On the Laplacian and signless Laplacian polynomials of graphs with semiregular automorphisms

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Abstract

A graph Γ is called n -Cayley graph over a group G if $\text{Aut}(\Gamma)$ has a semiregular subgroup isomorphic to G with n orbits (of equal size). In this paper, we give a decomposition of the Laplacian and signless Laplacian polynomials of n -Cayley graphs in terms of irreducible representations of G . Also, we construct several families of graphs with integral Laplacian and signless Laplacian spectrum.

Keywords n -Cayley graph · Laplacian eigenvalue · Signless Laplacian eigenvalue · Integral graph

Mathematics Subject Classification 05C50 · 05C25 · 05C31

1 Introduction

In this paper, a graph means a finite, undirected and simple graph and a group is finite unless specified otherwise. For a graph Γ , we use $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ to denote its vertex set, edge set and automorphism group. For the most part, our notation and terminology are standard and mainly taken from [5] (for graph theory) and [15] (for representation theory of finite groups). For the graph-theoretic and group-theoretic terminology not defined here, we refer the reader to [5, 15].

A graph Γ is called an n -Cayley graph over a group G if there exists a semiregular subgroup of $\text{Aut}(\Gamma)$ isomorphic to G with n orbits (of equal size). A graph is called a Cayley graph over a group G if its automorphism group contains a regular subgroup isomorphic to G . Indeed, every Cayley graphs is a 1-Cayley graph. Also, a Cayley graph over a finite group G having a subgroup H of index n is an n -Cayley graph over H [1, Lemma 8]. Also, n -Cayley graphs over cyclic groups are called n -circulant. In particular, 2-Cayley and 3-Cayley graphs over cyclic groups are called bicirculant

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and trirculant graphs [23], respectively. Unlike Cayley graphs, in general n -Cayley graphs are not vertex transitive for $n \geq 2$. Furthermore, there are vertex-transitive n -Cayley graphs which are not Cayley graphs such as generalized Petersen graphs.

Undirected and loop-free 2-Cayley graphs are by some authors called semi-Cayley graphs [2,27] and also bi-Cayley graphs [18]. In this paper, we follow [27] to use the term semi-Cayley.

n -Cayley graphs, in particular when $n = 2$ or $n = 3$, have been playing an important role in many classical fields of graph theory, such as strongly regular graphs [19, 20,22,23,27], symmetry properties of graphs [7,8,21] and the spectrum of graphs [1,3,4,11,12].

The following characterization of (undirected and without loop) n -Cayley graphs is given by the present author et. al in [1, Lemma 2]. Let T_{ij} , $1 \leq i, j \leq n$, be n^2 subsets of a group G such that $T_{ij}^{-1} = T_{ji}$ (here $X^{-1} = \{x^{-1} \mid x \in X\}$) and $1 \notin T_{ii}$ for all $1 \leq i, j \leq n$. Define a graph Γ with vertex set $G \times \{1, \dots, n\}$ and with vertices (g, i) , (h, j) adjacent if and only if $hg^{-1} \in T_{ij}$. Then, Γ is an n -Cayley graph over G , and conversely, every n -Cayley graph over G can be obtain in this way. In particular, let Γ be a semi-Cayley graph over a group G . Then, there exists subsets $T_{11} = R = R^{-1}$, $T_{22} = S = S^{-1}$, $1 \notin R \cup L$, $T_{12} = T$ and $T_{21} = T^{-1}$ of G such that $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq 2)$. Thus, the presentation of Γ only depends on R , S and T . So we denote Γ by $\text{SC}(G; R, S, T)$.

Let Γ be a graph with vertex set $\{v_1, \dots, v_n\}$. The adjacency matrix of Γ is an $n \times n$ matrix $A = [a_{ij}]$, where $a_{ij} = 1$ whenever v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The degree matrix of Γ is a diagonal $n \times n$ matrix $D = \text{diag}(d_1, \dots, d_n)$, where d_i is the number of vertices adjacent to v_i . The matrices $L = D - A$ and $Q = D + A$ are called *Laplacian* and *signless Laplacian* matrices of Γ , respectively. The characteristic polynomial of an $n \times n$ matrix X is $\chi_X(\lambda) = \det(\lambda I_n - X)$, where I_n is the $n \times n$ identity matrix and the roots of this polynomial are called eigenvalues of X . In this paper, the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of a graph Γ are eigenvalues of adjacency, Laplacian and signless Laplacian matrices of Γ , respectively.

It is known that numerous proofs in graph theory depend on the spectrum (the multiset of eigenvalues) of graphs and the spectrum of a graph is one of the most important algebraic invariants. The basic relationships between algebraic properties of these eigenvalues and the usual properties of graphs are available in [5]. Also, Laplacian eigenvalues of a graph can be used in several areas of mathematical research and have a physical interpretation in various physical and chemical theories. For a survey of known results about the spectrum of Laplacian matrices of graphs until 1991, we refer the reader to [26]. The adjacency and Laplacian matrix of a graph were much more investigated in the past than the signless Laplacian matrix. Since 2005, several papers on the signless Laplacian eigenvalues of a graph have been published. For a survey on this topic, we refer to [6].

There are several papers relating to the spectrum of Cayley graphs. Since Cayley graphs are regular, one can obtain their Laplacian and signless Laplacian from their adjacency spectrum. But n -Cayley graphs are not necessarily regular. So it seems that computing their Laplacian and signless Laplacian eigenvalues is important. Spectrum

of semi-Cayley graphs over abelian groups are computed in 2010 [11]. This result was improved for n -Cayley graphs over arbitrary groups in 2013 by the present author and Taeri [1]. In 2015, the Laplacian and signless Laplacian spectrum of semi-Cayley graphs over abelian groups is computed by Gao et al [12]. It seems that their argument cannot be extended to non-abelian groups or n -Cayley graphs for $n \geq 3$. In this paper, we improve this result. We find a factorization of the characteristic polynomial of $M_\alpha(\Gamma) := D + \alpha A$, where D and A are the degree and adjacency matrices of an n -Cayley graph Γ over a group G and α is a number, in terms of irreducible representations of G . In particular, we compute the Laplacian and signless Laplacian eigenvalues of semi-Cayley graphs and tri-Cayley graphs over abelian groups. Also, we compute the Laplacian and signless Laplacian eigenvalues of n -sunlet graph and Tutte's 8-edge graph as a result of our main result.

One of the themes that has arisen in the literature on Laplacian (signless Laplacian) eigenvalues for graphs is that of Laplacian (signless Laplacian) integral graphs—i.e., those graphs whose all of their Laplacian (signless Laplacian) eigenvalues are integers. In [9,13,14,24,25,29], various families of Laplacian and signless Laplacian integral graphs were identified. Also, [16,17,28,30] provide constructions for certain classes of Laplacian and signless Laplacian integral graphs. It is not difficult to see that any regular graph that is adjacency integral (i.e., all eigenvalues of its adjacency matrix are integers) is necessarily Laplacian and signless Laplacian integral as well. One of the interesting challenges in dealing with Laplacian (signless Laplacian) integral graphs is that of describing, constructing and understanding families of Laplacian (signless Laplacian) integral graphs that are not regular. In this paper, we construct several Laplacian and signless Laplacian integral n -Cayley graphs.

2 Main results

The author et al. determined the characteristic polynomial of an n -Cayley graph over finite group G in [1, Theorem 6]. In this paper, we compute the Laplacian and signless Laplacian polynomial of n -Cayley graphs. First, let us recall some notations of [1]. Let G be a finite group and $\mathbb{C}[G]$ be the $\mathbb{C}$ -vector space of all complex-valued functions on G with basis $\{e_g \mid g \in G\}$, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

Let $\text{Irr}(G) = \{\varrho^{(1)}, \dots, \varrho^{(m)}\}$ be the set of all irreducible inequivalent complex matrix representations of G and d_k be the degree of ϱ_k , $k = 1, \dots, m$. Let $\varrho_{ij}^{(k)}(g)$ be the ij th entry of $\varrho^{(k)}(g)$, $1 \leq i, j \leq d_k$ and $\bar{\varrho}_{ij}^{(k)} = \overline{\sum_{g \in G} \varrho_{ij}^{(k)}(g) e_g}$, where $\bar{z}$ denotes the complex conjugate of the complex number z . For a $1 \times m$ vector v and $1 \leq i \leq n$, we define v^i to be a $1 \times nm$ vector with n blocks, whose i th block is v and other blocks are $0_{1 \times m}$. So e_g^i is the $1 \times n|G|$ vector with n blocks, where i th block is e_g , as defined, and other blocks are $0_{1 \times |G|}$ vectors. Let V be the vector space with basis $\{e_g^i \mid g \in G, 1 \leq i \leq n\}$, $\dim_{\mathbb{C}}(V) = n|G|$, and we may view the adjacency

matrix A of an n -Cayley graph $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ of G as a linear map $A : V \rightarrow V$. Keeping these notations, we have

Theorem 1 (1) $\mathcal{B} = \{\overline{\varrho_{ij}^{(l),k}} \mid 1 \leq i, j \leq d_l, 1 \leq l \leq m, 1 \leq k \leq n\}$ is a basis for V , where $\overline{\varrho_{ij}^{(l),k}} = \sum_{g \in G} \varrho_{ij}^{(l)}(g) e_g^k$ [1, Theorem 6].
 (2) $A \overline{\varrho_{ij}^{(l),k}} = \sum_{s=1}^n \sum_{t \in T_{ks}} \sum_{r=1}^{d_l} \varrho_{ri}^{(l)}(t) \overline{\varrho_{rj}^{(l),s}}$ [1, Corollary 5].
 (3) $[A]_{\mathcal{B}} = \text{diag}(I_{d_1} \otimes A_1, I_{d_2} \otimes A_2, \dots, I_{d_m} \otimes A_m)$, where $A_l = [A_{ij}^{(l)}]$ and $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \varrho^{(l)}(t)$ [1, Theorem 6].

Now, let $D = [d_{(g,i)(h,j)}]_{g,h \in G, 1 \leq i, j \leq n}$ be the degree matrix of Γ . Then, $d_{(g,i)(g,i)} = \sum_{j=1}^n |T_{ij}|$ for all $g \in G$ and $1 \leq i \leq n$, and $d_{(g,i)(h,j)} = 0$ whenever $g \neq h$ or $i \neq j$. We may view D as a linear map $D : V \rightarrow V$ where $D e_g^i = \sum_{j=1}^n \sum_{h \in G} d_{(h,j)(g,i)} e_h^j$. Then, we have the following lemma:

Lemma 1 (1) $D e_g^i = \delta_i e_g^i$, where $\delta_i = \sum_{j=1}^n |T_{ij}|$.
 (2) $D \overline{\varrho_{ij}^{(l),k}} = \delta_k \overline{\varrho_{ij}^{(l),k}}$.

Proof (1) immediately follows from the fact that $d_{(h,j)(g,i)} = 0$ if and only if $j \neq i$ or $h \neq g$. We have

$$\begin{aligned} D \overline{\varrho_{ij}^{(l),k}} &= D \left(\sum_{g \in G} \overline{\varrho_{ij}^{(l)}(g)} e_g^k \right) \\ &= \sum_{g \in G} \overline{\varrho_{ij}^{(l)}(g)} D e_g^k \\ &= \sum_{g \in G} \overline{\varrho_{ij}^{(l)}(g)} \delta_k e_g^k \quad (\text{By (1)}) \\ &= \delta_k \sum_{g \in G} \overline{\varrho_{ij}^{(l)}(g)} e_g^k \\ &= \delta_k \overline{\varrho_{ij}^{(l),k}}, \end{aligned}$$

which proves (2). $\square$

Now, we are ready to prove the main result of the paper. Let $\alpha \in \mathbb{C}$ and $M_\alpha = D + \alpha A$, where D and A are the degree and adjacency matrices of graph Γ . Note that M_{-1} and M_{+1} are Laplacian and signless Laplacian matrices of Γ , respectively.

Theorem 2 Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ be an n -Cayley graph over a finite group G , $\text{Irr}(G) = \{\varrho^{(1)}, \dots, \varrho^{(m)}\}$, d_l be the degree of $\varrho^{(l)}$, and $A_l = [A_{ij}^{(l)}]$ is $nd_l \times nd_l$ block matrix with $d_l \times d_l$ blocks $A_{ij}^{(l)}$, $1 \leq i, j \leq n$, where $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \varrho^{(l)}(t)$ and $D_l = \text{diag}(\delta_1, \dots, \delta_n) \otimes I_{d_l}$, where $\delta_i = \sum_{j=1}^n |T_{ij}|$. Then, $M_\alpha(\Gamma)$ is similar to

$$\text{diag}(I_{d_1} \otimes M_{1,\alpha}, I_{d_2} \otimes M_{2,\alpha}, \dots, I_{d_m} \otimes M_{m,\alpha}),$$

where $M_{l,\alpha} = D_l + \alpha A_l$. In particular, $\chi_{M_\alpha(\Gamma)}(\lambda) = \prod_{l=1}^m \chi_{M_{l,\alpha}}(\lambda)^{d_l}$.

Proof By the proof of [1, Theorem 6], $V = \bigoplus_{l=1}^m \bigoplus_{j=1}^{d_l} V_j^{(l)}$, where $V_j^{(l)}$ is a vector space with basis $\mathcal{C}_j^{(l)} = \langle \bar{\varrho}_{ij}^{(l),k} \mid 1 \leq i \leq d_l, 1 \leq k \leq n \rangle$. By Lemma 1, $V_j^{(l)}$ is D -invariant subspace of V and $\left[D|_{V_j^{(l)}} \right]_{\mathcal{C}_j^{(l)}} = \left[D|_{V_{j'}^{(l)}} \right]_{\mathcal{C}_{j'}^{(l)}}$ for all $1 \leq j, j' \leq d_l$. So by the primary decomposition theorem, $[D]_{\mathcal{B}} = \text{diag}(I_{d_1} \otimes D_1, I_{d_2} \otimes D_2, \dots, I_{d_m} \otimes D_m)$, where $D_l = \left[D|_{V_1^{(l)}} \right]_{\mathcal{C}_1^{(l)}}$, $1 \leq l \leq m$. By ordering

$$\varrho_{11}^{(l),1}, \varrho_{21}^{(l),1}, \dots, \varrho_{d_l 1}^{(l),1}, \varrho_{11}^{(l),2}, \dots, \varrho_{d_l 1}^{(l),2}, \dots, \varrho_{11}^{(l),n}, \dots, \varrho_{d_l 1}^{(l),n}$$

of the generators of $\mathcal{C}_1^{(l)}$, by Lemma 1, $D_l = \text{diag}(\delta_1, \dots, \delta_n) \otimes I_{d_l}$. Now, the result follows from Theorem 1(3). $\square$

Corollary 1 Keeping the notations of Theorem 2,

- (1) the set of eigenvalues of $M_\alpha(\Gamma)$ is $\bigcup_{l=1}^m \xi_l$, where ξ_l is the set of eigenvalues of $M_{l,\alpha}$. In particular, each eigenvalue of matrix $[a_{ij}]$, where $a_{ii} = \delta_i + \alpha|T_{ii}|$ and $a_{ij} = \alpha|T_{ji}|$, where $1 \leq i, j \leq n$ and $j \neq i$, is an eigenvalue of $M_\alpha(\Gamma)$,
- (2) if the eigenvalue λ occurs with multiplicity $m_l(\lambda)$ in $M_{l,\alpha}$, then the multiplicity of λ in $M_\alpha(\Gamma)$ is $\sum_{l=1}^m d_l m_l(\lambda)$,
- (3) $M_\alpha(\Gamma)$ is integral if and only if $M_{l,\alpha}$ is integral for all $l = 1, \dots, m$.

Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$. Let for all $g \in G$ and all $1 \leq i, j \leq n$, we have $g^{-1}T_{ij}g = T_{ij}$ (in other words T_{ij} 's are a union of conjugacy classes of G). Then, we call Γ is a quasiabelian n -Cayley graph.

Theorem 3 Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ be a quasiabelian n -Cayley graph over G , $\text{Irr}(G) = \{\varrho^{(1)}, \dots, \varrho^{(m)}\}$ and η_l be the corresponding character of $\varrho^{(l)}$. Put $c_{ij}^{(l)} = \frac{\sum_{t \in T_{ji}} \eta_l(t)}{d_l}$ and $B_l = [c_{ij}^{(l)}]_{1 \leq i, j \leq n}$. Then, $\chi_{M_\alpha(\Gamma)}(\lambda) = \prod_{l=1}^m \chi_{N_{l,\alpha}}(\lambda)^{d_l^2}$, where $N_{l,\alpha} = \text{diag}(\delta_1, \dots, \delta_n) + \alpha B_l$.

Proof By Theorem 2, $\chi_{M_\alpha(\Gamma)}(\lambda) = \prod_{l=1}^m \chi_{M_{l,\alpha}}(\lambda)^{d_l}$ where $M_{l,\alpha} = D_l + \alpha A_l$, $D_l = \text{diag}(\delta_1, \dots, \delta_n) \otimes I_{d_l}$, $A_l = [A_{ij}^{(l)}]_{1 \leq i, j \leq n}$ and $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \varrho^{(l)}(t)$. Since $g^{-1}T_{ji}g = T_{ji}$, for every $g \in G$, we have

$$\begin{aligned}
\varrho^{(l)}(g)A_{ij}^{(l)}\varrho^{(l)}(g)^{-1} &= \sum_{t \in T_{ji}} \varrho^{(l)}(g)\varrho^{(l)}(t)\varrho^{(l)}(g^{-1}) \\
&= \sum_{t \in T_{ji}} \varrho^{(l)}(gtg^{-1}) \\
&= \sum_{t \in T_{ji}} \varrho^{(l)}(t) \\
&= A_{ij}^{(l)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
A_{ij}^{(l)} &= \frac{1}{|G|} \sum_{g \in G} \varrho^{(l)}(g)A_{ij}^{(l)}\varrho^{(l)}(g)^{-1} \\
&= \frac{1}{d_l} \text{tr}(A_{ij}^{(l)})I_{d_l} \quad (\text{By Schur's Lemma}) \\
&= \frac{1}{d_l} \left(\sum_{t \in T_{ji}} \text{tr}(\varrho^{(l)}(t)) \right) I_{d_l} \\
&= \frac{1}{d_l} \left(\sum_{t \in T_{ji}} \eta_l(t) \right) I_{d_l}
\end{aligned}$$

Then, $A_{ij}^{(l)} = c_{ij}^{(l)} I_{d_l}$ and $A_l = [c_{ij}^{(l)} I_{d_l}]_{1 \leq i, j \leq n} = B_l \otimes I_{d_l}$. Hence, by Theorem 2, $\chi_{M_\alpha(\Gamma)}(\lambda) = \prod_{l=1}^m \chi_{N_{l,\alpha}}(\lambda)^{d_l^2}$, where $N_{l,\alpha} = \text{diag}(\delta_1, \dots, \delta_n) + \alpha B_l$. $\square$

Example 1 Let n be an odd integer, $\Gamma = \text{SC}(G; R, S, T)$, where $G = \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle \cong D_{2n}$ is a dihedral group of order $2n$, $R = \{a^k, a^{n-k}\}$ for some $1 \leq k \leq (n-1)/2$, $S = \{ba^j \mid 0 \leq j \leq n-1\}$ and $T = \{1\}$. Then,

(1) Laplacian eigenvalues of Γ are $2, 0, n+1 \pm \sqrt{n^2+1}$ with multiplicity 1 and

$$\frac{n+4-2\cos\frac{2km\pi}{n} \pm \sqrt{(n-2+2\cos\frac{2km\pi}{n})^2+4}}{2}, \quad 1 \leq m \leq \frac{n-1}{2}$$

with multiplicity 4.

(2) Signless Laplacian eigenvalues of Γ are $n+3 \pm \sqrt{(n-2)^2+1}, 3 \pm \sqrt{5}$ with multiplicity 1 and

$$\frac{n+4+2\cos\frac{2km\pi}{n} \pm \sqrt{(n-2-2\cos\frac{2km\pi}{n})^2+4}}{2}, \quad 1 \leq m \leq \frac{n-1}{2}$$

with multiplicity 4.

Proof By [15, p. 182], R, S and T are conjugacy classes of G , and irreducible characters of G are

$$\begin{aligned}\mu_1 : G &\rightarrow \mathbb{C}; \quad b^i a^j \mapsto 1, \\ \mu_2 : G &\rightarrow \mathbb{C}; \quad b^i a^j \mapsto (-1)^i, \\ \theta_m : G &\rightarrow \mathbb{C}; \quad 1 \mapsto 2, a^j \mapsto \omega^{jm} + \omega^{-jm}, ba^j \mapsto 0, \quad 1 \leq m \leq (n-1)/2,\end{aligned}$$

where $\omega = \exp(2\pi i/n)$. Putting $\alpha = -1$ and $\alpha = 1$ in Theorem 3, by an easy calculation, one can get the Laplacian and signless Laplacian eigenvalues of Γ , respectively. $\square$

Since every semi-Cayley graph over an abelian group is a quasiabelian semi-Cayley graph and all character degrees of abelian groups are 1, one can get the main result of [12], see [12, Theorem 1], by putting $\alpha = 1, -1$ in the following corollary.

Corollary 2 *Let $\Gamma = \text{SC}(G; R, S, T)$ be a quasiabelian semi-Cayley graph a group G with irreducible characters μ_i , $i = 1, \dots, |G|$ and d_i be the degree of μ_i . Then, eigenvalues of $M_\alpha(\Gamma)$ are*

$$\frac{a_l \pm \sqrt{b_l + 4\alpha^2 c_l^2}}{2},$$

where $a_l = |R| + |S| + 2|T| + \alpha(\frac{\mu_l(R) + \mu_l(S)}{d_l})$, $b_l = |R| - |S| + \alpha(\frac{\mu_l(R) - \mu_l(S)}{d_l})$ and $c_l = \frac{|\mu_l(T)|}{d_l^2}$.

The n -sunlet graph on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n .

Corollary 3 *Let Γ be an n -sunlet graph. Then, $M_\alpha(\Gamma)$ has eigenvalues*

$$2 + \alpha \cos \frac{2\pi l}{n} \pm \sqrt{\left(1 + \alpha \cos \frac{2\pi l}{n}\right)^2 + \alpha^2},$$

where $l = 0, \dots, n-1$.

Proof It is easy to see that $\Gamma = \text{SC}(G, R, S, T)$, where $G = \langle a \rangle \cong \mathbb{Z}_n$, $R = \{a, a^{-1}\}$, $S = \emptyset$ and $T = \{1\}$. On the other hand, irreducible characters of G are $\mu_0, \dots, \mu_{n-1}$, where $\mu_l(a^k) = \cos \frac{2\pi kl}{n} + i \sin \frac{2\pi kl}{n}$. Now, the result follows from Corollary 2. $\square$

Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq 3)$ be a 3-Cayley graph. Then, $T_{ij}^{-1} = T_{ji}$ and $1 \notin T_{ii}$. We denote this graph by $\text{TC}(G; T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{23})$, where T_{ij} 's are the same as defined in Γ .

Corollary 4 *Let $\Gamma = \text{TC}(G; T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{23})$, where G is a finite abelian group with irreducible characters $\mu_1, \dots, \mu_{|G|}$. Then,*

- (1) *Laplacian eigenvalues of Γ are the roots of $\lambda^3 - a_l \lambda^2 + b_l \lambda - c_l = 0$, $1 \leq l \leq |G|$, where*

$$\begin{aligned}
 a_l &= -\sum_{i=1}^3 \mu_l(T_{ii}) + \sum_{i=1}^3 \delta_i, \\
 b_l &= \sum_{i,j=1, i < j}^3 \left[\mu_l(T_{ii})\mu_l(T_{jj}) - |\mu_l(T_{ij})|^2 \right] \\
 &\quad + \sum_{i,j=1, i < j}^3 \delta_i \delta_j - \sum_{i,j=1, i \neq j}^3 \delta_i \mu_l(T_{jj}), \\
 c_l &= -\left[\mu_l(T_{11})\mu_l(T_{22})\mu_l(T_{33}) - \mu_l(T_{11})|\mu_l(T_{23})|^2 - \mu_l(T_{33})|\mu_l(T_{12})|^2 \right. \\
 &\quad \left. - \mu_l(T_{22})|\mu_l(T_{13})|^2 + 2\operatorname{Re}(\overline{\mu_l(T_{13})}\mu_l(T_{12})\mu_l(T_{23})) \right] + \delta_1 \delta_2 \delta_3 \\
 &\quad - \sum_{i,j,k=1, i < j, k \neq i,j}^3 \delta_i \delta_j \mu_l(T_{kk}) + \sum_{i,j,k=1, i \neq j \neq k \neq i}^3 \delta_i \mu_l(T_{jj})\mu_l(T_{kk}) \\
 &\quad - \sum_{i,j,k=1, i \neq j < k \neq i}^3 \delta_i |\mu_l(T_{jk})|^2,
 \end{aligned}$$

- (2) *Signless Laplacian eigenvalues of Γ are the roots of $\lambda^3 - a'_l \lambda^2 + b'_l \lambda - c'_l = 0$, $1 \leq l \leq |G|$, where*

$$\begin{aligned}
 a'_l &= \sum_{i=1}^3 \mu_l(T_{ii}) + \sum_{i=1}^3 \delta_i, \\
 b'_l &= \sum_{i,j=1, i < j}^3 \left[\mu_l(T_{ii})\mu_l(T_{jj}) - |\mu_l(T_{ij})|^2 \right] + \sum_{i,j=1, i < j}^3 \delta_i \delta_j \\
 &\quad + \sum_{i,j=1, i \neq j}^3 \delta_i \mu_l(T_{jj}), \\
 c'_l &= \mu_l(T_{11})\mu_l(T_{22})\chi_l(T_{33}) - \chi_l(T_{11})|\chi_l(T_{23})|^2 \\
 &\quad - \chi_l(T_{33})|\chi_l(T_{12})|^2 - \chi_l(T_{22})|\chi_l(T_{13})|^2 \\
 &\quad + 2\operatorname{Re}(\overline{\chi_l(T_{13})}\chi_l(T_{12})\chi_l(T_{23})) + \delta_1 \delta_2 \delta_3 + \sum_{i,j,k=1, i < j, k \neq i,j}^3 \delta_i \delta_j \mu_l(T_{kk}) \\
 &\quad + \sum_{i,j,k=1, i \neq j \neq k \neq i}^3 \delta_i \mu_l(T_{jj})\mu_l(T_{kk}) - \sum_{i,j,k=1, i \neq j < k \neq i}^3 \delta_i |\mu_l(T_{jk})|^2,
 \end{aligned}$$

where $\delta_1 = |T_{11}| + |T_{12}| + |T_{13}|$, $\delta_2 = |T_{12}| + |T_{22}| + |T_{23}|$ and $\delta_3 = |T_{13}| + |T_{23}| + |T_{33}|$.

Proof Since the eigenvalues of a 3×3 matrix X are the roots of $\lambda^3 - \text{tr}(X)\lambda^2 + (\text{tr}(X^2) - \text{tr}(X)^2)\lambda - \det(X) = 0$, (1) and (2) are direct consequences of Theorem 3. $\square$

The adjacency eigenvalues of the Tutte's 8-cage graph are $(-3)^{[1]}$, $(-2)^{[9]}$, $0^{[10]}$, $2^{[9]}$, $3^{[1]}$, see [10]. Hence, since it is a bipartite 3-regular graph, its Laplacian and signless Laplacian eigenvalues are $(6)^{[1]}$, $(5)^{[9]}$, $3^{[10]}$, $1^{[9]}$, $0^{[1]}$. We recompute this as an example of Corollary 4 as follows:

Example 2 Let Γ be the Tutte's 8-cage graph. Then, $\Gamma = \text{TC}(G; T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{23})$ where $G = \langle a \rangle \cong \mathbb{Z}_{10}$, $T_{11} = \{a, a^9\}$, $T_{22} = \{a^5\}$, $T_{33} = \{a^3, a^7\}$, $T_{12} = \{1\}$, $T_{13} = \emptyset$ and $T_{23} = \{1\}$. Since the irreducible characters of G are $\mu_1, \dots, \mu_{10}$, where $\mu_l(a^j) = \omega^{(l-1)j}$ and $\omega = \exp(\pi i/5)$, using Corollary 4, the Laplacian characteristic polynomial of Γ is the product of polynomials $\lambda^3 - 4\lambda^2 + 3\lambda = \lambda(\lambda - 1)(\lambda - 3)$, $(\lambda^3 - 9\lambda^2 + 23\lambda - 15)^8 = (\lambda - 1)^8(\lambda - 3)^8(\lambda - 5)^8$ and $\lambda^3 - 14\lambda^2 + 63\lambda - 90 = (\lambda - 3)(\lambda - 5)(\lambda - 6)$. Hence, the Laplacian eigenvalues of Γ are $(6)^{[1]}$, $(5)^{[9]}$, $3^{[10]}$, $1^{[9]}$, $0^{[1]}$.

Some constructions of integral graphs from integral Cayley graphs

In this section, we assume that G is a group with $\text{Irr}(G) = \{\varrho^{(1)}, \dots, \varrho^{(m)}\}$ and $\varrho^{(1)}$ is the trivial representation of G . Also, if $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$, then $\delta_i = \sum_{j=1}^n |T_{ij}|$, $1 \leq i \leq n$. We know that $|S|$ is an eigenvalue of $\text{Cay}(G, S)$. We call this eigenvalue *trivial eigenvalue* of $\text{Cay}(G, S)$. Also, the remaining eigenvalues of $\text{Cay}(G, S)$ are eigenvalues of $\varrho^{(l)}(S)$ with multiplicity d_l , where $l = 2, \dots, m$, see [1, Corollary 7]. In what follows, using Theorem 2, we construct some Laplacian and signless Laplacian integral graphs of order $n|G|$, where $n \geq 2$.

Corollary 5 Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$, where $T_{ij} = G$ for all $i \neq j$. Then, Laplacian eigenvalues of Γ are 0 with multiplicity 1, $n|G|$ with multiplicity $n - 1$ and $|T_{ii}| + (n - 1)|G| - \theta$, where θ is a non-trivial eigenvalue of $\text{Cay}(G, T_{ii})$, $i = 1, \dots, n$. In particular, if $\text{Cay}(G, T_{ii})$ is integral for all $i = 1, \dots, n$, then Γ is Laplacian integral.

Proof Since for all $l = 2, \dots, m$, we have $\sum_{g \in G} \varrho^{(l)}(g) = 0_{d_l}$, where d_l is the degree of $\varrho^{(l)}$ and 0_{d_l} is the $d_l \times d_l$ zero matrix, by Theorem 2 the Laplacian eigenvalues of Γ are the eigenvalues of $n|G|I_n - |G|J_n$, where J_n is the $n \times n$ all ones matrix, and eigenvalues of $\text{diag}(\delta_1 I_{d_1} - \varrho^{(1)}(T_{11}), \delta_2 I_{d_2} - \varrho^{(1)}(T_{22}), \dots, \delta_n I_{d_n} - \varrho^{(1)}(T_{nn}))$ with multiplicity d_l , where $l = 2, \dots, m$. On the other hand, $\delta_i = |T_{ii}| + (n - 1)|G|$ and J_n has eigenvalues n with multiplicity 1 and 0 with multiplicity $n - 1$. This leads us to the result. $\square$

Corollary 6 Let $\Gamma = \text{SC}(G; R, S, G)$. Then, signless eigenvalues of Γ are

$$|G| + |R| + |S| \pm \sqrt{(|R| - |S|)^2 + |G|^2},$$

$|R|+|G|+\theta$ and $|S|+|G|+\gamma$, where θ and γ are non-trivial eigenvalues of $\text{Cay}(G, R)$ and $\text{Cay}(G, S)$, respectively. In particular, if $\text{Cay}(G, R)$ and $\text{Cay}(G, S)$ are integral and $|R| = |S|$, then Γ has integral signless Laplacian eigenvalues.

Proof Since for all $l = 2, \dots, m$, we have $\sum_{g \in G} \varrho^{(l)}(g) = 0_{d_l}$, where d_l is the degree of $\varrho^{(l)}$ and 0_{d_l} is the $d_l \times d_l$ zero matrix, by Theorem 2 signless Laplacian eigenvalues Γ are the eigenvalues of

$$\begin{bmatrix} 2|R|+|G| & |G| \\ |G| & 2|S|+|G| \end{bmatrix}, \begin{bmatrix} \delta_1 I_{d_l} + \varrho^{(l)}(R) & 0_{d_l} \\ 0_{d_l} & \delta_2 I_{d_l} + \varrho^{(l)}(S) \end{bmatrix}, \quad l = 2, \dots, m,$$

where $\delta_1 = |R|+|G|$ and $\delta_2 = |S|+|G|$. The eigenvalues of the right-hand side matrix are λ and μ , where λ and μ are eigenvalues of $\delta_1 I_{d_l} + \varrho^{(l)}(R)$ and $\delta_2 I_{d_l} + \varrho^{(l)}(S)$, respectively. These leads us to the result. $\square$

Corollary 7 Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$, where $T_{ii} = R$ and $T_{ij} = \{1\}$ for all $1 \leq i \leq j \leq n$ and $j \neq i$. Then, eigenvalues of $M_\alpha(\Gamma)$ are $|R|+n-1+\alpha(n+\lambda-1)$ with multiplicity 1 and $|R|+n-1+\alpha(\lambda-1)$ with multiplicity $n-1$, where λ is an eigenvalue of $\text{Cay}(G, R)$. In particular, if $\text{Cay}(G, R)$ is integral, then Γ is Laplacian and signless Laplacian integral.

Proof By Theorem 2, eigenvalues of $M_\alpha(\Gamma)$ are eigenvalues of

$$(|R|+n-1)I_{nd_l} + \alpha A_l$$

with multiplicity d_l , $l = 1, \dots, m$, where $A_l = [a_{ij}]_{1 \leq i, j \leq n}$, $a_{ii} = \varrho^{(l)}(R)$ and $a_{ij} = I_{d_l}$ for $i \neq j$. Let λ be an eigenvalue of $\varrho^{(l)}(R)$ with eigenvector v . Then, $u = (u_1, \dots, u_n)$ is an eigenvector of A_l with eigenvalue $n + \lambda - 1$ whenever $u_1 = \dots = u_n = v$. Also, the $n-1$ independent vectors

$$(-(n-1)v, v, \dots, v), (v, -(n-1)v, v, \dots, v), \dots, (v, \dots, v, -(n-1)v, v),$$

are eigenvectors of A_l corresponding to $\lambda - 1$. This shows that eigenvalues of A_l are $n + \lambda - 1$ with multiplicity 1 and $\lambda - 1$ with multiplicity $n-1$, where λ is an eigenvalue of $\varrho^{(l)}(R)$. Hence, eigenvalues of $M_\alpha(\Gamma)$ are $|R|+n-1+\alpha(n+\lambda-1)$ with multiplicity 1 and $|R|+n-1+\alpha(\lambda-1)$ with multiplicity $n-1$, where λ is an eigenvalue of $\text{Cay}(G, R)$. $\square$

Corollary 8 Let $\Gamma = \text{Cay}(G; T_{ij} \mid 1 \leq i, j \leq n)$ where $T_{ij} = R$ for all $1 \leq i, j \leq n$. Then, eigenvalues of $M_\alpha(\Gamma)$ are $n(|R|+\alpha\theta)$ with multiplicity 1, where θ is an eigenvalue of $\text{Cay}(G, R)$ and $n|R|$ with multiplicity $(n-1)|G|$. In particular, if $\text{Cay}(G, R)$ is integral, then Γ is Laplacian and signless Laplacian integral.

Proof By Theorem 2, eigenvalues of $M_\alpha(\Gamma)$ are the eigenvalues of $n|R|I_{nd_l} + \alpha J_n \otimes \varrho^{(l)}(X)$ with multiplicity d_l , where $l = 1, \dots, m$. Eigenvalues of J_n are 0 with multiplicity $n - 1$ and n with multiplicity 1, which leads us to the result. $\square$

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