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Genus polynomials of ladder-like sequences of graphs

Yichao Chen[1] · Jonathan L. Gross[2] · Toufik Mansour[3] · Thomas W. Tucker[4]

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Abstract

Production matrices have become established as a general paradigm for calculating the genus polynomials for linear sequences of graphs. Here we derive a formula for the production matrix of any of the linear sequences of graphs that we call *ladder-like*, where any connected graph H with two 1-valent root vertices may serve as a *super-rung* throughout the ladder. Our main theorem expresses the production matrix for any ladder-like sequence as a linear combination of two fixed 3×3 matrices, taken over the ring of polynomials with integer coefficients. This leads to a formula for the genus polynomials of the graphs in the ladder-like sequence, based on the two *partial genus polynomials* of the super-rung. We give a closed formula for these genus polynomials, for the case in which all imbeddings of the super-rung H are planar. We show that when the super-rung H has Betti number at most one, all the genus polynomials in the sequence are log-concave.

Keywords Linear sequences of graphs · String operations · Imbedding types · Genus polynomials · Partial genus polynomials · Production matrices

1 Introduction

Given any graph (H, u, v) whose root vertices u and v are both 1-valent, we construct a sequence of graphs

$$(L_1^H, u_1, v_1),\ (L_2^H, u_2, v_2),\ (L_3^H, u_3, v_3),\ \ldots \tag{1.1}$$

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✉ Yichao Chen
chengraph@163.com

Extended author information available on the last page of the article

recursively, as follows:

- The graph (L_1^H, u_1, v_1) is isomorphic to (H, u, v).
- We construct the graph $(L_{n+1}^H, u_{n+1}, v_{n+1})$ from the graph (L_n^H, u_n, v_n) and a copy of the graph (H, u, v), in which the roots u and v are renamed u_{n+1} and v_{n+1}, respectively, by joining the vertex u_n to the new root u_{n+1} and joining the vertex v_n to the new root v_{n+1}, so that the vertices u_n and v_n (which were roots of L_n^H) become 3-valent in L_{n+1}^H and the vertices u_{n+1} and v_{n+1} (the roots of L_{n+1}^H) become 2-valent.
- Only the vertices u_{n+1} and v_{n+1} are regarded as roots of L_{n+1}^H.

We call each of the graphs L_n^H a ***ladder-like graph with super-rung*** H, and we call the sequence (1.1) a ***ladder-like sequence***.

In Fig. 1 we see a particular such graph H and the ladder-like graph L_4^H. We have suppressed the vertex names that are not roots and are not essential to what follows. We represent H by a "blob with two pendant edges."

Ladder-like sequences are a special case of *linear sequences of graphs*, which are the focus of many papers since [4] gave the two initial examples and Stahl [14] expanded the idea. The most general formulation to date is given by [1]. Here we will concentrate only on ladder-like sequences.

The ***genus polynomial*** of a graph G is the generating function

$$\Gamma_G(z) = g_0(G) + g_1(G)z + g_2(G)z^2 + \cdots$$

where $g_i(G)$ is the number of different cellular imbeddings of G in the orientable surface S_i, of genus i. We are concerned here exclusively with finite graphs, in which case the genus polynomial is a finite polynomial. The smallest number i such that $g_i(G)$ is nonzero is called the ***minimum genus*** of the graph, in which case we write $\gamma_{\min}(G) = i$. The largest number j such that $g_j(G)$ is nonzero is called the ***maximum genus*** [12], and we write $\gamma_{\max}(G) = j$.

Since calculation of the minimum genus is NP-hard [15], it follows that calculation of the genus polynomial is at least NP-hard. By way of contrast, the maximum genus can be calculated in polynomial time [5]. We also know that every surface whose genus lies between the minimum genus and the maximum genus, there is at least one

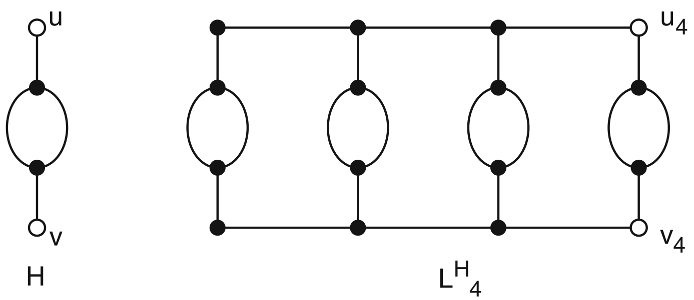

Fig. 1 A ladder-like graph with super-rung H

imbedding [3]. It has been conjectured [10] that the genus polynomial of every graph is log-concave. Most of the families of graphs for which this conjecture has been confirmed are linear sequences.

In this paper, our main result, Theorem 4.1, gives a recurrence relation for the genus polynomial of every graph in any ladder-like sequence of graphs, based on the two *partial genus polynomials* for the super-rung H, no matter how high the degrees of these two polynomials. For the case in which $\gamma_{\max}(H) = 0$, we derive conditions on H under which every graph in the corresponding ladder-like sequence has a log-concave genus polynomial.

Partial genus polynomials are developed in full generality in [8]. General background in topological graph theory is provided by [9]. We denote the *valence of a vertex* v in the graph G by $val_G(v)$ or $val(v)$.

2 Imbedding types for graphs

When the graph G has one or more root vertices, the imbeddings of G are partitioned into ***imbedding types*** (abbreviated as ***i-types***), according to the incidence of face-boundary walks (abbr. ***fb-walks***) at a designated set of root vertices. An i-type is denoted by a string of cyclic sequences, such that for each fb-walk that is incident on any of the root vertices, the i-type includes a cyclic string that lists the incidences of (designated) roots in the order in which they occur in an orientation-respecting traversal. This fully general system of notation for i-types was introduced in [8], which refined the earlier notational system of [7].

The i-types for a rooted graph $(G, v_1, v_2, \ldots, v_r)$ follow two rules, both of which are self-evident from the definitions:

(1) Each i-type partitions the multi-set of occurrences of root vertices along the fb-walks into cycles.
(2) The total number of times that a given root vertex occurs in any i-type equals its valence.

To simplify discussion in what follows, we describe three notational conventions established by [8], which give us a ***standard notation for an i-type***.

(1) When representing an fb-walk by a cycle, we choose the starting point such that the written form of the cycle is lexicographically least.
(2) Within the sequence of cycles that represents an i-type, if $j < k$, then a j-cycle precedes a k-cycle
(3) Within the sequence of cycles that represents an i-type, the cyclic strings of the same length are ordered lexicographically.

Proposition 2.1 *Each of the imbeddings of the graph* $(H, 0, 1)$ *is either of i-type* $(0)(1)$ *or of i-type* (01).

Proof The only possible ways to partition the multi-set $\{0, 1\}$ into cycles are $(0)(1)$ and (01). □

Proposition 2.2 *Using 0 for u_n and 1 for v_n as notations for the root vertices of the ladder-like graph L_n^H, the possible i-types, for $n \geq 2$ are*

$$(00)(11),\ (01)(01),\ \textit{and}\ (0011).$$

Proof For two 2-valent roots, here are the ten possible i-types, each a cyclic partition of a multi-set of two 0's and two 1's.

$$\begin{array}{ccccc} (0)(0)(1)(1) & (0)(0)(11) & (0)(1)(01) & (1)(1)(00) & (0)(011) \\ (1)(001) & (00)(11) & (01)(01) & (0011) & (0101) \end{array}$$

Since the roots 0 and 1 are both 2-valent and form a cutset, it follows, as illustrated in Fig. 2, that an i-type for L_n^H has no 1-cycles. That eliminates the first six cyclic partitions shown above. The i-type (0101) cannot occur, because it would represent an fb-walk that goes in the same direction twice at each of two cutpoints (i.e., 0 and 1) instead of once each way at each cutpoint. This leaves only the i-types $(00)(11)$, $(01)(01)$, and (0011). In previous papers, the following mnemonic notations were used for these three respective i-types:

$$ss^0,\ dd'',\ \text{and}\ ss^1.$$

□

The partitioning of the set of imbeddings into i-types leads to a partitioning of the genus polynomial for each of the graphs into a set of partial genus polynomials. The larger our designated set of roots, the great the number of i-types. A formula for the number of i-types of any number of roots of any valences is given by Theorem 6.3 of [8]. Tables 6.1 and 6.2 of that paper give the numbers of possible i-types for some smaller numbers of roots and some smaller valences. Unsurprisingly, the number of i-types grows rapidly with increasing numbers of roots or increasing valences.

The partition of the genus polynomial is given by a column vector called the ***pgd-vector*** ("pgd" stands for "partitioned genus distribution") with one coordinate for each i-type. That coordinate is a ***partial genus polynomial*** that enumerates the number of imbeddings of that i-type in the orientable surfaces of varying genera. The coordinates

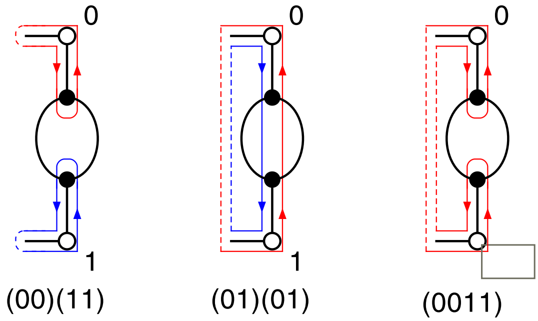

Fig. 2 Root incidences along the fb-walks of the imbeddings of a ladder-like graph

of a pgd-vector are ordered according to the lexicographic ordering of the induced partitions for the set of i-types.

Example 2.1 Suppose that we take the "blob" in Fig. 1 to be a 2-cycle. Since H has two 3-valent vertices and two 1-valent vertices, the total number of imbeddings of H is $(3-1)!(3-1)!(1-1)!(1-1)! = 4$. Two of them are of i-type (0)(1) and two are of i-type (01). The partial genus polynomials are

$$\Gamma_H^{(0)(1)}(z) = 2 \quad \text{and} \quad \Gamma_H^{(01)}(z) = 2.$$

The pgd-vector of H is

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Example 2.2 Using the graph H of the Example 2.1 as the super-rung, the graph L_2^H has four 3-valent vertices, so there are $2^4 = 16$ imbeddings. They partition into four of i-type (00)(11) in S_1, four of i-type (01)(01) in S_0, and eight of i-type (0011) in S_1. Thus, the pgd-vector of L_2^H is

$$\begin{bmatrix} 4z \\ 4 \\ 8z \end{bmatrix}.$$

3 Production matrices

In general, a linear sequence of graphs is specified by an initial graph and a topological operation by which each graph in the sequence is transformed into the next graph. We require that each graph in the sequence has the same number of roots of each valence. In a ladder-like sequence, the two roots of each sequence are both 2-valent. The ***production matrix*** $M_{L^H}(z)$ for the ladder-like sequence L_n^H is a matrix such that for all $n \geq 2$, multiplying the pgd-vector for L_n^H by $M_{L^H}(z)$ produces the pgd-vector for L_{n+1}^H. Each column of the production matrix represents a rule, called a ***production***, that describes, for each i-type of L_n^H and for any given imbedding of that i-type, the number of imbeddings of each i-type of L_{n+1}^H that are derivable from the given imbedding.

Example 2.2, continued For instance, let us consider the set of imbeddings of L_{n+1}^H that can be derived from an imbedding of L_n^H of i-type (00)(11). There are two corners at the root u_n at which to attach the edge whose other endpoint is u_{n+1}, and two corners at v_n at which to attach the edge whose other endpoint is v_{n+1}. Since there are two possible rotations at each of the two 3-valent vertices of the next copy of the super-rung H, that makes a total of $2^4 = 16$ imbeddings of L_{n+1}^H that are derivable from any given imbedding of L_n^H, no matter what its i-type. As is happens, four of those imbeddings of L_n^H are of i-type (00)(11), with a genus increment of one, four of i-type

(01)(01) with no genus increment, and four of i-type (0011) with a genus increment of one. Accordingly, we write the following production:

$$(01)(01) \rightarrow 4z(00)(11) + 4(01)(01) + 8z(0011).$$

The power of the indeterminate z indicates the genus increment. For the other two i-types, we have the following two productions:

$$(00)(11) \rightarrow 8z(00)(11) + 8z(0011) \quad \text{and}$$
$$(0011) \rightarrow 8(01)(01) + 8z(0011).$$

In a number of papers on linear families of graphs, the productions have been derived by face-tracing. A more recently invented method involving the use of string operations is described in the next section.

If there are n i-types, then we can represent the collective action of the productions by an $n \times n$ production matrix. For Example 2.2, with three i-types, the production matrix is

$$\begin{bmatrix} 8z & 4z & 0 \\ 0 & 4 & 8 \\ 8z & 8z & 8z \end{bmatrix}.$$

4 General production matrix for ladder-like graphs

The following theorem, our main theorem, presents a pair of 3×3 matrices and establishes a way to express the production matrix for any ladder-like sequence as a linear combination of those two matrices, such that the coefficients of the two matrices are the partial genus polynomials of the super-rung.

Theorem 4.1 *Let $(H, 0, 1)$ be any graph with two 1-valent root vertices. Let $p(z)$ and $q(z)$ be the partial genus polynomials for H of i-types (0)(1) and (01), respectively. Then the production matrix for the ladder-like sequence $L_1^H, L_2^H, L_3^H, \ldots$ is*

$$M_{L^H}(z) = p(z) \begin{bmatrix} 4z & 2z & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 4z \end{bmatrix} + q(z) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4z & 2z & 0 \end{bmatrix}. \tag{4.1}$$

Proof We give a proof with a completely symbolic calculation of the production matrix for a sequence of ladder-like graphs, using symbolic manipulation rules first described in [8]. An alternative proof with pictures is provided by [2].

We view the construction of L_{n+1}^H from L_n^H as involving two edge-adding steps. In the first step, we take an imbedding of L_n^H with the three i-types

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1}), \ (\overline{01})(\overline{01}), \ \text{and} \ (\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$$

and we add the edge $\overline{0}0$ to join the imbedding of L_n^H to an imbedding of H with the two i-types $(0)(1)$ and (01). To complete this step, we suppress all instances of vertex $\overline{0}$, since it plays no further role in the construction of the graph L_{n+1}^H.

To add an edge uv to an i-type, we have two simple string-processing rules, which first appeared in [8], in which the symbols P and Q denote sequences of vertices that do not include any occurrences of u or v.

Rule 1 To add edge uv within a face

$$(uPvQ) \to (uPv)(vQu) \tag{4.2}$$

Rule 2 To add edge uv between two distinct faces of a connected graph

$$(uP)(vQ) \to z(uPuvQv) \tag{4.3}$$

We have the extra z in the consequent of Rule (4.3), since adding an edge between faces of a connected graph requires adding a handle, which increases the genus by 1. When joining a face of an imbedded graph to a face of another imbedded graph, no extra handle is needed

NOTE Whenever there are multiple occurrences of u or v in any i-types, we must apply these rules for each pair of occurrences of u and v.

In particular, when we add the edge $0\overline{0}$ to the i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})(0)(1)$ where $u = \overline{0}$ and $v = 0$, we use Rule (4.3), with $P = \overline{0}$ and Q the empty string. The vertices 0 and $\overline{0}$ are initially in imbeddings in different surfaces, and we omit the z, since joining an imbedding in any given surface with a handle to an imbedding in the sphere does not involve an increase of genus.

In the i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})(0)(1)$, there are two occurrences of $\overline{0}$ and one occurrence of 0. Applying Rule (4.3), we obtain the production

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})(0)(1) \to 2(\overline{0}\,\overline{0}\,\overline{0}\,0\,0)(\overline{1}\,\overline{1})(1).$$

Suppressing $\overline{0}$ (i.e., terminating its designation as a root) is represented by the production

$$2(\overline{0}\,\overline{0}\,\overline{0}\,0\,0)(\overline{1}\,\overline{1})(1) \to 2(00)(\overline{1}\,\overline{1})(1).$$

Combining these two productions yields the production

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})(0)(1) \to 2(00)(\overline{1}\,\overline{1})(1).$$

Since there are $p(z)$ imbeddings of H of i-type $(0)(1)$, a single imbedding of L_n^H of i-type $(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})$ gives rise to $p(z)$ imbeddings:

$$(\overline{0}\,\overline{0}\,1)(\overline{1}\,\overline{1})(0)(1) \to 2p(z)(00)(\overline{1}\,\overline{1})(1).$$

Similarly, adding an edge into the two other i-types of L_n^H, we obtain the productions

$$(\overline{0}\,\overline{1})(\overline{0}\,\overline{1})(0)(1) \to 2p(z)(00\overline{1})(\overline{1})(1) \text{ and}$$
$$(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})(0)(1) \to 2p(z)(00\overline{1}\,\overline{1})(1).$$

There are $q(z)$ imbeddings of H of i-type (01). By applying the string-processing rules in similar fashion to the operation of adding edge $0\overline{0}$ between L_n^H and H-type (01), we obtain the productions

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1})(01) \to 2q(z)(010)(\overline{1}\,\overline{1}),$$
$$(\overline{0}\,\overline{1})(\overline{0}\,\overline{1})(01) \to 2q(z)(010\overline{1})(\overline{1}), \text{ and}$$
$$(\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})(01) \to 2q(z)(010\overline{1}\,\overline{1}).$$

We can summarize this with a 6×3 matrix with 3×3 blocks $2p(z)I$ and $2q(z)I$, where I is the 3×3 identity matrix, whose columns are labeled by the three i-types for L_n^H

$$(\overline{0}\,\overline{0})(\overline{1}\,\overline{1}),\ (\overline{0}\,\overline{1})(\overline{0}\,\overline{1}),\ \text{and } (\overline{0}\,\overline{0}\,\overline{1}\,\overline{1})$$

on $\overline{0}$, $\overline{1}$ and whose rows are labeled by the six intermediate i-types on $\overline{1}$, 0, 1:

$$\begin{matrix} (00)(\overline{1}\,\overline{1})(1) & (00\overline{1})(\overline{1})(1) & (00\overline{1}\,\overline{1})(1) \\ (010)(\overline{1}\,\overline{1}) & (010\overline{1})(\overline{1}) & (010\overline{1}\,\overline{1}). \end{matrix}$$

The second step of the construction of L_{n+1}^H is to add the edge $\overline{1}1$, in order to get a final imbedding of L_{n+1}^H. This gives us a 3×6 matrix, whose columns are labeled by the six intermediate i-types on $\overline{1}$, 0, and 1 and whose rows are the three imbedding types for L_{n+1}^H using 0 and 1.

For step (2), we add edge $\overline{1}1$ first to the first three intermediate i-types $(00)(\overline{1}\,\overline{1})(1)$, $(00\overline{1})(\overline{1})(1)$, $(00\overline{1}\,\overline{1})(1)$ and we suppress all instances of $\overline{1}$.

$$(00)(\overline{1}\,\overline{1})(1) \to 2zp(z)(00)(11)$$
$$(00\overline{1})(\overline{1})(1) \to zp(z)(00)(11) + zp(z)(0011)$$
$$(00\overline{1}\,\overline{1})(1) \to 2p(z)(0011)$$

The result is the matrix $p(z)A$, where

$$A = \begin{bmatrix} 2z & z & 0 \\ 0 & 0 & 0 \\ 0 & z & 2z \end{bmatrix}.$$

When we add edge $\overline{1}1$ to the second three intermediate i-types $(010)(\overline{1}\,\overline{1})$, $(010\overline{1})(\overline{1})$, and $(010\overline{1}\,\overline{1})$, we obtain:

$$\begin{aligned}
&(010)(\overline{1}\,\overline{1}) \to 2zq(z)(0011)\\
&(010\overline{1})(\overline{1}) \to q(z)(01)(01) + zq(z)(0011)\\
&(010\overline{1}\,\overline{1}) \to 2q(z)(01)(01)
\end{aligned}$$

The result is the matrix $q(z)B$, where

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 2z & z & 0 \end{bmatrix}.$$

We conclude that the production matrix for the operation of constructing L_{n+1}^H from L_n^H is $2p(z)A + 2q(z)B$. □

Example 4.1 The easiest example is the usual ladder graph sequence, in which case the super-rung H is isomorphic to the complete graph K_2. Then $p(z) = 0$ and $q(z) = 1$, and Theorem 4.1 implies that the production matrix for the ladders is

$$M_{L^{K_2}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4z & 2z & 0 \end{bmatrix}. \tag{4.4}$$

We could delete the first row and the first column, which correspond to i-type (00)(11), since that i-type does not occur among the imbedding of the ladders. We thereby obtain this familiar production matrix for the ladders:

$$\begin{bmatrix} 2 & 4 \\ 2z & 0 \end{bmatrix}.$$

Remark 4.1 We observe in Eq. (4.1) that when $p(z) = 0$ for the super-rung H, the production matrix is $q(z)$ times the 3×3 production matrix (4.4) given in Example 4.1 for the usual ladder sequence. This is what we expect, for topological reasons, as follows. If there are no imbeddings for H of type $(0)(1)$, then no path P between 0 and 1 passes through a vertex lying on a cycle of H. Thus, every edge of H incident to a vertex on path P separates H. If the removal of an edge from H separates H into components H_1, H_2, then by Theorem 5 of [6], the genus polynomial for H is a constant times the product of the genus polynomials for H_1 and H_2. Thus, the production matrix is just that for the usual ladder with an extra factor $q(z)$ equal to the product of all the genus polynomials of the components of $H - P$ times a constant. By contrast, we note that $q(z) = 0$ is impossible, since the graph H is connected, so there is always a face of type (01).

Example 4.2 Let H be the graph of Fig. 1. We easily calculate the partial genus polynomials $p(z) = 2$ and $q(z) = 2$. It follows from Theorem 4.1 that the production matrix is

$$M_{L^H}(z) = 2\begin{bmatrix} 4z & 2z & 0 \\ 0 & 0 & 0 \\ 0 & 2z & 4z \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4z & 2z & 0 \end{bmatrix} = \begin{bmatrix} 8z & 4z & 0 \\ 0 & 4 & 8 \\ 8z & 8z & 8z \end{bmatrix}. \tag{4.5}$$

Corollary 4.2 *Let H be any super-rung. Then the determinant of the production matrix $M_{L^H}(z)$ for the ladder-like sequence $L_1^H,\ L_2^H,\ L_3^H,\ \ldots$ is zero.*

Proof It is easily checked that the sum of the first and third columns of $M_{L^H}(z)$ is twice the second column, so the determinant is zero. □

5 Genus polynomials for ladder-like graphs

In this section, we calculate genus polynomials for the graphs in any ladder-like sequence, based on the partial genus polynomials for the super-rung.

5.1 A recursion for the pgd-vectors of a ladder-like sequence

In the initial graph L_1^H, the root vertices 0 and 1 are 1-valent. To remedy this inconvenience, we define the ***extended super-rung*** H^+ to be the graph obtained by attaching a pendant edge at the root vertex 0 and another pendant edge at the root vertex 1. In the graph H^+, the root vertices 0 and 1 are 2-valent. We will use the pgd-vector of the graph H^+ as the initial pgd-vector in our recursion for the pgd-vectors of the graphs in the ladder-like sequence with super-rung H.

Proposition 5.1 *Let the partial genus polynomials of the super-rung $(H, 0, 1)$ be $p(z)$ for i-type (0)(1) and $q(z)$ for i-type (01). Then the partial genus polynomials of the extended super-rung $(H^+, 0, 1)$ are $p(z)$ for i-type (00)(11), 0 for i-type (01)(01), and $q(z)$ for i-type (0011).*

Proof Clearly, any imbedding of H of type (0)(1) induces an imbedding of H^+ of type (00)(11), in the same surface. Similarly, any imbedding of H of type (01) induces an imbedding of H^+ of type (0011) in that same surface. The conclusion follows. □

Corollary 5.2 *Let $M_{L^H}(z)$ be the production matrix (4.1), and let $[p(z)\ 0\ q(z)]^{\mathrm{T}}$ be the pgd-vector for the extended super-rung H^+. We define*

$$X_1 = \begin{bmatrix} p(z)/4 \\ 0 \\ q(z)/4 \end{bmatrix}.$$

Then the pgd-vector of the graph L_n^H is given by $M_{L^H}(z)^{n-1}X_1$, for $n \geq 2$.

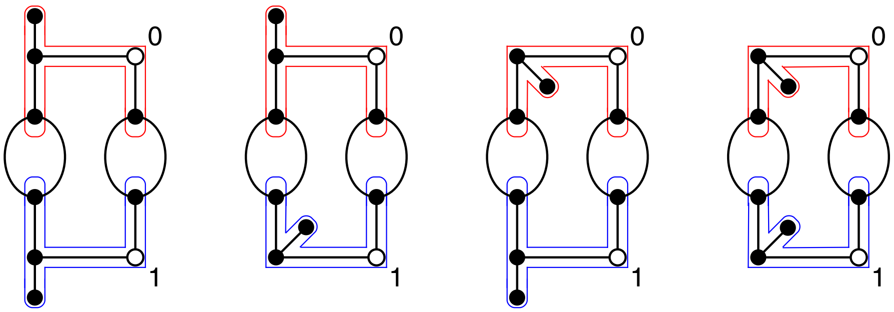

Fig. 3 Adding pendant edges quadruples each imbedding of L_n^H

Proof As illustrated in Fig. 3 for L_2^H, the extension of L_n^H by two pendant edges has four imbeddings for each imbedding of L_n^H. □

Example 4.2, continued The ladder-like sequence of Fig. 3 has the initial pgd-vector $[2\ 0\ 2]^{\mathrm{T}}$ and the production matrix (4.5). Therefore, the pgd-vector of the graph L_2^H is

$$\begin{bmatrix} 8z & 4z & 0 \\ 0 & 4 & 8 \\ 8z & 8z & 8z \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4z \\ 4 \\ 8z \end{bmatrix} \tag{5.1}$$

and the pgd-vector of the graph L_3^H is

$$\begin{bmatrix} 8z & 4z & 0 \\ 0 & 4 & 8 \\ 8z & 8z & 8z \end{bmatrix} \begin{bmatrix} 4z \\ 4 \\ 8z \end{bmatrix} = \begin{bmatrix} 16z + 32z^2 \\ 16 + 64z \\ 32z + 96z^2 T \end{bmatrix}. \tag{5.2}$$

5.2 Formulas for the genus and partial genus polynomials

In this subsection, we derive closed formulas for the partial genus polynomials of a ladder-like sequence and for the genus polynomial. For the special case in which the partial genus polynomials of the super-rung H are the constants $p(z) = a$ and $q(z) = b$, we prove log-concavity of the genus polynomials of the ladder-like graphs L_n^H whenever $a \le b$.

We recall that ***Chebyshev polynomials of the second kind*** are defined, for $r \ge 0$, by the formula

$$U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$$

Equivalently, $U_r(x)$ is a polynomial of degree r in x with integer coefficients, given by the recurrence

$$U_0(x) = 1,$$

$$U_1(x) = 2x, \text{ and}$$
$$U_r(x) = 2xU_{r-1}(x) - U_{r-2}(x).$$

Chebyshev polynomials were invented for the needs of approximation theory, and they are also widely used in various other branches of mathematics, including combinatorics, number theory, and algebra (see [13]). We further recall that the generating function for the Chebyshev polynomials is given by

$$\sum_{n\geq 0} U_n(x)t^n = \frac{1}{1-2xt+t^2}. \tag{5.3}$$

Theorem 5.3 *Let $[p(z)\ q(z)]^{\mathrm{T}}$ be the pgd-vector of the super-rung H. Then for all $n \geq 2$, the pgd-vector $M_{L^H}(z)^{n-1}X_1$ of the graph L_n^H is given by*

$$\frac{v^{n-1}(z)}{4}\begin{bmatrix} p(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2p(z)(2zp(z)+q(z))}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{C)}\right) \\ \frac{4q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right) \\ q(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right) \end{bmatrix},$$

where

$$v(z) = 2\sqrt{2z(2zp^2(z)+p(z)q(z)-q^2(z))}.$$

Proof Let $L^H(t)$ be the generating function for the pgd-vectors for the graph L_n^H. According to Corollary 5.2, we can express $L^H(t)$ as follows:

$$L^H(t) = \sum_{n\geq 1} M_{L^H}(z)^{n-1}X_1t^{n-1},$$

which is equivalent to

$$L^H(t) = (I - tM_{L^H}(z))^{-1}X_1.$$

We define

$$d = 1 - 2(4zp(z)+q(z))t + 8z(2zp^2(z)+p(z)q(z)-q^2(z))t^2.$$

Starting with $M_{L^H}(z)$ from (4.1), and using a mathematical computational engine, we can obtain the inverse matrix

$$(I - tM_{L^H}(z))^{-1}$$
$$= \frac{1}{d}\begin{bmatrix} 1-2t(2zp(z)+q(z))-8zq^2(z)t^2 & 2ztp(z)(1-4ztp(z)t) & 8zt^2p(z)q(z) \\ 16zt^2q^2(z) & (1-4ztp(z))^2 & 4tq(z)(1-4ztp(z)) \\ 4ztq(z)(1-2tq(z)) & 2zt(p(z)+q(z)-4ztp^2(z)) & (1-2tq(z))(1-4ztp(z)) \end{bmatrix}.$$

Hence, by using the pgd-vector $X_1 = [p(z)/4, 0, q(z)/4]^{\mathrm{T}}$ for the graph H^+, we can write the generating function $L^H(t)$ in the form

$$L^H(t) = [A(t), B(t), C(t)]^{\mathrm{T}} \equiv \frac{1}{4d}\left[p(z) - 4ztp^2(z) - 2tp(z)q(z), 4tq^2(z), q(z) - 2tq^2(z)\right]^{\mathrm{T}}.$$

We notice that d can be written as $d = 1 - \frac{4zp(z)+q(z)}{v(z)}(v(z)t) + (v(z)t)^2$. Thus, by (5.3), we obtain

$$\frac{1}{d} = \sum_{n\geq 0} U_n\left(\frac{4zp(z)+q(z)}{v(z)}\right) v^n(z)t^n.$$

Hence, the coefficient of t^{n-1} in the generating functions

$$A(t) = \frac{p(z)}{4d} - \frac{(4zp^2(z) + 2p(z)q(z))t}{4d},$$
$$B(t) = \frac{q^2(z)t}{d}, \text{ and}$$
$$C(t) = \frac{q(z)}{4d} - \frac{q^2(z)t}{2d},$$

is given by

$$\frac{v^{n-1}(z)}{4}\left(p(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2p(z)(2zp(z)+q(z))}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right)\right),$$
$$v^{n-2}(z)q^2(z)U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right), \text{ and}$$
$$\frac{v^{n-1}(z)}{4}\left(q(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right)\right),$$

respectively. It follows that the coefficient of t^{n-1} in the generating function $L^H(t)$ is given by

$$M_{L^H}(z)^{n-1}X_1 = \frac{v^{n-1}(z)}{4}\begin{bmatrix} p(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2p(z)(2zp(z)+q(z))}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right) \\ \frac{4q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right) \\ q(z)U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{2q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right) \end{bmatrix},$$

which completes the proof. □

As a corollary of the above theorem and its proof, we derive a recurrence relation and an explicit formula for the genus polynomial of L_n^H.

Theorem 5.4 *Let $[p(z)\ q(z)]^{\mathrm{T}}$ be the pgd-vector of the super-rung H, and let*

$$v(z) = 2\sqrt{2z(2zp^2(z) + p(z)q(z) - q^2(z))}.$$

Then for all $n \geq 2$, the genus polynomial $g_n(z)$ of the graph L_n^H is given by

$$\frac{v^{n-1}(z)}{4}\Bigg[(p(z)+q(z))U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{4zp^2(z)+2p(z)q(z)-4q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right)\Bigg].$$

Moreover, the sequence satisfies the recurrence relation

$$g_n(z) = 2(4zp(z)+q(z))g_{n-1}(x) - 8z(2zp^2(z)+p(z)q(z)-q^2(z))g_{n-2}(x)$$

with

$$g_2(z) = zp^2(z) + 2zp(z)q(z) + q^2(z),$$
$$g_3(z) = 4z^2p^3(z) + 12z^2p^2(z)q(z) + 12zp(z)q^2(z) + 2zq^3(z) + 2q^3(z).$$

Proof From the definitions, we calculate that $g_n(z) = (1, 1, 1)M_{L^H}(z)^{n-1}X_1$. Thus, by Theorem 5.3, we obtain the formula

$$g_n(z) = \frac{v^{n-1}(z)}{4}\Bigg[(p(z)+q(z))U_{n-1}\left(\frac{4zp(z)+q(z)}{v(z)}\right) - \frac{4zp^2(z)+2p(z)q(z)-4q^2(z)}{v(z)}U_{n-2}\left(\frac{4zp(z)+q(z)}{v(z)}\right)\Bigg].$$

From the recurrence relation $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for the Chebyshev polynomials, we obtain

$$\frac{g_n(z)}{v^{n-1}(z)} = 2\frac{4zp(z)+q(z)}{v(z)}\frac{g_{n-1}(z)}{v^{n-2}(z)} - \frac{g_{n-2}(z)}{v^{n-3}(z)},$$

which leads to the recurrence

$$g_n(z) = 2(4zp(z)+q(z))g_{n-1}(x) - 8z(2zp^2(z)+p(z)q(z)-q^2(z))g_{n-2}(x),$$

as required. □

Corollary 5.5 *Suppose that the maximum genus of the super-rung H is 0, so that its two partial genus polynomials $p(z) = a$ and $q(z) = b$, are constants. If $a \leq b$, then the genus polynomial of each of the graphs L_n^H is log-concave.*

Proof Theorem 5.4 establishes that the genus polynomial of the graph L_n^H satisfies the recurrence

$$g_n(z) = (8az+2b)g_{n-1}(z) - (16a^2z^2+8abz-8b^2z)g_{n-2}(z) \quad \text{for } n \geq 4, \text{ with}$$

$$g_2(z) = (a^2 + 2ab)z + b^2 \quad \text{and}$$
$$g_3(z) = (4a^3 + 12a^2b)z^2 + (12ab^2 + 2b^3)z + 2b^3. \tag{5.4}$$

By induction on n, it is not hard to see that the degree of the polynomial $g_n(z)$ is $n-1$ with leading coefficient $4^{n-2}a^{n-1}(a+nb)$. Then by [11], the polynomial $g_n(z)$ is LC when $a \leq b$. □

Example 4.2, continued For the super-rung H of this example, we have $p(z) = 2$ and $q(z) = 2$. Substituting these values into the recursion (5.4), we obtain the recursion

$$\Gamma_{L_n^H}(z) = (16z+4)\Gamma_{L_{n-1}^H}(z) - 64z^2\Gamma_{L_{n-2}^H}(z) \quad \text{for } n \geq 4, \text{ with}$$
$$\Gamma_{L_2^H}(z) = 12z + 4 \quad \text{and}$$
$$\Gamma_{L_3^H}(z) = 128z^2 + 112z + 16.$$

We see that the initial values as given by Theorem 5.4 and its corollary agree with what we calculated in (5.1) and (5.2). Moreover, the recursion above gives the genus polynomial

$$\begin{aligned}\Gamma_{L_4^H}(z) &= (16z+4)(128z^2+112z+16) - 64z^2(12z+4)\\ &= (2048z^3 + 2304z^2 + 704z + 64) - (768z^3 + 256z^2)\\ &= 1280z^3 + 2048z^2 + 704z + 64\end{aligned}$$

which equals the sum of the coordinates of the pgd-vector for L_4^H, as calculated via Corollary 5.2.

$$\begin{bmatrix} 8z & 4z & 0\\ 0 & 4 & 8\\ 8z & 8z & 8z \end{bmatrix}\begin{bmatrix} 16z + 32z^2\\ 16 + 64z\\ 32z + 96z^2 \end{bmatrix} = \begin{bmatrix} 256z^3z + 384z^2 + 64z\\ 768z^2 + 512z + 64\\ 1024z^3 + 896z^2 + 128z \end{bmatrix}. \tag{5.5}$$

5.3 Some ladder-like sequences with log-concave genus polynomials

We establish a few kinds of super-rungs for which the corresponding ladder-like graphs have log-concave genus polynomials.

The simplest families of ladder-like graphs to study are those having a super-rung $(H, 0, 1)$ with partial genus polynomials $p(z)$ and $q(z)$ for i-types (0)(1) and (01), respectively, such that $p(z) = a$ and $q(z) = b$, for constants a, b. This means the super-rung H is a ***planar-only*** graph—that is, all of its imbeddings are planar, which is equivalent to saying $\gamma_{\max}(H) = 0$. In what follows, a ***cycle*** in the graph G is a connected subgraph, each of whose vertices has valence 2, and a ***path*** is a connected subgraph with two vertices of valence one and the others of valence two.

We view imbeddings of the planar-only graph H as being built up from an imbedding of a smaller subgraph of H, starting from a single edge with endpoints 0 and 1, using edge additions within a face or adding an edge to a new vertex of valence one.

Remark 5.1 Within this process, there are no edge additions between two faces, since such an addition would increase the genus by one, and no subsequent edge additions can decrease the genus.

Proposition 5.6 *Let* $(H, 0, 1)$ *be a planar-only graph with root vertices* $0, 1$ *of valence one.*

(a) *Every subgraph of* H *is planar-only.*
(b) *Any two distinct cycles* C *and* C' *of* H *are mutually disjoint.*
(c) *Any imbedding of* H *obtained by adding an edge to a connected subgraph imbedding of i-type* $(0)(1)$ *also has i-type* $(0)(1)$.
(d) *Suppose that adding an edge* uv *transforms an imbedding of* H *of i-type* (01) *into one of i-type* $(0)(1)$. *Then the edge* uv *lies on a cycle* C *that intersects every path in* $H + uv$ *between* 0 *and* 1.

Proof For (a), this follows from the observation that edge adding cannot decrease the genus of an imbedding.

For (b), we suppose, by way of contradiction, that cycles C and C' in H intersect, and we suppose that some edge uv is in C, but not in C'. Then we extend uv (as a path) in one directions along the cycle C until the first intersection with a vertex u' of the cycle C', and we extend uv in the other direction until we meet a vertex v' of C'. If $u' = v'$, then $C \cup C' \cong B_2$. Otherwise, if A is one of the arcs from u' to v' in C, then $A \cup C' \cong D_3$. That is, the super-rung H contains a homeomorphic copy either of the bouquet B_2 or of the dipole D_3, both of which have imbeddings in the torus, contradicting (a).

For (c), we simply observe that if 0 and 1 are on separate faces of an imbedding, then the only kind of edge addition that can put them on the same face must join two faces, and such edge addition do not occur, as per Remark 5.1.

For (d), we observe that combining the premise of (d) with Rule 4.2 implies the transformation

$$(uPvQ) \to (uP)(vQ)$$

of i-types, for some sequences P and Q of vertices of H with $0 \in P$ and $1 \in Q$. Since the sequence P describes a walk from u to v. It follows that there is a cycle C containing uv and some of the edges along walk P. By the Jordan curve theorem, the cycle C separates the plane into two components. The face with fb-walk (uP) lies in one component and face with fb-walk (vQ) lies in the other component, since they lie on opposite sides of the shared edge uv. In particular, root vertex 0 lies in one component and root vertex 1 in the other, so that every path between 0 and 1 intersects C. □

Theorem 5.7 *Let* $(H, 0, 1)$ *be a planar-only graph with 1-valent roots* $0, 1$. *Let* P *be a path between* 0 *and* 1.

(a) *Then the intersection of any cycle of* H *with the path* P *is empty, a single vertex, or a path.*

(b) *Let r be the number of cycles with one-point intersection, and let s be the number of cycles that intersect P in a path. Then the partial genus polynomials are*

$$p(z) = (6^r 4^s - 4^r 2^s)N \text{ for i-type (0)(1), and}$$
$$q(z) = 4^r 2^s N \text{ for i-type (01)},$$

where $N = \Pi_v(d_v - 1)!/6^r 4^s$.

Proof Let H' be the union of the path P and all of the cycles of H that intersect path P. Then the subgraph H' has exactly r 4-valent vertices, exactly $2s$ 3-valent vertices, and exactly two 1-valent vertices, namely 0 and 1; all other vertices of subgraph H' are 2-valent. It follows that the subgraph H' has $(3!)^r(2)^{2s}$ imbeddings. Let N be the number of imbeddings to which each imbedding of H' extends, and let d_v denote the valence of a vertex v.

The vertices of H contribute the following factors to the value of N.

$$\begin{cases} (d_v - 1)! & \text{if } v \text{ is not in } H; \\ \dfrac{(d_v - 1)!}{3!} & \text{if } v \text{ is 4-valent in } H; \\ \dfrac{(d_v - 1)!}{2!} & \text{if } v \text{ is 3-valent in } H; \end{cases}$$

Thus,

$$N = \frac{\prod_{v \in V_H}(d_v - 1)!}{6^r 4^s}.$$

By part (c) of Proposition 5.6, every imbedding of i-type (0)(1) in H' extends only to imbeddings of i-type (0)(1) in H. By part (d), there is no edge addition that transforms an i-type (01) imbedding of H' into an i-type (0)(1) imbedding of H, since otherwise, there is a cycle not in H' that intersects the path P.

It remains, therefore, to calculate the number of imbeddings of H' of each of the i-types (0)(1) and (01). We have an i-type (01) imbedding of H' if and only if at every intersection of path P with each cycle of H', the path P remains on the same side of that cycle. At each of the r intersections of P with a cycle at a single vertex (a 4-valent vertex of H'), there are 4 rotations at that vertex for which P remains on the same side of the cycle. For each of the s path intersections of path P with a cycle, there are 2 pairs of rotations at the endpoints of that path (which are 3-valent vertices) such that P remains on the same side of the cycle. Thus, there are $4^r 2^s$ such rotations in all. The remaining $6^r 2^{2s} - 4^r 2^s$ imbeddings of H' are the imbeddings with i-type (0)(1). □

Corollary 5.8 *Let* $(H, 0, 1)$ *be a planar-only super-rung, so that its partial genus polynomials are constants. Let* $p(z) = a$ *be the partial genus polynomial for i-type (0)(1), and let* $q(z) = b$ *be the partial genus polynomial for i-type (01). Furthermore, let P be a path in H between root vertex 0 and root vertex 1, let r be the number of cycles of H at which the incidence of P is a single vertex, and let s be the number of cycles*

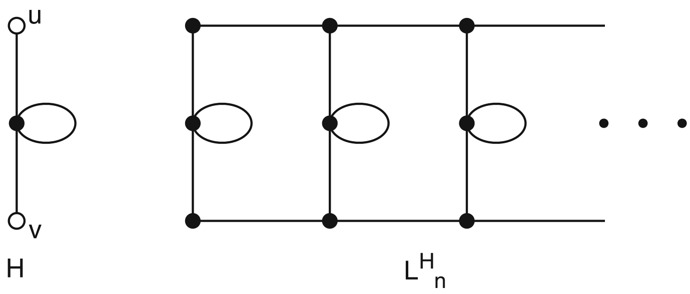

Fig. 4 A ladder-like graph with log-concave genus polynomial

at which the incidence of P is a subpath of P with at least one edge. If $r + s \leq 1$, then the genus polynomial of every ladder-like graph with super-rung H is log-concave.

Proof It follows from Theorem 5.7 that if $6^r 4^s - 4^r 2^s \leq 4^r 2^s$, then $a \leq b$. Since the inequality $r + s \leq 1$ implies that $6^r 4^s - 4^r 2^s \leq 4^r 2^s$, it follows, in turn, from Corollary 5.5, that the genus polynomials of the ladder-like graphs with super-rung H are log-concave whenever $r + s \leq 1$. □

Corollary 5.9 *Let $(H, 0, 1)$ be a super-rung whose Betti number $\beta(H)$ is at most one. Then the genus polynomial of every graph in the corresponding sequence of ladder-like graphs is log-concave.*

Example 5.1 It follows from Corollary 5.9 that the genus polynomials of the graphs in Figs. 1 and 4 are log-concave.

6 Summary

This paper continues the pursuit of families of graphs for which genus polynomials can be calculated by recursions or closed formulas. It establishes a formula for the genus polynomials of the graphs in the ladder-like sequence, combining the partial genus polynomials of the super-rung. It derives sufficient conditions on the super-rung H under which each of the genus polynomials of the graphs in the ladder-like sequence is log-concave, and it exhibits examples of super-rungs that satisfy these conditions. In particular, these genus polynomials are log-concave whenever $\beta(H) \leq 1$.

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Affiliations

Yichao Chen[1] · Jonathan L. Gross[2] · Toufik Mansour[3] · Thomas W. Tucker[4]

Jonathan L. Gross
gross@cs.columbia.edu

Toufik Mansour
tmansour@univ.haifa.ac.il

Thomas W. Tucker
ttucker@colgate.edu

1 School of Mathematics and Physics, SuZhou University of Science and Technolgy, SuZhou 215009, China

2 Department of Computer Science, Columbia University, New York, NY 10027, USA

3 Department of Mathematics, University of Haifa, 3498838 Haifa, Israel

4 Department of Mathematics, Colgate University, Hamilton, NY 13346, USA