms of the braid group action. Recall that for $1\le l\le m-1$ by $S_{l,m-l}$ we denoted the set of $(l,m-l)$-shuffles. We extend this notation to $0\le l\le m$, by defining $S_{0,m}=\{e\}=S_{m,0}$, where e is the identity permutation of S_m. In what follows, by S_m we understand the symmetric group of $\{2,\ldots,m+1\}$. Finally, the length of a permutation $\sigma\in S_{l,m-l}$ is the length of any reduced expression for σ in terms of the generators $s_l=(l,l+1)$, $1\le l\le m-1$. We will denote it by $r(\sigma)$; by convention, $r(e)=0$.

Proposition 6.4 *The algebra $T(V)$ in ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$ built in Proposition 6.3 admits a Hopf algebra structure in the braided category ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$. The coalgebra structure is defined by the comultiplication $\underline{\underline{\Delta}}$, $\underline{\underline{\Delta}}(\kappa)=\kappa\underline{\otimes}1=1\underline{\otimes}\kappa$, for all $\kappa\in k$, and*

$$\begin{aligned}\underline{\underline{\Delta}}(v^{\overleftarrow{m}})=\sum_{l=0}^{m}\sum_{\sigma^{-1}\in S_{l,m-l}}q^{\frac{r(\sigma^{-1})}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]}\\ v^{\sigma(1)}\otimes(\cdots\otimes(v^{\sigma(l-1)}\otimes v^{\sigma(l)})\cdots)\underline{\otimes}v^{\sigma(l+1)}\otimes(\cdots\otimes(v^{\sigma(m-1)}\otimes v^{\sigma(m)})\cdots),\end{aligned} \tag{6.4}$$

for all $m \geq 1$ and $v^1, \ldots, v^m \in V$ (if a component of the $\underline{\otimes}$ monomial does not make sense then it is equal to the unit of k), and counit $\underline{\underline{\varepsilon}}$ determined by $\underline{\underline{\varepsilon}}(\kappa) = \kappa$, for all $\kappa \in k$, and $\underline{\underline{\varepsilon}}(v^1 \otimes (\cdots \otimes (v^{m-1} \otimes v^m)\cdots)) = 0$, for all $m \geq 1$ and $v^1, \ldots, v^m \in V$.

The antipode $\underline{\underline{S}}$ of $T(V)$ is completely determined by $\underline{\underline{S}}(\kappa) = \kappa$, for all $\kappa \in k$, and

$$\underline{\underline{S}}(v^{\overleftarrow{m}}) = (-1)^m q^{\frac{m(m-1)}{2n}} v^m \otimes (\cdots \otimes (v^2 \otimes v^1)\cdots), \tag{6.5}$$

for all $m \geq 1$ and $v^1, \ldots, v^m \in V$.

Proof We specialize Proposition 5.9 for $H = k_\Phi[C_n]$ and V as in Lemma 6.1. The defining relations for $\underline{\underline{\varepsilon}}$ are immediate, as well as that for $\underline{\underline{\Delta}}$ restricted to k. We prove now by mathematical induction on $m \geq 1$ that $\underline{\underline{\Delta}}$ restricted to $T^{\otimes m)}(V)$ has the form stated in (6.4). For $m = 1$, this reduces to $\underline{\underline{\Delta}}(v) = v\underline{\otimes}1 + 1\underline{\otimes}v$, for all $v \in V$, which is just the definition of $\underline{\underline{\Delta}}$ restricted to V. To see that m implies $m+1$, we proceed as follows. Firstly, from (3.4) and $K \cdot (v^2 \otimes (\cdots \otimes (v^m \otimes v^{m+1})\cdots)) = q^{\frac{m'}{n}} v^2 \otimes (\cdots \otimes (v^m \otimes v^{m+1})\cdots)$, for all $m \geq 1$ and $v^2, \ldots, v^{m+1} \in V$, we get that

$$c_{T(V),T(V)}(v^1 \underline{\otimes} v^{\overleftarrow{2,m+1}}) = q^{\frac{m'}{n}} v^{\overleftarrow{2,m+1}} \underline{\otimes} v^1,$$

for all $m \geq 1$ and $v^1, \ldots, v^m \in V$. Secondly, by the definition of Φ and the above formula for c we deduce that

$$(1\underline{\otimes}v^1)(v^{\overleftarrow{2,m+1}}\underline{\otimes}v^{\overleftarrow{m+2,m+p+1}}) = q^{m'\left[\frac{p'+1}{n}\right]-\left[\frac{m'+p'}{n}\right]+\frac{m'}{n}} v^{\overleftarrow{2,m+1}}\underline{\otimes}(v^1 \otimes v^{\overleftarrow{m+2,m+p+1}}),$$
$$(v^1\underline{\otimes}1)(v^{\overleftarrow{2,m+1}}\underline{\otimes}v^{\overleftarrow{m+2,m+p+1}}) = q^{-\left[\frac{m'+p'}{n}\right]} v^{\overleftarrow{m+1}}\underline{\otimes}v^{\overleftarrow{m+2,m+p+1}},$$

for all $m, p \geq 1$ and $v^1, \ldots, v^{m+p} \in V$. Once more, the product is made in the tensor product algebra $T(V)\underline{\otimes}T(V)$, built within the braided category ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$.

Now we use that $\underline{\underline{\Delta}}$ is an algebra morphism in ${}^{k_\Phi[C_n]}_{k_\Phi[C_n]}\mathcal{YD}$ and the mathematical induction to compute that

$$\begin{aligned}
\underline{\underline{\Delta}}(v^{\overleftarrow{m+1}}) &= \underline{\underline{\Delta}}(v^1)\underline{\underline{\Delta}}(v^{\overleftarrow{2,m+1}}) \\
&= \sum_{l=0}^{m} \sum_{\sigma^{-1}\in S_{l,m-l}} q^{\frac{r(\sigma^{-1})}{n}+l\left[\frac{l-l}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]}(v^1\underline{\otimes}1 + 1\underline{\otimes}v^1) \\
&\quad \left(v^{\sigma(2)} \otimes (\cdots \otimes (v^{\sigma(l)} \otimes v^{\sigma(l+1)})\cdots)\underline{\otimes}v^{\sigma(l+2)}\right. \\
&\quad \left.\otimes(\cdots \otimes (v^{\sigma(m)} \otimes v^{\sigma(m+1)})\cdots)\right) \\
&= \sum_{l=0}^{m} \sum_{\sigma^{-1}\in S_{l,m-l}} q^{\frac{r(\sigma^{-1})}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]-\left[\frac{l'+(m-l)'}{n}\right]} \\
&\quad v^1 \otimes (v^{\sigma(2)} \otimes (\cdots \otimes (v^{\sigma(l)} \otimes v^{\sigma(l+1)})\cdots))\underline{\otimes}v^{\sigma(l+2)}
\end{aligned}$$

$$\otimes(\cdots\otimes(v^{\sigma(m)}\otimes v^{\sigma(m+1)})\cdots)$$

$$+\sum_{l=0}^{m}\sum_{\sigma^{-1}\in S_{l,m-l}} q^{\frac{r(\sigma^{-1})+l'}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]-\left[\frac{l'+(m-l)'}{n}\right]+l'\left[\frac{1+(m-l)'}{n}\right]}$$

$$v^{\sigma(2)}\otimes(\cdots\otimes(v^{\sigma(l)}\otimes v^{\sigma(l+1)})\cdots)\underline{\otimes}v^{1}\otimes(v^{\sigma(l+2)}$$

$$\otimes(\cdots\otimes(v^{\sigma(m)}\otimes v^{\sigma(m+1)})\cdots)).$$

So we have two double sums, each of them having 2^m summands. Now, for the first double sum we can write its general term under the form

$$q^{E_1}v^{\sigma_1(1)}\otimes(\cdots\otimes(v^{\sigma_1(l)}\otimes v^{\sigma_1(l+1)})\cdots)\underline{\otimes}v^{\sigma_1(l+2)}\otimes(\cdots\otimes(v^{\sigma_1(m)}\otimes v^{\sigma_1(m+1)})\cdots),$$

where $\sigma_1^{-1} := \begin{pmatrix}1 & 2 & \cdots & m+1\\ 1 & \sigma^{-1}(2) & \cdots & \sigma^{-1}(m+1)\end{pmatrix}$. It is clear that $\sigma_1^{-1}\in S_{l+1,m-l}$ with $r(\sigma_1^{-1})=r(\sigma^{-1})$. Also,

$$\begin{aligned}E_1 &= \frac{r(\sigma^{-1})}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]-\left[\frac{l'+(m-l)'}{n}\right]\\ &= \frac{r(\sigma_1^{-1})}{n}+l\left[\frac{l-1}{n}\right]+\left[\frac{l}{n}\right]-m\left[\frac{m-1}{n}\right]-\left[\frac{m}{n}\right]+(m+1)\left[\frac{m-l}{n}\right]\\ &\equiv \frac{r(\sigma_1^{-1})}{n}+(l+1)\left[\frac{l}{n}\right]-(m+1)\left[\frac{m}{n}\right]+(m+1)\left[\frac{m-l}{n}\right] \pmod n,\end{aligned}$$

where the congruence modulo n is due to (6.3).

Analogously, we can write a summand of the second double sum under the general form

$$q^{E_2}v^{\sigma_2(1)}\otimes(\cdots\otimes(v^{\sigma_2(l-1)}\otimes v^{\sigma_2(l)})\cdots)\underline{\otimes}v^{\sigma_2(l+1)}\otimes(\cdots\otimes(v^{\sigma_2(m)}\otimes v^{\sigma_2(m+1)})\cdots),$$

with $\sigma_2^{-1}=\begin{pmatrix}1 & \cdots & l & l+1 & l+2 & \cdots & m+1\\ \sigma^{-1}(2) & \cdots & \sigma^{-1}(l+1) & 1 & \sigma^{-1}(l+2) & \cdots & \sigma^{-1}(m+1)\end{pmatrix}\in S_{l,m+1-l}$.

By [17, Lemma 4.7], for $w\in S_n$ and $1\le l\le n-1$ we have $r(ws_l)=r(w)+1$ if and only if $w(l)<w(l+1)$. By using inductively this result and the fact that

$$\sigma_2^{-1}=\begin{pmatrix}1 & 2 & \cdots & m+1\\ 1 & \sigma^{-1}(2) & \cdots & \sigma^{-1}(m+1)\end{pmatrix}s_1\cdots s_l\in S_{m+1}$$

we get that $r(\sigma_2^{-1})=r(\sigma^{-1})+l$, and consequently, a reduced expression for σ_2^{-1} can be obtained by multiplying to the right a reduced expression for σ^{-1} with $s_1\cdots s_l$ in S_{m+1}. Hence, we have that

$$E_2=\frac{r(\sigma^{-1})+l'}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]$$

$$
\begin{aligned}
&\quad - \left[\frac{l' + (m-l)'}{n}\right] + l'\left[\frac{1 + (m-l)'}{n}\right]\\
&\equiv \frac{r(\sigma^{-1}) + l}{n} + l\left[\frac{l-1}{n}\right] - m\left[\frac{m-1}{n}\right]\\
&\quad + (m-l+1)\left[\frac{m-l}{n}\right] - \left[\frac{m}{n}\right] + l\left[\frac{m-l+1}{n}\right] \pmod n\\
&\equiv \frac{r(\sigma_2^{-1})}{n} + l\left[\frac{l-1}{n}\right] - (m+1)\left[\frac{m}{n}\right] + (m+1)\left[\frac{m-l+1}{n}\right] \pmod n.
\end{aligned}
$$

Otherwise stated, we have proved that all the summands of the two double sums considered above are also summands of the double sum

$$
\sum_{l=0}^{m+1} \sum_{\theta^{-1}\in S_{l,m-l+1}} q^{\frac{r(\theta^{-1})}{n} + l\left[\frac{l-1}{n}\right] - (m+1)\left[\frac{m}{n}\right] + (m+1)\left[\frac{m-l+1}{n}\right]}
$$
$$
v^{\theta(1)} \otimes (\cdots \otimes (v^{\theta(l-1)} \otimes v^{\theta(l)}) \cdots) \underline{\otimes} v^{\theta(l+1)} \otimes (\cdots \otimes (v^{\theta(m)} \otimes v^{\theta(m+1)}) \cdots),
$$

which means that the latter double sum contains the two mentioned double sums. Actually, it is the sum of the two because in both cases we have 2^{m+1} summands. This completes the induction.

Finally, by definition $\underline{\underline{S}}(\kappa) = \kappa$, for all $\kappa \in k$. We have $\underline{\underline{S}}(v) = -v$, for all $v \in V$, and

$$
\underline{\underline{S}}(v^{\overleftarrow{m+1}}) = -q^{\frac{m'}{n}} \underline{\underline{S}}(v^{\overleftarrow{2,m+1}}) \odot v^1,
$$

for all $m \geq 1$ and $v^1, \ldots, v^{m+1} \in V$. Thus, the formula in (6.5) is a consequence of the mathematical induction and of the explicit definition of $\odot$ in the statement. □

By using the biproduct quasi-Hopf algebra construction, to the triple $(V, C_n, \mathfrak{q})$ we associate a quasi-Hopf algebra with projection $H(n, q, V) := T(V) \times k_\Phi[C_n]$, where $q = \mathfrak{q}^n$ and Φ is as in (6.1). We next describe this structure.

Recall that, for $\kappa \in k\backslash\{0\}$ and $a \in \mathbb{N}\backslash\{0\}$, $(a)_\kappa := \sum_{j=0}^{a-1} \kappa^j = \begin{cases} a & \text{, if } \kappa = 1 \\ \frac{\kappa^a - 1}{\kappa - 1} & \text{, if } \kappa \neq 1 \end{cases}$.

If $v^{\overleftarrow{m}} = v^1 \otimes (v^2 \otimes (\cdots \otimes (v^{m-1} \otimes v^m) \cdots))$ then $v^{\overleftarrow{m}_\tau} := v^m \otimes (v^{m-1} \otimes (\cdots \otimes (v^2 \otimes v^1) \cdots))$. Also, the Heaviside symbol $[i > j]$ stands for the integer 1 if $i > j$ and for 0 otherwise.

Theorem 6.5 *Let k be a field containing a primitive root of unity $\mathfrak{q}$ of degree n^2, $n \geq 2$, V a k-vector space and C_n the cyclic group of order n generated by g. If $q = \mathfrak{q}^n$ then the quasi-Hopf algebra structure of $H(n, q, V) = T(V) \times k_\Phi[C_n]$ is the following.*

The multiplication is given by

$$(\overleftarrow{v^{m}} \times g^{s})(\overleftarrow{v^{m+1,m+p}} \times g^{t}) = q^{p\left(s+\left[\frac{m+p'}{n}\right]\right)+m\left[\frac{m'+p'}{n}\right]}$$
$$\overleftarrow{v^{m+p}} \times \left(\left(1-\frac{p'}{n}+\frac{p'}{n}q^{-m}\right)g^{s+t}+\frac{1-q^{-m}}{n}\sum_{i=1}^{n-1}\left(1-(p'+1)_{q^{i}}\right)g^{i+s+t}\right),$$

for all $m, p \in \mathbb{N}$ *and* $0 \le s, t \le n-1$*, where, by convention,* $\overleftarrow{v^{0}} = 1$*, the unit of* k*. It is unital with unit* $1 \times \mathbf{1}$*.*

The comultiplication Δ *is completely determined by*

$$\Delta(\overleftarrow{v^{m}} \times g^{j}) = q^{-m\left[\frac{m-1}{n}\right]}\sum_{l=0}^{m} q^{l\left[\frac{l-1}{n}\right]+(m-l)\left[\frac{m-l}{n}\right]}\sum_{\sigma^{-1}\in S_{l,m-l}} q^{\frac{r(\sigma^{-1})}{n}}$$
$$\sum_{s,t=0}^{n-1} q^{\frac{(m-l)s}{n}+(l+s)\left[\frac{m-l+t}{n}\right]}$$
$$q^{(s+t)j-m\left[\frac{s+t}{n}\right]} v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)})\cdots) \times 1_{s}$$
$$\underline{\otimes} v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)})\cdots) \times 1_{t},$$

and is counital with counit given by $\underline{\underline{\varepsilon}}(\overleftarrow{v^{m}} \times g^{j}) = \delta_{m,0}\overleftarrow{v^{m}}$*, for all* $m \in \mathbb{N}$*,* $j \in \{0,\dots,n-1\}$ *and* $v^{1},\dots,v^{m} \in V$*.*

The antipode s *is defined, for all* $v^{1},\dots,v^{m} \in V$ *and* $0 \le l \le n-1$*, by*

$$s(\overleftarrow{v^{m}} \times g^{l}) = (-1)^{m} q^{-\frac{m(m+1)}{2n}-ml}\,\overleftarrow{v^{m_{\tau}}} \times \left(\sum_{i=0}^{n-1} q^{-i\left[\frac{m}{n}\right]-\frac{im}{n}-i(l+[i>n-m'])}1_{i}\right).$$

The distinguished elements α *and* β *that together with* s *define the antipode of* $H(n,q,V)$ *are* $1 \times g^{-1}$ *and* $1 \times \mathbf{1}$*, respectively.*

Proof We have

$$\begin{aligned}(1_{j})_{1} \cdot \overleftarrow{v^{m+1,m+p}} \times (1_{j})_{2}1_{l} &= \frac{1}{n}\sum_{i=0}^{n-1} q^{(n-j)i} g^{i} \cdot \overleftarrow{v^{m+1,m+p}} \times g^{i}1_{l}\\ &= \frac{1}{n}\sum_{i=0}^{n-1} q^{(n-j)i+pi+li}\,\overleftarrow{v^{m+1,m+p}} \times 1_{l}\\ &= \delta_{j,(p+l)'}\overleftarrow{v^{m+1,m+p}} \times 1_{l},\end{aligned}$$

and therefore

$$(\overleftarrow{v^{m}} \times 1_{j})(\overleftarrow{v^{m+1,m+p}} \times 1_{l}) = (x^{1} \cdot \overleftarrow{v^{m}}) \odot (x^{2}(1_{j})_{1} \cdot \overleftarrow{v^{m+1,m+p}}) \times x^{3}(1_{j})_{2}1_{l}$$
$$= \delta_{j,(p+l)'}(x^{1} \cdot \overleftarrow{v^{m}}) \odot (x^{2} \cdot \overleftarrow{v^{m+1,m+p}}) \times x^{3}1_{l}$$

$$= \delta_{j,(p+l)'} q^{-m'\left[\frac{p'+l}{n}\right]} \overleftarrow{v^{m}} \odot \overleftarrow{v^{m+1,m+p}} \times 1_l$$
$$= \delta_{j,(p+l)'} q^{m\left[\frac{m'+p'}{n}\right]+p\left[\frac{m+p'}{n}\right]-m\left[\frac{p'+l}{n}\right]} \overleftarrow{v^{m+p}} \times 1_l.$$

From here, we get

$$\begin{aligned}(\overleftarrow{v^{m}} \times g^s)(\overleftarrow{v^{m+1,m+p}} \times g^t) &= \sum_{j,l=0}^{n-1} q^{sj+tl} (\overleftarrow{v^{m}} \times 1_j)(\overleftarrow{v^{m+1,m+p}} \times 1_l) \\ &= \sum_{l=0}^{n-1} q^{s(p+l)'+tl+m\left[\frac{m'+p'}{n}\right]+p\left[\frac{m+p'}{n}\right]-m\left[\frac{p'+l}{n}\right]} \overleftarrow{v^{m+p}} \times 1_l \\ &= q^{p\left(s+\left[\frac{m+p'}{n}\right]\right)+m\left[\frac{m'+p'}{n}\right]} \overleftarrow{v^{m+p}} \times \left(\sum_{l=0}^{n-1} q^{-m\left[\frac{p'+l}{n}\right]} 1_l\right) g^{s+t},\end{aligned}$$

and since

$$\begin{aligned}\sum_{l=0}^{n-1} q^{-m\left[\frac{p'+l}{n}\right]} 1_l &= \sum_{l=0}^{n-p'-1} 1_l + q^{-m} \sum_{l=n-p'}^{n-1} 1_l \\ &= q^{-m}\mathbf{1} + \frac{1-q^{-m}}{n} \sum_{i=0}^{n-1} \left(\sum_{l=0}^{n-p'-1} (q^i)^{n-l}\right) g^i \\ &= \left(1 - \frac{p'}{n} + \frac{p'}{n} q^{-m}\right)\mathbf{1} + \frac{1-q^{-m}}{n} \sum_{i=1}^{n-1} \left(1-(p'+1)_{q^i}\right) g^i\end{aligned}$$

we conclude that

$$\begin{aligned}&(\overleftarrow{v^{m}} \times g^s)(\overleftarrow{v^{m+1,m+p}} \times g^t) \\ &\quad = q^{p\left(s+\left[\frac{m+p'}{n}\right]\right)+m\left[\frac{m'+p'}{n}\right]} \overleftarrow{v^{m+p}} \times \left(\left(1 - \frac{p'}{n} + \frac{p'}{n} q^{-m}\right) g^{s+t}\right. \\ &\qquad \left. + \frac{1-q^{-m}}{n} \sum_{i=1}^{n-1} \left(1-(p'+1)_{q^i}\right) g^{i+s+t}\right),\end{aligned}$$

as stated. For the computation of $\Delta(\overleftarrow{v^{m}} \times g^s)$, we proceed in a similar manner. First, we use (4.13) to calculate

$$\begin{aligned}&\Delta(\overleftarrow{v^{m}} \times 1_i) \\ &\quad = \sum_{l=0}^{m} \sum_{\sigma^{-1} \in S_{l,m-l}} q^{\frac{r(\sigma^{-1})}{n}+l\left[\frac{l-1}{n}\right]-m\left[\frac{m-1}{n}\right]+m\left[\frac{m-l}{n}\right]} y^1 X^1\end{aligned}$$

$$
\begin{aligned}
&\cdot(v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)}) \cdots))\\
&\times y^2 Y^1 (x^1 X^2 \cdot (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots)))_{[-1]} x^2 X_1^3 (1_i)_1\\
&\underline{\otimes} y_1^3 Y^2 \cdot (x^1 X^2 \cdot (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots)))_{[0]}\\
&\times y_2^3 Y^3 x^3 X_2^3 (1_i)_2\\
= &\sum_{l=0}^{m} \sum_{\sigma^{-1} \in S_{l,m-l}} \sum_{s,t=0}^{n-1} q^{\frac{r(\sigma^{-1})}{n} + l\left[\frac{l-1}{n}\right] - m\left[\frac{m-1}{n}\right] + (m-l)\left[\frac{m-l}{n}\right] + l\left[\frac{m-l+i}{n}\right] - (m-l)\left[\frac{s+t}{n}\right]}\\
&y^1 \cdot (v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)}) \cdots))\\
&\times y^2 Y^1 (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots))_{[-1]} 1_s (1_i)_1\\
&\underline{\otimes} y_1^3 Y^2 \cdot (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots))_{[0]} \times y_2^3 Y^3 1_t (1_i)_2\\
= &\sum_{l=0}^{m} \sum_{\sigma^{-1} \in S_{l,m-l}}\\
&\qquad \sum_{\{0 \le s,t \le n-1 \mid (s+t)' = i\}} q^{\frac{r(\sigma^{-1})}{n} + l\left[\frac{l-1}{n}\right] - m\left[\frac{m-1}{n}\right] + (m-l)\left[\frac{m-l}{n}\right] + l\left[\frac{m-l+s+t}{n}\right] - m\left[\frac{s+t}{n}\right]}\\
&y^1 \cdot (v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)}) \cdots)) \times y^2 Y^1 K^{m-l+n\left[\frac{m-l}{n}\right]} 1_s\\
&\underline{\otimes} y_1^3 Y^2 \cdot (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots)) \times y_2^3 Y^3 1_t\\
= &\sum_{l=0}^{m} \sum_{\sigma^{-1} \in S_{l,m-l}} \sum_{\{0 \le s,t \le n-1 \mid (s+t)' = i\}} q^{\frac{r(\sigma^{-1}) + (m-l)s}{n} + l\left[\frac{l-1}{n}\right] - m\left[\frac{m-1}{n}\right] + (m-l)\left[\frac{m-l}{n}\right] + s\left[\frac{m-l+t}{n}\right]}\\
&q^{l\left[\frac{m-l+s+t}{n}\right] - m\left[\frac{s+t}{n}\right]} y^1 \cdot (v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)}) \cdots)) \times y^2 1_s\\
&\underline{\otimes} y_1^3 \cdot (v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots)) \times y_2^3 1_t\\
= &\sum_{l=0}^{m} \sum_{\sigma^{-1} \in S_{l,m-l}} q^{\frac{r(\sigma^{-1})}{n} + l\left[\frac{l-1}{n}\right] - m\left[\frac{m-1}{n}\right] + (m-l)\left[\frac{m-l}{n}\right]}\\
&\qquad \sum_{\{0 \le s,t \le n-1 \mid (s+t)' = i\}} q^{\frac{(m-l)s}{n} + (l+s)\left[\frac{m-l+t}{n}\right]}\\
&q^{-m\left[\frac{s+t}{n}\right]} v^{\sigma(1)} \otimes (\cdots \otimes (v^{\sigma(l-1)} \otimes v^{\sigma(l)}) \cdots) \times 1_s\\
&\underline{\otimes} v^{\sigma(l+1)} \otimes (\cdots \otimes (v^{\sigma(m-1)} \otimes v^{\sigma(m)}) \cdots) \times 1_t.
\end{aligned}
$$

This leads to the claimed formula for $\Delta(\overleftarrow{v^m} \times g^j)$ since $g^j = \sum_{i=0}^{n-1} q^{ij} 1_i$.

So it remains to prove the formula for s. For this, notice that $S(1_l) = 1_{n-l}$, for all $0 \le l \le n-1$, where by convention $1_n = 1_0$. Therefore, the element p_R in $k_\Phi[C_n]^{\otimes 2}$ is

$$
\begin{aligned}
p_R &= \sum_{i,j,l=0}^{n-1} q^{-i\left[\frac{j+l}{n}\right]} 1_i \otimes 1_j 1_{n-l} \\
&= \mathbf{1} \otimes 1_0 + \left(\sum_{i=0}^{n-1} q^{-i} 1_i\right) \otimes \left(\sum_{l=1}^{n-1} 1_{n-l}\right) \\
&= \mathbf{1} \otimes 1_0 + g^{-1} \otimes (\mathbf{1} - 1_0) = \mathbf{1} \otimes 1_0 + g^{-1} \otimes \mathbf{1} - g^{-1} \otimes 1_0,
\end{aligned}
$$

where in the last but one equality we used the fact that $g \sum_{i=0}^{n-1} q^{-i} 1_i = \mathbf{1}$ in $k_\Phi[C_n]$. Thus,

$$
\begin{aligned}
&X^1 p_1^1 \otimes X^2 p_2^1 \otimes X^3 p^2 \\
&\quad = X^1 \otimes X^2 \otimes X^3 1_0 + X^1 g^{-1} \otimes X^2 g^{-1} \otimes X^3 - X^1 g^{-1} \otimes X^2 g^{-1} \otimes X^3 1_0 \\
&\quad = \mathbf{1} \otimes \mathbf{1} \otimes 1_0 + \sum_{i,j,l=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]-i-j} 1_i \otimes 1_j \otimes 1_l - g^{-1} \otimes g^{-1} \otimes 1_0.
\end{aligned}
$$

Remark also that K is invertible with inverse $K^{-1} = \sum_{j=0}^{n-1} q^{-j} 1_j$, and this allows to prove that $K^a = \sum_{j=0}^{n-1} q^{aj} 1_j$, for any integer number a. Hence, for $a \in \mathbb{Z}$ and $v^1, \ldots, v^m \in V$,

$$
\begin{aligned}
S(K^a) &= \sum_{j=0}^{n-1} q^{\frac{aj}{n}} 1_{n-j} = 1_0 + \sum_{i=1}^{n-1} q^{\frac{a(n-i)}{n}} 1_i = 1_0 + q^a (K^{-a} - 1_0) \\
&= (1 - q^a) 1_0 + q^a K^{-a}, \\
(1 \times K^a)(\overleftarrow{v^m} \times 1_0) &= \sum_{i,j=0}^{n-1} q^{\frac{a(i+j)'}{n}} 1_i \cdot \overleftarrow{v^m} \times 1_j 1_0 = q^{\frac{am'}{n}} \overleftarrow{v^m} \times 1_0.
\end{aligned}
$$

Finally, for $m \in \mathbb{N}$ and $j \in \{0, \ldots, n-1\}$ the equation $(m' + t)' = j$ has a unique solution in $\{0, \ldots, n-1\}$. Namely, if $j \in \{0, \ldots, m'-1\}$ then $t = n + j - m'$, and if $j \in \{m', \ldots, n + m' - 1\}$ then $t = j - m'$. By using all these facts and the formula for the antipode s of a biproduct quasi-Hopf algebra found in Corollary 4.5, we get that

$$
\begin{aligned}
&s(\overleftarrow{v^m} \times g^l) \\
&\quad = (1 \times S(K^{m+n\left[\frac{m}{n}\right]} g^l) g^{-1})(\underline{\underline{S}}(\overleftarrow{v^m}) \times 1_0) \\
&\qquad - (1 \times S(g^{-1} K^{m+n\left[\frac{m}{n}\right]} g^l) g^{-1})(g^{-1} \cdot \underline{\underline{S}}(\overleftarrow{v^m}) \times 1_0) \\
&\qquad + \sum_{i,j,t=0}^{n-1} q^{i\left[\frac{j+t}{n}\right]-i-j} (1 \times S(1_i K^{m+n\left[\frac{m}{n}\right]} g^l) g^{-1})(1_j \cdot \underline{\underline{S}}(\overleftarrow{v^m}) \times 1_t) \\
&\quad = (-1)^m q^{\frac{m(m-1)}{2n}} (1 \times (1 - q^m) 1_0 + 1 \times q^m K^{-m-n\left[\frac{m}{n}\right]-n(l+1)})(\overleftarrow{v^{m_\tau}} \times 1_0)
\end{aligned}
$$

$$+(-1)^{m+1}q^{\frac{m(m-1)}{2n}}(1\times(1-q^m)1_0+1\times q^mK^{-m-n[\frac{m}{n}]-nl})(q^{-m'}v^{\overleftarrow{m}_\tau}\times 1_0)$$

$$+(-1)^m q^{\frac{m(m-1)}{2n}}\sum_{i,t=0}^{n-1}q^{i[\frac{t+m}{n}]-m+\frac{im}{n}+il}(1\times 1_{n-i})(v^{\overleftarrow{m}_\tau}\times 1_t)$$

$$=(-1)^m q^{\frac{m(m-1)}{2n}}\left(q^m 1\times K^{-m-n[\frac{m}{n}]-n(l+1)}-1\times K^{-m-n[\frac{m}{n}]-nl}\right)(v^{\overleftarrow{m}_\tau}\times 1_0)$$

$$+(-1)^m q^{\frac{m(m-1)}{2n}}\left(\sum_{t=0}^{n-1}q^{-m}(1\times 1_0)(v^{\overleftarrow{m}_\tau}\times 1_t)\right.$$

$$\left.+\sum_{j=1}^{n-1}\sum_{t=0}^{n-1}q^{-j[\frac{t+m}{n}]-\frac{jm}{n}-jl}(1\times 1_j)(v^{\overleftarrow{m}_\tau}\times 1_t)\right)$$

$$=(-1)^m q^{\frac{m(m-1)}{2n}-m}v^{\overleftarrow{m}_\tau}\times 1_{n-m'}$$

$$+(-1)^m q^{\frac{m(m-1)}{2n}}\sum_{j=1}^{n-1}\sum_{\{0\le t\le n-1|(m'+t)'=j\}}q^{-j[\frac{t+m}{n}]-\frac{jm}{n}-jl}v^{\overleftarrow{m}_\tau}\times 1_t$$

$$=(-1)^m q^{\frac{m(m-1)}{2n}-m}v^{\overleftarrow{m}_\tau}\times 1_{n-m'}+(-1)^m q^{\frac{m(m-1)}{2n}}$$

$$\sum_{j=1}^{m'-1}q^{-j[\frac{m}{n}]-\frac{jm}{n}-j(l+1)}v^{\overleftarrow{m}_\tau}\times 1_{n+j-m'}$$

$$+(-1)^m q^{\frac{m(m-1)}{2n}}\sum_{j=m'}^{n-1}q^{-j[\frac{m}{n}]-\frac{jm}{n}-jl}v^{\overleftarrow{m}_\tau}\times 1_{j-m'}$$

$$=(-1)^m q^{-\frac{m(m+1)}{2n}-ml}\sum_{i=0}^{n-m'-1}q^{-i[\frac{m}{n}]-\frac{im}{n}-il}v^{\overleftarrow{m}_\tau}\times 1_i$$

$$+(-1)^m q^{\frac{m(m-1)}{2n}-m}v^{\overleftarrow{m}_\tau}\times 1_{n-m'}$$

$$+(-1)^m q^{-\frac{m(m+1)}{2n}-ml}\sum_{i=n-m'+1}^{n-1}q^{-i[\frac{m}{n}]-\frac{im}{n}-i(l+1)}v^{\overleftarrow{m}_\tau}\times 1_i$$

$$=(-1)^m q^{-\frac{m(m+1)}{2n}-ml}v^{\overleftarrow{m}_\tau}\times\left(\sum_{i=0}^{n-1}q^{-i[\frac{m}{n}]-\frac{im}{n}-i(l+[i>n-m'])}1_i\right),$$

as stated (for the third equality we used that $(1\times 1_0)(v^{\overleftarrow{m}}\times 1_0)=\delta_{m',0}v^{\overleftarrow{m}}\times 1_0$). □

We end by specializing Theorem 6.5 for $V=kv$, a one dimensional vector space. In this situation $\{v_m\}_{m\in\mathbb{N}}$ is a basis for $T(V)$, where $v_m := v\otimes(v\otimes(\cdots\otimes(v\otimes v)\cdots))\in T^{m)}(V)$; by convention $v_0=1$, the unit of k. It follows that $\{v_m g^l \mid m\in\mathbb{N},\ 0\le l\le n-1\}$ is a basis for $H(n,q,kv)$, where we identify $v_m\equiv v_m\times\mathbf{1}$ and $g^l\equiv 1\times g^l$, and therefore $v^m\times g^l=(v_m\times\mathbf{1})(1\times g^l)\equiv v_m g^l$. With these identifications in mind, we have that

$$g^l v_m \equiv (1 \times g^l)(v_m \times \mathbf{1}) = \sum_{j=0}^{n-1} q^{l(m+j)'} v_m \times 1_j = q^{lm} v_m \times \sum_{j=0}^{n-1} g^{lj} 1_j$$
$$= q^{lm} v_m \times g^l \equiv q^{lm} v_m g^l,$$

for all $m \in \mathbb{N}$ and $0 \leq l \leq n-1$. Therefore, $H(n, q, kv)$ is the unital associative algebra generated by $\{v_m\}_{m\in\mathbb{N}}$ and g with relations

$$v_m v_p = q^{p\left[\frac{m+p'}{n}\right]+m\left[\frac{m'+p'}{n}\right]} \left(\left(1 - \frac{p'}{n} + \frac{p'}{n} q^{-m}\right) v_{m+p} + \frac{1-q^{-m}}{n} \sum_{l=1}^{n-1} \left(1 - (p'+1)_{q^l}\right) v_{m+p} g^l\right), \tag{6.6}$$

$$g^a g^b = g^{a+b}, \quad g^n = 1, \quad g v_m = q^m v_m g, \tag{6.7}$$

for all $m \in \mathbb{N}$ and $0 \leq l, a, b \leq n-1$. The unit is $1 = v_0$.

In order to give a nicer form for the comultiplication of $H(n, q, kv)$, we need a preliminary result. We believe that it was proved already somewhere else, but because we were not able to find a reference we decided to include its proof here. Recall that $(0)!_{\mathfrak{q}} := 1$ and $(p)!_{\mathfrak{q}} = (1)_{\mathfrak{q}}(2)_{\mathfrak{q}} \cdots (p)_{\mathfrak{q}}$ is the $\mathfrak{q}$-factorial of p, $p \in \mathbb{N}$, and that $\binom{p}{s}_{\mathfrak{q}} = \frac{(p)!_{\mathfrak{q}}}{(s)!_{\mathfrak{q}}(p-s)!_{\mathfrak{q}}}$, with $0 \leq s \leq p$, are the so-called Gauss polynomials.

Lemma 6.6 *We have* $\sum_{w\in S_{l,m-l}} \mathfrak{q}^{r(w)} = \binom{m}{m-l}_{\mathfrak{q}}$*, for all* $m \in \mathbb{N}$ *and* $0 \leq l \leq m$.

Proof For simplicity, denote $\lambda_m(\mathfrak{q}, l) := \sum_{w\in S_{l,m-l}} \mathfrak{q}^{r(w)}$. As we observed, any $(l, m-l)$ shuffle is completely determined by a subset $\{i_1, \ldots, i_l\}$ of $\{1, \ldots, m\}$, arranged in ascending order. Actually, any $(l, m-l)$ shuffle is of the form

$$\begin{pmatrix} 1 & \cdots & l & l+1 & \cdots & l+i_k-k & l+i_k-k+1 & \cdots & i_l & i_l+1 & \cdots & m \\ i_1 & \cdots & i_l & 1 & \cdots & i_k-1 & i_k+1 & \cdots & i_l-1 & i_l+1 & \cdots & m \end{pmatrix},$$

for some $1 \leq i_1 < \cdots < i_k < \cdots < i_l \leq m$. Consequently, the inversions of it are

$$(k \;\; l+1), \;\; \ldots, \;\; (k \;\; l+i_k-k), \;\; 1 \leq k \leq l,$$

and so these are in number of $i_1 + \cdots i_l - \frac{l(l+1)}{n}$. According to [17, Lemma 4.7], the length of a permutation is equal to the number of its inversions, and so

$$\lambda_m(\mathfrak{q}, l) = \mathfrak{q}^{-\frac{l(l+1)}{2}} \sum_{1\leq i_1<\cdots<i_l\leq m} \mathfrak{q}^{i_1+\cdots+i_l}$$
$$= \mathfrak{q}^{-\frac{l(l+1)}{2}} \sum_{i_1=1}^{m-l+1} q^{i_1} \sum_{i_1+1\leq i_2<i_3<\cdots<i_l\leq m} \mathfrak{q}^{i_2+\cdots i_l}$$

$$= \mathfrak{q}^{-\frac{l(l+1)}{2}} \sum_{i_1=1}^{m-l+1} q^{l i_1} \sum_{1\le j_1<j_2<\cdots<j_{l-1}\le m-i_1} \mathfrak{q}^{j_1+\cdots j_{l-1}}$$
$$= \sum_{i=1}^{m-l+1} \mathfrak{q}^{(i-1)l} \lambda_{m-i}(\mathfrak{q}, l-1),$$

for all $m \in \mathbb{N}$ and $1 \le l \le m$. This recurrence together with $\lambda_m(\mathfrak{q}, 0) = 1$ and the Pascal identity, see [16, Proposition IV.2.1],

$$\binom{n}{k}_{\mathfrak{q}} = \binom{n-1}{k}_{\mathfrak{q}} + q^{n-k} \binom{n-1}{k-1}_{\mathfrak{q}},$$

valid for any $0 \le k \le n$ in $\mathbb{N}$, allows to obtain in an inductive way the formula for $\lambda_m(\mathfrak{q}, l)$ stated above. We leave this detail to the reader. □

One can present now the quasi-Hopf algebra structure of $H(n, q, kv)$.

Corollary 6.7 *For any $n \in \mathbb{N}$, $n \ge 2$ and $\mathfrak{q}$, a primitive root of unity of degree n^2 in k denote by $H_{\aleph}(n, q)$ the k-algebra generated by $\{v_m\}_{m\in\mathbb{N}}$ and g with relations (6.6), (6.7) and unit $1 = v_0$, where $q = \mathfrak{q}^n$. Then, $H_{\aleph}(n, q)$ is a quasi-Hopf algebra with projection, via the quasi-coalgebra structure given, for all $m \in \mathbb{N}$, by*

$$\Delta(v_m) = \frac{q^{-m\left[\frac{m-1}{n}\right]}}{n^2} \sum_{l=0}^{m} \binom{m}{m-l}_{\mathfrak{q}} q^{l\left[\frac{l-1}{n}\right]+(m-l)\left[\frac{m-l}{n}\right]}$$
$$\sum_{a,b,s,t=0}^{n-1} q^{\frac{(m-l)s}{n}+(l+s)\left[\frac{m-l+t}{n}\right]-m\left[\frac{s+t}{n}\right]}$$
$$q^{-sa-tb} v_l g^a \otimes v_{m-l} g^b, \quad \varepsilon(v_m) = \delta_{m,0},$$
$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1,$$
$$\Phi = \frac{1}{n^3} \sum_{i,j,l,a,b,c=0}^{n-1} q^{i\left[\frac{j+l}{n}\right]-ia-jb-jc} g^a \otimes g^b \otimes g^c,$$

and antipode s determined by $s(g) = g^{-1}$ and

$$s(v_m) = \frac{(-1)^m}{n} q^{-\frac{m(m+1)}{2}} v_m \left(\sum_{j=0}^{n-1} \left(\sum_{i=0}^{n-1} q^{-i\left[\frac{m}{n}\right]-\frac{im}{n}-i(j+[i>n-m'])} \right) g^j \right),$$

for all $m \in \mathbb{N}$, and distinguished elements $\alpha = g^{-1}$ and $\beta = 1$.

Proof We have $H_{\aleph}(n, q) = H(n, q, kv)$, so everything follows from the comments made after Theorem 6.5 and the formula in Lemma 6.6. □

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