On some distance-regular graphs with many vertices

Dean Crnković¹ · Sanja Rukavina¹ · Andrea Švob¹

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Abstract

We construct distance-regular graphs, including strongly regular graphs, admitting a transitive action of the Chevalley groups $G_2(4)$ and $G_2(5)$, the orthogonal group $O(7, 3)$ and the Tits group $T = {}^2F_4(2)'$. Most of the constructed graphs have more than 1000 vertices, and the number of vertices goes up to 28431. Some of the obtained graphs are new.

Keywords Strongly regular graph · Distance-regular graph · Orthogonal group · Chevalley group · Tits group

Mathematics Subject Classification 05E30 · 05E18

1 Introduction

A construction of distance-regular graphs (DRGs), and especially strongly regular graphs (SRGs), from finite groups gave an important contribution to the graph theory and the design theory (see [4,6,19]). Recently, in [11] the authors found new SRGs admitting a transitive action of some finite simple groups. Research presented in this paper shows how one can use groups as a tool to produce wide range of regular graphs. Moreover, in that way an insight to particular groups is given and some DRGs (and especially SRGs) with many vertices are constructed. In the literature, and also implemented in Sagemath (see [8,23]), there are a few examples of SRGs and DRGs with many vertices that do not belong to some infinite family of graphs. The expression “many vertices” in this paper refers to the number of vertices that is out of range given

✉ Andrea Švob
asvob@math.uniri.hr

Dean Crnković
deanc@math.uniri.hr

Sanja Rukavina
sanjar@math.uniri.hr

¹ Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia

in [5,6]. For the sake of completeness of the presented results, we also include obtained graphs on less than 1000 vertices.

We assume that the reader is familiar with the basic facts of the group theory, the theory of strongly regular graphs and the theory of distance-regular graphs. We refer the reader to [9,20] for relevant background reading in the group theory, to [4,25] for the theory of strongly regular graphs, and to [6,13] for the theory of distance-regular graphs.

In this paper we study the exceptional groups of Lie type $G_2(4)$ and $G_2(5)$, the orthogonal group $O(7, 3)$ and the Tits group T , which are the simple groups of orders 251596800, 5859000000, 4585351680 and 17971200, respectively. We refer the reader to [9,28] for more details about these groups.

Using the method outlined in Sect. 3, we constructed and classified SRGs and DRGs of diameter $d \geq 3$ from above-mentioned simple groups as follows:

- up to 20000 vertices admitting a primitive action of the group $G_2(4)$, and up to 13000 vertices admitting an imprimitive action of the group $G_2(4)$,
- up to 400000 vertices admitting a primitive action of the group $G_2(5)$, and up to 15000 vertices admitting an imprimitive action of the group $G_2(5)$,
- up to 250000 vertices admitting a primitive action of the group $O(7, 3)$, and up to 10000 vertices admitting an imprimitive action of the group $O(7, 3)$,
- up to 12000 vertices admitting a primitive action of the group T , and up to 7000 vertices admitting an imprimitive action of the group T .

Based on [4,5,17] and the other literature, we conclude that the constructed strongly regular graphs with parameters (28431, 3150, 621, 315) and (28431, 2880, 324, 288) are the first known examples of SRGs with these parameters.

Note that in the paper we use the notation from [9] to describe the structures of the groups. To find the graphs and compute their full automorphism groups, we used programmes written for Magma [3] and GAP [15]. The constructed SRGs and DRGs can be found at the link:

http://www.math.uniri.hr/~asvob/DRGs_manyVertices.zip.

2 Preliminaries

In this section we define coherent configurations and association schemes, which are the tools for the construction of graphs presented in this paper. We also give basic definitions and properties of DRGs and SRGs.

Definition 1 A coherent configuration on a finite non-empty set Ω is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ a set of non-empty relations on Ω , such that the following axioms hold.

- $\sum_{i=0}^d R_i$ is the identity relation, where $\{R_0, R_1, \dots, R_t\} \subseteq \{R_0, R_1, \dots, R_d\}$.
- $\mathcal{R}$ is a partition of Ω^2 .
- For every relation $R_i \in \mathcal{R}$, its converse $R_i^T = \{(y, x) : (x, y) \in R_i\}$ is in $\mathcal{R}$.

- (iv) There are constants p_{ij}^k known as the intersection numbers of the coherent configuration $\mathcal{R}$, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k .

We say that a coherent configuration is homogeneous if it contains the identity relation, i.e., if $R_0 = I$. If $\mathcal{R}$ is a set of symmetric relations on Ω , then a coherent configuration is called symmetric. A symmetric coherent configuration is homogeneous (see [7]). Symmetric coherent configurations are introduced by Bose and Shimamoto in [2] and called association schemes. An association scheme with relations $\{R_0, R_1, \dots, R_d\}$ is called a d -class association scheme.

Let Γ be a graph with diameter d , and let $\delta(u, v)$ denote the distance between vertices u and v of Γ . The i th-neighborhood of a vertex v is the set $\Gamma_i(v) = \{w : \delta(v, w) = i\}$. Similarly, we define Γ_i to be the i th-distance graph of Γ , that is, the vertex set of Γ_i is the same as for Γ , with adjacency in Γ_i defined by the i th-distance relation in Γ . We say that Γ is distance-regular if the distance relations of Γ give the relations of a d -class association scheme, that is, for every choice of $0 \leq i, j, k \leq d$, all vertices v and w with $\delta(v, w) = k$ satisfy $|\Gamma_i(v) \cap \Gamma_j(w)| = p_{ij}^k$ for some constant p_{ij}^k . In a distance-regular graph, we have that $p_{ij}^k = 0$ whenever $i + j < k$ or $k < |i - j|$. A distance-regular graph Γ is necessarily regular with degree p_{11}^0 ; more generally, each distance graph Γ_i is regular with degree $k_i = p_{ii}^0$.

An equivalent definition of distance-regular graphs is the existence of the constants $b_i = p_{i+1,1}^i$ and $c_i = p_{i-1,1}^i$ for $0 \leq i \leq d$ (notice that $b_d = c_0 = 0$). The sequence $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, where d is the diameter of Γ is called the intersection array of Γ . Clearly, $b_0 = k$, $b_d = c_0 = 0$, $c_1 = 0$.

A regular graph is strongly regular with parameters (v, k, λ, μ) if it has v vertices, degree k , and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by $\text{SRG}(v, k, \lambda, \mu)$. A strongly regular graph is a distance-regular graph with diameter 2 whenever $\mu \neq 0$. The intersection array of an SRG is given by $\{k, k - 1 - \lambda; 1, \mu\}$.

3 SRGs and DRGs constructed from the groups

Let G be a finite permutation group acting on the finite set Ω . This action induces the action of the group G on the set $\Omega \times \Omega$. For more information see [27]. The orbits of this action are the sets of the form $\{(\alpha g, \beta g) : g \in G\}$. If G is transitive, then $\{(\alpha, \alpha) : \alpha \in \Omega\}$ is one such orbit. If the rank of G is r , then it has r orbits on $\Omega \times \Omega$. Let $|\Omega| = n$ and Δ_i is one of these orbits. We say that the $n \times n$ matrix A_i , with rows and columns indexed by Ω and entries

$$A_i(\alpha, \beta) = \begin{cases} 1, & \text{if } (\alpha, \beta) \in \Delta_i \\ 0, & \text{otherwise.} \end{cases}$$

is called the adjacency matrix for the orbit Δ_i .

Let $A_0, \dots, A_{r-1}$ be the adjacency matrices for the orbits of G on $\Omega \times \Omega$. These satisfy the following conditions.

- (i) $A_0 = I$, if G is transitive on Ω . If G has s orbits on Ω , then I is a sum of s adjacency matrices.
- (ii) $\sum_i A_i = J$, where J is the all-one matrix.
- (iii) If A_i is an adjacency matrix, then so is its transpose A_i^T .
- (iv) If A_i and A_j are adjacency matrices, then their product is an integer-linear-combination of adjacency matrices.

If A_i is symmetric, then the corresponding orbit is called self-paired. Further, if $A_i = A_j^T$, then the corresponding orbits are called mutually paired.

The graphs obtained in this paper are constructed using the method described in [10] which can be rewritten in terms of coherent configurations in the following way.

Theorem 1 *Let G be a finite permutation group acting transitively on the set Ω and $A_0, \dots, A_d$ be the adjacency matrices for orbits of G on $\Omega \times \Omega$. Let $\{B_1, \dots, B_t\} \subseteq \{A_1, \dots, A_d\}$ be a set of adjacency matrices for the self-paired or mutually paired orbits. Then, $M = \sum_{i=1}^t B_i$ is the adjacency matrix of a regular graph Γ . The group G acts transitively on the set of vertices of the graph Γ .*

Using this method one can construct all regular graphs admitting a transitive action of the group G . We will be interested only in those regular graphs that are distance-regular, and especially strongly regular.

The running time complexity of the algorithm used for the construction of the graphs depends on a number of parameters, such as the size of the used group and subgroup, the number of orbits of the stabilizer and the number of self-paired and mutually paired orbits in a particular case. For example, one of the cases that we considered but were not able to solve is the following. The group $O(7, 3)$ has the subgroup of order 113824 and index 331695. The number of orbits of the stabilizer is 102 which makes the computation impossible to finish in a reasonable time. Moreover, working with a structure defined on 331695 elements is difficult because of the used memory space.

3.1 SRGs and DRGs from the group $G_2(4)$

The group $G_2(4)$ is the simple group of order $251596800 = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$. Up to conjugation it has 4300 subgroups, 8 of which are maximal. It belongs to Chevalley's exceptional groups of Lie type. The group $G_2(4)$ has 9 conjugacy classes of subgroups up to the index 13000. In Table 1 we give the list of all the subgroups $H_i^1 \leq G_2(4)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Using the method described in Theorem 1, we obtained all DRGs with at most 20000 vertices on which the group $G_2(4)$ acts primitively, and all DRGs with at most 13000 vertices on which the group $G_2(4)$ acts imprimitively (transitively but not primitively).

Theorem 2 *Up to isomorphism there are exactly five strongly regular graphs and exactly two distance-regular graphs of diameter $d \geq 3$ with at most 20000 vertices, admitting a primitive action of the group $G_2(4)$, and exactly one strongly regular graph and exactly one distance-regular graph of diameter $d \geq 3$ with at*

most 13000 vertices, admitting an imprimitive action of the group $G_2(4)$. The SRGs have parameters $(416, 100, 36, 20)$, $(1365, 340, 83, 85)$, $(2080, 1008, 480, 496)$, $(2016, 975, 462, 480)$ and $(4095, 2046, 1021, 1023)$, and the DRGs have 1365 and 6825 vertices, respectively. Details about the obtained strongly regular graphs are given in Table 2, and details about the obtained DRGs with $d \geq 3$ are given in Table 3.

Remark 1 The SRGs Γ_2^1 and Γ_3^1 are connected with the generalized hexagon $GH(4, 4)$ of order $(4, 4)$ [or shortly $H(4)$]. It is well known that the distance 3 graph of the point graph of a generalized hexagon $GH(4, 4)$ is strongly regular. Moreover, according to [26, Corollary 3.5.7] the generalized hexagon $H(q)$ is self-dual if and only if $q = 3^h$. Therefore, the generalized hexagon $H(4)$ is not self-dual. The graph Γ_3^1 is the distance 3 graph of the point graph of the generalized hexagon $H(4)$, and the strongly regular graph Γ_2^1 is the distance 3 graph of the point graph of the dual of the generalized

Table 1 Subgroups of the group $G_2(4)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^1	J_2	604800	416	3	Yes
H_2^1	$2^{2+8} : (3 \times A_5)$	184320	1365	4	Yes
H_3^1	$2^{4+6} : (A_5 \times 3)$	184320	1365	4	Yes
H_4^1	$U(3, 4) : Z_2$	124800	2016	3	Yes
H_5^1	$3 \cdot L_3(4) : 2_3$	120960	2080	4	Yes
H_6^1	$2^6 : (2^4 : A_5)$	61440	4095	8	No
H_7^1	$(2^2 \times ((2^4 : 2) : 2)) : (A_4 \times A_4)$	36864	6825	12	No

Table 2 SRGs constructed from the group $G_2(4)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^1 = \Gamma(G_2(4), H_1^1)$	$(416, 100, 36, 20)$	$G_2(4) : Z_2$
$\Gamma_2^1 = \Gamma(G_2(4), H_2^1)$	$(1365, 340, 83, 85)$	$G_2(4) : Z_2$
$\Gamma_3^1 = \Gamma(G_2(4), H_3^1)$	$(1365, 340, 83, 85)$	$O(7, 4) : Z_2$
$\Gamma_4^1 = \Gamma(G_2(4), H_4^1)$	$(2016, 975, 462, 480)$	$O(7, 4) : Z_2$
$\Gamma_5^1 = \Gamma(G_2(4), H_5^1)$	$(2080, 1008, 480, 496)$	$O(7, 4) : Z_2$
$\Gamma_6^1 = \Gamma(G_2(4), H_6^1)$	$(4095, 2046, 1021, 1023)$	$O(13, 2)$

Table 3 DRGs constructed from the group $G_2(4)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_7^1 = \Gamma(G_2(4), H_7^1)$	1365	3	$\{20, 16, 16; 1, 1, 5\}$	$G_2(4) : Z_2$
$\Gamma_8^1 = \Gamma(G_2(4), H_3^1)$	1365	3	$\{20, 16, 16; 1, 1, 5\}$	$G_2(4) : Z_2$
$\Gamma_9^1 = \Gamma(G_2(4), H_7^1)$	6825	6	$\{8, 4, 4, 4, 4, 4; 1, 1, 1, 1, 1, 2\}$	$G_2(4) : Z_2$

hexagon $H(4)$. For more information see [14]. The graphs Γ_3^1 and Γ_6^1 are $O(7, 4)$ and $O(13, 2)$ graphs, respectively (see [5]). The graphs Γ_1^1 , Γ_3^1 , Γ_4^1 , Γ_5^1 and Γ_6^1 are rank 3 graphs. Strongly regular graph Γ_1^1 is known as $G_2(4)$ graph. It is locally the Janko graph (see [18]), and the second subconstituent of the Suzuki tower.

Remark 2 SRGs denoted by Γ_2^1 , Γ_3^1 and Γ_6^1 can be constructed from symmetric incidence matrices with all-one diagonal of symmetric block designs with parameters $(1365, 341, 85)$ and $(4095, 2047, 1023)$. The full automorphism groups of the designs and the corresponding SRGs do not have to be the same. In Table 4 we give the details about the obtained designs.

Remark 3 The graphs Γ_7^1 and Γ_8^1 belong to the family of graphs of Lie type from Chevalley groups (see [6]). The graph Γ_9^1 is the generalized dodecagon of order $(4, 1)$ (see [12]).

3.2 SRGs and DRGs from the group $G_2(5)$

The group $G_2(5)$ is the simple group of order $5859000000 = 2^6 \cdot 3^5 \cdot 5^6 \cdot 7 \cdot 31$. Up to conjugation it has 824 subgroups, 7 of which are maximal. It belongs to Chevalley's exceptional groups of Lie type. The group $G_2(5)$ has 7 conjugacy classes of subgroups up to the index 15000. In Table 5 we give the list of all the subgroups $H_i^2 \leq G_2(5)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Using the method described in Theorem 1, we obtained all DRGs with at most 400000 vertices on which the group $G_2(5)$ acts primitively, and all DRGs with at most 15000 vertices on which the group $G_2(5)$ acts imprimitively.

Theorem 3 *Up to isomorphism there are exactly six strongly regular graphs and exactly two distance-regular graphs of diameter $d \geq 3$ with at most 400000 vertices, admitting a primitive action of the group $G_2(5)$. The SRGs have parameters $(3906, 780, 154, 156)$, $(7875, 1550, 325, 300)$, $(7875, 3224, 1348, 1300)$, $(7750,$*

Table 4 Symmetric block designs constructed from the group $G_2(4)$

Design $\mathcal{D}$	Parameters	$\text{Aut}(\mathcal{D})$
$\mathcal{D}_1^1 = \Gamma(G_2(4), H_2^1)$	2-(1365, 341, 58)	$G_2(4) : Z_2$
$\mathcal{D}_2^1 = \Gamma(G_2(4), H_3^1)$	2-(1365, 341, 58)	$PSL(6, 4) : S_3$
$\mathcal{D}_3^1 = \Gamma(G_2(4), H_6^1)$	2-(4095, 2047, 1023)	$PSL(12, 2)$

Table 5 Subgroups of the group $G_2(5)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^2	$5^{1+4} : GL_2(5)$	1500000	3906	4	Yes
H_2^2	$5^{2+3} : GL_2(5)$	1500000	3906	4	Yes
H_3^2	$3.U_3(5) : 2$	756000	7750	4	Yes
H_4^2	$L_3(5) : 2$	744000	7875	5	Yes

3024, 1148, 1200) and (7750, 1575, 300, 325), and the DRGs have 3906 vertices. Details about the obtained SRGs are given in Table 6 and about the obtained DRGs with $d \geq 3$ are given in Table 7.

Remark 4 The SRGs Γ_1^2 and Γ_2^2 are connected with the generalized hexagon $GH(5, 5)$ of order $(5, 5)$ [or shortly $H(5)$]. The graph Γ_2^2 is the distance 3 graph of the point graph of the generalized hexagon $H(5)$, and the strongly regular graph Γ_1^2 is the distance 3 graph of the point graph of the dual of the generalized hexagon $H(5)$. The graph Γ_2^2 is a rank 3 graph and known as the $O(7, 5)$ graph. Further, strongly regular graphs Γ_1^2 and Γ_2^2 can be constructed from a symmetric incidence matrix with all-one diagonal of a symmetric block design with parameters 2 -(3906, 781, 156). The full automorphism groups of symmetric block designs in these cases are the same as the automorphism groups of the corresponding SRGs. The SRGs with parameters (7875, 1550, 325, 300), (7875, 3224, 1348, 1300), (7750, 3024, 1148, 1200) and (7750, 1575, 300, 325) belong to families of SRGs related to non-singular quadrics in $PG(2m, q)$, described in [17, Chapters 7C, 7D], for $m = 3$ and $q = 5$. We checked imprimitive representations of the group $G_2(5)$ up to the degree 15000, and proved that there are no SRGs in these cases.

Remark 5 The distance-regular graphs Γ_7^2 and Γ_8^2 belong to the family of graphs of Lie type from Chevalley groups. See [6] for more information. We checked imprimitive representations of the group $G_2(5)$ up to the degree 15000, and proved that there are no DRGs in these cases.

Table 6 SRGs constructed from the group $G_2(5)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^2 = \Gamma(G_2(5), H_1^2)$	(3906, 780, 154, 156)	$G_2(5)$
$\Gamma_2^2 = \Gamma(G_2(5), H_2^2)$	(3906, 780, 154, 156)	$O(7, 5) : Z_2$
$\Gamma_3^2 = \Gamma(G_2(5), H_3^2)$	(7750, 1575, 300, 325)	$O(7, 5)$
$\Gamma_4^2 = \Gamma(G_2(5), H_3^2)$	(7750, 3024, 1148, 1200)	$O(7, 5)$
$\Gamma_5^2 = \Gamma(G_2(5), H_4^2)$	(7875, 3224, 1348, 1300)	$O(7, 5)$
$\Gamma_6^2 = \Gamma(G_2(5), H_4^2)$	(7875, 1550, 325, 300)	$O(7, 5)$

Table 7 DRGs constructed from the group $G_2(5)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_7^2 = \Gamma(G_2(5), H_1^2)$	3906	3	{30, 25, 25; 1, 1, 6}	$G_2(5)$
$\Gamma_8^2 = \Gamma(G_2(5), H_2^2)$	3906	3	{30, 25, 25; 1, 1, 6}	$G_2(5)$

Table 8 Subgroups of the group $O(7, 3)$

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^3	$2U_4(3) : 2$	13063680	351	3	Yes
H_2^3	$3^5 : U_4(2) : 2$	12597120	364	3	Yes
H_3^3	$L_4(3) : 2$	12130560	378	3	Yes
H_4^3	$G_2(3)$	4245696	1080	3	Yes
H_5^3	$3^{3+3} : L_3(3)$	4094064	1120	4	Yes
H_6^3	$2^6 : A_7$	161280	28431	12	Yes

3.3 SRGs and DRGs from the group $O(7, 3)$

The group $O(7, 3)$ is the simple group of order $4585351680 = 2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$. Up to conjugation it has 7735 subgroups, 15 of which are maximal. The group $O(7, 3)$ has 15 conjugacy classes of subgroups up to the index 10000. In Table 8 we give the list of all the subgroups $H_i^3 \leq O(7, 3)$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Using the method described in Theorem 1, we obtained all DRGs with at most 250000 vertices on which the group $O(7, 3)$ acts primitively, and all DRGs with at most 10000 vertices on which the group $O(7, 3)$ acts imprimitively.

Theorem 4 *Up to isomorphism there are exactly seven SRGs and exactly one DRG of diameter $d \geq 3$ with at most 250000 vertices, admitting a primitive action of the orthogonal group $O(7, 3)$. The SRGs have parameters $(351, 126, 45, 45)$, $(364, 120, 38, 40)$, $(378, 117, 36, 36)$, $(1080, 351, 126, 108)$, $(1120, 390, 146, 130)$, $(28431, 2880, 324, 288)$ and $(28431, 3150, 621, 315)$, and the DRG with $d \geq 3$ has 1120 vertices. Details about the obtained SRGs are given in Table 9, and details about the obtained DRG with $d \geq 3$ are given in Table 10.*

Remark 6 To the best of our knowledge the strongly regular graphs with parameters $(28431, 3150, 621, 315)$ and $(28431, 2880, 324, 288)$ are the first known examples of SRGs with these parameters. The graphs $\Gamma_1^3, \Gamma_2^3, \Gamma_3^3, \Gamma_4^3$ and Γ_5^3 are rank 3 graphs. The graph Γ_1^3 is $NO^-(7, 3)$ graph, Γ_2^3 is $O(7, 3)$ graph, the graph Γ_3^3 is $NO^+(7, 3)$ graph, the graph Γ_4^3 is $NO^+(8, 3)$ graph, and the graph Γ_5^3 is $O^+(8, 3)$ graph. The orthogonal group $O^+(8, 3)$ acts primitively on the SRGs $\Gamma_4^3, \Gamma_5^3, \Gamma_6^3$ and Γ_7^3 . Further, strongly regular graphs Γ_1^3 and Γ_3^3 can be constructed from symmetric incidence matrices with all-zero diagonal of symmetric block designs with parameters 2-(351, 126, 45) and 2-(378, 117, 36), respectively. Similarly, the strongly regular graph Γ_2^3 can be constructed from the symmetric incidence matrix with all-one diagonal of a symmetric block design with parameters 2-(364, 121, 40). The full automorphism groups of symmetric block designs in these cases are the same as the automorphism groups of the corresponding SRGs. We checked imprimitive representations of the group $O(7, 3)$ up to the degree 10000, and proved that there are no SRGs in these cases.

Table 9 SRGs constructed from the group $O(7, 3)$

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^3 = \Gamma(O(7, 3), H_1^3)$	(351, 126, 45, 45)	$O(7, 3) : Z_2$
$\Gamma_2^3 = \Gamma(O(7, 3), H_2^3)$	(364, 120, 38, 40)	$O(7, 3) : Z_2$
$\Gamma_3^3 = \Gamma(O(7, 3), H_3^3)$	(378, 117, 36, 36)	$O(7, 3) : Z_2$
$\Gamma_4^3 = \Gamma(O(7, 3), H_4^3)$	(1080, 351, 126, 108)	$O^+(8, 3) : E_4$
$\Gamma_5^3 = \Gamma(O(7, 3), H_5^3)$	(1120, 390, 146, 130)	$O^+(8, 3) : D_8$
$\Gamma_6^3 = \Gamma(O(7, 3), H_6^3)$	(28431, 3150, 621, 315)	$O^+(8, 3) : S_3$
$\Gamma_7^3 = \Gamma(O(7, 3), H_6^3)$	(28431, 2880, 324, 288)	$O^+(8, 3) : S_3$

Table 10 DRG constructed from the group $O(7, 3)$, $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_8^3 = \Gamma(O(7, 3), H_5^3)$	1120	3	{39, 36, 27; 1, 4, 13}	$O(7, 3) : Z_2$

Remark 7 The graph Γ_8^3 is the dual polar graph $B_3(3)$ (see [6]). We checked imprimitive representations of the group $O(7, 3)$ up to the degree 10000, and proved that there are no DRGs in these cases.

3.4 SRGs and DRGs from the Tits group T

The Tits group ${}^2F_4(2)'$ or T , named after Jacques Tits, is a finite simple group of order $17971200 = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. Up to conjugation it has 434 subgroups, 8 of which are maximal. It is sometimes considered as the 27th sporadic group since the Tits group itself is not a group of Lie type even the group ${}^2F_4(2)$ (which is not simple group) is a group of Lie type. The Tits group T has 10 conjugacy classes of subgroups up to the index 7000. In Table 11 we give the list of all the subgroups $H_i^4 \leq T$ which lead to the construction of SRGs or DRGs of diameter $d \geq 3$.

Using the method described in Theorem 1, we obtained all DRGs with at most 12000 vertices on which the group T acts primitively, and all DRGs with at most 7000 vertices on which the group T acts imprimitively.

Table 11 Subgroups of the group T

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^4	$L_3(3) : 2$	11232	1600	4	Yes
H_2^4	$2.2^8 : 5 : 4$	10240	1755	5	Yes
H_3^4	$2.2^8 : S_3$	6144	2925	6	Yes
H_4^4	$2^5 : ((2^4 : 5) : 2)$	5120	3510	10	No

Table 12 SRGs constructed from the group T

Graph Γ	Parameters	$\text{Aut}(\Gamma)$
$\Gamma_1^4 = \Gamma(T, H_1^4)$	(1600, 351, 94, 72)	T
$\Gamma_2^4 = \Gamma(T, H_4^4)$	(3510, 693, 180, 126)	$Fi_{22}.2$

Table 13 DRGs constructed from the group T , $d \geq 3$

Graph Γ	Number of vertices	Diameter	Intersection array	$\text{Aut}(\Gamma)$
$\Gamma_3^4 = \Gamma(T, H_2^4)$	1755	4	{10, 8, 8, 8; 1, 1, 1, 5}	$T : 2$
$\Gamma_4^4 = \Gamma(T, H_3^4)$	2925	4	{12, 8, 8, 8; 1, 1, 1, 3}	$T : 2$

Theorem 5 *Up to isomorphism there is exactly one SRG and there are exactly two DRGs of diameter $d \geq 3$ with at most 12000 vertices, admitting a primitive action of the group T and exactly one SRG with at most 7000 vertices, admitting an imprimitive action of the group T . The SRGs have parameters (1600, 351, 94, 72) and (3510, 693, 180, 126), and DRGs have 1755 or 2925 vertices. Details about the obtained strongly regular graphs are given in Table 12, and details about the obtained DRGs with $d \geq 3$ are in Table 13.*

Remark 8 The SRGs with parameters (1600, 351, 94, 72) and (3510, 693, 180, 126) are described in [21]. The graph Γ_2^4 can be constructed from the group Fi_{22} as a rank 3 graph. Moreover, it belongs to the Fisher tower (see [16,24]).

Remark 9 The graph Γ_3^4 is the generalized octagon of order (2, 4), and the graph Γ_4^4 is the generalized octagon of order (4, 2). For more information about graph Γ_3^4 we refer the reader to [1]. We checked imprimitive representations of the group T up to the degree 7000, and proved that there are no DRGs in these cases.

Remark 10 All SRGs and DRGs constructed in this paper are vertex-transitive, but some of them are also edge-transitive. The graphs that are not edge-transitive are Γ_2^1 , Γ_8^1 and Γ_1^2 .

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Appendix

We give here a construction of new examples of SRGs with parameters (28431, 3150, 621, 315) and (28431, 2880, 324, 288) using GAP [15,22,29]. To simplify the description we give a construction of the SRGs from their automorphism group $O^+(8, 3)$.

```
LoadPackage("atlasrep");;
```

```

LoadPackage("grape");
G:=AtlasGroup("O8+(3)");
H:=AtlasSubgroup(G,10);
o:=Set(Orbits(H)[2]);
oo:=Orbit(G,o,OnSets);
G28431:=Action(G,oo,OnSets);
H28431:=Action(H,oo,OnSets);
List(Orbits(H28431),x->x[1]);#[2,4,102,164,205]
OrbitLengths(H28431);#[3150,22400,960,960,960]
G1:=NullGraph(G28431);
AddEdgeOrbit(G1,[1,2]);
LocalParameters(G1,1);#[[0,0,3150],[1,621,2528],
[315,2835,0]]
G2:=NullGraph(G28431);
AddEdgeOrbit(G2,[1,102]);
AddEdgeOrbit(G2,[1,164]);
AddEdgeOrbit(G2,[1,205]);
LocalParameters(G2,1);#[[0,0,2880],[1,324,2555],
[288,2592,0]]

```

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