Automorphisms and isomorphisms of some p -ary bent functions

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Abstract

In the predecessor to this paper Dempwolff (Comm Algebra 34(3):1077–1131, 2006), group-theoretic methods were used to solve automorphism and equivalence questions for (certain) ordinary bent functions, i.e., bent functions over $\mathbb{F}_2$. Here, we consider the same problems for p -ary bent functions, p an odd prime and solve these questions for functions analogous to those which appear in Dempwolff (Comm Algebra 34(3):1077–1131, 2006). Although our general analysis is similar to the approach of Dempwolff (Comm Algebra 34(3):1077–1131, 2006), it turns out that the odd characteristic leads to simplifications: Often, the double derivative can be computed (cf. Lemma 2.10) and factorizations of the automorphism group (cf. Lemma 2.3) can be established resulting in restrictions for automorphisms and equivalence maps.

Keywords Bent function · Automorphism group · EA equivalence · Cohomology

1 Introduction

Equivalence and automorphism problems of bent functions are known to be difficult. Since the number of bent functions grows exponentially with the number of variables, simple invariants like the algebraic degree will usually not suffice to solve equivalence questions. So stronger methods seem to be necessary. In [9], the author considered classes of ordinary bent functions (bent functions over $\mathbb{F}_2$) for which such questions can be successfully treated with group-theoretic methods. For these examples, group theory works, since large groups of automorphisms are visible due to the special nature of the examples.

In this article, we consider the analogous p -ary bent functions for primes $p > 2$. Now the analysis becomes easier, since the invariant “double derivative” (cf. Sect. 2.3)

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can be computed. Moreover, the odd characteristic gives rise to a factorization lemma (Lemma 2.3), which allows to reduce the problem of EA equivalence to the more restricted problem of linear equivalence (see Cor. 2.4).

In the next section, we review preparatory results mainly from group theory. These facts are basic, but an exception is Sect. 2.5 on the H^1 -cohomology of symplectic groups, where we collect details needed for the specific Example 3.8.

In Sect. 3, we introduce the examples, which will be investigated in the subsequent sections with regard to automorphisms and equivalence. This section starts with a general construction of bent function. In Examples 3.4, 3.6 and 3.8, we consider special cases of this construction, cases in which large groups of automorphisms are visible by virtue of the construction. Example 3.13 introduces a very simple type of Maiorana functions, again admitting large groups of automorphisms.

In Sect. 4, Theorem 4.1 and Corollaries 4.2 and 4.3 solve the automorphism and equivalence problem of the examples from 3.4, 3.6 and 3.8. Theorem 5.1 and Corollary 5.2 of Sect. 5 handle these questions for bent functions from Example 3.13.

Our proofs are mainly of group-theoretic nature, but (as already mentioned) we require only basic knowledge of group theory and a few facts from cohomology theory. Standard texts like [1, 16] cover the needed background. Our notation follows the conventions of group theory; in particular, we write the symbol of an homomorphism on the right-hand side of the argument.

2 Preliminaries

2.1 Bent functions

Let p be a prime, $V = V(n, p)$ an n -dimensional $\mathbb{F}_p$ -space and $(\cdot, \cdot) : V \times V \rightarrow \mathbb{F}_p$ a non-degenerate symmetric bilinear form. The function $f : V \rightarrow \mathbb{F}_p$ is a p -ary bent function or short a bent function if for all $v \in V$

$$(B1) \quad \widehat{f}(v) \overline{\widehat{f}(v)} = p^n,$$

where

$$\widehat{f}(v) = \sum_{x \in V} \zeta_p^{f(x) - (x, v)}$$

is the *Walch coefficient* at v and $\zeta_p = \exp(2\pi i/p)$. For a function $f : V \rightarrow \mathbb{F}_p$, define the complex $p^n \times p^n$ -matrix H_f by

$$H_f(x, y) = \zeta_p^{f(x-y)}, \quad x, y \in V.$$

Then, f is bent (cf. [19, Prop. 4]) iff:

$$(B2) \quad H_f H_f^* = p^n \mathbf{1}_{p^n \times p^n},$$

where H_f^* is the complex matrix, whose entries are obtained by complex conjugation and transposition of H_f .

- Remark 2.1** (a) We use criterion (B2) in the form $\sum_{x \in V} \zeta_p^{f(x) - f(x+b)} = 0$ for all $0 \neq b \in V$.
- (b) For $p = 2$, bent functions on $V(n, 2)$ only exist if n is even, while for $p > 2$ bent functions exist in any dimension. A quadratic form Q on $V(n, p)$ is bent, iff the associated bilinear form (obtained by polarization) is non-degenerate.

2.2 Equivalence and automorphisms of functions

Let F be a field and V be an F -space. For $v \in V$, we define the *translation* $\tau_v : V \rightarrow V$ with v by $\tau_v : x \mapsto x + v$. An *affine transformation* α on V has the form $\alpha = \Phi \circ \tau_v$, $\Phi \in \text{End}(V)$. We denote by $\text{AGL}(V)$ the group of invertible affine transformation. Clearly, $\text{AGL}(V)$ is the semidirect product of $\tau(V)$ with $\text{GL}(V)$: $\text{AGL}(V) = \text{GL}(V)\tau(V)$. If $V = V_1 \oplus V_2$ is a decomposition and $\Phi \in \text{GL}(V)$ fixes V_1 and V_2 , we write $\Phi = \text{diag}(\Phi_1, \Phi_2)$ where $\Phi_i = \Phi|_{V_i}$. More generally, we write $\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$ with $\Phi_{ij} \in \text{Hom}_F(V_i, V_j)$, such that for $v_i \in V_i$ we have $(v_1 + v_2)\Phi = v_1\Phi_{11} + v_2\Phi_{21} + v_1\Phi_{12} + v_2\Phi_{22}$. Let $(\cdot, \cdot) : V \times V \rightarrow F$ be a non-degenerate symmetric bilinear form on V and W, U be subspaces, which are dual with respect to $(\cdot, \cdot)$. For $\Phi \in \text{End}(U)$, we denote by $\Phi^* \in \text{End}(W)$ the adjoint operator, i.e., $(u\Phi, w) = (u, w\Phi^*)$ for all $u \in U, w \in W$. Similarly, if $\Phi \in \text{Hom}(U, W)$, then $\Phi^* \in \text{Hom}(U, W)$ is the unique operator with $(u_1\Phi, u_2) = (u_1, u_2\Phi^*)$ for all $u_1, u_2 \in U$.

Let V, W be F -spaces and $f : V \rightarrow W$ a function. We call $\mathcal{U} = \mathcal{U}(f) = V \times W$ the *ambient space* of f , and $\Gamma(f) = \{(x, f(x)) \mid x \in V\} \subseteq \mathcal{U}$ is the *graph* of f . Let $g : V \rightarrow W$ be another function. We call f and g *CCZ equivalent*, if there exists $\alpha \in \text{AGL}(\mathcal{U})$ with $\Gamma(g) = \Gamma(f)\alpha$. A CCZ isomorphism $\alpha = \Phi \circ \tau_u$, $\Phi \in \text{GL}(\mathcal{U})$, $u \in \mathcal{U}$ is an *EA isomorphism* (EA = extended affine) if W is invariant under Φ , and α is an *affine isomorphism* if V is invariant under Φ too. For our purposes, it is convenient to call an affine isomorphism *linear* if its translation part is trivial (i.e., $u = 0$). The groups of EA automorphisms and affine automorphisms are denoted by $\mathbf{EA}(f)$ and $\mathbf{A}(f)$, respectively.

Alternatively, an element $\alpha \in \mathbf{EA}(f)$ can be uniquely described by $\Phi_{11} \in \text{GL}(V)$, $\Phi_{22} \in \text{GL}(W)$, $v \in V, w \in W$, and $\Phi_{12} \in \text{Hom}(V, W)$ such that for all $x \in V$ the equation

$$f(x\Phi_{11} + v) = f(x)\Phi_{22} + x\Phi_{12} + w$$

holds; α lies in $\mathbf{A}(f)$ if $\Phi_{12} = 0$.

Our investigation of bent functions will only concern EA equivalence, since **CCZ equivalent bent functions are already EA equivalent** (cf. [4]).

We denote by $\mathbf{E}(f)$ the subgroup of elements in $\mathbf{EA}(f)$, whose linear parts act trivially on W and $\mathcal{U}/W$. Clearly, $\mathbf{E}(f)$ is a normal subgroup of $\mathbf{EA}(f)$. If $\Phi \circ \tau_u \in$

$\mathbf{E}(f)$, $\Phi \in \text{GL}(\mathcal{U})$, $u \in \mathcal{U}$, then $(x, z)\Phi = (x, z + x\Phi_{12})$ with $\Phi_{12} \in \text{Hom}(V, W)$. With this notation, we have:

Lemma 2.2 *Let $p > 0$ be the characteristic of the finite vector space V .*

- (a) $\mathbf{E}(f)$ is an elementary abelian p -group.
- (b) The mapping $\mathbf{E}(f) \ni \Phi \circ \tau_u \mapsto u + (0 \times W) \in \mathcal{U}/(0 \times W)$ is a monomorphism.
- (c) Let f be bent and $\Phi \circ \tau_u \in \mathbf{E}(f)$. Then, $\Phi \circ \tau_u = 1$ iff $\Phi = 1$.

Proof Assertion (a) is a consequence of (b).

For (b), we observe that the mapping is a homomorphism and $\alpha = \Phi \circ \tau_u$ lies in the kernel of this mapping iff $u = (0, c) \in 0 \times W$. Then, $\alpha : \Gamma(f) \ni (x, f(x)) \mapsto (x, f(x) + x\Phi_{12} + c) \in \Gamma(f)$. This shows $f(x) = f(x) + x\Phi_{12} + c$, therefore $0 = x\Phi_{12} + c$ for all x . Then, $c = 0$ and $\Phi_{12} = 0$, i.e., $\Phi \circ \tau_u = 1$.

To (c): Finally, if $\tau_u \in \mathbf{E}(f)$, then $f(x + v) = f(x) + c$ for $u = (v, c)$. By Remark 2.1(a) first $v = 0$ and then $c = 0$, i.e., $u = 0$. $\square$

Under mild assumptions, we have the following factorization of $\mathbf{EA}(f)$.

Lemma 2.3 *Let $p > 2$ be a prime, V, W finite-dimensional $\mathbb{F}_p$ -spaces and $f : V \rightarrow W$, a function with $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in V$. Set $\mathcal{U} = \mathcal{U}(f)$. Then, $\mathbf{EA}(f) = (\mathbf{A}(f) \cap \text{GL}(\mathcal{U}))\mathbf{E}(f)$ and $\mathbf{A}(f) \leq \text{GL}(\mathcal{U})$, if f is bent.*

Proof We will prove this lemma by an Frattini argument. By assumption, $\mathbf{A}(f)$ contains the element $\iota = \text{diag}(-1_V, 1_W)$. A quick computation shows $\alpha^{-1}\iota\alpha \equiv \iota \pmod{\mathbf{E}(f)}$ for $\alpha \in \mathbf{EA}(f)$. So $\langle \iota, \mathbf{E}(f) \rangle \trianglelefteq \mathbf{EA}(f)$ and $\langle \iota \mathbf{E}(f) \rangle \leq Z(\mathbf{EA}(f)/\mathbf{E}(f))$. Let $g \in \mathbf{EA}(f)$. Then, $\langle \iota^g \rangle \in \text{Syl}_2(\langle \iota, \mathbf{E}(f) \rangle)$ by Lemma 2.2. Therefore, there exists $h \in \langle \iota, \mathbf{E}(f) \rangle$ with $\langle \iota^{gh} \rangle = \langle \iota \rangle$. This shows $gh \in C_{\mathbf{EA}(f)}(\iota)$, and we conclude $\mathbf{EA}(f) = C_{\mathbf{EA}(f)}(\iota)\mathbf{E}(f)$.

We claim $\mathbf{A}(f) \cap \text{GL}(\mathcal{U}) = C_{\mathbf{EA}(f)}(\iota)$: Clearly, $\mathbf{A}(f) \cap \text{GL}(\mathcal{U}) \leq C_{\mathbf{EA}(f)}(\iota)$. Let $\alpha = \Phi \circ \tau_u \in C_{\mathbf{EA}(f)}(\iota)$. We have $\Phi^{-1}\Phi^\iota = \tau_u(\tau_{-u})^\iota \in \text{GL}(\mathcal{U}) \cap \tau(U) = 1$ forcing $\Phi \in C_{\text{GL}(\mathcal{U})}(\iota)$ and $u \in 0 \times W = C_{\mathcal{U}}(\iota)$. Also Φ leaves invariant the decomposition $\mathcal{U} = V \oplus W = [\mathcal{U}, \iota] \oplus C_{\mathcal{U}}(\iota)$. As $(0, 0) \in \Gamma(f)$, we finally obtain $u = 0$. Hence, $\mathbf{A}(f) \cap \text{GL}(\mathcal{U}) = C_{\mathbf{EA}(f)}(\iota)$.

Let f be bent and $\alpha = \Phi \circ \beta \in \mathbf{A}(f)$ with $\Phi \in \mathbf{A}(f) \cap \text{GL}(\mathcal{U})$ and $\beta \in \mathbf{E}(f)$. The linear parts of α and Φ are fixing V . Thus, the linear part of β fixes V too, i.e., we have $\beta = \tau_u$ for some $u \in \mathcal{U}$ and $u = 0$ by Lemma 2.2(c), i.e., $\mathbf{A}(f) \leq \text{GL}(\mathcal{U})$. $\square$

A consequence of this lemma is a simplified criterion for EA equivalence:

Corollary 2.4 *Let $p > 2$ be a prime, V, W finite-dimensional $\mathbb{F}_p$ -spaces and $f, g : V \rightarrow W$, functions with $f(0) = g(0) = 0$ and $f(-x) = f(x)$, $g(-x) = g(x)$ for all $x \in V$. Let f and g be EA equivalent. Then, they are linear equivalent.*

Proof We know that $\iota \in \mathbf{A}(f) \cap \mathbf{A}(g)$, where ι is defined as in the proof of Lemma 2.3. Let $\psi \in \text{AGL}(\mathcal{U})$, such that the linear part of ψ fixes W and that $\Gamma(g) = \Gamma(f)\psi$. Then, $\mathbf{EA}(g) = \psi^{-1}\mathbf{EA}(f)\psi$, $\mathbf{E}(g) = \psi^{-1}\mathbf{E}(f)\psi$, and $\iota^\psi \in \mathbf{E}(g)\langle \iota \rangle$. We can adjust ψ by some element in $\mathbf{E}(g)\langle \iota \rangle$, such that $\psi \in C_{\text{AGL}(\mathcal{U})}(\iota)$. We have seen in the proof of Lemma 2.3 that $\psi = \Phi \circ \tau_w$, where $\Phi \in \text{GL}(\mathcal{U})$ fixes $V \times 0$ and $0 \times W$ and $w \in W$. Since $\Gamma(g) = \Gamma(f)\psi$ and as $f(0) = g(0) = 0$, we have $w = 0$. The proof is complete. $\square$

Remark 2.5 Let p be an odd prime V, W finite-dimensional $\mathbb{F}_p$ -spaces and $f : V \rightarrow W$ a function satisfying the assumptions of Lemma 2.3. In particular, if $G = \mathbf{A}(f) \cap \mathrm{GL}(\mathcal{U})$ and $\mathcal{U} = V \times W$, then G leaves the spaces $V \times 0$ and $0 \times W$ invariant. An element of $\mathbf{E}(f)$ is denoted by $\beta(v)$, with a unique $v \in V$ (Lemma 2.2), when

$$(x, z)\beta(v) = (x + v, x\lambda_v + g(v) + z), \quad (x, z) \in \mathcal{U}, \quad (1)$$

with $\lambda_v \in \mathrm{Hom}(V, W)$ and $g(v) \in W$. In fact, as $(v, g(v)) = (0, 0)\beta(v) \in \Gamma(f)$, we obtain

$$g(v) = f(v).$$

As $\beta(u + v) = \beta(u) \circ \beta(v)$, there exists a subspace $V_0 \subseteq V$, such that $\mathbf{E}(f) = \beta(V_0)$. Moreover, $\lambda : V_0 \rightarrow \mathrm{Hom}(V, W)$ is a homomorphism and we have

$$x\lambda_v = f(x + v) - f(x) - f(v),$$

since $(x + v, f(x + v)) = (x, f(x))\beta(v) = (x + v, x\lambda_v + f(v) + f(x))$. In particular, if $W = \mathbb{F}_p$, then the restriction of f to V_0 is a quadratic form. Acting with G on $\mathbf{E}(f)$ by conjugation, we see

$$\Phi^{-1}\beta(v)\Phi = \beta(v\Phi).$$

Example 2.6 Let Q be a non-degenerate quadratic form on $V = V(n, p)$ and denote by $(\cdot, \cdot)$ the (non-degenerate) bilinear form obtained from polarization of Q . An element $\Phi \in \mathrm{GL}(V)$ is called a *similarity*, if there exists $r = r_\Phi \in \mathbb{F}_p^*$, such that $Q(x\Phi) = rQ(x)$ for all $x \in V$. One calls r_Φ the *multiplier* of Φ , and we denote by $\mathrm{GO}(V, Q)$ the group of similarities. If $r_\Phi = 1$ one calls Φ an isometry; the group of isometries $\mathrm{O}(V, Q)$ is the *orthogonal group (on V)*. Clearly, the group

$$H = \{\mathrm{diag}(\Phi, r_\Phi) \in \mathrm{GL}(\mathcal{U}) \mid \Phi \in \mathrm{GO}(V, Q)\} \simeq \mathrm{GO}(V, Q),$$

$\mathcal{U} = V \times \mathbb{F}_p$ is a subgroup of $\mathbf{A}(Q)$. Assume now $p = 2$, and let $0 \neq v \in V$. Define a symplectic transvection $t(v) \in \mathrm{GL}(V)$ by $xt(v) = x + (v, x)v$. Then, $t(v) \in \mathrm{O}(V, Q)$ if $Q(v) = 1$. If, however, v is singular (i.e., $Q(v) = 0$), then $t(v)$ lies in the symplectic group $\mathrm{Sp}(V) = \{\Phi \in \mathrm{GL}(V) \mid (x\Phi, y\Phi) = (x, y), x, y \in V\}$ but not in $\mathrm{GO}(V, Q)$. We extend $t(v)$ to $\mathcal{U}$ by $(x, z)t(v) = (x + (x, v)v, z)$ and if v is singular we set $\alpha(v) = t(v) \circ \tau_v \in \mathrm{AGL}(\mathcal{U})$. We observe $\alpha(v) \in \mathbf{A}(Q)$ and that $\mathrm{Sp}(V) \simeq \langle H, \alpha(v) \mid 0 \neq v \in V \text{ singular} \rangle \leq \mathbf{A}(Q)$.

By Lemma 2.2 and Remark 2.5, $\mathbf{E}(Q) = \beta(V) \simeq (V, +)$. Here, $\beta(v) = \lambda_v \circ \tau_{(v, Q(v))}$ with $(x, z)\lambda_v = (x, (x, v) + z)$. Hence, for $p > 2$ the group $\mathbf{EA}(V)$ contains a subgroup isomorphic to $\mathrm{GO}(V, Q)V$ (semidirect product), while for $p = 2$ the group $\mathbf{EA}(V)$ contains a subgroup isomorphic to $\mathrm{Sp}(V)V$ (semidirect product).

Remark 2.7 (a) The classification of non-degenerate quadratic forms over prime field $\mathbb{F}_p$ distinguishes the following types [14, Thm. 12.9]:

- (1) $n = 2m$, $Q^+(X) = \sum_{i=1}^m X_i X_{n+1-i}$ (hyperbolic type).
- (2) $n = 2m$, $Q^-(X) = \sum_{i=1}^{m-1} X_i X_{n+1-i} + X_m^2 + aX_m X_{m+1} + bX_{m+1}^2$ (elliptic type). Here, $X^2 + aX + b$ is an irreducible polynomial over $\mathbb{F}_p$.
- (3) $n = 2m + 1$, $p > 2$, $Q(X) = \sum_{i=1}^m X_i X_{n+1-i} + X_{m+1}^2$.

Assume $p = 2$. Then, $(X_m + 1)(X_{m+1} + 1) = X_m^2 + X_m X_{m+1} + X_{m+1}^2 + 1$ (as $X_i^2 \equiv X_i$ as a function over $\mathbb{F}_2$), which shows that Q^+ and Q^- are affine equivalent as bent functions.

- (b) Let $p > 2$. The mapping $\text{GO}(V, Q) \ni \Phi \mapsto r_\Phi \in \mathbb{F}_p$ of similarities on their multipliers is a homomorphism, whose image is $\mathbb{F}_p^*$ if n is even and $(\mathbb{F}_p^*)^2$ if n is odd (cf. [15, Prop. 5(b)]).
- (c) Let (V, Q) be of hyperbolic type, $V = W \oplus W'$ a decomposition into totally singular spaces. Let r be a generator of $\mathbb{F}_p^*$. Then, $\text{diag}(1_W, r1_{W'})$ is a similarity with multiplier r .

Proposition 2.8 *Let p be a prime and let Q be a non-degenerate quadratic form on $V = V(n, p)$.*

(a)

$$\mathbf{A}(Q) \simeq \begin{cases} \text{GO}(V, Q), & p > 2, \\ \text{Sp}(V), & p = 2, \end{cases}$$

where n is even for $p = 2$ and $\text{Sp}(V)$ is the symplectic group with respect to the bilinear form obtained by polarization of Q .

- (b) $\mathbf{EA}(Q) \simeq \mathbf{A}(Q)\mathbf{E}(Q)$, $\mathbf{E}(Q) = \beta(V) \simeq (V, +)$.

Proof Let $(\cdot, \cdot)$ be the bilinear form obtained by polarization from Q and $\mathcal{U} = V \times \mathbb{F}_p$ the ambient space of Q .

Let $\alpha = \Phi \circ \tau_u$ be an element in $\mathbf{EA}(Q)$ with $\Phi \in \text{GL}(\mathcal{U})$, and $u = (v, b) \in \mathcal{U}$. Assume $(x, z)\Phi = (x\Phi_1, xL_w + az)$ with $xL_v = (x, v)$, $\Phi_1 \in \text{GL}(V)$, $w \in V$, and $a \in \mathbb{F}_p^*$, i.e., $Q(x\Phi_1 + v) = aQ(x) + xL_w + b$ for all x . Then, $Q(v) = b$ (set $x = 0$), so that $Q(x\Phi_1) + (x\Phi_1, v) = aQ(x) + xL_w$. Replace x by $x + y$ and we get

$$(x\Phi_1, y\Phi_1) = a(x, y)$$

for all x, y . So Φ_1 is a similarity with respect to $(\cdot, \cdot)$, i.e., $\Phi_1 \in \text{GO}(V, Q)$ if $p > 2$ and $\Phi_1 \in \text{Sp}(V)$ if $p = 2$. But we know by Example 2.6 that there exists $H \leq \mathbf{A}(Q) \cap \text{GL}(\mathcal{U})$, such that $H \simeq \text{GO}(V, Q)$ if $p > 2$ and $H \simeq \text{Sp}(V)$ if $p = 2$. So adjusting α with an element from H , we may assume $\Phi_1 = 1$. So $Q(x + v) = aQ(x) + xL_w + b$ for all x , which leads to $\alpha = \beta(v)$. By Example 2.6, (a) and (b) follow. $\square$

Remark 2.9 (a) Let f and g be two EA equivalent functions, then $\mathbf{EA}(f) \simeq \mathbf{EA}(g)$: Namely, if $\Psi : f \rightarrow g$ is an EA equivalence map, then $\mathbf{EA}(g) = \Psi^{-1}\mathbf{EA}(f)\Psi$.

- (b) Let n be even and Q^+ and Q^- be the two non-degenerate quadratic forms from Remark 2.7 on $V = V(n, p)$. We have already seen that the two forms are affine equivalent for $p = 2$. For $p > 2$, however, $|\text{GO}(V, Q^+)| \neq |\text{GO}(V, Q^-)|$ (cf. [16, Bem. II, 10.16]), so by Proposition 2.8 Q^+ and Q^- are not EA equivalent if $p > 2$.

- (c) If $p > 2$, then $H = \mathbf{A}(Q) \leq \mathrm{GL}(U)$ leaves invariant the decomposition $\mathcal{U} = V \oplus \mathcal{U}_0$ in accordance with Lemma 2.3. For $p = 2$, however, the assertions of this lemma do not hold: Here, $H_0 = \mathbf{A}(Q) \simeq \mathrm{Sp}(V)$ lies in $\mathrm{AGL}(\mathcal{U})$, but not in $\mathrm{GL}(\mathcal{U})$. On the other hand, $\mathrm{Sp}(V) \simeq H_1 = \langle H, \alpha(v) \circ \beta(v) \mid 0 \neq v \in V \text{ singular} \rangle$ lies in $\mathrm{GL}(\mathcal{U})$ but acts indecomposably on $\mathcal{U}$, i.e., $\mathcal{U}_0$ is the socle of the $\mathbb{F}_2 H_1$ -module $\mathcal{U}$. Note that H_0 and H_1 represent the two conjugacy classes of complements of $\mathbf{E}(Q)$ in $\mathbf{EA}(Q)$ (it is well known that $H^1(\mathrm{Sp}(2n, 2), V(2n, 2)) \simeq \mathbb{F}_2$).

2.3 Derivatives

Let V, W be two F -spaces and $f : V \rightarrow W$ a function. Let $v \in V$. We call the function $D_v f : V \rightarrow W$ defined by $D_v f(x) = f(x + v) - f(x)$ the *derivative of f in direction of v* . We need the following properties.

- Lemma 2.10** (a) For all $x \in V$, we have $D_v D_v f(x) = D_{2v} f(x) - 2D_v f(x)$.
 (b) Let $v, a \in V, b \in W$ and $\Phi_{11} \in \mathrm{End}(V), \Phi_{22} \in \mathrm{End}(W)$ and $\Phi_{12} \in \mathrm{Hom}(V, W)$. Set $g(x) = f(x\Phi_{11} + a)\Phi_{22} + x\Phi_{12} + b$. Then

$$D_v g(x) = D_{v\Phi_{11}} f(x\Phi_{11} + a)\Phi_{22} + v\Phi_{12}.$$

Proof (a)

$$\begin{aligned} D_v D_v f(x) &= D_v(f(x + v) - f(x)) \\ &= f(x + 2v) - f(x + v) - (f(x + v) - f(x)) \\ &= (f(x + 2v) - f(x)) - 2(f(x + v) - f(x)) \\ &= D_{2v} f(x) - 2D_v f(x) \end{aligned}$$

- (b) follows by a similar routine calculation. $\square$

Remark 2.11 Let $p > 2$ be a prime, $f : V = V(n, p) \rightarrow \mathbb{F}_p$ a function and $\alpha \in \mathbf{EA}(f)$. Then, there exist $\Phi \in \mathrm{GL}(V), a \in V, L \in \mathrm{Hom}(V, \mathbb{F}_p)$ and $b, c \in \mathbb{F}_p, c \neq 0$, such that $\alpha = \Phi^* \tau_{(a,b)}$, with $\Phi^* : \mathcal{U} = V \times \mathbb{F}_p \ni (x, z) \mapsto (x\Phi, zc + xL)$, i.e., $f(x\Phi + a) = f(x)c + xL + b$ for all $x \in V$. By Lemma 2.10

$$D_{v\Phi}^2 f(x\Phi + a) = D_v^2 f(x)c$$

for all $x \in V$. Therefore, Φ leaves invariant the set $\{v \in V \mid D_v^2 f = 0\}$ of vectors with a trivial double derivative.

2.4 Representations of cyclic groups

Let q be a prime power. An operator $\sigma \in \mathrm{GL}(n, q)$ of order $q^n - 1$ is called a *Singer cycle*, and we call the group generated by σ a *Singer group*. The following result is well known; see, for instance, [16, II Satz 7.3].

Lemma 2.12 *For every n and every prime power q , the group $\mathrm{GL}(U)$, $U = V(n, q)$, contains Singer groups. Moreover:*

- (a) *Every two Singer groups are conjugate in $\mathrm{GL}(U)$.*
- (b) *Let $S = \langle \sigma \rangle$ be a Singer group. Then, $N_{\mathrm{GL}(U)}(S) = S\langle \phi \rangle$, ϕ has order n and can be chosen such that $\sigma^\phi = \sigma^q$.*
- (c) *Let $A \leq \mathrm{GL}(U)$ be an irreducible abelian group. Then, A is cyclic and $C_{\mathrm{GL}(U)}(A)$ is a Singer group.*

Let p be a prime and n a positive integer. We call $0 \leq k, m < p^n - 1$ equivalent and write $k \sim m$, iff there exist a nonnegative integer s , such that

$$m \equiv p^s k \pmod{p^n - 1}.$$

Usually, one calls the equivalence classes *cyclotomic cosets of p modulo n* . We call a cyclotomic coset *long*, if it has size n , and *short* otherwise.

Let $F = \mathbb{F}_{p^n}$, i.e., $Z = (F^*, \cdot)$ is a cyclic group of order $p^n - 1$. For $1 \leq k < p^n - 1$, define a representation $D^k : Z \rightarrow \mathrm{GL}_{\mathbb{F}_p}(F)$, by $uD^k(b) = b^k u$, $u \in F$, $b \in Z$. Clearly, the eigenvalues of $D^k(b)$ lie in F ; i.e., the degree of the minimal polynomial of a $D^k(b)$ is a divisor d of n . As a direct consequence of Lemma 2.12, one has:

Lemma 2.13 *Let ζ be a generator of Z .*

- (a) *The following statements are equivalent.*
 - (1) *F as a $\mathbb{F}_p Z$ -module for the representation D^k splits into a direct sum of n/d isomorphic, simple modules all of degree d .*
 - (2) *The minimal polynomial of $D^k(\zeta)$ has degree d .*
- (b) *Every irreducible representation of Z over $\mathbb{F}_p$ is isomorphic to a simple constituent of some representation D^k , $0 \leq k < p^n - 1$.*
- (c) *The representation D^k is irreducible, iff k lies in a long coset modulo p , and D^k is faithful, iff $(k, p^n - 1) = 1$.*
- (d) *The irreducible constituents of D^k and D^ℓ are isomorphic, iff $\ell \equiv p^a k \pmod{p^n - 1}$ for some integer a , i.e., the irreducible $\mathbb{F}_p Z$ -representations are in one-to-one correspondence with the cyclotomic cosets of p modulo n .*

For $0 \leq k < n$ and $a \in F$, define the $\mathbb{F}_p$ -linear mapping $T_k(a) \in \mathrm{End}(F)$ by

$$xT_k(a) = ax^{p^k}.$$

It is well known that an $\mathbb{F}_p$ -linear operator Φ on F can be written uniquely in the form

$$\Phi = T((a)) = \sum_{i=0}^{n-1} T_i(a_i),$$

with $a = (a_0, \dots, a_{n-1}) \in F^n$. A Singer group Z can be identified with $T_0(F^*)$. Also $T_0(\zeta)$ is a Singer cycle for ζ primitive in F . Set $\Phi = T_1(1)$. Then, $T_0(\zeta)^\Phi = \Phi^{-1}T_0(\zeta)\Phi = T_0(\zeta)^p$ and $N_{\mathrm{GL}(F)}(Z) = Z\langle \Phi \rangle$.

The space $E = \text{End}_{\mathbb{F}_p}(F)$ is a $\mathbb{F}_p Z$ -module for the representation $D^{(k,\ell)} : Z \rightarrow \text{GL}(E)$, where $\Phi D^{(k,\ell)}(u) = T_0(u^k)^{-1} \Phi T_0(u^\ell)$, $\Phi \in E$, $u \in Z$, which is induced by D^k and D^ℓ . The next result is [10, Lemma 2.15]:

Lemma 2.14 *We use the preceding notation and consider $E = \text{End}_{\mathbb{F}_p}(F)$ as a $\mathbb{F}_p Z$ -module for the representation $D^{(k,\ell)}$.*

(a)

$$E = T_0(F) \oplus T_1(F) \oplus \cdots \oplus T_{n-1}(F)$$

is a decomposition into $\mathbb{F}_p Z$ -modules.

(b) *The $\mathbb{F}_p Z$ -module $T_i(F)$ affords the representation $D^{\ell-p^i k}$, with the identification of F and $T_i(F)$ via $a \mapsto T_i(a)$ and the action $T_0(u^k)^{-1} T_i(a) T_0(u^\ell) = T_i(a D^{\ell-p^i k}(u)) = T_i(u^{\ell-p^i k} a)$ for $a \in F$, $u \in F^*$.*

2.5 On the H^1 -cohomology of $\text{Sp}(2n, q)$

For the investigation of Example 3.8, we need facts about the H^1 -cohomology of $\text{Sp}(2n, 3^a)$. They follow from the more general main result of this section:

Theorem 2.15 *Let q be a prime power, $F = \mathbb{F}_q$ and $\tau \in \text{Aut}(F)$. Let $G = \text{Sp}(2n, F)$, $n \geq 2$ be the symplectic group with the natural FG -module $V = V(2n, q)$. Denote by V^τ the Frobenius twist of V under τ . Then, $H^1(G, V^\tau \otimes V) = 0$ for $n \geq 2$ except for $\tau = 1$ and $(n, q) = (2, 2), (2, 3)$.*

Remark 2.16 (a) Let G be a finite group and $c : G \rightarrow V$ a 1-cocycle into the FG -module V . We write $c = c_v$, if c is a 1-coboundary of the form $c(g) = vg - v$. Let $g, h \in G$ and $c(h) = 0$. The following is obvious:

- (1) $c(g) \in C_V(h)$ if g and h commute.
- (2) Let $c = c_v$. Then $v \in C_V(h)$.

(b) Let V be the standard FG -module for $G = \text{Sp}(2m, q)$, $F = \mathbb{F}_q$. The following information can be found for instance in the proof of [21, Thm. 5.1]: The symmetric square $S^2(V)$ is simple, if q is odd. If q is even, then the socle $\text{soc}((S^2(V))) \simeq V^\iota$ (ι the Frobenius automorphism in $\text{Aut}(\mathbb{F}_q)$) and $S^2(V)/\text{soc}((S^2(V))) \simeq \wedge^2(V)$. Denote by F the trivial $F\text{Sp}(V)$ -module. The module $\wedge^2(V)$ is the direct sum of F and a simple module, if $(q, m) = 1$. If $(q, m) \neq 1$, then $\wedge^2(V)$ has a socle series $0 \subset \text{soc}^1(\wedge^2(V)) \subset \text{soc}^2(\wedge^2(V)) \subset \wedge^2(V)$, such that $\text{soc}^1(\wedge^2(V)) \simeq \wedge^2(V)/\text{soc}^2(\wedge^2(V)) \simeq F$, and $\text{soc}^2(\wedge^2(V))/\text{soc}^1(\wedge^2(V))$ is simple. Finally, $V \otimes V \simeq \wedge^2(V) \oplus S^2(V)$ if q is odd. If q is even, one has submodules $0 \subset A \subset S \subset V \otimes V$ such that $A \simeq V \otimes V/S \simeq \wedge^2(V)$ and $V \otimes V/A \simeq S^2(V)$ and S is dual to $S^2(V)$.

In the proof of the next lemma, we will denote by VV either the symmetric square $S^2(V)$ of V or $\wedge^2(V)$ the alternating square of V . Let $V = U \perp W$ be an orthogonal decomposition. Denote for $X, Y \in \{U, W\}$ by XY the submodule of VV , which is

generated by the $x \cdot y$ (symmetric square) or $x \wedge y$ (alternating square), $x \in X$, $y \in Y$. For a non-degenerate subspace X , denote by $\text{Sp}(X)$ the subgroup of $\text{Sp}(V)$, which induces the symplectic group on X and acts trivially on $X^\perp$. Note that $XX^\perp \simeq X \otimes X^\perp$ as a $F[\text{Sp}(X) \times \text{Sp}(X^\perp)]$ -module.

Lemma 2.17 *Let $q \leq 9$ be a prime power and $G = \text{Sp}(2n, q)$, $V = V(2n, q)$, $n \geq 2$. Then, $H^1(G, S^2(V)) = H^1(G, \wedge^2(V)) = 0$ except for $H^1(G, \wedge^2(V)) = \mathbb{F}_q$, $(n, q) = (2, 2), (2, 3)$.*

Proof We have $H^1(\text{Sp}(V), VV) \simeq \text{Ext}_{F\text{Sp}(V)}(F, VV)$. For $n \leq 3$, the groups $\text{Ext}_{F\text{Sp}(V)}(F, VV)$ can be quickly computed with MAGMA [3], showing for $n \leq 3$ the assertion of the lemma. The remaining cases are verified by induction.

First, we assume $q = p = 2$ or 3 . Let $\dim V \geq 8$ and $V = U \perp W$, $\dim U = 2$. Set $K = O^p(\text{Sp}(U)) \simeq C_3$ for $p = 2$ and $\simeq Q_8$ for $p = 3$ and let $c : \text{Sp}(V) \rightarrow V$ a 1-cocycle. We have $VV = UU \oplus UW \oplus WW$. Denote by c_{XY} the part of c , which lies in XY .

We have $c_{WW}(K) = 0$ as WW is the trivial K -module and as K has no subgroup of index p . As K is a p' -group, we can adjust c by a coboundary c_w , $w \in UU \oplus UW$, such that $c(K) = 0$. By induction, we can adjust c a coboundary c_z , $z \in WW$, such that $c_{WW}(\text{Sp}(W)) = 0$. Since $c_z(K) = 0$, we still have $c(K) = 0$. Then, $c_{UW}(\text{Sp}(W)) = 0$ by (1), as K has a trivial centralizer in UW and $c_{UU}(\text{Sp}(W)) = 0$ as $\text{Sp}(W)$ has no subgroup of index p . Hence, $c(\text{Sp}(W)) = 0$.

Let $W = W_1 \perp W_2$, $\dim W_1 = 4$ and set $R = U \perp W_1$. We have $c_{RW_2}(\text{Sp}(R)) = 0$, as $c_{RW_2}(\text{Sp}(W_2)) = 0$ and $c(\text{Sp}(W_2)) = 0$ (note $W_2 \subseteq W$). Also $c_{W_2W_2}(\text{Sp}(R)) = 0$, as W_2W_2 is a trivial R -module and $\text{Sp}(R)$ has no subgroup of index p . As $K \times \text{Sp}(W_1) \leq \text{Sp}(R) \cap (K \times \text{Sp}(W))$, one has $c_{RR}(K \times \text{Sp}(W_1)) = 0$. We distinguish the cases (i) $p = 3$, $VV = S^2(V)$, (ii) $p = 2$, $VV = S^2(V)$, and (iii) $VV = \wedge^2(V)$.

By Remark 2.16(b): In case (i), the centralizer of $K \times \text{Sp}(W_1)$ in RR is trivial. In case (ii), the centralizer of $K \times \text{Sp}(W_1)$ in RR is $C_{UU}(K)$. In case (iii) the centralizer of $K \times \text{Sp}(W_1)$ in RR is $UU \oplus C_{W_1W_1}(\text{Sp}(W_1))$. In the latter case, we have $\dim UU = \dim C_{W_1W_1}(\text{Sp}(W_1)) = C_{RR}(\text{Sp}(R)) = 1$ and $C_{RR}(\text{Sp}(R))$ lies diagonally in $UU \oplus C_{W_1W_1}(\text{Sp}(W_1))$.

We know that the restriction $\tilde{c}$ of R to c_{RR} is a 1-coboundary. By (2), we conclude in case (i) $\tilde{c}(R) = 0$. Also (2) implies $\tilde{c}(z) = vz - v$, $z \in \text{Sp}(R)$ for some $v \in UU$ in case (ii) and in case (iii) we have $\tilde{c}(z) = vz - v$ for some $v \in UU \oplus C_{W_1W_1}(\text{Sp}(W_1))$. Note that in case (iii) $vz - v = (v + v_1)z - (v + v_1)$ for $v_1 \in C_{RR}(\text{Sp}(R))$ and z in R . Hence, we can choose $v \in UU$ in case (ii) as well as in case (iii). Set $c' = c$ in case (i) and $c' = c - c_v$ in cases (ii) and (iii). By definition $c'_{RR}(\text{Sp}(R)) = 0$ and as $\text{Sp}(W)$ centralizes UU , we have $c'(\text{Sp}(W)) = c(\text{Sp}(W)) = 0$. Again, $c'_{W_2W_2}(\text{Sp}(R)) = 0$, as W_2W_2 is a trivial R -module and $\text{Sp}(R)$ has no subgroup of index p . Then, $c'_{RW_2}(\text{Sp}(R)) = 0$, since $c'(\text{Sp}(W_2)) = 0$ and as $\text{Sp}(W_2)$ has a trivial centralizer in RW_2 . We conclude $c'(R) = 0$. Hence $c' = 0$, as $\text{Sp}(W)$ and $\text{Sp}(R)$ generate $\text{Sp}(V)$.

For $4 \leq q \leq 9$, we can start our induction already with $\dim V \geq 6$ and $K = \text{Sp}(U)$. All arguments of the previous section carry over. $\square$

I owe Sin [23] the proofs of the next two results.

Lemma 2.18 *Let $n > 2$, $q = p^t > 4$ a prime power, and $1 \neq \tau \in \text{Aut}(F)$, where $F = \mathbb{F}_q$. Let $G = \text{Sp}(2n, q)$, $n > 1$ and V be the standard FG -module. Then, $H^1(G, V^\tau \otimes V) = 0$.*

Proof (P. Sin) Let $e_1, \dots, e_n, f_1, \dots, f_n$ be a symplectic basis with $(e_i, f_j) = \delta_{i,j}$, i.e., the associated Gram matrix has the shape $\begin{pmatrix} 0 & 1_{n \times n} \\ -1_{n \times n} & 0 \end{pmatrix}$. Let $P = G_W$ be the stabilizer of the isotropic space spanned by the e_i 's. Then, $P = AL$ is a semidirect product, with

$$L = \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix} \mid g \in \text{GL}(W) \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} 1_{n \times n} & 0 \\ S & 1_{n \times n} \end{pmatrix} \mid S = S^t \right\}$$

is elementary abelian and in fact is an F -vector space with scalar multiplication induced by $Z(L) \simeq F^*$. The action of L on A by conjugation corresponds to the action of $\text{GL}(n, F)$ on the space $\text{Sym}(n, n, F)$ of symmetric $m \times m$ -matrices by the representation $G \times \text{Sym}(n, n, F) \ni (g, S) \mapsto g^t S g \in \text{Sym}(n, n, F)$.

As a FP -module V is the nonsplit extension of W with W^* (FP and FL have the same simple modules as $O_p(P) = A$). Thus, the FP -module $V^\tau \otimes V$ has four composition factors:

$$W^\tau \otimes W, \quad W^\tau \otimes W^*, \quad (W^*)^\tau \otimes W, \quad (W^*)^\tau \otimes W^*, \quad (2)$$

which are simple by the Steinberg tensor product theorem (W^* is the dual of W). The restriction map $H^1(G, V^\tau \otimes V) \rightarrow H^1(P, V^\tau \otimes V)$ is injective as $|G : P|$ is coprime to p , and we shall prove that its codomain is zero. From the long exact sequence of cohomology, it suffices to show that $H^1(P, X) = 0$ for each composition factor X in (2). We apply the inflation restriction sequence

$$0 \rightarrow H^1(L, X^A) \rightarrow H^1(P, X) \rightarrow H^1(A, X)^L. \quad (3)$$

In the first term of (3), we have $X = X^A$. For each X in (2), we follow easily from the fact that $Z(L)$ acts non-trivially on X (this argument fails for $q = 4$). For the third term in (3), since A is elementary abelian and acts trivially on X , we have

$$H^1(A, X)^L \simeq \text{Hom}_{\mathbb{F}_p L}(A, X).$$

As an $\mathbb{F}_p L$ -module, A is simple and the modules X are also simple $\mathbb{F}_p L$ -modules except when τ is an involution and $W^\tau \otimes W$ and $(W^*)^\tau \otimes W^*$ split into two simple $\mathbb{F}_p L$ -modules of the same dimension. Comparing dimensions, we see that the third term of (3) is zero for all X in (2). $\square$

P. Sin argues similarly in the case $q = 4$ and, however, replaces the parabolic subgroup P by a stabilizer of a 1-space, and he obtains:

Lemma 2.19 *Let $G = \text{Sp}(2n, 4)$, $n \geq 1$, τ the involution in $\text{Aut}(\mathbb{F}_4)$ and V be the standard FG -module. Then, $H^1(G, V^\tau \otimes V) = 0$.*

Alternatively, one can follow the pattern of the proof of Lemma 2.17 to obtain a verification of Lemma 2.19.

Proof (Of Theorem 2.15) For $\tau \neq 1$, the assertion of the theorem follows from Lemmas 2.18 and 2.19. So assume $\tau = 1$. We have $H^1(G, S^2(V)) = 0$ for $q \leq 9$ by Lemma 2.17 and for $q > 9$ by [26]. Also $H^1(G, \wedge^2(V)) = 0$ for $q \leq 9$ (except $(n, q) = (2, 2), (2, 3)$) by Lemma 2.17 and for $q > 9$ we have by [17] $H^1(G, L) = 0$, where L is the non-trivial simple composition factor of $\wedge^2(V)$. Together with Remark 2.16(b) and the long exact sequence of cohomology, we have again $H^1(G, \wedge^2(V)) = 0$. If q is odd, we are done by Remark 2.16(b) and for q even we use again the long exact sequence of cohomology in combination with Remark 2.16(b). $\square$

3 Some examples

In this section, we introduce classes of p -ary bent functions closely related to quadratic forms. In the subsequent sections, we shall consider equivalence of these bent functions and deal with their automorphism groups. For $p = 2$, these examples were already investigated in [9] and their affine equivalence and affine automorphisms were determined.

3.1 Perturbations of quadratic forms

A construction of bent functions. We recall two constructions of bent functions. For $i = 1, 2$, let V_i be finite-dimensional $\mathbb{F}_p$ -spaces and $g_i : V_i \rightarrow \mathbb{F}_p$ bent functions. We denote by $g_1 \oplus g_2 : V_1 \oplus V_2 \rightarrow \mathbb{F}_p$, the *direct sum of g_1 and g_2* , the bent function defined by $(g_1 \oplus g_2)(x_1 + x_2) = g_1(x_1) + g_2(x_2)$, $x_i \in V_i$ (cf. [19, Prop. 3]).

Let W be a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on W , let $(\cdot, \cdot)$ be the polarization of Q and let $W = W_1 \oplus W_2$ be a decomposition into two maximal totally singular subspaces. For a permutation ρ of W_2 and a function $h : W_2 \rightarrow \mathbb{F}_p$, define the *Maiorana–McFarland bent function* or *Maiorana function* $\mu = \mu_{\rho, h} : W \rightarrow \mathbb{F}_p$ by $\mu(x_1 + x_2) = (x_1, x_2^\rho) + h(x_2)$ (cf. [7]).

Let V be a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V and W_0 be a non-degenerate subspace of V . Let $W_0^\perp = W_1 \oplus W_2$ be a decomposition into two maximal totally singular subspaces. Let $g, k : W_0 \rightarrow \mathbb{F}_p$ be arbitrary bent functions and $\mu = \mu_{\rho, h} : W_0^\perp = W_1 \oplus W_2 \rightarrow \mathbb{F}_p$ a Maiorana function. Assume that ρ fixes 0. Define $f = f(g, k, \mu) : V \rightarrow \mathbb{F}_p$ by

$$f(x) = \begin{cases} g(x_0), & x \in W_0 \oplus W_1, \\ (k \oplus \mu)(x), & x \notin W_0 \oplus W_1, \end{cases}$$

where x_0 is the projection of $x \in W_0 \oplus W_1$ into W_0 . We have:

Proposition 3.1 $f(g, k, \mu)$ is a bent function.

Proof Let $0 \neq b \in V$. We have to show $S(f, b) = \sum_{x \in V} \zeta_p^{f(x) - f(x+b)} = 0$. We write elements x, b in V as $x = x_0 + x_1 + x_2$, $b = b_0 + b_1 + b_2$ with $x_i, b_i \in W_i$, $i = 0, 1, 2$. Set $f' = k \oplus \mu$.

CASE $b_2 = 0$. We have for $x_2 = 0$

$$\begin{aligned} \sum_{x_0, x_1} \zeta_p^{f(x) - f(x+b)} &= |W_1| \sum_{x_0} \zeta_p^{g(x_0) - g(x_0+b_0)} \\ &= |W_1| \sum_{x_0} \zeta_p^{k(x_0) - k(x_0+b_0)} = \sum_{x_0, x_1} \zeta_p^{f'(x) - f'(x+b)} \end{aligned}$$

since $0^\rho = 0$. As f and f' coincide on $V - (W_0 \oplus W_1)$, we get $S(f, b) = S(f', b) = 0$.

CASE $b_2 \neq 0$. Then, $S(f, b) = A + B + C$ with

$$\begin{aligned} A &= \sum_{x_0, x_1, x_2 \neq 0, -b_2} \zeta_p^{k(x_0) - k(x_0+b_0) + \mu(x_1+x_2) - \mu(x_1+b_1+x_2+b_2)}, \\ B &= \sum_{x_0, x_1} \zeta_p^{g(x_0) - k(x_0+b_0) - \mu(x_1+b_1+b_2)}, \quad (x_2 = 0), \end{aligned}$$

and

$$C = \sum_{x_0, x_1} \zeta_p^{k(x_0) + \mu(x_1 - b_2) - g(x_0+b_0)}, \quad (x_2 = -b_2).$$

Now $\mu(x_1 + x_2) - \mu(x_1 + b_1 + x_2 + b_2) = (x_1, x_2^\rho - (x_2 + b_2)^\rho) + c$ with a function $c = c(x_2)$ in x_2 . As a function of x_1 , this expression has the form $x_1 \lambda + c$, λ a non-trivial functional on W_1 , since ρ is a permutation. So for a fixed $x_2 \neq 0, -b_2$, we get

$$\begin{aligned} \sum_{x_0, x_1} \zeta_p^{k(x_0) - k(x_0+b_0) + \mu(x_1+x_2) - \mu(x_1+b_1+x_2+b_2)} &= \left(\zeta_p^c \sum_{x_1} \zeta_p^{x_1 \lambda} \right) \\ &\cdot \left(\sum_{x_0} \zeta_p^{k(x_0) - k(x_0+b_0)} \right) = 0, \end{aligned}$$

i.e., $A = 0$. We have $W_1 = W' \oplus \mathbb{F}_p c_1$ with $W' = W_1 \cap (\mathbb{F}_p b_2^\rho)^\perp$, $(c_1, b_2^\rho) = 1$. Write $x_1 = y + a c_1$, $y \in W'$, $a \in \mathbb{F}_p$. Then, $\mu(x_1 + b_1 + b_2) = a + (b_1, b_2^\rho) + h(b_2)$. So

$$B = \left(|W'| \zeta_p^{(b_1, b_2^\rho) + h(b_2)} \sum_{a \in \mathbb{F}_p} \zeta_p^a \right) \left(\sum_{x_0 \in W_0} \zeta_p^{g(x_0) - k(x_0+b_0)} \right) = 0.$$

Similarly, one shows $C = 0$. □

Remark 3.2 (a) One referee of this paper points out a connection between the functions $f(g, k, \mu)$ and a criterion for bentness: $f : \mathbb{F}_p^n \times \mathbb{F}_p^s \rightarrow \mathbb{F}_p$ is bent, if for all $y \in \mathbb{F}_p^s$

the function $f_y : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ defined by $f_y(x) = f(x, y)$ is s -plateaued and if the supports of the functions f_y are pairwise disjoint (see, for instance, for [7] for the definition of “ s -plateaued” and the criterion). For $f = f(g, k, \mu)$ and $x_2 \in W_2$, clearly $f_{x_2}(x_0, x_1) = f(x_0, x_1, x_2)$ defines a $\dim W_2$ -plateaued function. So for many choices of g, k and μ , the criterion is applicable.

- (b) Proposition 3.1 allows the construction of many bent functions. We consider the case, where g, k and μ are quadratic forms closely related to Q . In such cases, large parts of the automorphism group of Q are also automorphisms of f and one can hope in turn to compute the full automorphism group of f .

We will pick out some special examples, which we call of *parabolic type* in order to be consistent with the denomination of analogical examples in [9]. There is no special reason for this choice of examples other than to exhibit some group-theoretic machinery, which sometimes can solve equivalence and automorphism problems.

Assume the notation of Proposition 3.1 and set $W = W_0 \oplus W_1$. Let $\Pi \in \text{GL}(W_0)$ and $a, b \in \mathbb{F}_p, b \neq 0$. Define $Q_{W, \Pi, a}^b : V \rightarrow \mathbb{F}_p$ by

$$Q_{W, \Pi, a}^b(x_0 + x_1 + x_2) = \begin{cases} Q(x_0 \Pi), & x_2 = 0, \\ b(Q(x_0) + (x_1, x_2)) + a, & x_2 \neq 0. \end{cases} \quad (4)$$

Note that $b(Q(x_0) + (x_1, x_2)) = bQ(x_0 + x_1 + x_2)$. By Proposition 3.1, $Q_{W, \Pi, a}^b$ is bent. It has by construction a “large” group of automorphisms:

Corollary 3.3 *Let $f = Q_{W, \Pi, a}^b$ be defined as above.*

- $\mathbf{A}(f)$ contains a group $\mathcal{Q} \simeq O_p(\text{O}(V, Q)_W)$ (the p -radical of the stabilizer of W in $\text{O}(V, Q)$).
- $\mathbf{A}(f)$ contains a group $H \simeq \text{GL}(W_1)$. Moreover, H leaves the decomposition $V = W_0 \oplus W_1 \oplus W_2$ invariant, $C_V(H) = W_0$, H induces the natural representation of $\text{GL}(W_1)$ on W_1 and the dual representation on W_2 .
- $\beta(\text{rad}(W)) \leq \mathbf{E}(f)$.

Proof (a) Let $\Phi \in O_p(\text{O}(V, Q)_W)$. If $x \notin W$, then $x\Phi = x + w$, $w \in W$ and $Q(x) = Q(x\Phi)$, showing $f(x) = bQ(x) + a = bQ(x\Phi) + a = f(x\Phi)$. If $x \in W$, then $x = x_0 + x_1$, $x_i \in W_i$, and $x\Phi = x_0 + x'_1$ for some $x'_1 \in W_1$. Again $f(x) = f(x_0 + x_1) = f(x_0 + x'_1) = f(x\Phi)$ and $\text{diag}(\Phi, 1) \in \mathbf{A}(f)$ follows.

- (b) Let $W = W_0 \oplus W_1$ and $\Pi \in \text{GL}(W_0)$ as in the definition of $f = Q_{W, \Pi, a}^b$. One observes that $\Phi : \text{GL}(W_1) \ni \sigma \mapsto \Phi(\sigma) = \text{diag}(1_{W_0}, \sigma, (\sigma^{-1})^*) \in \text{O}(V, Q)$ is an injection. A routine calculation shows $f(x\Phi(\sigma)) = f(x)$ for $x \in V$. So $H = \{\text{diag}(\Phi(\sigma), 1) \mid \sigma \in \text{GL}(W_1)\} \leq \mathbf{A}(f)$.

- (c) Let u be in $\text{rad}(W)$. We have $f(u) = 0$. If we set $x\lambda_u = f(x + u) - f(x) - f(u) = f(x + u) - f(u)$, we get $x\lambda_u = 0$ if $x \in W$ and $x\lambda_u = b(u, x)$ for $x \notin W$. So $x\lambda_u = b(u, x)$ holds globally. Equation (1) of Remark 2.5 becomes $(x_0 + x_1 + x_2, z)\beta(u) = (x_0 + (x_1 + u) + x_2, z + x\lambda_u)$. We compute $(x, f(x))\beta(u) = (x + u, f(x + u))$ and obtain $\beta(u) \in \mathbf{E}(f)$. □

Some special cases of Construction (4) will be studied in the sequel.

Example 3.4 Let Q be a non-degenerate quadratic form on $V = V(n, p)$. Let W be a subspace such that $\text{rad}(W) = W^\perp$ is a non-trivial totally singular space. For $a, b \in \mathbb{F}_p, b \neq 0$, define $Q_{W,a}^b : V \rightarrow \mathbb{F}_p$ as $Q_{W,1_{W_0},a}^b$, i.e.,

$$Q_{W,a}^b(x) = \begin{cases} Q(x), & x \in W, \\ bQ(x) + a, & x \notin W. \end{cases}$$

We call $Q_{W,a}^b$ a

standard parabolic bent function.

Clearly, for $a = 0$ and $b = 1$, we have $Q = Q_{W,a}^b$, while for $W = W^\perp$ of dimension $n/2$ (n has to be even) $Q_{W,0}^b = bQ$ is again a quadratic form. Set $f = Q_{W,a}^b$. Let $p > 2$ and $K = O(V, Q)_W$ be the stabilizer of W in the orthogonal group. Clearly, any transformation $\Phi \in K$ fixes $\text{rad}(W)$ too. We consider K as a subgroup of $\text{GL}(\mathcal{U})$, $\mathcal{U} = V \times \mathbb{F}_p$ by extending Φ to $\text{diag}(\Phi, 1)$. It is easy to see that $K \leq \mathbf{A}(f)$. We have already seen in Corollary 3.3, that $\mathbf{E}(f)$ contains a group $\beta(\text{rad}(W)) \simeq (\text{rad}(W), +)$.

Assume next $b = 1$. Let $w \in W$. We observe $f(x+w) - f(x) - f(w) = (x, w)$ for all $x \in V$. Define $\beta(w)$ as in Eq. (1) of Remark 2.5, and we have $\beta(w) \in \mathbf{E}(f)$, i.e., $\beta(W) \leq \mathbf{E}(f)$.

Remark 3.5 Clearly, we have $Q_{W,a}^b = bQ + (1-b)Q\chi_W + a(1-\chi_W)$, where χ_W is the characteristic function of W . Choosing Q as in Remark 2.7(1)–(3) and $W = \langle e_1, \dots, e_k, \dots, e_{n-k} \rangle, k \leq \lfloor \frac{n}{2} \rfloor$, this allows us to express $Q_{W,a}^b$ as a polynomial. If $W = W^\perp$, then $Q\chi_{W^\perp} = 0$, and if W is a hyperplane, then $\chi_W = \text{const} \cdot \prod_{0 \neq e \in \mathbb{F}_p} (X_n - e)$. So if $p > 2$, then in the following cases $Q_{W,a}^b$ is a quadratic form:

- (1) $W = W^\perp$ (n is even) and $a = 0$ (i.e., $Q_{W,0}^b = bQ$).
- (2) W is a hyperplane, $b = 1$ and $p = 3$.
- (3) $(a, b) = (0, 1)$ (i.e., $Q_{W,a}^b = Q$).

We call $Q_{W,a}^b$ a *proper* standard parabolic bent function, if it is not a quadratic form. Finally, if $W = W^\perp$, then $Q_{W,a}^b$ is equivalent to $Q_{W,b^{-1}a}^1$. So we will make the convention $b = 1$, if $W = W^\perp$.

Example 3.6 Let Q be a non-degenerate quadratic form on $V = V(n, p)$. Let W be a subspace such that $W_1 = \text{rad}(W) = W^\perp$ is a non-trivial totally singular space and $V = W_0 \oplus W_1 \oplus W_2$ with $W = W_0 \oplus W_1$ and $W_2 \subseteq W_0^\perp$ totally singular. For $\Pi \in \text{GL}(W_0)$, define $Q_{W,\Pi} : V \rightarrow \mathbb{F}_p$ as $Q_{W,\Pi,0}^1$, i.e.,

$$Q_{W,\Pi}(x_0 + x_1 + x_2) = \begin{cases} Q(x_0\Pi), & x_2 = 0, \\ Q(x_0) + (x_1, x_2), & x_2 \neq 0. \end{cases}$$

We call a bent function $Q_{W,\Pi}$ a

twisted parabolic bent function.

By Corollary 3.3, we already know

$$\mathcal{Q} = \mathcal{O}_p(\mathrm{GO}(V, \mathcal{Q})_W) \leq \mathbf{A}(\mathcal{Q}_{W,\Pi}), \quad \beta(W_1) \leq \mathbf{E}(\mathcal{Q}_{W,\Pi})$$

and

$$H \simeq \mathrm{GL}(W_1) \leq \mathrm{GO}(V, \mathcal{Q})_{W_0, W_1, W_2} \cap \mathbf{A}(\mathcal{Q}_{W,\Pi}).$$

Remark 3.7 If $\Pi \in \mathrm{O}(W_0, \mathcal{Q}_{W_0})$, then $\mathcal{Q} = \mathcal{Q}_{W,\Pi}$ is a quadratic form. Conversely, if $\mathcal{Q}_{W,\Pi}$ is a quadratic form, then $\Pi \in \mathrm{O}(W_0, \mathcal{Q}_{W_0})$: Let $[\cdot, \cdot]$ be the bilinear form associated with this quadratic form. Then, by definition

$$[w, w'] = (w\Pi, w'\Pi) \quad w, w' \in W_0.$$

Let $0 \neq v \in W_2$. Then $v, v + w', v + w + w' \notin W$, so that $[w, v] = \mathcal{Q}(v + w) - \mathcal{Q}(w\Pi) - \mathcal{Q}(v)$ and $[w, v + w'] = \mathcal{Q}(v + w + w') - \mathcal{Q}(w\Pi) - \mathcal{Q}(v + w')$ holds. Expanding the second equation and using the bilinearity of $[\cdot, \cdot]$, one obtains

$$(w, w') = [w, w'] = (w\Pi, w'\Pi)$$

and $\Pi \in \mathrm{O}(W_0, \mathcal{Q}_{W_0})$ follows. We call $\mathcal{Q}_{W,\Pi}$ *proper*, if this function is not a quadratic form.

We give a variation of these constructions, which rests on fact that a certain cohomology group does not vanish.

Some indecomposable $\mathrm{SL}(2, 3^n)$ -modules

Let $F = \mathbb{F}_q$, $q = 3^n$, $n \geq 2$, let σ be the inverse of the Frobenius automorphism of F and set $V = F^4$. We define in $\mathrm{GL}(V)$ the elements $h(v) = \mathrm{diag}(v^\sigma, v^{-\sigma}, v, v^{-1}) \in \mathrm{GL}(V)$, $v \in F^*$, $w_1 = \mathrm{diag}(J, J)$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$x_+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u^\sigma & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u^{2\sigma} & -u^\sigma & u & 1 \end{pmatrix},$$

$u \in F$ and set $x_-(-u) = x_+(u)^{w_1}$. Define $H = \langle x_+(u), x_-(u) \mid u \in F \rangle$. A routine computation shows that the generators of H satisfy the Steinberg relations for $\mathrm{SL}(2, q)$, i.e., $H \simeq \mathrm{SL}(2, q)$ by [24, Thm. 8]. Let $M = V(2, q)$ be the natural $\mathbb{F}_q \mathrm{SL}(2, q)$ -module. Then, $M^\sigma \simeq W$, $W = F \times F \times 0 \times 0$ and $V/W \simeq M$. So V is the extension of the module M^σ by M . The explicit description of the non-vanishing elements in $\mathrm{Ext}_{F\mathrm{SL}(2, q)}(M^\sigma, M) \simeq H^1(\mathrm{SL}(2, q), M^\sigma \otimes M^*) \simeq H^1(\mathrm{SL}(2, q), M^\sigma \otimes M)$ (here $M^* \simeq M$, where M^* is the dual module for M) in [2, Lemma 3.3] shows that V is an indecomposable $F\mathrm{SL}(2, q)$ -module.

Example 3.8 Set as above $F = \mathbb{F}_q$, $q = 3^n$, $n \geq 2$, $V = F^4$ and consider this indecomposable FH -module, $H = \mathrm{SL}(2, q)$ as an $\mathbb{F}_3 H$ -module. We do this by replacing each matrix entry x of F by $T_0(x)$, for instance $h(v) = \mathrm{diag}(T_0(v^\sigma), T_0(v^{-\sigma}), T_0(v), T_0(v^{-1}))$. Conjugating by $\mathrm{diag}(T_1(1), T_1(1), 1, 1)$, the generators get the shape:

$$h(v) = \mathrm{diag}(T_0(v), T_0(v^{-1}), T_0(v), T_0(v^{-1})), \quad w(1) = \mathrm{diag}(J, J),$$

$$J = \begin{pmatrix} 0 & T_0(-1) \\ T_0(1) & 0 \end{pmatrix},$$

and

$$x_+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ T_0(u) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_1(u^2) & T_1(-u) & T_0(u) & 1 \end{pmatrix}.$$

We define on the $\mathbb{F}_3$ -space V a quadratic form by $Q(x_1, x_2, x_3, x_4) = \mathrm{Tr}(x_1 x_4 - x_2 x_3)$, where $\mathrm{Tr} : F \rightarrow \mathbb{F}_3$ is the trace map. Then, $W = F \times F \times 0 \times 0$ and $\widehat{W} = 0 \times 0 \times F \times F$ are totally singular spaces. The element

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ T((a)) & T((b)) & 1 & 0 \\ T((c)) & T((d)) & 0 & 1 \end{pmatrix}$$

lies in $O(V, Q)$ and hence in the 3-radical $\mathcal{Q} = O_3(O(V, Q)_W)$ of the parabolic subgroup $O(V, Q)_W$, iff $T((d)) = T((a))^*$, $T((b))^* = -T((b))$, and $T((c))^* = -T((c))$. Here, $T((x))^* = \sum_{i=0}^{n-1} T_i(x_{n-i}^{3^{n-i}})$ if $x = (x_0, \dots, x_{n-1})$. Note that $T((x))^*$ is the adjoint operator to $T((x))$ with respect to the bilinear form $F^2 \ni (x, y) \mapsto \mathrm{Tr}(xy)$ on F . A computation shows that the $h(v)$'s and $w(1)$ lie in $O(V, Q)$ but not $x_+(u)$, $u \neq 0$, since Q takes nonzero values on the orbit of $v_4 = (0, 0, 0, 1)$ under $X_+ = \langle x_+(u) \mid u \in F \rangle$. Nevertheless, X_+ normalizes the 3-radical $\mathcal{Q}$, i.e., $\mathcal{Q}H$ is a group, where $H = \langle X_+, w_1, h(F^*) \rangle$.

Any element $v \in V - W$ is mapped under $\mathcal{Q}$ to every element $v + w \in v + W$ with $Q(v) = Q(v + w)$. If $v_3 = (0, 0, 1, 0)$, then $C_H(v_3) = C_H(v_3 + W) = X_+$, so that $V - W$ splits under $\mathcal{Q}H$ into three orbits B_0, B_1, B_2 of the same length. We choose the notation, such that $v_4 = (0, 0, 0, 1) \in B_0$.

Let σ be an element in $\mathrm{Sym}(3)$. Define a function $f = Q^\sigma : V \rightarrow \mathbb{F}_3$ by $f(v) = 0$ for $v \in W$ and $f(v) = i^\sigma$ if $v \in B_i \cap (V - W)$. Then, f is a 3-ary bent function of Maiorana type: Let $0 \neq u \in \widehat{W}$. Then, f is constant on $[u, \mathcal{Q}] + u = (W \cap \langle u \rangle^\perp) + u$. But $W \cap \langle u \rangle^\perp$ is a hyperplane of W . So $W \ni x \mapsto f(x, u) - f(0, u) \in \mathbb{F}_3$ is a linear functional (we identify V with $W \times \widehat{W}$ and W and $\widehat{W}$ with F^2). Hence, there exists a unique $0 \neq u^\pi \in \widehat{W}$, such that for all $w \in W$ the equation (*) $f(w, u) = (w, u^\pi)$ holds ($(\cdot, \cdot)$ is the bilinear form induced by Q). Setting $0^\pi = 0$, then π is a permutation on $\widehat{W}$ and equation (*) holds for all $(w, u) \in V$.

We call $f = Q^\sigma$ of *special parabolic type*.

Set $\mathcal{U}_0 = 0 \times 0 \times 0 \times 0 \times \mathbb{F}_p$. By construction $H\mathcal{Q} \leq \mathbf{A}(f)$, where we extend the action of $H\mathcal{Q}$ on V to $\mathcal{U}$ by setting $(H\mathcal{Q})_{\mathcal{U}_0} = 1$. We observe that Θ with $\Theta_V = \text{diag}(T_1(1), T_1(1), T_1(1), T_1(1))$, $\Theta_{\mathcal{U}_0} = 1$ fixes v_3 and normalizes $H\mathcal{Q}$. Hence, $\Theta \in \mathbf{A}(f)$. Define $z(-1) \in \text{GL}(\mathcal{U})$ by $z(-1)_V = \text{diag}(1, -1, -1, 1)$ and $z(-1)_{\mathcal{U}_0} = 1$. Then, $z(-1)$ fixes v_4 and normalizes $H\mathcal{Q}$. Hence, $\Theta, z(-1) \in \mathbf{A}(f)$, as v_3 and v_4 lie in the same $\mathcal{Q}H$ -orbit. If n is even, we pick $v \in F^*$ of order 4 and define the element $z(v) \in \text{GL}(\mathcal{U})$ by $z(v)_V = \text{diag}(T_0(-1), T_0(v), T_0(v), T_0(1))$, and $z(v)_{\mathcal{U}_0} = -1$, i.e., $|z(v)| = 4$. We see that $z(v)$ normalizes $H\mathcal{Q}$, and we recall $v_4 \in B_0$. As $(\pm k, 0, 0, 0) + v_4$ (where $\text{Tr}(k) = 1$) represent the orbits B_1 and B_2 and as $z(v)$ interchanges these elements, we get $z(v) \in \mathbf{A}(f)$, iff σ fixes 0. So $N_{\mathbf{A}(f)}(H\mathcal{Q})/H\mathcal{Q}$ contains a subgroup isomorphic to $\text{Aut}(F)C_4$ if n is even and σ fixes 0. Otherwise, $N_{\mathbf{A}(f)}(H\mathcal{Q})/H\mathcal{Q}$ contains a subgroup isomorphic to $\text{Aut}(F)C_2$.

We claim that $\beta(W) \leq \mathbf{E}(f)$: As $\mathcal{Q} \leq \mathbf{A}(f)$, we see that for $(x, y, 0, 1) \in W + v_4$ the function f takes the value $f(x, y, 0, 1) = e \text{Tr}(x) + b$ for some $b, e \in \mathbb{F}_3, e \neq 0$. The scalars b, e depend only on the choice of σ . Define for $w \in W$ the element $\beta(w) = \tau_w \circ \lambda_w \in \text{AGL}(\mathcal{U})$, where $\lambda_w \in \text{GL}(\mathcal{U})$ is defined by $(v, z)\lambda_w = (v, z + e(v, w))$. If $w = (w_1, w_2, 0, 0)$, then

$$\beta(w) : (x, y, 0, 1, e \text{Tr}(x) + b) \mapsto (x + w_1, y + w_2, 0, 1, e \text{Tr}(x) + b + e \text{Tr}(w_1)).$$

So $\beta(w)$ leaves the set $\{(v, f(v)) \in \Gamma(f) \mid v \in W + v_4\}$ invariant. Also the set $\{(w, f(w)) \in \Gamma(f) \mid w \in W\}$ is invariant under $\beta(w)$. Obviously, the group $\mathcal{Q}$ centralizes $\beta(W)$ and a computation shows that H normalizes $\beta(W)$. We deduce that $\Gamma(f)$ is invariant under $\beta(W)$, as $\Gamma(f) - \{(w, f(w)) \in \Gamma(f) \mid v \in W\}$ is the same as $\{(v, f(v)) \in \Gamma(f) \mid v \in W + v_4\}H\mathcal{Q}$. We remark that by Lemma 2.2 $\beta(w)$ has to agree with the transformation defined by Equation (1) of Remark 2.5.

Proposition 3.9 *Let $f = Q^\sigma$ be special parabolic as in Example 3.8. Then, f is not a quadratic form.*

Proof Assume that f is a quadratic form. Define W and $\widehat{W}$ as in Example 3.8. Since $f(0) = Q(0) = 0$ and as f vanishes on W , both f and Q are of hyperbolic type. By Witt's theorem [14, Thm. 12.10], there exist $\Phi \in \text{GL}(V)$, such that $f(v) = Q(v\Phi)$ for $v \in V$ and as Q and f vanish on W , we can even assume that $\Phi_W = 1_W$. Write $v \in V$ as $v = w + \widehat{w}$, with $w \in W, \widehat{w} \in \widehat{W}$. In particular, $\widehat{w}\Phi = \widehat{w}\Phi_{21} + \widehat{w}\Phi_0$ with $\Phi_{21} \in \text{Hom}(\widehat{W}, W)$ and $\Phi_0 \in \text{GL}(\widehat{W})$. So $f(v) = Q(v\Phi) = (w + \widehat{w}\Phi_{21}, \widehat{w}\Phi_0)$. Here, $(\cdot, \cdot)$ is the bilinear form associated with Q . Denote by $[\cdot, \cdot]$ the bilinear form associated with f . Then:

$$[w_1 + \widehat{w}_1, w_2 + \widehat{w}_2] = (w_1 + \widehat{w}_1\Phi_{21}, \widehat{w}_2\Phi_0) + (w_2 + \widehat{w}_2\Phi_{21}, \widehat{w}_1\Phi_0) \quad (5)$$

Set $H_0 = \langle h(v), w(1), \Theta \mid v \in F^* \rangle$. This group leaves the decomposition $V = W \oplus \widehat{W}$ invariant. We identify W and $\widehat{W}$ with F^2 in the obvious way. Then, H_0 has on both spaces the same representation (i.e., $\Phi_W = \Phi_{\widehat{W}}$ for $\Phi \in H_0$). We claim that this representation is absolutely irreducible: The subgroup $h(F^*)$ induces on $F \times 0$

and $0 \times F$ a Singer group, and we can assume that this Singer group induces the representations D^1 and D^{-1} , i.e., the $\mathbb{F}_3 h(F^*)$ -modules $F \times 0$ and $0 \times F$ are simple and non-equivalent. Also $h(F^*)\langle \Theta \rangle$ induces the normalizer of a Singer group, so that both modules are absolutely irreducible under this group (cf. Lemma 2.12 and [1, (25.8)]). Finally, $w(1)$ interchanges both modules showing the claim.

The action of H_0 will help us to pin down the concrete form of Φ . For $h \in H_0$, we write $(w + \widehat{w})h = wh_W + \widehat{w}h_{\widehat{W}}$, where h_W is the restriction of h to W and $h_{\widehat{W}}$ the restriction of h to $\widehat{W}$. Using Eq. (5), we have $(wh_W, \widehat{w}\Phi_0 h_{\widehat{W}}) = (w, \widehat{w}\Phi_0) = [w, \widehat{w}] = [wh_W, \widehat{w}h_{\widehat{W}}] = (wh_W, \widehat{w}h_{\widehat{W}}\Phi_0)$. We deduce $h_{\widehat{W}}\Phi_0 = \Phi_0 h_{\widehat{W}}$ and as H_0 induces an absolutely irreducible representation on $\widehat{W}$ we get $\Phi_0 = c1_{\widehat{W}}$, $c \in \mathbb{F}_3^*$. Replacing f by $c^{-1}f$, we can assume even $c = 1$, i.e., $\Phi_0 = 1$. Moreover, Eq. (5) becomes

$$[w_1 + \widehat{w}_1, w_2 + \widehat{w}_2] = (\widehat{w}_1, w_2) + (w_1, \widehat{w}_2) + (\widehat{w}_1, \widehat{w}_2\Psi),$$

where $\Psi = \Phi_{21} + \Phi_{21}^*$. We even may assume $\Phi_{21} = \frac{1}{2}\Psi = -\Psi$, since $\Psi^* = \Psi$ and $(\widehat{w}_1, \widehat{w}_2\Phi_{21}) + (\widehat{w}_1\Phi_{21}, \widehat{w}_2) = (\widehat{w}_1, \widehat{w}_2\Psi) = (\widehat{w}_1, \widehat{w}_2\frac{1}{2}\Psi) + (\widehat{w}_1\frac{1}{2}\Psi, \widehat{w}_2)$. Then

$$(\widehat{w}_1 h_{\widehat{W}}, \widehat{w}_2 \Psi h_W) = (\widehat{w}_1, \widehat{w}_2\Psi) = [\widehat{w}_1, \widehat{w}_2] = [\widehat{w}_1 h_{\widehat{W}}, \widehat{w}_2 h_{\widehat{W}}] = (\widehat{w}_1 h_{\widehat{W}}, \widehat{w}_2 h_{\widehat{W}}\Psi).$$

Thus, $\Psi h_W = h_{\widehat{W}}\Psi$. So Ψ can be identified with $c_1 1_{F^2}$, $c_1 \in \mathbb{F}_3$ as H_0 is absolutely irreducible (on $F \times F$) and then $\Phi_{21} = -c_1 1_{F^2}$. But then Q vanishes on the set $N\Phi^{-1}$, $N = \{(u^2 + k, -u, u, 1) \mid u \in F\} \subseteq f^{-1}(0)$ (here we assume $f(k, 0, 0, 1) = 0$). But $N\Phi^{-1} = \{(u^2 + c_1 u + k, c_1 - u, u, 1) \mid u \in F\}$, showing $N\Phi^{-1} \not\subseteq Q^{-1}(0)$. $\square$

By [2], one knows that $\text{Ext}_{\mathbb{F}_5\text{SL}(2,5)}(M, M) \neq 0$, for $M = V(2, 5)$ the natural $\mathbb{F}_5\text{SL}(2, 5)$ -module. This means that there are indecomposable four-dimensional $\mathbb{F}_5\text{SL}(2, 5)$ -modules with two composition factors M . This gives rise to further bent functions:

Example 3.10 Let $V = V(4, 5)$. We define in $\text{GL}(V)$ the elements $h = \text{diag}(3, 2, 3, 2) \in \text{GL}(V)$, $w_1 = \text{diag}(J, J)$, $J = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$,

$$x_+ = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 3 & 1 & 1 \\ 3 & 4 & 0 & 1 \end{pmatrix}$$

and set $H = \langle h, w_1, x_+ \rangle$. A computer calculation with MAGMA [3] shows $H \cong \text{SL}(2, 5)$. Set $W = \mathbb{F}_5 \times \mathbb{F}_5 \times 0 \times 0$. Then, W is H -invariant and $V - W$ splits into five orbits $B_0, \dots, B_4$ under H , each of it has length 120. In particular, V is an indecomposable $\mathbb{F}_5 H$ -module.

For any permutation σ on $\{0, \dots, 4\}$, define $f_\sigma : V \rightarrow \mathbb{F}_5$ by $f_\sigma(v) = 0$ if $v \in W$ and $f_\sigma(v) = i$ if $v \in B_{i\sigma} \cap (V - W)$. Computing the Walsh coefficients with MAGMA (it is enough to compute the Walsh coefficients for representatives of H -orbits) shows

that f_σ is a 5-ary bent function. Again using MAGMA, one can compute the CCZ automorphism group and thus $\mathbf{EA}(f_\sigma)$. A procedure for this calculation is described in [12]. One gets $\mathbf{EA}(f_\sigma) = \mathbf{A}(f_\sigma) \leq \mathrm{GL}(\mathcal{U})$, $\mathcal{U} = V \times \mathbb{F}_5$ and $\mathbf{A}(f_\sigma) \simeq \mathrm{GL}(2, 5)$ (40 times) or $\mathbf{A}(f_\sigma)$ is isomorphic to HC with $C \simeq C_4$, $H \cap C = Z(H)$ and $[H, C] = 1$ (80 times).

Remark 3.11 The construction of this example resembles closely the construction of the previous example. Nevertheless, f_σ is not EA equivalent to a bent function f of parabolic type: the 5-radical of $\mathbf{A}(f_\sigma)$ and that of $\mathbf{E}(f_\sigma)$ are trivial in contrast to $O_p(\mathbf{A}(f)) \neq 1$, $\mathbf{E}(f) \neq 1$.

Remark 3.12 Let f be standard parabolic, twisted parabolic, or special parabolic with respect to the subspace W of the quadratic space V . We have shown in Examples 3.4, 3.6 and 3.8 that $\mathbf{A}(f) \cap \mathrm{GL}(\mathcal{U})$ contains a subgroup, such that W is the unique maximal subspace of V invariant under this group.

3.2 Maiorana functions with large cyclic automorphism groups

Example 3.13 Let n be a positive integer, p a prime, $F = \mathbb{F}_{p^n}$ and $0 < \ell < p^n - 1$ a number coprime to $p^n - 1$. Let Tr be the absolute trace on F . Set $V = F \times F$ and define $\mu_\ell : V \rightarrow \mathbb{F}_p$ by

$$\mu_\ell(x, y) = \mathrm{Tr}(xy^\ell).$$

Clearly, μ_ℓ is a bent function of Maiorana type and this function is even a quadratic form, if ℓ is a p -power.

Let $\bar{\ell}$ be the inverse of ℓ modulo $p^n - 1$ and define for $a \in F^*$ the operator $z_a = \mathrm{diag}(T_0(a), T_0(a^{-\bar{\ell}}), 1)$. Then, z_a lies in $\mathbf{A}(\mu_\ell)$, since $\mu_\ell((x, y)z_a) = \mu_\ell(x, y)$, i.e., $\mathcal{Z} = \{z_a \mid a \in F^*\}$ is a cyclic subgroup of $\mathbf{A}(\mu_\ell)$ of order $p^n - 1$. As $\mu_\ell(x^p, y^p) = \mu_\ell(x, y)$, we see that $\Theta = \mathrm{diag}(T_1(1), T_1(1), 1)$ of order n lies in $\mathbf{A}(\mu_\ell)$ and normalizes $\mathcal{Z}$. For $u \in \mathbb{F}_p^*$, we have $\mu_\ell(ux, y) = u\mu_\ell(x, y)$, so that $\pi_u = \mathrm{diag}(T_0(u), 1_F, u)$ lies in $\mathbf{A}(\mu_\ell)$ and centralizes $\mathcal{Z}$. Set $\mathcal{P} = \{\pi_u \mid u \in \mathbb{F}_p^*\}$. Then, $\mathcal{Z}\mathcal{P}\langle\Theta\rangle \simeq (C_{p^n-1} \times C_{p-1})C_n$.

4 Automorphism groups and equivalence of parabolic bent functions

In this section, we consider parabolic p -ary bent functions $f : V = V(n, p) \rightarrow \mathbb{F}_p$, which are standard parabolic, twisted parabolic or special parabolic. For the description of the automorphism group of a twisted parabolic bent function $Q_{W, \Pi}$, we use an operator $\tilde{\Pi}$, which is derived from Π . The definition of $\tilde{\Pi}$ can be found before Lemma 4.8. We set

$$C_{\mathrm{GO}(V, Q)}^*(\tilde{\Pi}) = \left\{ \Phi \in \mathrm{GO}(V, Q) \mid r_\Phi^2 \tilde{\Pi} \Phi = \Phi \tilde{\Pi} \right\}.$$

We extend Π to V by setting $x\Pi = x$ for $x \in W_1 \oplus W_2$. It will be clear from the context, when we consider Π as an operator on W_0 or on V . The spaces W_0, W_1, W_2 have the meaning as in Example 3.6.

Theorem 4.1 *Let $p > 2$ be a prime, V a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V and f a proper parabolic bent function on V with respect to W . Then, $\mathbf{EA}(f) = \mathbf{E}(f)\mathbf{A}(f)$ and $\mathbf{A}(f) \leq \mathrm{GL}(\mathcal{U})$, $\mathcal{U} = V \times \mathbb{F}_p$ the ambient space of f . Moreover:*

- Let $f = Q_{W,a}^b$ be a standard parabolic bent function. Then, $\mathbf{E}(f) = \beta(W)$ if $b = 1$, while $\mathbf{E}(f) = \beta(\mathrm{rad}(W))$ otherwise. Also $\mathbf{A}(f) = \mathrm{GO}(V, Q)_W$ if $a = 0$ and $\mathbf{A}(f) = \mathrm{O}(V, Q)_W$ otherwise.
- Let $f = Q_{W,\Pi}$ be a twisted parabolic bent function. Then, $\mathbf{E}(f) = \beta(\tilde{W})$ with $\tilde{W} = C_W(\tilde{\Pi}) \cap \{w \in W \mid Q(w) = Q(w\Pi)\}$ and $\mathbf{A}(f) = LQ$, $Q \simeq O_p(\mathrm{GO}(V, Q)_W)$, where $L = \mathrm{GO}(V, Q)_{W_0W_1W_2} \cap C_{\mathrm{GO}(V,Q)}^*(\tilde{\Pi})$.
- Let $f = Q^\sigma$ be a special parabolic bent function on $V = V(4n, 3)$. Then, $\mathbf{E}(f) = \beta(W)$. The group $\mathbf{A}(f)$ has a normal subgroup HQ with $Q \simeq O_p(\mathrm{O}(V, Q)_W)$ and $H \simeq \mathrm{SL}(2, 3^n)$. Also $\mathbf{A}(f)/HQ \simeq C_4 \cdot \mathrm{Aut}(\mathbb{F}_{3^n})$ if n is even and σ fixes 0 and otherwise $\mathbf{A}(f)/HQ \simeq C_2 \cdot \mathrm{Aut}(\mathbb{F}_{3^n})$.

The answer to the equivalence problem is given by the next two corollaries.

Corollary 4.2 *Let $p > 2$ be a prime and V a finite-dimensional $\mathbb{F}_p$ -space. For $1 \leq i \leq 2$, assume that Q_i is a non-degenerate quadratic form on V and that f_i is a proper parabolic bent function (standard, twisted or special) with respect to Q_i and the subspace W^i . Let f_1 and f_2 be EA equivalent. Then:*

- f_1 and f_2 are linear equivalent.
- Q_1 and Q_2 have the same type.
- W^1 and W^2 are isometric.
- Either both f_1 and f_2 are standard or twisted parabolic or both f_1 and f_2 are special parabolic.

By the preceding corollary, we can assume for two EA equivalent parabolic bent functions that both are associated with the same quadratic form and the same subspace W .

Corollary 4.3 *Let $p > 2$ be a prime, V a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V and let f and f' be a proper parabolic bent function (standard, twisted or special) with respect to the subspace W .*

- Let $f = Q_{W,a}^b$ and $f' = Q_{W,a'}^{b'}$. Then, f is EA equivalent to f' , iff $b = b'$ and if either $\dim V$ is even and $a\mathbb{F}_p^* = a'\mathbb{F}_p^*$ or if $\dim V$ is odd and $a(\mathbb{F}_p^*)^2 = a'(\mathbb{F}_p^*)^2$.
- Let $f = Q_{W,\Pi}$ and $f' = Q_{W,\Pi'}$ be twisted parabolic bent functions. Then, f is EA equivalent to f' iff there exist $\Phi_1, \Phi_2 \in \mathrm{GO}(W_0, Q)$, such that Φ_1 has the same multiplier as Φ_2 and $\Phi_1\Pi = \Pi'\Phi_2$.
- Let $f = Q_{W,a}^b$ and $f' = Q_{W,\Pi}$. Then, f is EA equivalent to f' , iff $a = 0$ and Π lies $\mathrm{GO}(W_0, Q_{W_0})$ and has multiplier b^{-1} .
- Let $f = Q^\sigma$ and $f' = Q^{\sigma'}$ be special parabolic on $V = V(4n, 3)$. Then, f is EA equivalent to f' iff either n is even and $\sigma' \in \langle (2, 3) \rangle \sigma \langle (2, 3) \rangle$ or if n is odd and $\sigma' \in \langle (2, 3) \rangle \sigma$.

4.1 Derivatives of parabolic bent functions

Lemma 4.4 *Let $p > 2$ be a prime, $V = V(n, p)$, $n \geq 2$ and Q a non-degenerate quadratic form on V . Let $f : V \rightarrow \mathbb{F}_p$ be a standard parabolic or twisted parabolic bent function with respect to the subspace W of V .*

- (a) *Let $f = Q_{W,a}^b$ a proper standard parabolic bent function. Let $0 \neq v \in V$. Then, $D_v^2 f = 0$ iff $v \in W$ and $Q(v) = 0$.*
- (b) *Let $f = Q_{W,\Pi}$ be a proper twisted parabolic bent function. Let $0 \neq v \in V$. Then, $D_v^2 f = 0$ iff $v \in W$ and $Q(v) = Q(v\Pi) = 0$.*

Proof (a) Let $(\cdot, \cdot)$ be the bilinear form associated with Q . Set $f = Q_{W,a}^b$ and assume first $v \in W$. For $x \in V$ either both $x, x+v \in W$ or both $x, x+v \notin W$. Hence, $D_v f(x) = Q(v) + (v, x)$ if $x \in W$ and $D_v f(x) = b(Q(v) + (v, x))$ otherwise. Thus, by Lemma 2.10 $D_v^2 f(x) = 2Q(v)$ if $x \in W$ and $D_v^2 f(x) = 2bQ(v)$ otherwise and $D_v^2 f(x) = 0$ iff $Q(v) = 0$.

Assume now $v \notin W$. If $x \in W$, then $x+v, x+2v \notin W$. Moreover

$$D_v f(x) = bQ(x+v) + a - Q(x) = (b-1)Q(x) + bQ(v) + b(v, x) + a.$$

So

$$\begin{aligned} D_v^2 f(x) &= D_{2v} f(x) - 2D_v f(x) \\ &= (b-1)Q(x) + bQ(2v) + b(2v, x) + a \\ &\quad - 2((b-1)Q(x) + bQ(v) + b(v, x) + a) \\ &= -(b-1)Q(x) + 2bQ(v) - a. \end{aligned}$$

Then, $D_v^2 f \neq 0$, if $2bQ(v) \neq a$. So assume $2bQ(v) = a$. If $W^\perp$ is a proper subspace of W and $b \neq 1$, the function $(b-1)Q$ takes at least two values on W and we conclude $D_v^2 f \neq 0$. So we assume $W = W^\perp$ (and then $b = 1$ and $a \neq 0$ by our assumptions) or we assume $W \neq W^\perp$ and $b = 1$.

We claim that we can pick $x \notin W$, such that $x+v, x+2v \notin W$: This is obvious if $W + \langle v \rangle$ is a proper subspace of V or if $V = W + \langle v \rangle$ and $p > 3$. So assume $V = W + \langle v \rangle$ and $p = 3$. Then, f is a quadratic form by Remark 3.5, contradicting the fact that f is proper parabolic. Our claim holds.

Hence, $D_w f(x) = b(Q(w) + (w, x))$ for $w = v, 2v$ and $D_v^2 f(x) = b(Q(2v) + (2v, x) - 2(Q(v) + (v, x))) = 2bQ(v)$. So $D_v^2 f \neq 0$ if $Q(v) \neq 0$. So we assume $Q(v) = 0$. But this forces $a = 0$, ruling out the case $W = W^\perp$. So $W \neq W^\perp$, $b = 1$ and $Q(v) = a = 0$, which was not allowed too.

- (b) We can write

$$Q_{W,\Pi}(x) = \begin{cases} Q(x\Pi), & x \in W, \\ Q(x), & x \notin W. \end{cases}$$

Set $f = Q_{W,\Pi}$ and assume first $v \in W$. For $x \in V$ either both $x, x+v \in W$ or both $x, x+v \notin W$. Hence, $D_v f(x) = Q(v\Pi) + (v\Pi, x\Pi)$ if $x \in W$ and $D_v f(x) =$

$Q(v) + (v, x)$ otherwise. Thus, by Lemma 2.10 $D_v^2 f(x) = 2Q(v\Pi)$ if $x \in W$ and $D_v^2 f(x) = 2Q(v)$ otherwise and $D_v^2 f = 0$ iff $Q(v) = Q(v\Pi) = 0$.

Assume now $v \notin W$. If $x \in W$, then $x + v, x + 2v \notin W$. Moreover,

$$D_v f(x) = Q(x + v) - Q(x\Pi) = Q(x) + Q(v) + (v, x) - Q(x\Pi).$$

So

$$\begin{aligned} D_v^2 f(x) &= D_{2v} f(x) - 2D_v f(x) \\ &= Q(x) + Q(2v) + (2v, x) - Q(x\Pi) \\ &\quad - 2(Q(x) + Q(v) + (v, x) - Q(x\Pi)) \\ &= Q(x\Pi) - Q(x) + 2Q(v). \end{aligned}$$

As the function $Q(x\Pi) - Q(x)$ takes at least two values on W_0 , we conclude $D_v^2 f \neq 0$. $\square$

Lemma 4.5 *Let $f = Q^\sigma$ be a special parabolic bent function on $V = V(4n, 3)$, $n \geq 2$ with respect to the subspace W . Let $v \in V$. Then, $D_v^2 f = 0$ iff $v \in W$.*

Proof We assume as in Example 3.8 that $V = F \times F \times F \times F$, $W = F \times F \times 0 \times 0$ and $\widehat{W} = 0 \times 0 \times F \times F$. We know that f is a function of Maiorana type with

$$f(w + \widehat{w}) = (w, \widehat{w}^\pi),$$

where $w \in W$, $\widehat{w} \in \widehat{W}$ and π is a permutation on $\widehat{W}$.

Let $v \in W$. Then, $D_v f(w + \widehat{w}) = (w + v, \widehat{w}^\pi) - (w, \widehat{w}^\pi) = (v, \widehat{w}^\pi)$ and

$$D_v^2 f(w + \widehat{w}) = (2v, \widehat{w}^\pi) - 2(v, \widehat{w}^\pi) = 0.$$

Let $v \in V - W$. By Remark 2.11, it is enough to show $D_v^2 f \neq 0$ for representatives from the orbits B_i , $0 \leq i \leq 2$. Let $v = v_4 = (0, 0, 0, 1)$.

Assume first $0^\sigma \neq 0$. Pick $k \in F^*$, such that $\text{Tr}(k)^\sigma = 0$. Then

$$D_v f(k, 0, 0, 0) = \text{Tr}(k)^\sigma - 0 = 0$$

and $D_{2v} f(k, 0, 0, 0) = \text{Tr}(-k)^\sigma - 0 \neq 0$. Hence $D_v^2 f(k, 0, 0, 0) \neq 0$, i.e., $D_v^2 f \neq 0$.

Assume now $0^\sigma = 0$. Since Q^σ is equivalent to Q^{id} , we may assume that σ is the trivial permutation. Pick $a \in F$, such that $\text{Tr}(a) = 1$ and set $x = (a, 0, 0, 1)$. Then, $D_v f(x) = f(a, 0, 0, 2) - f(a, 0, 0, 1) = -\text{Tr}(a) - \text{Tr}(a) = \text{Tr}(a)$ and $D_{2v} f(x) = f(a, 0, 0, 0) - f(a, 0, 0, 1) = -\text{Tr}(a)$. Hence, $D_v^2 f(x) = D_{2v} f(x) - 2D_v f(x) = -\text{Tr}(a) - \text{Tr}(a) = 1 \neq 0$, i.e., $D_v^2 f \neq 0$. In a similar fashion, one shows $D_v^2 f \neq 0$, if v has the shape $(b, 0, 0, 1)$, $b \neq 0$. $\square$

Corollary 4.6 *Let $p > 2$ be a prime, V a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V , W a subspace and f a proper parabolic bent function with respect to W . Let $\Phi \in \mathbf{A}(f) \cap \text{GL}(\mathcal{U})$, $\mathcal{U} = V \times \mathbb{F}_p$. Then, Φ fixes the spaces W and $\text{rad}(W)$.*

Proof The assertion follows from Lemmas 4.4 and 4.5, if $W = \text{rad}(W)$. So assume $W \neq \text{rad}(W)$. Then, $f = Q_{W,a}^b$ or $= Q_{W,\Pi}$. Assume $k = \dim \text{rad}(W)$. By Lemma 4.4, $\text{rad}(W)$ is the only k -flat on which the double derivative vanishes. Also in the case $f = Q_{W,\Pi}$, the space W is the set of all $v \in V$, such that $f(x+v) - f(x) - f(v) = 0$ for all $x \in \text{rad}(W)$. The assertion follows by Remark 2.11. $\square$

4.2 The groups $\mathbf{A}(f)$

Let $f : V = V(n, p) \rightarrow \mathbb{F}_p$ be a bent function of parabolic type. In this subsection, we set $\mathcal{U} = V \times \mathbb{F}_p$, $\mathcal{U}_0 = 0 \times \mathbb{F}_p$ and identify V in the obvious way with $V \times 0$. By Lemma 2.3, $\mathbf{A}(f) \leq \text{GL}(\mathcal{U})$ and this group fixes the decomposition $\mathcal{U} = V \oplus \mathcal{U}_0$.

Lemma 4.7 *Let $p > 2$ be a prime, $V = V(n, p)$, $n \geq 2$ and Q a non-degenerate quadratic form on V . Let $f = Q_{W,a}^b$ be a proper standard parabolic bent function with respect to the subspace W of V . Then, $\mathbf{A}(f) = \text{GO}(V, Q)_W$ if $a = 0$ and $\mathbf{A}(f) = \text{O}(V, Q)_W$ otherwise.*

Proof Let $\Phi \in \mathbf{A}(f)$. Then, for all $x \in V$, we have

$$f(x\Phi) = cf(x),$$

where c is the scalar induced by Φ on $0 \times \mathbb{F}_p$. By Corollary 4.6, Φ fixes W and $\text{rad}(W)$.

Let $x \in W$. Then, $Q(x\Phi) = cQ(x)$. If $x \notin W$, then $bQ(x\Phi) + a = c(bQ(x) + a)$ or

$$Q(x\Phi) = cQ(x) + C, \quad C = b^{-1}a(c - 1).$$

If $a = 0$, the assertion of the lemma already follows. So assume $a \neq 0$. Replace x by $2x$ and $4Q(x\Phi) = 4cQ(x) + C$ follows. Hence, $C = 0$, i.e., $c = 1$ if $p > 3$. So assume $p = 3$. Then, $\dim V/W \geq 2$. So there exist $x, y \notin W$ with $x + \varepsilon y \notin W$, $\varepsilon = \pm 1$. This leads to

$$Q(x\Phi) + Q(y\Phi) + \varepsilon(x\Phi, y\Phi) = c(Q(x) + Q(y) + \varepsilon(x, y)) + C$$

So $Q(x\Phi) + Q(y\Phi) = c(Q(x) + Q(y)) + C$ showing again $C = 0$, i.e., $c = 1$. The proof is complete. $\square$

Let V, W, W_0, W_1 and Π be as in Example 3.6. Extending Π to a transformation in $\text{GL}(V)$ as above, we set

$$\tilde{\Pi} = \Pi\Pi^*.$$

Lemma 4.8 *Let $p > 2$ be a prime, $V = V(n, p)$, $n \geq 2$ and Q a non-degenerate quadratic form on V . Let $f = Q_{W,\Pi}$ be a proper twisted parabolic bent function*

with respect to the subspace W of V . Then, $\mathbf{A}(f) = L\mathcal{Q}$, $\mathcal{Q} \simeq O_p(\mathrm{GO}(V, \mathcal{Q})_W)$ and $L = \mathrm{GO}(V, \mathcal{Q})_{W, W_0, W_1} \cap C_{\mathrm{GO}(V, \mathcal{Q})}^*(\tilde{\Pi})$.

Proof By Corollary 4.6, $\Phi \in G = \mathbf{A}(f)$ fixes W and $\mathrm{rad}(W)$. Let $\mathcal{Q}$ be the normal subgroup of operators of G , whose restrictions to $\mathrm{rad}(W)$, $W/\mathrm{rad}(W)$ and V/W are in each case the identity. Then, the restriction of such an operator to V is unipotent, i.e., the order of Φ is a p -power, as its restriction to V is faithful. The restriction to $\mathcal{U}_0$ lies in $\mathrm{GL}(\mathcal{U}_0) \simeq \mathbb{F}_p^*$. Hence, the restriction to $\mathcal{U}_0$ is the identity, i.e.,

$$f(x\Phi) = f(x) \quad \text{for } x \in V \quad \text{and} \quad \Phi \in \mathcal{Q}.$$

So $\mathcal{Q}(x\Phi) = \mathcal{Q}(x)$ for $x \notin W$ and $\mathcal{Q}(x\Phi\Pi) = \mathcal{Q}(x\Pi)$ for $x \in W$. But for $x \in W$, we have by definition of Φ the equation $\mathcal{Q}(x\Pi) = \mathcal{Q}(x\Pi\Phi)$. So $\Phi \in \mathrm{GO}(V, \mathcal{Q})$ and by Corollary 3.3 we obtain $\mathcal{Q} \simeq O_p(\mathrm{GO}(V, \mathcal{Q})_W)$.

Let $\tau_1 \in \mathrm{GL}(\mathcal{U})$ be the identity on W_0 and $\mathcal{U}_0$ and let τ_1 induce -1 on $W_1 \oplus W_2$. Let $\tau_2 \in \mathrm{GL}(\mathcal{U})$ be the identity on $W_1 \oplus W_2$ and $\mathcal{U}_0$, and let τ_2 induce -1 on W_0 . Then, $T = \langle \tau_1, \tau_2 \rangle$ is a four group in G , such that

$$T\mathcal{Q}/\mathcal{Q} \leq Z(G/\mathcal{Q}),$$

since G fixes W , W_1 and V . Moreover, T is a Sylow 2-subgroup of $T\mathcal{Q}$. A similar Frattini argument as in the proof of Lemma 2.3 leads to

$$G = C_G(T)\mathcal{Q}.$$

Let $\Phi \in C_G(T)$. As $C_V(\tau_1) = W_0$ and $C_V(\tau_2) = W_1 \oplus W_2$, the operator Φ leaves the decomposition $\mathcal{U} = W_0 \oplus (W_1 \oplus W_2) \oplus \mathcal{U}_0$ invariant. Suppose

$$f(x\Phi) = cf(x), \quad x \in V.$$

As the restriction of f to $W_1 \oplus W_2$ is $\mathcal{Q}$, we see that the restriction of Φ to $W_1 \oplus W_2$ is an element in $\mathrm{GO}(W_1 \oplus W_2, \mathcal{Q})$ with multiplier c . Let $0 \neq x_2 \in W_2$. Then $\mathcal{Q}(y\Phi) = \mathcal{Q}((x_2 + y)\Phi) = f((x_2 + y)\Phi) = cf(x_2 + y) = c\mathcal{Q}(y)$ for all $y \in W_0$, since $\mathcal{Q}(x_2\Phi) = \mathcal{Q}(x_2) = 0$ and $(x_2\Phi, y\Phi) = (x_2, y) = 0$. Thus, Φ is an element in $\mathrm{GO}(V, \mathcal{Q})$ with multiplier c .

On the other hand, $\mathcal{Q}(y\Phi\Pi) = f(y\Phi) = cf(y) = c\mathcal{Q}(y\Pi) = \mathcal{Q}(y\Pi\Phi)$. So for $y, y' \in W_0$ and as Φ^{-1} has multiplier c^{-1} :

$$\begin{aligned} c(y', y\Pi\Pi^*) &= c(y'\Pi, y\Pi) = (y'\Pi\Phi, y\Pi\Phi) \\ &= (y'\Phi\Pi, y\Phi\Pi) = c^{-1}(y', y\Phi\Pi\Pi^*\Phi^{-1}) \end{aligned}$$

So $c^2\tilde{\Pi} = \Phi\tilde{\Pi}\Phi^{-1}$, and we conclude $C_G(T) \leq \mathrm{GO}(V, \mathcal{Q})_{W, W_0, W_1} \cap C_{\mathrm{GO}(V, \mathcal{Q})}^*(\tilde{\Pi})$. The verification of the reverse inclusion is routine. $\square$

Lemma 4.9 Let $f = \mathcal{Q}^\sigma$ be a special parabolic 3-ary bent function on $V = V(4n, 3)$, $n \geq 2$, and assume the notation of Example 3.8. Then, $N_{\mathbf{A}(f)}(\mathcal{Q}H) = \mathcal{Q}H\langle\Theta, z(v)\rangle$ if n is even and σ fixes 0. Otherwise, $N_{\mathbf{A}(f)}(\mathcal{Q}H) = \mathcal{Q}H\langle\Theta, z(-1)\rangle$.

Proof We denote by L_V the restriction of a subgroup $L \leq \mathbf{A}(f)$ to V . Of course, this restriction map is faithful. First, we claim:

(*) $N_{\mathrm{GL}(V)}((\mathcal{Q}H)_V) = (\mathcal{Q}H\langle\Theta, z(v)\rangle)_V$ if n is even and $= (\mathcal{Q}H\langle\Theta, z(-1)\rangle)_V$ if n is odd.

By Dickson's theorem, every cyclic group of order $3^n - 1$ in $(\mathcal{Q}H/\mathcal{Q})_V \simeq \mathrm{SL}(2, F)$ is conjugate to $(h(F^*)\mathcal{Q}/\mathcal{Q})_V$ and so by the Schur–Zassenhaus theorem $h(F^*)_V$ represents the only conjugacy class of cyclic groups of order $3^n - 1$ in $(\mathcal{Q}H)_V$. Then, $N_{\mathrm{GL}(V)}((\mathcal{Q}H)_V) = (\mathcal{Q}H)_V N_{\mathrm{GL}(V)}(h(F^*)_V)$ by a Frattini argument. Note that this normalizer leaves invariant $W = C_V(\mathcal{Q}_V)$. We even can write $N_{\mathrm{GL}(V)}((\mathcal{Q}H)_V) = (\mathcal{Q}H\langle\Theta\rangle)_V C_{\mathrm{GL}(V)}(h(F^*)_V) \cap \mathrm{GL}(V)_W$, since $h(F^*)\langle\Theta\rangle$ can be considered as the normalizer of a Singer group of $\mathrm{GL}_{\mathbb{F}_3}(F)$. We have to determine $C_{\mathrm{GL}(V)}(h(F^*)_V) \cap \mathrm{GL}(V)_W$. Let α be an element of this group.

- (i) We may assume that $\alpha = \tau\delta$, with $\delta = \mathrm{diag}(T_0(uv), T_0(v), T_0(u), T_0(1))$, $u, v \in F^*$ and

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ T_0(b_1) & 0 & 1 & 0 \\ 0 & T_0(b_2) & 0 & 1 \end{pmatrix}.$$

The element α fixes W and the homogeneous components of $h(F^*)_V$, which are $F \times 0 \times F \times 0$ and $0 \times F \times 0 \times F$. Adjusting α by an element from $h(F^*)$, we may therefore assume that $\alpha_{V/W} = \mathrm{diag}(T_0(u), 1)$, $u \in F^*$, which in turn forces $\alpha_W = \mathrm{diag}(T_0(uv), T_0(v))$, $v \in F^*$. Note that α_W induces via conjugation on $(H\mathcal{Q}/\mathcal{Q})_W$ the same automorphism as $\alpha_{V/W}$ induces on $(H\mathcal{Q}/\mathcal{Q})_{V/W}$, i.e., these elements differ by an operator of the form $\mathrm{diag}(T_0(v), T_0(v))$. So $\tau = \alpha\delta^{-1}$ acts trivially on W and V/W as well as the homogeneous components of $h(F^*)_W$. Since τ lies in the centralizer of $h(F^*)_V$, a simple calculation shows assertion (i).

- (ii) $uv \in \mathbb{F}_3$.

Write $T \in \mathcal{Q}$ as in Example 3.8. Then, $T_V^\alpha = T_V^\delta \in \mathcal{Q}_V$, which implies

$$T_0(uv)T((d))T_0(u^{-1}) = (T_0(u^{-1})T((a))T_0(uv))^* = T((d))T_0(v),$$

i.e., $T_0(uv)$ lies in the center of $\mathrm{End}_{\mathbb{F}_3}(F)$ and assertion (ii) follows.

- (iii) $\tau \in \mathcal{Q}_V$ and $\delta \in \langle z(v)_V \rangle$.

A computation shows

$$x_+(1)^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_0((b_2 - b_1)uv) + T_1(uv) & T_1(-v) & u & 1 \end{pmatrix}.$$

Then, $x_+(1)^\alpha x_+(-u) \in \mathcal{Q}_V$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ T_0((b_2 - b_1)uv) + T_1(u^2 - uv) & T_1(u - v) & 0 & 1 \end{pmatrix}.$$

This forces $u = v$ and $b_1 = b_2$. So $u^2 \in \mathbb{F}_3$ by (ii), $\tau \in \mathcal{Q}_V$ and $\delta = \langle z(\pm v)_V \rangle$. Claim (iii) follows and thus Claim (*).

Clearly, $N_{\mathbf{A}(f)}(\mathcal{Q}H)_V \leq N_{\mathrm{GL}(V)}(\mathcal{Q}H)_V$. We already know that $z(-1) \in \mathbf{A}(f)$ and $z(v) \in \mathbf{A}(f)$ if n is even and σ fixes 0. So the assertion of the lemma holds, if n is odd or n is even and σ fixes 0.

So consider the case that σ moves 0, n is even and assume $z \in \mathbf{A}(f)$, $z_V = z(v)_V$, and $z_{\mathcal{U}_0} = \pm 1$. Set $N = f^{-1}(0) \cap (v_4 + (F \times F \times F \times 0))$. The action of $x_+(F)\mathcal{Q}$ shows that $N = \{(k + y + u^2, x - u, u, 1) \mid x, u \in F, y \in F_0\}$, where F_0 is the kernel of the trace map and $(k, 0, 0, 1)$ is a representative of the $x_+(F)\mathcal{Q}$ -orbit N . Then, $\mathrm{Tr}(k) \neq 0$, as $f(v_4) = 0$ iff σ fixes 0. N is invariant under z , which shows that $(-k, 0, 0, 1) \in N$ or $-k \in k + F_0$ forcing $\mathrm{Tr}(k) = 0$, a contradiction. The proof is complete. $\square$

Lemma 4.10 *Let f be a special parabolic 3-ary bent function on $V = V(4n, 3)$, $n \geq 2$ and assume the notation of Example 3.8. Then, $\mathcal{Q}H \trianglelefteq \mathbf{A}(f)$.*

Proof We know that $G = \mathbf{A}(f) \leq \mathrm{GL}(\mathcal{U})$ and that W is G -invariant by Corollary 4.6.

We claim $O_3(G) = \mathcal{Q}$: Let $\Phi \in O_3(G)$. Then, $C_X(\Phi) \neq 0$ for $X \in \{W, V/W, \mathcal{U}_0\}$ as $O_3(G)$ is a 3-group [1, (5.15)]. Also every such X is an irreducible H -module, which forces $\Phi_X = 1_X$. Therefore, for all $v \in V$, we have $f(v\Phi) = f(v)$ and $v\Phi - v \in W$. We have seen in the proof of the previous lemma that f takes on $v_4 + W$ every value of $\mathbb{F}_3$ precisely $|W|/3$ times. So this fact holds too for any $v \in V - W$ (as H acts transitively on the non-trivial cosets of V/W). Since $\mathcal{Q} \leq O_3(G)$ and $|[V, \mathcal{Q}]| = |W|/3$, we deduce from $f(v\Phi) = f(v)$ that $\mathcal{Q}(v\Phi) = \mathcal{Q}(v)$ holds too, hence our claim.

For $X = W$ or $X = V/W$, let N_X be the kernel of the action of G on X . Of course, $N_W \cap N_{V/W} = \mathcal{Q}$.

(i) $N_W = N_{V/W} = \mathcal{Q}$.

The group G/N_X acts transitively on $X - 0$, and since $HN_X/N_X \simeq H \simeq \mathrm{SL}(2, 3^n)$, we deduce from Hering's theorem [20] that G has a normal subgroup S , such that

$$S/N_X \simeq \mathrm{SL}(2a, 3^b) \quad \text{or} \quad \mathrm{Sp}(2a, 3^b), \quad n = ab \quad (6)$$

and $H \simeq HN_X/N_X \leq S/N_X$.

If $\Phi \in N_{V/W}$ and Φ_W centralizes H_W , then $[H, \Phi] \leq N_{V/W}$ and $[H, \Phi]_W = 1$ or $[H, \Phi] \leq N_W \cap N_{V/W}$. So $\Phi \in N_G(H\mathcal{Q})$. But $N_{V/W} \cap N_G(H\mathcal{Q}) = \mathcal{Q}$ by Lemma 4.9. So only elements from $\mathcal{Q}$ are those elements in $N_{V/W}$, which centralize H_W . But any non-trivial normal subgroup of G/N_W contains also non-trivial elements,

which centralize S/N_W and thus H_W . This forces $N_{V/W} = Q$ and by symmetry claim (i) follows.

(ii) G has a normal subgroup S , such that $S/Q \simeq \mathrm{SL}(2a, 3^b)$ or $\mathrm{Sp}(2a, 3^b)$ for $n = ab$.

This follows from (i) and Eq. (6).

(iii) $a = 1$.

Suppose $a > 1$. Since $\mathrm{Sp}(2a, 3^b) \leq \mathrm{SL}(2a, 3^b)$, we can limit our attention to the case $\bar{S} = S/Q \simeq \mathrm{Sp}(2a, 3^b)$. Let $E = \{\Phi \in \mathrm{GL}(V) \mid \Phi_W = 1_W, \Phi_{V/W} = 1_{V/W}\}$. Then, E becomes an $\mathbb{F}_3 S$ -module under conjugation by S . Clearly, Q lies in the kernel of this representation, so that we can and do consider E as an $\mathbb{F}_3 \bar{S}$ -module. Let L be the natural $\mathbb{F}_{3^b} \bar{S}$ -module, and let ϕ be the Frobenius automorphism of $\mathbb{F}_{3^b}$. Denote by L^{ϕ^i} the i th Frobenius twist of L . Then, by [2, p. 218] E is the direct sum of the modules $L \otimes L^{\phi^i}$, $0 \leq i < b$. Now we follow [11, Sect. 2] (the restriction there to even characteristic is not essential). The module $M_i = L \otimes L^{\phi^i}$ is simple even as an $\mathbb{F}_3 \bar{S}$ -module for $i > 0$ and $i \neq \frac{b}{2}$ (b even). If b is even, then $L \otimes L^{\phi^{b/2}} \simeq M_{b/2} \oplus M_{b/2}$, $M_{b/2}$ a simple $\mathbb{F}_3 \bar{S}$ -module. Set $\mathcal{A} = \wedge^2(L)$ and $\mathcal{S} = S^2(L)$ (second term of the symmetric algebra over L). Then, $\mathcal{S}$ is simple and $L \otimes L \simeq \mathcal{A} \oplus \mathcal{S}$. Finally, we have $L \otimes L^{\phi^i} \simeq L \otimes L^{\phi^{b-i}}$ as $\mathbb{F}_3 \bar{S}$ -modules and $M_i \simeq M_j$ for $i, j > 0$ iff $j = i$ or $j = b - i$.

Then, by [11, Remark 2.5], we have decompositions of $\mathbb{F}_3 \bar{S}$ -modules

$$Q \simeq \mathcal{A} \oplus \bigoplus_{i=1}^{\lfloor b/2 \rfloor} M_i \quad \text{and} \quad E/Q \simeq \mathcal{S} \oplus \bigoplus_{i=1}^{\lfloor b/2 \rfloor} M_i.$$

Set $\widehat{W} = 0 \times 0 \times F \times F$. For $g \in S$ and $\widehat{w} \in \widehat{W}$ write $\widehat{w}g = \widehat{w}B(g) + \widehat{w}g_{\widehat{W}}$ with $\widehat{w}B(g) \in W$ and $\widehat{w}g_{\widehat{W}} \in \widehat{W}$. Then, $B(g_1 g_2) = (g_1)_{\widehat{W}} B(g_2) + B(g_1)(g_2)_W$, i.e., if we set $C(g) = g_{\widehat{W}}^{-1} B(g)$, then $C : \bar{S} \rightarrow \mathrm{End}_{\mathbb{F}_3}(F^2) \simeq E$ is a 1-cocycle of $\bar{S}$ into E and C induces a 1-cocycle into E/Q too. The restriction of this cocycle to the subgroup to HQ/Q is not a coboundary. Therefore, the restriction of C to $\bar{S}$ is not a coboundary too. But $H^1(\bar{S}, E/Q) = 0$ by Theorem 2.15 (or Lemma 2.17 if $\bar{S} \simeq \mathrm{Sp}(4, 3)$). So (iii) holds.

By (iii), we have $S = QH$, and the lemma follows $\square$

4.3 The groups $E(f)$

In this subsection, $p > 2$ is an odd prime.

Lemma 4.11 *Let V be a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V and $f : V \rightarrow \mathbb{F}_p$ be a proper parabolic bent function with respect to the subspace W of V . Let $\beta(v) \in \mathbf{E}(f)$. Then, $v \in W$.*

Proof Let $M \subseteq V$ be the subspace with $\beta(M) = \mathbf{E}(f)$ (c.f. Remark 2.5). Assume $M \not\subseteq W$. We know from Remark 3.12 that $\mathbf{A}(f) \cap \mathrm{GL}(\mathcal{U})$ contains a subgroup H , such that W is the unique maximal H -space in V . But M is an H -space too and therefore $M = V$. By Remark 2.5 f is a quadratic form, a contradiction. $\square$

Lemma 4.12 *Let V be a finite-dimensional $\mathbb{F}_p$ -space, Q a non-degenerate quadratic form on V and $f : V \rightarrow \mathbb{F}_p$ be a proper parabolic bent function with respect to the subspace W of V .*

- (a) *Let $f = Q_{W,a}^b$ be standard parabolic. Then, $\mathbf{E}(f) = \beta(W)$ if $b = 1$ and $\mathbf{E}(f) = \beta(\text{rad}(W))$ otherwise.*
- (b) *Let $f = Q_{W,\Pi}$ be twisted parabolic. Then, $\mathbf{E}(f) = \beta(\tilde{W})$, where $\tilde{W} = C_W(\tilde{\Pi}) \cap \{w \in W \mid Q(w) = Q(w\Pi)\}$.*
- (c) *Let f be special parabolic. Then, $\mathbf{E}(f) = \beta(W)$.*

Proof The assertion of the lemma follows from Examples 3.4, 3.6 and 3.8, and Lemmas 2.2 and 4.11, if $W = \text{rad}(W)$ or if $f = Q_{W,a}^1$; in particular, assertion (c) holds. So either $f = Q_{W,a}^b$, $b \neq 1$, or $f = Q_{W,\Pi}$. Let $\beta(v) \in \mathbf{E}(f)$. Then, $v \in W$ (Lemma 4.11) and there exists a linear functional $\lambda_v : V \rightarrow \mathbb{F}_3$, such that

$$f(x+v) = f(x) + x\lambda_v + f(v), \quad \text{all } x \in V.$$

Assume first $f = Q_{W,a}^b$. For $x \in W$, we get $Q(x) + Q(v) + (v, x) = Q(x+v) = Q(x) + x\lambda_v + f(v)$ or (as $f(v) = Q(v)$)

$$(v, x) = x\lambda_v.$$

For $x \notin W$, we get $b(Q(x) + Q(v) + (v, x)) + a = bQ(x) + a + x\lambda_v + Q(v)$ or

$$b(v, x) = x\lambda_v + (1-b)Q(v).$$

Replacing x by $-x$ and subtracting, we get $b(v, x) = x\lambda_v$ for all $x \in V - W$. Since $V - W$ generates V , we have $b(v, x) = x\lambda_v$ for $x \in W$ too. As $b \neq 1$, this forces $v \in \text{rad}(W)$ and assertion (a) holds.

Assume now $f = Q_{W,\Pi}$. For $x \in W$, we get $Q(x\Pi) + Q(v\Pi) + (v\Pi, x\Pi) = Q((x+v)\Pi) = Q(x\Pi) + x\lambda_v + f(v)$ or (as $f(v) = Q(v\Pi)$)

$$(v\Pi\Pi^*, x) = x\lambda_v.$$

For $x \notin W$, we have $Q(x) + Q(v) + (v, x) = Q(x+v) = Q(x) + x\lambda_v + f(v)$ or

$$(v, x) = x\lambda_v + Q(v\Pi) - Q(v).$$

Replacing x by $-x$ and subtracting, we get $(v, x) = x\lambda_v$ for all $x \in V - W$ and therefore even for all x . Then, $Q(v\Pi) = Q(v)$ holds too. This forces $v \in \tilde{W}$. Conversely, it is easy to see that elements in $\tilde{W}$ define elements in $\mathbf{E}(f)$. $\square$

4.4 Proof of Theorem 4.1 and its Corollaries

Proof (Of Theorem 4.1) This theorem follows from Lemmas 4.7, 4.8, 4.9, 4.10 and 4.12. $\square$

Proof (Of Corollary 4.2) By Corollary 2.4, f_1 and f_2 are linear equivalent. Assertion (a) follows.

By Remark 3.12 and Theorem 4.1 $\mathbf{A}(f_i) \leq \mathrm{GL}(\mathcal{U})$, these groups leave invariant the decomposition $\mathcal{U} = V \oplus \mathcal{U}_0$ (with the usual conventions) and W^i is the unique maximal $\mathbf{A}(f_i)$ -space in V ($i = 1, 2$). Moreover, if $\dim V/W^i = m_i$, then $\mathbf{A}(f_i)$ induces on V/W^i the group $\mathrm{GL}(m_i, p)$, if f_i is standard or twisted, whereas a subgroup of $\Gamma\mathrm{L}(2, 3^m)$, $m \geq 2$. $\dim V = 4m$ is induced, if f_i is special parabolic. Assertion (d) follows.

So let f_1 and f_2 be standard or twisted parabolic. By Lemma 4.4 and Remark 2.11, we have $\dim \mathrm{rad}(W^1) = \dim \mathrm{rad}(W^2)$ and $\dim W^1 = \dim W^2$. If $\mathrm{rad}(W^i) \subset W^i$, then f_i is a quadratic form on $W^i/\mathrm{rad}(W^i)$ which has the same type as Q_i . The linear equivalence of f_1 and f_2 shows that the spaces $W^1/\mathrm{rad}(W^1)$ and $W^2/\mathrm{rad}(W^2)$ are isometric. Hence, Q_1 and Q_2 have the same type and Assertions (b) and (c) follow. $\square$

Proof (Of Corollary 4.3) (a), (b) and (c). Let $W_1 = \mathrm{rad}(W)$ be the radical of W , $W = W_1 \oplus W_0$ and let $W_2 \subseteq W_0^\perp$ be a totally singular complement of W in V .

We first show that the given conditions are sufficient for equivalence. We start by considering $Q_{W,a}^b$ and $Q_{W,a'}^b$. Let Φ_0 be a similarity of W_0 with multiplier r . Define $(w_0 + w_1 + w_2)\Phi = w_0\Phi_0 + w_1 + rw_2$ for $w \in W$ and $w_i \in W_i$. By Remark 2.7(c), Φ is a similarity with multiplier r and one observes $Q_{W,a}^b(x\Phi) = rQ_{W,ar^{-1}}^b(x)$. So by Remark 2.7(b), $Q_{W,a}^b$ is linear equivalent to $Q_{W,a'}^b$ if $aF^* = a'F^*$ in case $\dim V$ is even and if $a(F^*)^2 = a'(F^*)^2$ in case $\dim V$ is odd.

Consider next twisted parabolics $Q_{W,\Pi}$ and $Q_{W,\Pi'}$, and assume that there exist similarities $\Phi_1, \Phi_2 \in \mathrm{GO}(W_0, Q)$ with multiplier r such that $\Phi_1\Pi = \Pi'\Phi_2$. Define $\Phi \in \mathrm{GL}(V)$ by $\Phi = \mathrm{diag}(\Phi_1, 1_{W_1}, r1_{W_2})$. One observes $rQ_{W,\Pi'}(x) = Q_{W,\Pi}(x\Phi)$, i.e., both functions are linear equivalent.

Finally, consider the standard parabolic $Q_{W,0}^b$ and the twisted parabolic $Q_{W,\Pi}$ and assume that $\Pi \in \mathrm{GO}(W_0, Q_{W_0})$ has multiplier b^{-1} . Then, $bQ_{W,\Pi}(x) = Q_{W,0}^b(x)$ for all $x \in V$. Again both functions are linear equivalent.

We turn now to the necessity condition for equivalence of two parabolic bent functions (standard or twisted) f and f' . By Corollary 4.2, they are linear equivalent. Moreover, by Corollary 2.4 and its proof, there exists $\Phi \in \mathrm{GL}(\mathcal{U})$ such that $V\Phi = V$, $\mathcal{U}_0\Phi = \mathcal{U}_0$ ($\mathcal{U}$ and $\mathcal{U}_0$ as usual) and $\Gamma(f') = \Gamma(f)\Phi$. This implies $\Phi^{-1}\mathbf{A}(f)\Phi = \mathbf{A}(f')$, and this group is isomorphic to a subgroup of $\mathrm{GO}(V, Q)_W$. By Theorem 4.1, Φ normalizes a group QH , where Q acts trivially on $\mathrm{rad}(W)$, $W/\mathrm{rad}(W)$ and V/W and H induces the identity on $W/W_1 (\simeq W_0)$ and the restriction H_V lies in the stabilizer of W in $\mathrm{O}(V, Q)$ and by Theorem 4.1 $H_{W_1} \simeq H_{V/W} \simeq \mathrm{GL}(W_1)$. Let $S \leq H$ be a cyclic group of order $|W_1| - 1$. It induces a Singer group on W_1 . Our usual Frattini argument shows that we can assume that Φ centralizes S . So Φ fixes W_1 , $W_0 = C_V(S)$ and the unique S -complement W_2 of W in V (W_1 and W_2 are not isomorphic as $\mathbb{F}_p S$ -modules!). Then we adjust Φ by an element in S so that we have $\Phi_{W_1} = 1_{W_1}$. Then, $\Phi_{W_2} = r1_{W_2}$, $r \in \mathbb{F}_p^*$, as Φ induces the trivial automorphism of $H_{V/W}$. Let $\Phi_{\mathcal{U}_0} = d \in \mathbb{F}_p^*$, i.e.,

$$f(w_0\Phi_{W_0} + w_1 + rw_2) = df'(w_1 + w_2 + w_2), \quad w_i \in W_i.$$

Assume first $f = Q_{W,a}^b$, $f' = Q_{W,a'}^{b'}$. If $w_2 = 0$, then

$$dQ(w_0) = Q(w_0\Phi_{W_0})$$

and if $w_2 \neq 0$, then

$$d(b'((w_1, w_2) + Q(w_0)) + a') = b((w_1, rw_2) + Q(w_0\Phi_{W_0})) + a.$$

So if $W \neq W_1 = \text{rad}(W)$, then Φ_{W_0} is a similarity with multiplier d . In this case, the second equation implies $r = d$, $b = b'$ and $a'd = a$. Assertion (a) follows in this case. But if $W = W_1$, then $a \neq 0 \neq a'$ and by convention $b = b' = 1$ and we are done again, as $\dim V$ is even.

Assume now that $f = Q_{W,\Pi}$ and $f' = Q_{W,\Pi'}$. Write $x \in V$ as $x = w_0 + w_1 + w_2$ with $w_i \in W_i$. If $w_2 = 0$, then $dQ(w_0\Pi') = df'(x) = f(x\Phi) = Q(w_0\Pi'((\Pi')^{-1}\Phi_{W_0}\Pi))$, i.e., $(\Pi')^{-1}\Phi_{W_0}\Pi$ lies in $\text{GO}(W_0, Q)$ and has multiplier d .

If $w_2 \neq 0$, then we get $Q(w_0\Phi_{W_0}) + (w_1, rw_2) = d(Q(w_0) + (w_1, w_2))$. So $r = d$ and Φ_{W_0} lies in $\text{GO}(W_0, Q)$ and has multiplier r . With $\Phi_1 = \Phi_{W_0}$ and $\Phi_2 = (\Pi')^{-1}\Phi_{W_0}\Pi$ assertion (b) follows.

Assume finally that $f = Q_{W,a}^b$ and $f' = Q_{W,\Pi}$. Then, $W_1 \neq 0 \neq W_0$. We have $Q(w_0\Phi_{W_0}) = f(w_0\Phi_{W_0} + w_1) = df'(w_0 + w_1) = dQ(w_0\Pi)$ and for $w_2 \neq 0$ we get

$$b(Q(w_0\Phi_{W_0}) + (w_1, rw_2)) + a = d(Q(w_0) + (w_1, w_2)).$$

Replacing w_1 by $-w_1$ and subtracting the equations, we obtain $(w_1, rw_2) = d(w_1, w_2)$ and $bQ(w_0\Phi_{W_0}) + a = dQ(w_0)$ for all $w_0 \in W_0$. So $a = 0$ and using the first equation, we have $dQ(w_0\Pi) = Q(w_0\Phi_{W_0}) = b^{-1}dQ(w_0)$. Hence, $\Pi \in \text{GO}(W_0, Q_{W_0})$ has multiplier b^{-1} .

(d) First, we would like to show that the conditions for σ and σ' force the equivalence of $f = Q^\sigma$ and $f' = Q^{\sigma'}$. Let $k_i \in F$ with $\text{Tr}(k_i) = i$, $i = 0, 1, 2$. Then, the $w_i = (k_i, 0, 0, 1)$'s are representatives of the orbits B_i of QH in $V - W$. Note that $Q^\sigma(w_i) = i^\sigma$. Clearly, $-Q^\sigma = Q^{(2,3)\sigma}$, i.e., Q^σ and $Q^{(2,3)\sigma}$ are linear equivalent. We are left with the case that n is even and $0^\sigma \neq 0 \neq 0^{\sigma'}$. As $\text{Sym}(3) = \langle (2, 3) \rangle \cup \langle (2, 3) \rangle \sigma \langle (2, 3) \rangle$, there exist $\pi_1, \pi_2 \in \langle (2, 3) \rangle$ with $\sigma\pi_1 = \pi_2\sigma'$. Also there exist $\epsilon_j \in \mathbb{F}_3^*$ with $i^{\pi_j} = \epsilon_j i$ for $i \in \mathbb{F}_3$, $j = 1, 2$. Let $\mathcal{U}$ and $\mathcal{U}_0$ be defined as usual. Define $\Phi \in \text{GL}(\mathcal{U})$ by $\Phi_V = 1_V$ and $\Phi_{\mathcal{U}_0} = \epsilon_1$ if $\epsilon_2 = 1$ and by $\Phi_V = z(v)_V$ and $\Phi_{\mathcal{U}_0} = \epsilon_1$ if $\epsilon_2 = -1$. Then, $\Phi_V \in N_{\text{GL}(V)}((QH)_V)$ (cf. Lemma 4.9) and Φ maps $(w_i, i^\sigma) = (k_i, 0, 0, 1, i^\sigma) \in \Gamma(f)$ onto $(\epsilon_2 k_i, 0, 0, 1, \epsilon_1 i^\sigma) = (w_{\epsilon_2 i}, \epsilon_1 i^\sigma)$. As $(w_{\epsilon_2 i}, \epsilon_1 i^\sigma) = (w_{i^{\pi_2}}, i^{\sigma\pi_1}) = (w_{i^{\pi_2}}, (i^{\pi_2})^{\sigma'})$, we get $(w_i, i^\sigma)\Phi \in \Gamma(f')$. Since Φ_V normalizes $(QH)_V$, we see that Φ maps $\Gamma(Q^\sigma)$ onto $\Gamma(Q^{\sigma'})$.

Assume now that $f = Q^\sigma$ and $f' = Q^{\sigma'}$ are EA equivalent. By Corollary 4.2, they are linear equivalent and there exists $\Phi \in \text{GL}(\mathcal{U})$ that fixes the spaces W, V and $\mathcal{U}_0$ and maps $\Gamma(f)$ onto $\Gamma(f')$. Also because of this corollary and the first part of this verification, we only have to consider the case that n is odd. Then, $\mathbf{A}(f) = \mathbf{A}(f') = \Phi^{-1}\mathbf{A}(f)\Phi$ by Theorem 4.1 and hence $\Phi_V \in N_{\text{GL}(V)}((Q)H_V)$. By Claim (*) in the

proof of Lemma 4.9 and Theorem 4.1, we may assume $\Phi_V = 1_V$ or $= z(v)_V$ (n is even). But as n is odd, we see $\Phi = \text{diag}(1_V, \pm 1)$ and assertion (c) follows. $\square$

Remark 4.13 Let V and W be as in Corollary 4.3. The equivalence classes of standard parabolic bent functions with respect to W are represented by $Q_{W,0}^1$ and $Q_{W,1}^b$, $b \in \mathbb{F}_p^*$ if $\dim V$ is even and by $Q_{W,0}^1$, $Q_{W,1}^b$, $Q_{W,\nu}^b$, for $b \in \mathbb{F}_p^*$ and some $\nu \in \mathbb{F}_p^* - (\mathbb{F}_p^*)^2$ if $\dim V$ is odd.

Let $V = V(4n, 3)$. The equivalence classes of special parabolic bent functions are represented by Q^1 and $Q^{(0,1)}$ if n is even and by Q^1 , $Q^{(0,1)}$ and $Q^{(0,2)}$ if n is odd.

5 Automorphism groups and equivalence of Maiorana functions

In this section, we will clarify the automorphism and equivalence problem for the Maiorana McFarland functions from Example 3.13. We shall prove:

Theorem 5.1 Let $p > 2$ be a prime, $V = F \times F$, $F = \mathbb{F}_{p^n}$, $0 < \ell < p^n - 1$ be coprime to $p^n - 1$ and let $\mu_\ell : V \rightarrow \mathbb{F}_p$ be the p -ary bent function defined by $\mu_\ell(x, y) = \text{Tr}(xy^\ell)$. Assume that ℓ is not congruent to a p -power modulo $p^n - 1$. Then, $\mathbf{EA}(\mu_\ell) = \mathbf{A}(\mu_\ell) \leq \text{GL}(\mathcal{U})$, $\mathcal{U} = \mathcal{U}(\mu_\ell)$. Moreover, $\mathbf{A}(\mu_\ell) = \mathcal{QZP}(\Theta)$ and $\mathcal{Q} = \mathcal{O}_p(\mathbf{A}(\mu_\ell))$ is described in Sect. 5.1.

Corollary 5.2 Let $p > 2$ be a prime, $V = F \times F$, $F = \mathbb{F}_{p^n}$, and let $0 < \ell, m < p^n - 1$ be coprime to $p^n - 1$. Define p -ary bent functions $\mu_\ell, \mu_m : V \rightarrow \mathbb{F}_p$ by $\mu_s(x, y) = \text{Tr}(xy^s)$ for $s = \ell$ or $= m$. The following hold:

- (a) Let $\ell \equiv mp^a \pmod{p^n - 1}$ for some number a . Then, μ_ℓ and μ_m are linear equivalent.
- (b) Let μ_ℓ and μ_m be EA equivalent. Then, $\ell \equiv mp^a \pmod{p^n - 1}$ for some number a .

We start with simple observations.

Lemma 5.3 Let p be an odd prime and $\ell \not\approx 1$. Then, $\mathbf{E}(\mu_\ell) = 1$ and $\mathbf{EA}(\mu_\ell) = \mathbf{A}(\mu_\ell) \leq \text{GL}(\mathcal{U})$, $\mathcal{U} = \mathcal{U}(\mu_\ell)$.

Proof Assume $\mathbf{E}(\mu_\ell) \neq 1$ and pick $1 \neq \beta(x_0, y_0) \in \mathbf{E}(\mu_\ell)$. We know

$$(x, y, z)\beta(x_0, y_0) = (x + x_0, y + y_0, z + (x, y)\lambda + \mu_\ell(x, y))$$

with $\lambda = \lambda_{x_0, y_0}$ a linear functional, which we write as $(x, y)\lambda = \text{Tr}(ax + by)$ (cf. Remark 2.5). So

$$\mu_\ell(x + x_0, y + y_0) = \mu_\ell(x, y) + \text{Tr}(ax + by) + \mu_\ell(x_0, y_0)$$

for all $x, y \in F$. We have $\pi_{-1}\beta(x_0, y_0)\pi_{-1} = \beta(-x_0, y_0)$ (π_{-1} is defined in Example 3.13). Moreover, $(x, y)\lambda_{-x_0, y_0} = \text{Tr}(ax - by)$. Replace $\beta(x_0, y_0)$ by $\beta(x_0, y_0)\pi_{-1}\beta(x_0, y_0)\pi_{-1}$ or by $\beta(x_0, y_0)^{-1}\pi_{-1}\beta(x_0, y_0)\pi_{-1}$, so that we may either

assume (1) $x_0 = b = 0 \neq y_0, a$ or (2) $y_0 = a = 0 \neq x_0, b$. In both cases, $\mu_\ell(x_0, y_0) = 0$.

CASE (1) For all $x, y \in F$ we have $\text{Tr}(x(y + y_0)^\ell) = \text{Tr}(xy^\ell) + \text{Tr}(ax)$. Set $y = -y_0$ and we see $a = y_0^\ell$. The equation becomes

$$\sum_{i=1}^{\ell-1} \binom{\ell}{i} \text{Tr}(xy^i y_0^{\ell-i}) = \text{Tr}\left(x \sum_{i=1}^{\ell-1} \binom{\ell}{i} y^i y_0^{\ell-i}\right) = 0.$$

This shows $\sum_{i=1}^{\ell-1} \binom{\ell}{i} y^i y_0^{\ell-i} = 0$ and that $\binom{\ell}{i} = 0, 1 \leq i < \ell$, i.e., $x \mapsto x^\ell$ is additive, so that $\ell \sim 1$, a contradiction.

CASE (2) For all $x, y \in F$, we have $\text{Tr}((x + x_0)y^\ell) = \text{Tr}(xy^\ell) + \text{Tr}(by)$, i.e., $\text{Tr}(x_0 y^\ell) = \text{Tr}(by)$, which forces $x_0 = b = 0$, a contradiction. So $\mathbf{E}(\mu_\ell) = 1$. The second assertion follows from Lemma 2.3. $\square$

Lemma 5.4 *Let $\mu = \mu_\ell$ and $(a, b), (x, y) \in F \times F$. Then,*

$$D_{(a,b)}\mu(x, y) = \text{Tr}\left(x \sum_{i=0}^{\ell-1} \binom{\ell}{i} b^{\ell-i} y^i + a \sum_{i=0}^{\ell} \binom{\ell}{i} b^{\ell-i} y^i\right)$$

and

$$D_{(a,b)}^2\mu(x, y) = \text{Tr}\left(x \sum_{i=0}^{\ell-2} \binom{\ell}{i} (2^{\ell-i} - 2) b^{\ell-i} y^i + a \sum_{i=0}^{\ell-1} \binom{\ell}{i} (2^{\ell-i+1} - 2) b^{\ell-i} y^i\right).$$

If ℓ is not a p -power, then $D_{(a,b)}^2\mu = 0$ iff $b = 0$.

Proof As the trace is additive, we have $D_{(a,b)}\mu(x, y) = \text{Tr}((x + a)(y + b)^\ell - xy^\ell)$ and the first assertion follows. By Lemma 2.10:

$$\begin{aligned} D_{(a,b)}^2\mu(x, y) &= \text{Tr}\left(x \sum_{i=0}^{\ell-1} \binom{\ell}{i} (2b)^{\ell-i} y^i + (2a) \sum_{i=0}^{\ell} \binom{\ell}{i} (2b)^{\ell-i} y^i\right) \\ &\quad - 2 \text{Tr}\left(x \sum_{i=0}^{\ell-1} \binom{\ell}{i} b^{\ell-i} y^i + a \sum_{i=0}^{\ell} \binom{\ell}{i} b^{\ell-i} y^i\right) \\ &= \text{Tr}\left(x \sum_{i=0}^{\ell-2} \binom{\ell}{i} (2^{\ell-i} - 2) b^{\ell-i} y^i + a \sum_{i=0}^{\ell-1} \binom{\ell}{i} (2^{\ell-i+1} - 2) b^{\ell-i} y^i\right) \end{aligned}$$

Assume now that ℓ is not a p -power. Clearly, $D_{(a,b)}^2\mu(x, y) = 0$ if $b = 0$.

So assume $b \neq 0$. We claim that $A(Y) = \sum_{i=0}^{\ell-2} \binom{\ell}{i} (2^{\ell-i} - 2) b^{\ell-i} Y^i \in F[Y]$ is not the trivial polynomial.

Let $\ell = \ell_k p^k + \ell_{k-1} p^{k-1} + \dots + \ell_0$ be the p -adic expansion of ℓ . As ℓ is not a p -power and coprime to $p^n - 1$ (in particular, ℓ is odd), we have that $\ell_k + \ell_{k-1} + \dots + \ell_0 \geq$

3. By replacing ℓ by $p^c \ell \pmod{p^n - 1}$ for a suitably chosen c if necessary, we may assume $\ell_0 = \max\{\ell_i \mid 0 \leq i \leq k\}$. Then, $\ell_0 > 0$ and according to the theorem of Lucas $\binom{\ell}{i} \not\equiv 0 \pmod{p}$ for $0 \leq i \leq \ell_0$. Two consecutive $(2^{\ell-i} - 2)$'s cannot both be congruent 0 modulo p . So for at least one $i \in \{0, \dots, \ell_0\}$, we have $\binom{\ell}{i}(2^{\ell-i} - 2)b^{\ell-i} \not\equiv 0 \pmod{p}$. This proves the claim.

Choose, $y \in F$, such that $A(y) \neq 0$. Then, $D_{(a,b)}^2 \mu(x, y)$ is the trace of $xA(y) + aB(y)$ with $B(y) = \sum_{i=0}^{\ell-1} \binom{\ell}{i}(2^{\ell-i+1} - 2)b^{\ell-i}y^i$. Choose $x \in F$, such that $\text{Tr}(xA(y) + aB(y)) \neq 0$, i.e., $D_{(a,b)}^2 \mu(x, y) \neq 0$ and $D_{(a,b)}^2 \mu$ is not the trivial function. $\square$

5.1 The p -radical of $A(\mu_\ell)$

Lemma 5.5 *Let $n > 1$ be an integer, p a prime and $\ell \in [1, p^n - 2] \cap \mathbb{Z}$ be invertible modulo $p^n - 1$.*

- (a) *Assume $1 + \ell \sim 1 + p^a \ell \sim 1 + p^x \ell$ with $a, x \in [0, n - 1] \cap \mathbb{Z}$, $a > 1$. Then, $x = 0$ or $x = a$.*
- (b) *Let $(n, p) \neq (6, 2)$. Then, there exists at most one $a \in [0, n - 1] \cap \mathbb{Z}$, such that there exists a proper divisor d of n , such that $(\ell + p^a)p^d \equiv \ell + p^a \pmod{p^n - 1}$.*

Proof (a) Suppose $1 + \ell \equiv p^b(1 + p^a \ell) \equiv p^y(1 + p^x \ell) \pmod{p^n - 1}$. Then, $a > 0$ forces $b > 0$ too and $\ell(p^{a+b} - 1) \equiv 1 - p^b \pmod{p^n - 1}$ and $\ell(p^{x+y} - 1) \equiv 1 - p^y \pmod{p^n - 1}$. Hence,

$$\begin{aligned} (1 - p^y)(p^{a+b} - 1) &\equiv \ell(p^{a+b} - 1)(p^{x+y} - 1) \\ &\equiv (1 - p^b)(p^{x+y} - 1) \pmod{p^n - 1}, \end{aligned}$$

which implies

$$p^{b+x+y} + p^{a+b} + p^y \equiv p^{a+b+y} + p^{x+y} + p^b \pmod{p^n - 1}.$$

Substitute $a + b$ by $\bar{a}$ and $x + y$ by $\bar{x}$. By [8, (2.3)] either $\bar{a} = \bar{x}$ and $b = y$ or $\bar{x} = y$ and the assertion follows..

- (b) Suppose $(\ell + p^a)p^d \equiv \ell + p^a$ and $(\ell + p^b)p^e \equiv \ell + p^b \pmod{p^n - 1}$ for proper divisors d and e of n . Then, $(\ell + p^a)(p^d - 1) \equiv 0 \pmod{p^n - 1}$ and $(\ell + p^b)(p^e - 1) \equiv 0 \pmod{p^n - 1}$. We use Zsigmondy's theorem [27]. Assume that $p^n - 1$ has a p -primitive prime divisor r . Then, r divides $\ell + p^a$ and $\ell + p^b$ and thus $p^a - p^b$ forcing $a = b$, i.e., the assertion.

We are left with the case $n = 2$ and $p + 1 = 2^c$. Then, $e = d = 1$ and so $(p^a - p^b)(p - 1) \equiv 0 \pmod{p^2 - 1}$, forcing $a = b$. $\square$

Remark 5.6 Let $\mathcal{U} = F \times F \times \mathbb{F}_p$, $F = \mathbb{F}_{p^n}$. For $0 \leq a < n$ and $\beta \in F^*$, define $t_a(\beta)$ in $\text{GL}(\mathcal{U})$ by

$$(x, y, z)t_a(\beta) = (x + yT_a(\beta), y, z) = (x + \beta y^{p^a}, y, z).$$

We set $\mathcal{T}_a = \{t_a(\beta) \mid \beta \in F\}$.

Suppose $0 \leq a, b < n$, $a \neq b$ and that there exists a $0 \leq c < n$, such that $\ell + p^a \equiv p^c \ell + p^{b+c} \pmod{p^n - 1}$. Then, c is uniquely determined: Assume $\ell + p^a \equiv p^d \ell + p^{b+d} \pmod{p^n - 1}$. This leads to $\ell(p^c - 1) \equiv p^a - p^{b+c} \pmod{p^n - 1}$ and $\ell(p^d - 1) \equiv p^a - p^{b+d} \pmod{p^n - 1}$. So we get

$$(p^c - 1)(p^a - p^{b+d}) \equiv \ell(p^c - 1)(p^d - 1) \equiv (p^d - 1)(p^a - p^{b+c}) \pmod{p^n - 1}$$

or $p^{a+c} + p^{b+d} \equiv p^{a+d} + p^{b+c} \pmod{p^n - 1}$, which forces $d = c$ (as $a \neq b$).

We claim that $\sigma_{a,b}(\beta) = t_a(\beta)t_b(-\beta^{p^{-c}})$ induces an element of $\mathbf{A}(\mu_\ell)$. We observe

$$\text{Tr}(y(T_a(\beta) + T_b(-\beta^{p^{-c}})) \cdot y^\ell) = \text{Tr}(\beta y^{p^a+\ell} - \beta^{p^{-c}} y^{p^{b+\ell}}) = 0,$$

which implies $\mu_\ell((x, y)\sigma_{a,b,c}(\beta)) = \mu_\ell(x, y)$. We set

$$\mathcal{Q}_{a,b} = \{\sigma_{a,b}(\beta) \mid \beta \in F\}.$$

One computes (or use Lemma 2.14) $z_u^{-1}t_a(\beta)z_u = t_a(u^{\bar{\ell}p^a+1}\beta)$. Thus, $Z = (F^*, \cdot)$ induces on $\mathcal{T}_a$ the representation $D^{\bar{\ell}p^a+1}$ (we identify $\beta \in F$ with $t_a(\beta)$). If $\ell + p^a \equiv p^c \ell + p^{b+c} \pmod{p^n - 1}$, then $1 + p^a \bar{\ell} \equiv p^c + p^{b+c} \bar{\ell} \pmod{p^n - 1}$. So Z induces the representation $D^{\bar{\ell}p^a+1}$ on $\mathcal{Q}_{a,b}$.

It can happen that the intersection of $\mathbf{A}(\mu_\ell)$ with a group $\mathcal{T}_a$ is non-trivial. This will be considered in the next lemma.

Lemma 5.7 *Let $0 \leq a < n$ and assume that there exists a proper divisor d of n , such that $(\ell + p^a)p^d \equiv \ell + p^a \pmod{p^n - 1}$. If d is chosen minimal with this property, then*

$$\mathbf{A}(\mu_\ell) \cap \mathcal{T}_a = \{t_a(\beta) \mid \beta \in \ker \text{Tr}_{n:d}\}.$$

In particular, $|\mathbf{A}(\mu_\ell) \cap \mathcal{T}_a| = p^{n-d}$.

Proof $t_a(\beta) \in \mathbf{A}(\mu_\ell) \cap \mathcal{T}_a$ is equivalent to

$$\text{Tr}(\beta y^{p^a+\ell}) = 0$$

for all $y \in F$. Hence, $\beta D^{p^a+\ell}(F^*) \subseteq \ker \text{Tr} \subset F$. By Lemma 2.13, we conclude that $p^a + \ell$ lies in a short cyclotomic coset of n modulo p . So there exists a proper divisor d of n , such that $(\ell + p^a)p^d \equiv \ell + p^a \pmod{p^n - 1}$, i.e., $D^{p^a+\ell}(F^*) \subseteq \mathbb{F}_{p^d}$. Moreover, $\dim \langle D^{p^a+\ell}(F^*) \rangle = d$, if d is chosen minimal with this property: By this assumption, $D^{p^a+\ell}(F^*)$ and so $E = \langle D^{p^a+\ell}(F^*) \rangle$ lies in no proper subfield of $\mathbb{F}_{p^d}$. But E is a ring, i.e., a subfield and $E = \mathbb{F}_{p^d}$ follows.

Let $\beta \in \ker \text{Tr}_{n:d} \subseteq \ker \text{Tr}$. Then, $\beta y^{p^a+\ell} \in \ker \text{Tr}_{n:d}$, as $\ker \text{Tr}_{n:d}$ is an $\mathbb{F}_{p^d}$ -space. Hence, $\text{Tr}(\beta y^{p^a+\ell}) = 0$ for all $y \in F$ and $t_a(\beta) \in \mathbf{A}(\mu_\ell)$.

On the other hand, $t_a(\beta) \in \mathbf{A}(\mu_\ell)$ implies $\beta D^{p^a+\ell}(F^*) \subseteq \ker \text{Tr}$, i.e., β lies in an $\mathbb{F}_{p^d}$ -subspace of $\ker \text{Tr}$ and we have already $\beta \in \ker \text{Tr}_{n:d}$ if $d = 1$. If $d > 1$, then

$\ker \text{Tr}_{n:d}$ is a maximal $\mathbb{F}_{p^d}$ -subspace of $\ker \text{Tr}$ (note that $\ker \text{Tr}$ is not an $\mathbb{F}_{p^d}$ -space), we obtain again $\beta \in \ker \text{Tr}_{n:d}$. $\square$

Let $p > 2$ be a prime and set $\mathcal{I} = [0, n-1] \cap \mathbb{Z}$. By Example 3.13, Remark 5.6 and Lemmas 5.5 and 5.7, there exists a partition $\mathcal{I} = \mathcal{P} \cup \mathcal{E} \cup (\mathcal{I} - (\mathcal{P} \cup \mathcal{E}))$, such that there exists a partition $\mathcal{P} = \{a_1, b_1\} \cup \dots \cup \{a_s, b_s\}$ into 2-sets, such that $|\mathcal{Q}_{a_i b_i}| = p^n$ for $1 \leq i \leq s$ and $\mathcal{Q}_a = \mathcal{T}_a \cap \mathbf{A}(\mu_\ell) \neq 1$ for $a \in \mathcal{E}$. Moreover, both sets $\mathcal{P}$ and $\mathcal{E}$ may be empty and $|\mathcal{E}| \leq 1$ by Lemma 5.5(b). We set

$$\mathcal{Q} = \prod_{i=1}^s \mathcal{Q}_{a_i, b_i} \cdot \prod_{a \in \mathcal{E}} \mathcal{Q}_a.$$

Lemma 5.8 *Let $\mathcal{Q}$ be the subgroup of operators $\Phi \in \mathbf{A}(\mu_\ell)$, such that Φ induces the identity on X , V/X and U/V . Then $\mathcal{Q} = \mathcal{Q}$.*

Proof Let Φ be an element of $\mathcal{Q}$. Our assumption forces $\Phi \in \mathcal{T} := \prod_{i=0}^{n-1} \mathcal{T}_i$. The group $Z = (F^*, \cdot)$ induces the representation D^1 on X , $D^{-\bar{\ell}}$ on Y , and $D^{1+p^a \bar{\ell}}$ on $\mathcal{T}_a$ by Remark 5.6. By Lemma 5.5, a representation $D^{1+p^a \bar{\ell}}$ occurs either once or twice in $\mathcal{T}$. Define $\mathcal{P}$ and a_i, b_i , $1 \leq i \leq s$ as above. Then, $\mathcal{T} = \prod_{i=1}^s (\mathcal{T}_{a_i} \mathcal{T}_{b_i}) \prod_{a \in \mathcal{I} - \mathcal{P}} \mathcal{T}_a$ is the decomposition of $\mathcal{T}$ into homogeneous components with respect to Z . Thus, $\mathcal{Q} = \prod_{i=1}^s (\mathcal{Q} \cap \mathcal{T}_{a_i} \mathcal{T}_{b_i}) \prod_{a \in \mathcal{I} - \mathcal{P}} (\mathcal{Q} \cap \mathcal{T}_a)$ by Clifford's Theorem. But obviously, $\mathcal{Q}_{a_i, b_i} = \mathcal{Q} \cap \mathcal{T}_{a_i} \mathcal{T}_{b_i}$ and $\mathcal{Q}_a = \mathcal{Q} \cap \mathcal{T}_a$ showing the assertion. $\square$

5.2 Proof of Theorem 5.1 and Corollary 5.2

Lemma 5.9 *Let p be a prime, $V = V(n, p)$, $n = ma > 1$, and $S \leq G = \text{GL}(V)$ a Singer group.*

- (a) *G contains precisely one conjugacy class $\mathcal{C}$ of subgroups isomorphic to $\text{SL}(m, p^a)$, which act transitively on $V - 0$.*
- (b) *There exists precisely one element in $\mathcal{C}$, whose normalizer contains S .*

Proof (a) follows from the classification of 2-transitive permutation groups [5], Hering's Theorem [20], but also from the fact that $\text{SL}(m, p^a)$ has only one simple module over $\mathbb{F}_p$ of size p^n , such that all non-trivial elements of the module are conjugate (cf. [22]).

- (b) Assume $S \leq N_G(H) \cap N_G(H_1)$ with H and $H_1 \in \mathcal{C}$. Let $H_1 = H^g$. Then, S and S^g are Singer groups in $N_G(H_1) \simeq \Gamma\text{L}(m, p^a)$. By Lemma 2.12, there exists $h \in N_G(H_1)$ with $S = S^{gh}$. But then $gh \in N_G(H)$, as $N_G(S) \leq N_G(H)$. Hence, $H_1 = H^g = H^{gh} = H$. $\square$

Identify $V = V(n, p)$ with $(F, +)$, $F = \mathbb{F}_{p^n}$. Then, $S = T_0(F^*)$ is a Singer group in $\text{GL}(V)$. Let $\mathcal{H}$ be the set of operators $T((b))$, $b = (b_0, \dots, b_{n-1}) \in F^n$ with $b_i = 0$ for $i \not\equiv 0 \pmod{a}$. In the obvious way, $\mathcal{H}$ can be identified with the set of $\mathbb{F}_{p^a}$ -linear endomorphisms of V . So the invertible elements of $\mathcal{H}$ form a group isomorphic to $\text{GL}(m, p^a)$ and the unique subgroup $H \simeq \text{SL}(m, p^a)$ has the property $S \leq N_{\text{GL}(V)}(H)$.

Lemma 5.10 *Let $V = V(n, q)$, $n > 1$ and $S = \text{SL}(V)$. Then, $\Gamma\text{L}(V)$ is the normalizer of S in the symmetric group $\text{Sym}(V)$.*

Proof We exclude the cases $(n, q) = (2, 2)$ and $(2, 3)$, which can be treated in an ad hoc fashion, for instance by computer calculation with GAP [25]. Set $N = N_{\text{Sym}(V)}(S)$ and $H = \Gamma\text{L}(V)$. Clearly, $H \leq N$. The group N fixes 0, the only fixed-point of S , and permutes the sets $\mathbb{F}_q v - \{0\}$, $0 \neq v \in V$: Assume $\alpha \in N$, such that $(Fv)\alpha \cap Fv \neq \{0\} \neq (Fv)\alpha \cap Fw$ for $Fv \neq Fw$. Note that $(Fv)\alpha - \{0\}$ is a set of imprimitivity of S , since $Fv - \{0\}$ is such a set. However, the pointwise stabilizer of Fv in S acts transitively on points $\neq Fv$, i.e., we find some $\beta \in S$, such that $(Fv)\alpha\beta \cap (Fv)\alpha \neq \{0\}$ and $(Fv)\alpha\beta \neq (Fv)\alpha$ as there are more than q points, a contradiction.

Set $\Omega = \{\mathbb{F}_q v \mid 0 \neq v \in V\}$. Let K be the kernel of the action of N on Ω . We use the bar convention for homomorphic images modulo K , in particular $\bar{N} = N/K$ and $\bar{S} = SK/K \simeq S/(S \cap K) \simeq \text{PSL}(V)$ is simple. Then, $\bar{N}$ is a subgroup of $\text{Aut}(\bar{S})$, as $(\bar{S}, \Omega)$ is a primitive permutation group. One knows $\text{Aut}(\bar{S}) = \bar{H}$ if $n = 2$ and $\text{Aut}(\bar{S}) = \bar{H}\langle\gamma\rangle$ if $n > 2$ (see [6, Thm. 12.5.1]). Here, γ is the graph automorphism of $\bar{S}$. But if $\bar{S}_\omega$, $\omega \in \Omega$, is a point stabilizer, then $\bar{S}_\omega^\gamma$ is the stabilizer of a hyperplane. Since these two groups have orbits of different sizes on Ω , we conclude $\bar{N} = \bar{H}$.

Let $\alpha \in N$. Adjusting α by an element from H , we may assume that α induces the trivial automorphism of $\bar{S}$, i.e., $[S, \alpha] = [\alpha, S] \leq Z(S)$. Then, by the Three-Subgroup Lemma [1, (8.7)] $[S, \alpha] = [[S, S], \alpha] = [[S, \alpha], S] = [[\alpha, S], S] = 1$, i.e., $\alpha \in C_{\text{Sym}(V)}(S)$. The stabilizer of a point $\mathbb{F}_q v$ in S induces $\text{GL}(1, q)$ on this point. So there exists a scalar $0 \neq a_v \in \mathbb{F}_q^*$, such that $(cv)\alpha = a_v(cv)$, $c \in \mathbb{F}_q$. If v and w are linearly independent, there exists $\Phi \in S$ with $v\Phi \in \mathbb{F}_q w$. Hence, $a_w(v\Phi) = (v\Phi)\alpha = (v\alpha)\Phi = (a_v v)\Phi$ and $a_w = a_v$. So $\alpha = a1_V$, $a \in \mathbb{F}_q^*$, i.e., $\alpha \in H$. $\square$

Proof (Of Theorem 5.1) Set $\mu = \mu_\ell$. By Lemma 2.3, $G = \mathbf{A}(\mu) \leq \text{GL}(\mathcal{U})$, $\mathcal{U} = F \times F \times \mathbb{F}_p$, $F = \mathbb{F}_{p^n}$. Set $X = F \times 0 \times 0$, $Y = 0 \times F \times 0$ and $\mathcal{U}_0 = 0 \times 0 \times \mathbb{F}_p$, i.e., $V = X \oplus Y$. By Lemma 5.4, X is invariant under G . Note that $C_M(O_p(G)) \neq 0$ by [1, (5.15)] and M is an irreducible $\mathcal{Z}$ -module for $M \in \{X, V/X, \mathcal{U}_0\}$. This shows $C_M(O_p(G)) = M$ and therefore $\mathcal{Q} = O_p(G)$ by Lemma 5.8.

By Lemma 5.8, $G/\mathcal{Q}$ is isomorphic to a subgroup of $\text{GL}(X) \times \text{GL}(V/X)$. Set $\mathcal{T} = \mathcal{Z}_0\mathcal{P}$, where $\mathcal{Z}_0$ is the subgroup of $\mathcal{Z}$ of order $p-1$. Then, $\mathcal{T}\mathcal{Q}/\mathcal{Q} \simeq \mathcal{Z}(\text{GL}(X) \times \text{GL}(V/X))$, i.e., $\mathcal{T}\mathcal{Q} \trianglelefteq G$. A Frattini argument shows $G = N_G(\mathcal{T})\mathcal{Q}$. Set $N = N_G(\mathcal{T})$. Then, $V = X \oplus Y$ is an N -invariant decomposition (as X and Y are the homogeneous components of $\mathcal{T}$ on V). Clearly, $\mathcal{Z}\mathcal{P}(\Theta) \leq N$ and $\mathcal{Z}(\Theta) \leq N_0$, where N_0 is the kernel of the action of N on $\mathcal{U}_0$.

We have to show $N_0 = \mathcal{Z}(\Theta)$. The group N_0 is faithfully represented on X as well as on Y , since $\mu(x, y) = \mu(x\Phi_X, y\Phi_Y)$ for all $x \in X$, $y \in Y$ and $\Phi \in N_0$. Assume $N_0 > \mathcal{Z}(\Theta) \simeq \Gamma\text{L}(1, p^n)$. By [5, 18] or [20] $N_0 \simeq \Gamma\text{L}(m, p^b)$, with $mb = n$, $m > 1$. The representation of the operator $z_a \in \mathcal{Z}$ as $\text{diag}(T_0(a), T_0(a^{-\bar{\ell}}), 1)$ leads to an obvious identification of X and Y with F . Let $S_0 \simeq \text{SL}(m, p^b) \trianglelefteq N_0$. The groups $(S_0)_X$ and $(S_0)_Y$ (for both groups the normalizer in $\text{GL}_{\mathbb{F}_p}(F)$ contains $T_0(F^*)$) have the same concrete description as a subgroup of $\text{GL}_{\mathbb{F}_p}(F)$, which is given after the proof of Lemma 5.9.

By Lemma 5.10 $N_{\text{Sym}(F)}((S_0)_Y) = (N_0)_Y$. For $\Phi \in S_0$, define Φ_X^* as the unique operator in $\text{GL}_{\mathbb{F}_p}(F)$, such that $\text{Tr}((x\Phi_X)y) = \text{Tr}(x(y\Phi_X^*))$ holds for all x, y in F . Since $T_0(F^*)$ is a Singer group of $\text{GL}_{\mathbb{F}_p}(F)$ contained in $(N_0)_X \cong (N_0)_Y$ and as $T_0(F^*)^* = T_0(F^*)$, we get $\Phi_X^* \in (S_0)_X^* = (S_0)_Y$ by Lemma 5.9. Let $\pi(\ell)$ be the permutation on F , which is induced by $x \mapsto x^\ell$. From $\mu(x, y) = \mu(x\Phi_X, y\Phi_Y)$, we deduce

$$\text{Tr}(x(y\pi(\ell))) = \text{Tr}(x\Phi_X(y\Phi_Y\pi(\ell))) = \text{Tr}\left(x\left(y\pi(\ell)\left(\pi(\ell)^{-1}\Phi_Y\pi(\ell)\Phi_X^*\right)\right)\right).$$

This forces $y' = y'\pi(\ell)^{-1}\Phi_Y\pi(\ell)\Phi_X^*$ for all $y' \in F$. So $1_F = \pi(\ell)^{-1}\Phi_Y\pi(\ell)\Phi_X^*$ or $\pi(\ell)^{-1}\Phi_Y\pi(\ell) = (\Phi_X^*)^{-1} \in (S_0)_Y$. Thus (we identify F and Y) $\pi(\ell) \in (N_0)_Y$, as we have seen above. But then $\pi(\ell)$ would be additive, a contradiction. $\square$

Proof (Of Corollary 5.2) Assume that $\mu = \mu_\ell$ and $\mu' = \mu_m$ are EA equivalent. By Corollary 2.4, there exists $\Phi \in \text{GL}(\mathcal{U})$, $\mathcal{U} = V \times \mathbb{F}_p$, fixing $V \times 0$ and $0 \times \mathbb{F}_p$, which sends the graph of μ onto the graph of μ' . In particular, $\mathbf{A}(\mu') = \Phi^{-1}\mathbf{A}(\mu)\Phi$. Our usual Frattini argument shows that we even may assume $\mathcal{Z}' = \mathcal{Z}^\Phi$, where $\mathcal{Z}$ and $\mathcal{Z}'$ are cyclic subgroups of $\mathbf{A}(\mu)$ and $\mathbf{A}(\mu')$, respectively, which correspond to the group $\mathcal{Z}$ of Example 3.13. Set $X = F \times 0 \times 0$ and $Y = 0 \times F \times 0$. Then, $\mathcal{Z}_W = \mathcal{Z}'_W = T_0(F^*)$, where we identify $W = X, Y$ in the obvious way with F .

Let z be a generator of $\mathcal{Z}$. Then, $z' = \Phi^{-1}z\Phi$ is a generator of $\mathcal{Z}'$. If $\omega, \omega^p, \dots, \omega^{p^{n-1}}$ are the eigenvalues of z on X (ω a primitive element in F), then $\omega^{-\bar{\ell}}, \omega^{-\bar{\ell}p}, \dots, \omega^{-\bar{\ell}p^{n-1}}$ are the eigenvalues of z on Y . Here, $\bar{\ell}$ is the inverse of ℓ modulo $p^n - 1$. The same must hold for z' , as these elements are conjugate by Φ and as this transformation fixes X and Y . On the other hand, z' as an element of $\mathcal{Z}'$ has on Y the eigenvalues $\omega^{-\bar{m}}, \omega^{-\bar{m}p}, \dots, \omega^{-\bar{m}p^{n-1}}$, where $\bar{m}$ is the inverse of m modulo $p^n - 1$. Hence, $-\bar{\ell}p^k \equiv -\bar{m} \pmod{p^n - 1}$ for some k , i.e., $-\bar{\ell} \sim -\bar{m}$ and $\ell \sim m$ follows. $\square$

6 Remarks to the case of even characteristic

As already mentioned, the predecessor of this paper is [9]. The results of [9] look a bit different than the findings the present paper. We discuss how [9] and the present paper compare and point out the slight differences.

It is well known that a function $f: V = V(n, 2) \rightarrow \mathbb{F}_2$ is bent iff the counterimage of 0, i.e., $B(f) = f^{-1}(0)$ is a difference set in V . In [9], we investigated instead of bent functions rather their associated difference set $B(f)$ (called *B-set* there). As a result, *B*-sets $B(f_1)$ and $B(f_2)$ are equivalent in the sense of [9], iff f_1 and f_2 are affine equivalent. In other words, the results of [9] relate to affine equivalence of functions and are therefore slightly more special than the related results in the present paper. So extra work is needed to generalize the work of [9] to the level of the present paper. We do not expect any difficulty but refrained from this task to keep the size of the present paper limited. However, there is a quick transfer of Corollary 5.2 to the case of even characteristic, namely Theorem 6.1.

The standard parabolic B -sets and the special parabolic B -sets are related to functions, which are exact analogs of standard parabolic and special parabolic bent functions in the present paper. However, the twisted parabolic B -sets correspond only to twisted parabolic bent functions $Q_{W,\Pi}$, where Π is chosen in such a way that $C_{\text{SO}(W/\text{rad}(W))}(\Pi) \simeq \text{SO}(a, p^b)$, $m = ab$, where $\text{SO}(W/\text{rad}(W)) = \text{SO}(m, p)$.

The big B -sets of cyclic trace type in [9] associated with the automorphism $x \mapsto x^k$ of $\mathbb{F}_{2^n}^*$ in $V(2n, 2)$ correspond to Maiorana functions μ_ℓ in the present paper, where $k\ell \equiv 1 \pmod{p^n - 1}$. The study B -sets of cyclic trace type was elaborate as it required the investigation of irreducible subgroups of $\text{GL}(2n, 2)$, which contain semiregular cyclic subgroups of order $2^n - 1$. The double derivative was allowed in the present paper to circumvent these obstacles. We note that double derivatives are useless in the even characteristic case, since they are trivial then. However, for *certain* numbers ℓ Gangopadhyay [13] is able to count the numbers of triples $v, w, x \in V(2n, 2)$, $v \neq w$ with $D_v D_w \mu_\ell(x) = 0$. As a consequence, he obtains inequivalence results for *some* functions μ_ℓ .

Theorem 6.1 *Let $n \geq 2$, $F = \mathbb{F}_{2^n}$ and $V = F \times F$, and let $0 < \ell, m < 2^n - 1$ be coprime to $2^n - 1$. Define bent functions $\mu_\ell, \mu_m : V \rightarrow \mathbb{F}_2$ by $\mu_s(x, y) = \text{Tr}(xy^s)$ for $s = \ell$ or $s = m$. Let μ_ℓ and μ_m be EA equivalent. Then, $\ell \equiv m2^a \pmod{2^n - 1}$ for some number a .*

Proof Assume first that $n \neq 6$. Then, there exists a p -primitive (cf. [27]) prime divisor r of $2^n - 1$, i.e., r divides $2^n - 1$ but not $2^k - 1$ for $1 \leq k < n$. Let r^a be the largest prime power dividing $2^n - 1$ and denote by μ_s either μ_ℓ or μ_m . The Sylow r -subgroup $\mathcal{Z}_r^{(s)}$ of $\mathcal{Z}^{(s)}$ (which plays the role of $\mathcal{Z}$ in Example 3.13) has order r^a . Let R be a Sylow r -subgroup of $\mathbf{EA}(\mu)$ containing $\mathcal{Z}_r^{(s)}$. We claim, $R \leq \text{GL}(\mathcal{U})$, ($\mathcal{U}$ the ambient space of μ), and $[\mathcal{U}, R] = \langle u(\phi - 1) \mid u \in \mathcal{U}, \phi \in R \rangle = F \times F \times 0$ and $C_{\mathcal{U}}(R) = 0 \times 0 \times \mathbb{F}_2$:

As r is coprime to 2, the subgroup $\text{GL}(\mathcal{U})$ contains Sylow r -subgroups of $\text{AGL}(\mathcal{U})$. They are isomorphic to $C_{r^a} \times C_{r^a}$ (Clearly, $\text{GL}(\mathcal{U})$ contains a subgroup $Z_1 \times Z_2 \simeq C_{2^{n-1}} \times C_{2^{n-1}}$, where the groups Z_i induce Singer groups on two disjoint n -subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$. Note also that r^{2a} is the maximal r -power dividing the order of $\text{GL}(\mathcal{U})$). In particular, Sylow r -subgroups are abelian. Let ϕ be in R . As ϕ centralizes $\mathcal{Z}_r$, this operator fixes $[\mathcal{U}, \mathcal{Z}_r^{(s)}] = F \times F \times 0$ and $C_{\mathcal{U}}(\mathcal{Z}_r^{(s)}) = 0 \times 0 \times \mathbb{F}_2$. Moreover, one must have $\phi = \Phi \circ \tau_u$, with $u \in 0 \times 0 \times \mathbb{F}_2$. As ϕ has r -power order also $u = 0$, forcing the claim.

Assume now that $\alpha \in \text{AGL}(\mathcal{U})$ induces an EA isomorphism from μ_ℓ to μ_m . Let $R \leq \mathbf{A}(\mu_\ell) \cap \text{GL}(\mathcal{U})$ be the Sylow r -subgroup containing $\mathcal{Z}_r^{(\ell)}$. By Sylow's theorem, we can assume $\mathcal{Z}_r^{(m)} \leq \alpha^{-1} R \alpha$. Then, $C_{\mathcal{U}}(R) \alpha = C_{\mathcal{U}}(\alpha^{-1} R \alpha) = C_{\mathcal{U}}(R)$ and $[\mathcal{U}, R] \alpha = [\mathcal{U}, \alpha^{-1} R \alpha] = [\mathcal{U}, R]$ by the claim above. So μ_ℓ and μ_m are affine equivalent. By [9, Thm. B], the assertion of the theorem follows.

For $n = 6$, we observe: $\dim \mathcal{U} = 13$, a Sylow 7-subgroup R of $\text{GL}(\mathcal{U})$ is elementary abelian of order 7^4 , $\dim[R, \mathcal{U}] = 12$ and $\dim C_{\mathcal{U}}(R) = 1$. We can now repeat the arguments from above with 7 in the role of the prime r . $\square$

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