The complex K^* ring of the complex projective Stiefel manifold

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Abstract

The topological K^* ring is one of the important tools used for understanding the topology of a manifold. We settle a problem of computing the K^* ring of complex projective Stiefel manifold using combinatorics. We calculate K^* ring of the right generalized projective Stiefel manifold, denoted by $P_\ell W_{n,k}$, which gives us description of complex K^* ring of complex projective Stiefel manifold $PW_{n,k}$, which is a particular case of $P_\ell W_{n,k}$.

Keywords Complex projective Stiefel manifolds · Complex K^* ring · Right generalized complex projective Stiefel manifold · Hodgkin spectral sequence · Generalized binomial coefficients

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1 Introduction

The complex projective Stiefel manifold, denoted by $PW_{n,k}$, is a well-known manifold. Cohomology of $PW_{n,k}$ was studied in detail in [1,6,7]. In [3], question about the parallelizability of $PW_{n,k}$ was settled and for doing so, a relation from K^* ring was used but till now K^* ring of $PW_{n,k}$ is not studied in detail as per our knowledge. (See [4] for the details of the topological K theory.) K theory plays an important role in the study of topology of a manifold, and we became curious after observing that K^* ring of such a well-known manifold is unknown. This motivated us to calculate $K^*(PW_{n,k})$. Purpose of this note is to have a discussion about complex K^* ring of $PW_{n,k}$. We consider a class of manifolds which are called right generalized complex projective Stiefel manifolds, denoted by $P_\ell W_{n,k}$ where $n, k \in \mathbb{N}$. These manifolds are obtained as

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quotients of the complex Stiefel manifold $W_{n,k}$ under an action of the circle group $\mathbb{S}^1$. We fix a primitive tuple $\ell = (l_1, \dots, l_k) \in \mathbb{Z}^k$ and the action is given by the formula $g \cdot [v_1, \dots, v_k] := [g^{l_1} v_1, \dots, g^{l_k} v_k]$ where $g \in \mathbb{S}^1$ and $[v_1, \dots, v_k] \in W_{n,k}$. Notice that $PW_{n,k} = P_{\mathbf{1}_k} W_{n,k}$ where $\mathbf{1}_k$ is the k -tuple $(1, 1, \dots, 1)$. We use Hodgkin spectral sequence to understand complex K^* ring of right generalized complex projective Stiefel manifold. We show that

Theorem 1.1 *For a primitive tuple $\ell := (l_1, \dots, l_k) \in \mathbb{Z}^k$, we have*

$$K^*(P_\ell W_{n,k}) = \Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}[y, y^{-1}]/\langle b \rangle$$

where $w_0, \dots, w_{k-2}$ and y are of degree one each, $z_i := y^{l_i} - 1$ for $1 \leq i \leq k$ and $b := \gcd(S_{n-k+1}^k(Z), \dots, S_{n-1}^k(Z), S_n^k(Z))$ with $Z = (z_1, \dots, z_k)$ and for $i = n - k + 1, \dots, n - 1$, $S_i^k(Z) := \sum_{1 \leq j_1 \leq \dots \leq j_i \leq k} z_{j_1} z_{j_2} \cdots z_{j_i}$.

An immediate corollary is the following statement for the complex projective Stiefel manifold $PW_{n,k}$. Particular case of the Corollary was proved in [3] (for $k = n - 1$ and $i = 2$) and was used to show that $PW_{n,n-1}$ is stably parallelizable.

Corollary 1.2

$$K^*(PW_{n,k}) = \Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}[z]/\langle b \rangle$$

where $w_0, \dots, w_{k-2}$ are of degree one and $z := y - 1$ is of degree zero and $b := \gcd(\binom{n}{i} z^i \mid n - k < i \leq n)$. In particular, $\binom{n}{i} z^i = 0$ for $n - k < i \leq n$.

We begin by reviewing idea of generalized binomial coefficients and the Hodgkin spectral sequence. We then discuss complex K^* ring of $P_\ell W_{n,k}$.

2 Generalized binomial coefficients

Let $z_1, \dots, z_k$ be set of variables. Let $Z := (z_1, \dots, z_k)$ and define

$$C_i^k(Z) := \sum_{1 \leq j_1 < \dots < j_i \leq k} z_{j_1} z_{j_2} \cdots z_{j_i},$$

$$S_i^k(Z) := \sum_{1 \leq j_1 \leq \dots \leq j_i \leq k} z_{j_1} z_{j_2} \cdots z_{j_i}$$

with the convention that $C_0^k(Z) = S_0^k(Z) = 1$ and $C_i^k(Z) = S_i^k(Z) = 0$ for $i < 0$. (See [10] for a motivation and the formal definition.) For any natural number m ,

$$\sum_{i=0}^m (-1)^i C_i^k(Z) S_{m-i}^k(Z) = 0. \quad (2.1)$$

In particular, $\mathbf{C}$ and $\mathbf{S}$ are inverses of each other where $\mathbf{C}$ is

$$\mathbf{C} := \begin{pmatrix} C_0^k(Z) & C_1^k(Z) & \cdots & C_n^k(Z) \\ 0 & C_0^k(Z) & \cdots & C_{n-1}^k(Z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & C_0^k(Z) & C_1^k(Z) \\ 0 & \cdots & 0 & C_0^k(Z) \end{pmatrix}$$

and $\mathbf{S}$ is given by

$$\mathbf{S} := \begin{pmatrix} S_0^k(Z) & -S_1^k(Z) & \cdots & (-1)^n S_n^k(Z) \\ 0 & S_0^k(Z) & \cdots & (-1)^{n-1} S_{n-1}^k(Z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & S_0^k(Z) & -S_1^k(Z) \\ 0 & \cdots & 0 & S_0^k(Z) \end{pmatrix}$$

Also, if we denote by Z_i , the tuple $(z_1, \dots, \hat{z}_i, \dots, z_k)$, then we see that

$$C_j^k(Z) = C_j^{k-1}(Z_i) + z_i C_{j-1}^{k-1}(Z_i), \quad S_j^k(Z) = S_j^{k-1}(Z_i) + z_i S_{j-1}^k(Z). \quad (2.2)$$

Let $X' - \mathbf{1}_{n-k} := (x'_1 - 1, \dots, x'_{n-k} - 1)$, where $\mathbf{1}_{n-k}$ denote the $(n-k)$ -tuple $(1, \dots, 1)$. It is a direct checking that

$$C_i^n(Z, X' - \mathbf{1}_{n-k}) = \sum_{j=0}^i C_j^k(Z) C_{i-j}^{n-k}(X' - 1). \quad (2.3)$$

Example 2.1 (1) We see that

$$C_i^k(\mathbf{1}_k) = \binom{k}{i}, \quad S_i^k(\mathbf{1}_k) = \binom{k+i-1}{i}.$$

(2) If $z_1 = z_2 = \cdots = z_k = z$. That is, $Z = z \cdot \mathbf{1}_k$, then $C_i^k(Z) = C_i^k(\mathbf{1}_k) \cdot z^i$ and $S_i^k(Z) = S_i^k(\mathbf{1}_k) \cdot z^i$. Hence, the Eq. 2.1 gives us

$$\sum_{i=0}^m (-1)^{m-i} \binom{k}{m-i} \binom{k+i-1}{i} = 0. \quad (2.4)$$

3 Hodgkin spectral sequence

Hodgkin considered a spectral sequence involving a generalized cohomology theory in [8] and as a particular case he obtained a method to calculate K^* ring of G/H

where G is compact, connected Lie group with $\pi_1(G)$ is torsion free and $H \subset G$ is a closed subgroup. We discuss the Hodgkin spectral sequence briefly here.

Suppose $\mathfrak{h}^*$ is a generalized cohomology theory and we have following commutative diagram of fiber product

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

Hodgkin considered a spectral sequence of the form $E_2^{*,*} = \text{Tor}_{\mathfrak{h}^*(B)}^*(\mathfrak{h}^*(X), \mathfrak{h}^*(Y))$ and he proved that after putting appropriate conditions, this spectral sequence converges to $\mathfrak{h}^*(X \times_B Y)$. As a particular case, let $\mathfrak{h}^*$ be the G -equivariant K theory K_G where G is a compact, connected Lie group with $\pi_1(G)$ is torsion free and B be a one point space. Then he proved that $E_2^{*,*} = \text{Tor}_{R(G)}^*(K_G^*(X), K_G^*(Y)) \implies K_G^*(X \times Y)$.

We take $Y = G$, $X = G/H$ where H is a closed subgroup of G then we get $K_G^*(B) = K_G^*(\text{point}) = RG$, $K_G^*(Y) = K_G^*(G) = K^*(G/G) = \mathbb{Z}$, $K_G^*(X) = K_G^*(G/H) = K_H^*(\text{point}) = RH$ and $K_G^*(G \times G/H) = K^*((G \times G/H)/G) = K^*(G/H)$. Hence, the spectral sequence has the form

$$E_2^{*,0} = \text{Tor}_{R(G)}^*(R(H), \mathbb{Z}) \implies K^*(G/H).$$

A relation between E_∞ term and K^* ring was studied, and a criterion was established in [2] to understand when E_∞ term uniquely determines K^* ring. We recall it briefly here:

Let $\mathcal{F}$ be the category of $\mathbb{Z}_2$ -graded algebras with a multiplicative filtration: $K = K^0 \oplus K^1$ with

$$\begin{aligned} 0 &= F_{-2} \subset F_0 \subset \cdots \subset F_{2n} \subset \cdots \subset K^0 = \bigcup_i F_{2i}, \\ 0 &= F_{-1} \subset F_1 \subset \cdots \subset F_{2n+1} \subset \cdots \subset K^1 = \bigcup_i F_{2i+1} \end{aligned}$$

and $F_i \cdot F_j \subset F_{i+j}$.

Let $\mathcal{G}$ be the category of $\mathbb{Z}$ -graded algebras $E = E^0 \oplus E^1 \oplus E^2 \oplus \cdots$

We have $T: \mathcal{F} \rightarrow \mathcal{G}$ taking K to E with $E^i = F_i/F_{i-2}$ and the product induced by the multiplicative filtration on K , and $L: \mathcal{G} \rightarrow \mathcal{F}$ taking E to K with $F_{2i} = E^0 \oplus E^1 \oplus \cdots \oplus E^{2i}$ and $F_{2i+1} = E^1 \oplus \cdots \oplus E^{2i+1}$.

Notice that $T \circ L = \text{Id}_{\mathcal{G}}$, and if we consider algebras that are quotients of universal algebras, then the following theorem holds. Recall that a universal algebra is a commutative algebra which by definition is a tensor product of an exterior algebra over $\mathbb{Z}$, finitely generated by elements of odd degree, with a polynomial algebra over $\mathbb{Z}$, finitely generated by elements of even degree.

Theorem 3.1 *Let K and E be quotients of a universal algebra U by ideals I_K and I_E , respectively, and $T(K) = E$. If I_E is generated by elements of degree 0 and 1 then $K = L(E)$.*

4 K^* ring of $P_\ell W_{n,k}$

Convention 4.1 Let RG denote the complex representation ring of a group G and regard the representation ring RH of H as an RG -module via the restriction homomorphism $R: RG \rightarrow RH$. We have the augmentation maps $\epsilon_{RG}: RG \rightarrow \mathbb{Z}$ and $\epsilon_{RH}: RH \rightarrow \mathbb{Z}$. Note that $\epsilon_{RH} \circ R = \epsilon_{RG}$, that is R preserves the augmentation. We shall omit the suffixes from our notation for the augmentation maps when there is no danger of confusion. We regard $\mathbb{Z}$ as a module over RG (resp. RH) via ϵ_{RG} (resp. ϵ_{RH}).

4.1 Representation rings of $U(n)$ and $\mathbb{S}^1$

We know that (See [9] for details)

$$RU(n) = \mathbb{Z}[\sigma_1, \dots, \sigma_{n-1}, \sigma_n^{\pm 1}].$$

Here, σ_1 denotes the isomorphism class of the standard representation, sometimes denoted std_n , of $U(n)$ on $\mathbb{C}^n$, $\sigma_j := \Lambda^j(\sigma_1)$, the class of the j -th exterior power on the standard representation. Note that σ_n is one dimensional, corresponding to the character $\det: U(n) \rightarrow U(1)$ and hence is invertible in $RU(n)$ with inverse σ_n^{-1} . To be explicit, we have, $\sigma_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k} = C_k^n(x_1, \dots, x_n)$ for $1 \leq k \leq n$. Here, x_i is the character given by projection of T onto i -th factor where $T = (\mathbb{S}^1)^n$ is the maximal torus of $U(n)$ consisting of diagonal matrices of $U(n)$. Define $\Lambda_i := C_i^n(x_1 - 1, \dots, x_n - 1)$. Then, $RU(n) = \mathbb{Z}[\Lambda_1, \dots, \Lambda_{n-1}, \sigma_n^{\pm 1}]$.

We know that $R\mathbb{S}^1 = \mathbb{Z}[y, y^{-1}]$.

Let $S = T \cap U(n - k)$, so that S is a maximal torus of $U(n - k)$. Let x'_i be the character given by projection of S onto i -th factor. Then,

$$RU(n - k) = \mathbb{Z}[\lambda_1, \dots, \lambda_{n-k-1}, \mu_{n-k}^{\pm 1}]$$

where $\lambda_i := C_i^{n-k}(x'_1 - 1, \dots, x'_{n-k} - 1)$ and $\mu_{n-k} := x'_1 \cdots x'_{n-k}$.

Recall that $P_\ell W_{n,k}$ is the quotient space of the action of $\mathbb{S}^1$ on complex Stiefel manifold $W_{n,k}$ where action is given by

$$g \cdot [v_1, \dots, v_k] := [g^{\ell_1} v_1, \dots, g^{\ell_k} v_k], \quad g \in \mathbb{S}^1, \quad [v_1, \dots, v_k] \in W_{n,k}$$

where $\ell := (\ell_1, \dots, \ell_k)$ is a fixed primitive tuple.

We take $G = U(n)$ and $H = \mathbb{S}^1 \times U(n - k)$. The inclusion $H \xrightarrow{i} U(n)$ induces the map $R: RU(n) \rightarrow RH$ as follows: For $\varepsilon \in \mathbb{S}^1$ and $(t_1, \dots, t_{n-k}) \in S$, we have $i(\varepsilon, (t_1, \dots, t_{n-k})) = (\varepsilon^{\ell_1}, \dots, \varepsilon^{\ell_k}, t_1, \dots, t_{n-k})$. Therefore,

$$R(x_i) = \begin{cases} y^{l_i} & \text{if } 1 \leq i \leq k \\ x'_{i-k} & \text{if } k+1 \leq i \leq n. \end{cases}$$

Convention 4.2 (1) For $1 \leq i \leq k$, we define $z_i := y^{l_i} - 1$.

(3) We call a $(k-j)$ -tuple Z' , a *subtuple* of a k -tuple $Z = (z_1, \dots, z_k)$ if Z' is obtained from Z by removing j coordinates, say $i_1, i_2, \dots, i_k$. That is, $Z' = (z_1, \dots, \hat{z}_{i_1}, z_{i_1+1}, \dots, \hat{z}_{i_j}, \dots, z_k)$ for $1 \leq i_1 < i_2 < \dots < i_j \leq k$.

Using Eqs. (2.3) and (2.1), we see that

Lemma 4.3 For $i = 1, \dots, n$, $R(\Lambda_i) = \sum_{j=0}^i C_j^k(Z) \lambda_{i-j}$ and $\lambda_i = \sum_{j=0}^i (-1)^j S_j^k(Z) R(\Lambda_{i-j})$ where Z is as defined in the Convention 4.2.

Immediate consequence of this Lemma is

Lemma 4.4 Let Γ be the subring of RH generated by $R(\Lambda_i)$ with $1 \leq i \leq n-k-1$ and $R(\sigma_n^{\pm 1} - 1)$. Then RH is a free Γ module with basis $\{y^i \mid i \in \mathbb{Z}\}$.

Proof We have $\Gamma = \mathbb{Z}[R(\Lambda_1), \dots, R(\Lambda_{n-k-1}), \text{Res}(\sigma_n^{\pm 1} - 1)]$.

Let $\Gamma_0 := \mathbb{Z}[\lambda_1, \dots, \lambda_{n-k-1}, y^{C_1^k(\ell)} \mu_{n-k} - 1, y^{-C_1^k(\ell)} \mu_{n-k}^{-1} - 1]$. Define $\phi: RH \rightarrow RH$ by

$$\phi(y) = y, \quad \phi(\lambda_i) = \text{Res}(\Lambda_i) \quad \text{for } 1 \leq i \leq n-k-1 \quad \text{and} \quad \phi(\mu_{n-k}^{\pm 1}) = \text{Res}(\sigma_n^{\pm 1}).$$

Then Lemma 4.3 implies that ϕ is an isomorphism and $\phi(\Gamma_0) = \Gamma$. The map ϕ intertwines the Γ_0 -module structure on RH and the Γ -module structure on RH . It is evident that $RH = \Gamma_0[\mathbb{Z}[y, y^{-1}]]$. Hence we have $RH = \Gamma[\mathbb{Z}[y, y^{-1}]]$. $\square$

Set $A := RG/I_S$ where $S = \mathbb{Z}[\Lambda_1, \dots, \Lambda_{n-k-1}, \sigma_n^{\pm 1}] \subset RG$ and I_S is the ideal of RG generated by the kernel of the augmentation map $S \rightarrow \mathbb{Z}$. Thus $A = RG \otimes_S \mathbb{Z}$ where $\mathbb{Z}$ is an S -module via the augmentation. Let $B := RH/I_\Gamma = RH \otimes_\Gamma \mathbb{Z}$. Note that $R: RG \rightarrow RH$ defines a ring homomorphism $\bar{R}: A \rightarrow B$. We shall denote by $\bar{\Lambda}_j \in A$ the image of $\Lambda_j \in RG$. However, we use the same symbol $\mu_{n-k}^{\pm 1}, \lambda_i, y \in B$ for the image of $\mu_{n-k}^{\pm 1}, \lambda_i, y \in RH$ where $1 \leq i \leq n-k$ under the respective natural quotient maps. Note that $A = \mathbb{Z}[\bar{\Lambda}_j; n-k \leq j < n]$ is a polynomial algebra and $B = \mathbb{Z}[y, y^{-1}]$. We have the following

Lemma 4.5 $\text{Tor}_{RG}^*(RH, \mathbb{Z}) \cong \text{Tor}_A^*(B; \mathbb{Z})$.

Proof By the above discussion, RH is free as a Γ -module. By the change of ring theorem [5, Ch. XVI, Theorem 6.1], we have a spectral sequence with $E_2^{*,*} = \text{Tor}_A^*(\text{Tor}_\Gamma^*(RH; \mathbb{Z}); \mathbb{Z})$ converging to $\text{Tor}_{RG}^*(RH; \mathbb{Z})$. The spectral sequence collapses as RH is Γ -free by Lemma 4.3. Since $\text{Tor}_\Gamma^0(RH; \mathbb{Z}) = RH \otimes_\Gamma \mathbb{Z} = RH/I_\Gamma = B$, we get $\text{Tor}_{RG}^*(RH; \mathbb{Z}) \cong \text{Tor}_A^*(B; \mathbb{Z})$. $\square$

The algebras A and B are augmented where the augmentations $\bar{\epsilon}_A: A \rightarrow \mathbb{Z}$ and $\bar{\epsilon}_B: B \rightarrow \mathbb{Z}$ are induced by the augmentations of RG and RH respectively. In particular, $\bar{\epsilon}_A = \bar{\epsilon}_B \circ \bar{R}$. We shall merely write $\bar{\epsilon}$ for either augmentations when there is no risk of confusion. Clearly $\bar{\epsilon}(\bar{\Lambda}_r) = 0$, $n - k \leq r < n$. We understand the A module structure on B in detail by calculating image $\bar{R}: A \rightarrow B$ explicitly.

Lemma 4.6 In B , $\lambda_i = (-1)^i S_i^k(Z)$ and $\lambda_{n-k} = y^{-C_1^k(\ell)} + \sum_{j=0}^{n-k-1} (-1)^{j+1} S_j^k(Z)$.

Proof Let $i < n - k$. We know from the Lemma 4.3 that

$$\lambda_i = \sum_{j=0}^i (-1)^j S_j^k(Z) R(\Lambda_{i-j}) = \sum_{j=0}^{i-1} (-1)^j S_j^k(Z) R(\Lambda_{i-j}) + (-1)^i S_i^k(Z).$$

By very definition, for $1 \leq n - k - 1$, $R(\Lambda_j) = 0$ in B and hence $\lambda_i = (-1)^i S_i^k(Z)$.

We know that $\mu_{n-k} = \prod_{j=1}^{n-k} x'_j$. Hence, $\mu_{n-k} = \sum_{j=0}^{n-k} \lambda_j$. This gives us,

$$y^{C_1^k(\ell)} \mu_{n-k} - 1 = y^{C_1^k(\ell)} \left(\lambda_{n-k} + \lambda_0 + \sum_{j=1}^{n-k-1} \lambda_j \right) - 1$$

Using value of λ_j in B and the fact that $y^{C_1^k(\ell)} \mu_{n-k} - 1 = 0$ in B , we have

$$\begin{aligned} \lambda_{n-k} &= -y^{-C_1^k(\ell)} \left(y^{C_1^k(\ell)} - 1 + y^{C_1^k(\ell)} \sum_{j=1}^{n-k-1} (-1)^j S_j^k(Z) \right) \\ &= y^{-C_1^k(\ell)} - 1 + \sum_{j=1}^{n-k-1} (-1)^{j+1} S_j^k(Z). \end{aligned}$$

This concludes the proof. $\square$

Lemma 4.7 For $i \geq n - k$,

$$\bar{R}(\bar{\Lambda}_i) = C_{i-(n-k)}^k(Z) [y^{-C_1^k(\ell)} + \sum_{j=0}^{n-k} (-1)^{j+1} S_j^k(Z)] + \sum_{J=n-k+1}^i C_{i-J}^k(Z) (-1)^{J+1} S_J^k(Z).$$

Proof Let $i \geq n - k$. We write $R(\lambda_i)$ as

$$R(\Lambda_i) = \sum_{J=0}^{n-k} C_{i-J}^k(Z) \lambda_J = C_{i-(n-k)}^k(Z) \lambda_{n-k} + \sum_{J=0}^{n-k-1} C_{i-J}^k(Z) \lambda_J.$$

Using Lemma 4.6, we see that

$$\bar{R}(\bar{A}_i) = C_{i-(n-k)}^k(Z) \left[y^{-C_1^k(\ell)} + \sum_{j=0}^{n-k-1} (-1)^{j+1} S_j^k(Z) \right] + \sum_{J=0}^{n-k-1} C_{i-J}^k(Z) (-1)^J S_J^k(Z).$$

Using 2.1, we see that

$$\sum_{J=0}^{n-k-1} C_{i-J}^k(Z) (-1)^J S_J^k(Z) = \sum_{J=n-k}^i C_{i-J}^k(Z) (-1)^{J+1} S_J^k(Z).$$

Substituting in the previous equation, we see that

$$\bar{R}(\bar{A}_i) = C_{i-(n-k)}^k(Z) \left[y^{-C_1^k(\ell)} + \sum_{j=0}^{n-k} (-1)^{j+1} S_j^k(Z) \right] + \sum_{J=n-k+1}^i (-1)^{J+1} C_{i-J}^k(Z) S_J^k(Z).$$

□

Remark 4.8 From Lemma 4.3, we see that for $i > n - k$,

$$0 = \sum_{j=0}^i (-1)^{i-j} S_{i-j}^k(Z) R(\Lambda_j).$$

Rewriting this, we see that

$$\bar{R}(\bar{A}_i) = - \sum_{j=n-k}^{i-1} (-1)^{i-j} S_{i-j}^k(Z) \bar{R}(\bar{A}_j) - (-1)^i S_i^k(Z).$$

So in B , we have a recurrence relationship which is given by

$$\bar{R}(\bar{A}_i) + \sum_{j=n-k}^{i-1} (-1)^{i-j} S_{i-j}^k(Z) \bar{R}(\bar{A}_j) = (-1)^{i-1} S_i^k(Z). \quad (4.1)$$

Before continuing with a general case, we illustrate our calculation for a particular case, namely, complex projective space, denoted by CP^{n-1} .

Example 4.9 [K^* ring of complex projective space] Consider $k = 1$. Then, $n - k = n - 1$, $C_1^k(\ell) = l_1$ and $Z = (z_1)$. So in this case, we have

$$\bar{R}(\bar{A}_{n-k}) = y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j S_j^1(Z) = y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j.$$

We use Koszul resolution $\Lambda_A^*(\xi_0) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$ to find $\text{Tor}_A^*(B, \mathbb{Z})$. That is, we consider the following resolution of $\mathbb{Z}$:

$$0 \longrightarrow \xi_0 A \xrightarrow{d_0^{(1)}} A \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where differentials mentioned above are defined on the generators as $\epsilon(\bar{\Lambda}_{n-k}) = 0$ and $d_0^{(1)}(\xi_0) = \bar{\Lambda}_{n-k}$. The $\text{Tor}_A^*(B; \mathbb{Z})$ is the homology of the chain complex $0 \rightarrow \xi_0 B \xrightarrow{d_0^{(1)}} B \rightarrow 0$ where $d_0^{(1)}(\xi_0) = y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j$. That is,

$$\begin{aligned} \text{Tor}_A^*(B; \mathbb{Z}) &= \text{Tor}_A^0(B; \mathbb{Z}) = \text{Ker}\{B \rightarrow 0\} / \text{Im}\{\xi_0 B \xrightarrow{d_0^{(1)}} B\} \\ &= \mathbb{Z}[y, y^{-1}] / \left\langle y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j \right\rangle. \end{aligned}$$

Since y^{l_1} is a unit in B , $\left\langle y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j \right\rangle = \left\langle y^{l_1} (y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j) \right\rangle$.

Further expansion shows us

$$\begin{aligned} y^{l_1} \left(y^{-l_1} - \sum_{j=0}^{n-1} (-1)^j z_1^j \right) &= 1 - y^{l_1} \sum_{j=0}^{n-1} (-1)^j z_1^j = 1 - (1 + z_1) \sum_{j=0}^{n-1} (-1)^j z_1^j \\ &= 1 - \left[\sum_{j=0}^{n-1} (-1)^j z_1^j + \sum_{j=1}^n (-1)^{j-1} z_1^j \right] \\ &= 1 - \left[1 + \sum_{j=1}^{n-1} ((-1)^j + (-1)^{j-1}) z_1^j + (-1)^{n-1} z_1^n \right] \\ &= 1 - [1 + (-1)^{n-1} z_1^n] = (-1)^n z_1^n \\ &= (-1)^n S_n^1((z_1)). \end{aligned}$$

Using Theorem 3.1, we conclude that

$$K^*(P_{l_1} W_{n,1}) = \text{Tor}_A^*(B; \mathbb{Z}) = \mathbb{Z}[y, y^{-1}] / \langle (y^{l_1} - 1)^n \rangle$$

In particular, when $l_1 = 1$, $P_{l_1} W_{n,1} = P W_{n,1} = C P^{n-1}$ and after simplifying, we see that

$$K^*(P W_{n,1}) = \mathbb{Z}[z] / \langle z^n \rangle$$

where $z = y - 1$.

Observation 4.10 For every $i = 1, \dots, k$, there is a natural projection map

$$\pi_i: P_\ell W_{n,k} \rightarrow P_{\ell_i} W_{n,1}, \quad [v_1, \dots, v_k] \mapsto [v_i]$$

Hence, $(y^{l_i} - 1)^n = 0$ for all $1 \leq i \leq k$ in $K^*(P_\ell W_{n,k})$.

From now onwards, we will assume $\mathbf{k} \geq 2$.

We use Koszul complex and using Lemma 4.7, we see that $\text{Tor}_A^*(B; \mathbb{Z})$ is the homology of the long exact sequence

$$0 \rightarrow (\xi_0 \wedge \dots \wedge \xi_{k-1})B \xrightarrow{d_{k-1}^{(k)}} \dots \xrightarrow{d_{k-2}^{(k)}} \bigoplus_{0 \leq i < j \leq k-1} (\xi_i \wedge \xi_j)B \xrightarrow{d_1^{(k)}} \bigoplus_{i=0}^{k-1} \xi_i B \xrightarrow{d_0^{(k)}} B \rightarrow 0$$

where

$$d_p^{(k)}(\xi_{i_0} \wedge \dots \wedge \xi_{i_p}) = \sum_{j=0}^p (-1)^{j+1} [\xi_{i_0} \wedge \dots \wedge \hat{\xi}_{i_j} \wedge \dots \wedge \xi_{i_p}] \quad \bar{R}(\bar{\Lambda}_{(n-k)+i_j}).$$

Using Eq. 4.1, we see that

$$\bar{R}(\bar{\Lambda}_i) + \sum_{j=n-k}^{i-1} (-1)^{i-j} S_{i-j}^k(Z) \bar{R}(\bar{\Lambda}_j) = (-1)^{i-1} S_i^k(Z).$$

After substituting $J = j - (n - k)$ and $I = i - (n - k)$, we get

$$\bar{R}(\bar{\Lambda}_{I+(n-k)}) + \sum_{J=0}^{I-1} (-1)^{I-J} S_{I-J}^k(Z) \bar{R}(\bar{\Lambda}_{J+(n-k)}) = (-1)^{I+(n-k)-1} S_{I+(n-k)}^k(Z).$$

In other words, for $I \leq k - 1$,

$$d_0^{(k)}(\xi_I) + \sum_{J=0}^{I-1} (-1)^{I-J} S_{I-J}^k(Z) d_0^{(k)}(\xi_J) = (-1)^{I+(n-k)-1} S_{I+(n-k)}^k(Z). \quad (4.2)$$

Define new variables $\zeta'_0, \zeta'_1, \dots, \zeta'_{k-1}$ by

$$\zeta'_I := \sum_{J=0}^I (-1)^{I-J} S_{I-J}^k(Z) \xi_J$$

for $0 \leq I \leq k-1$. Then, by Eq. 2.1, we see that $\xi_I = \sum_{j=0}^I C_j^k(Z) \zeta'_j$. Now define

$$\zeta_i^{(Z)} := \begin{cases} y^{C_1^k(\ell)} \zeta'_0 & \text{for } i = 0, \\ (-1)^{n-k+(i+1)} \zeta'_i & \text{otherwise.} \end{cases}$$

We see that $\text{Tor}_A^*(B; \mathbb{Z})$ is the homology of the long exact sequence (up to isomorphism) which is given by

$$0 \xrightarrow{d_k^{(k)}} (\zeta_0^{(Z)} \wedge \cdots \wedge \zeta_{k-1}^{(Z)}) B \xrightarrow{d_{k-1}^{(k)}} \cdots \xrightarrow{d_{k-2}^{(k)}} \bigoplus_{0 \leq i < j \leq k-1} (\zeta_i^{(Z)} \wedge \zeta_j^{(Z)}) B \xrightarrow{d_1^{(k)}} \bigoplus_{i=0}^{k-1} \zeta_i^{(Z)} B \xrightarrow{d_0^{(k)}} B \xrightarrow{d_{-1}^{(k)}} 0$$

where $d_0^{(k)}(\zeta_0^{(Z)}) := \bar{R}(\bar{A}_{n-k}) = 1 + y^{C_1^k(\ell)} \sum_{j=0}^{n-k} (-1)^{j+1} S_j^k(Z)$ and for $1 \leq i \leq k-1$, $d_0^{(k)}(\zeta_i^{(Z)}) := S_{i+(n-k)}^k(Z)$.

Lemma 4.11

$$d_0^{(k)}(\zeta_0^{(Z)}) = \sum_{j=1}^k a_j S_{n-k+j}^k(Z)$$

where $a_j \in \mathbb{Z}[y, y^{-1}]$ for all $i = 1, \dots, k$.

Proof Recall that

$$\begin{aligned} d_0^{(k)}(\zeta_0^{(Z)}) &= 1 + y^{C_1^k(\ell)} \sum_{j=0}^{n-k} (-1)^{j+1} S_j^k(Z) \\ &= 1 + \left(\sum_{i=0}^{\infty} C_i^k(Z) \right) \left(\sum_{j=0}^{n-k} (-1)^{j+1} S_j^k(Z) \right) = 1 + \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} (-1)^{j+1} C_i^k(Z) S_j^k(Z). \end{aligned}$$

We know that for any natural number m , $\sum_{i+j=m} (-1)^j C_i^k(Z) S_j^k(Z) = 0$. After rearranging terms and canceling them using the relation, we have

$$\begin{aligned} d_0^{(k)}(\zeta_0^{(Z)}) &= \sum_{I=1}^k \sum_{j=0}^{n-k} (-1)^{j+1} C_{n-k+I-j}^k(Z) S_j^k(Z) = \sum_{I=1}^k \sum_{j=n-k+1}^{n-k+I} (-1)^j C_{n-k+I-j}^k(Z) S_j^k(Z) \\ &= \sum_{j=1}^k (-1)^{n-k+j} \left(\sum_{i=0}^{k-j} C_i^k(Z) \right) S_{n-k+j}^k(Z). \end{aligned}$$

Hence, we have the result. $\square$

This allows us to conclude that

Corollary 4.12

$$\gcd(d_0^{(k)}(\zeta_0^{(Z)}), d_0^{(k)}(\zeta_1^{(Z)}), \dots, d_0^{(k)}(\zeta_{k-1}^{(Z)})) = \gcd(S_{n-k+1}^k(Z), \dots, S_{n-1}^k(Z), S_n^k(Z)).$$

Let $b := \gcd(d_0^{(k)}(\zeta_i^{(Z)}) \mid 0 \leq i \leq k-1)$. It follows that $b = \sum_{i=0}^{k-1} b_i(d_0^{(k)}(\zeta_i^{(Z)}))$ for some $b_i \in \mathbb{Z}[y, y^{-1}]$. Note that $(b_0, \dots, b_{k-1})$ is a primitive vector.

Lemma 4.13 *We can choose $\eta_0, \dots, \eta_{k-2}, \zeta$ such that $d_0^{(k)}(\eta_j) = 0$ for all $0 \leq j < k-1$ and the Koszul complex corresponding to $\zeta_0^{(z)}, \zeta_1^{(z)}, \dots, \zeta_{k-1}^{(z)}$ is isomorphic to the Koszul complex corresponding to $\eta_0, \dots, \eta_{k-2}, \zeta$.*

Proof We accomplish the task in two steps. First we find ζ so that $d_0^{(k)}(\zeta) = b$. Later we choose $\eta_0, \dots, \eta_{k-2}$ which would satisfy conditions mentioned in the statement of the Lemma.

As observed above $(b_0, \dots, b_{k-1})$ is a primitive vector and hence we can find a matrix $B = (b_{i,j}) \in GL_k(\mathbb{Z}[y, y^{-1}])$ with $b_{k,i} = b_i$ (See Lang- Algebra, page 547, exercise 25). Let B_0 be the block diagonal matrix $\text{diag}(1, B) \in GL_{k+1}(\mathbb{Z}[y, y^{-1}])$. Consider the algebra homomorphism $f_{B_0}: B \rightarrow B$ defined on the generators $\zeta_0^{(Z)}, \zeta_1^{(Z)}, \dots, \zeta_{k-1}^{(Z)}$ by multiplication by B_0 . It is clear that f_{B_0} is an algebra isomorphism. Let $U_j := f_{B_0}(\zeta_j^{(Z)}) = \sum_{p=0}^{k-1} b_{p,j} \zeta_p^{(Z)}$ for $0 \leq j \leq k-2$ and $\zeta := f_{B_0}(\zeta_{k-1}^{(Z)})$. Then we have

$$d_0^{(k)}(\zeta) = \sum_{i=0}^{k-1} b_{k,i} d_0^{(k)}(\zeta_i^{(Z)}) = \sum_{i=0}^{k-1} b_i (d_0^{(k)}(\zeta_i^{(Z)})) = b.$$

Choose α_j such that $d_0^{(k)}(U_j) = \alpha_j \cdot b$ for $j = 0, 1, \dots, k-2$.

Define $\eta_i := U_i - \alpha_i \zeta$ for $0 \leq i \leq k-2$ then $d_0^{(k)}(\eta_i) = 0$. The transformation is invertible and hence we have a proof. $\square$

Hence,

$$\text{Tor}_A^*(B; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2}) \otimes \text{Tor}_{A'}^*(B; \mathbb{Z})$$

where $\Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2})$ denotes the graded exterior algebra with generators w_i of degree 1 and $A' = A/\langle \eta_0, \dots, \eta_{k-2} \rangle = \mathbb{Z}[\zeta]$.

Now, $\text{Tor}_{A'}^*(B; \mathbb{Z})$ is the homology of the complex $0 \rightarrow \zeta B \xrightarrow{d_0^{(k)}} B \rightarrow 0$ which is given by $\text{Tor}_{A'}^*(B; \mathbb{Z}) = \text{Tor}_{A'}^0(B; \mathbb{Z}) = B/\langle b \rangle$.

Before writing explicit description for K^* ring, we notice the following

Theorem 4.14 *For $0 \leq j \leq k-2$, $S_{n-k+(j+1)}^{k-j}(Z') \in \langle S_{n-k+1}^k(Z), \dots, S_{n-1}^k(Z), S_n^k(Z) \rangle$ where Z' is a $(k-j)$ -subtuple of Z and for $1 \leq i \leq k-1$, $S_{n-k+i}^k(Z) \in \langle S_{n-k+1}^k(Z'), \dots, S_{n-1}^k(Z') \rangle$.*

Proof Proof follows from Eq. (2.2) which says that $S_j^k(Z) - z_i S_{j-1}^k(Z) = S_j^{k-1}(Z_i)$. Following diagram is a pictorial representation of the proof.

□

Immediate consequence of the Theorem is

Corollary 4.15 (1) For a $(k - j)$ -subtuple Z' of Z

$$\gcd(S_{n-k+1}^k(Z), \dots, S_{n-1}^k(Z), S_n^k(Z)) = \gcd(S_{n-k+1}^k(Z'), \dots, S_{n-1}^2(Z')).$$

(2) For $n - k < i \leq n$, $\binom{n}{i} Z^i = 0$ in $K^*(PW_{n,k})$.

Proof By Observation 4.10, we know that in $K(PW_{n,k})$, $z^n = 0$. Also, for $Z = (z, \dots, z) = z \cdot \mathbf{1}_k$,

$$\begin{aligned} S_{n-k+(j+1)}^{k-j}(Z') &= \binom{k-j+n-k+(j+1)-1}{n-k+(j+1)} z^{n-k+(j+1)} \\ &= \binom{n}{n-k+(j+1)} z^{n-k+(j+1)}. \end{aligned}$$

This allows us to conclude the proof.

□

We conclude the section with the following theorem which is obtained using Theorem 4.14, Corollary 4.15 and Theorem 3.1.

Theorem 4.16 For a primitive tuple $\ell := (l_1, \dots, l_k) \in \mathbb{Z}^k$, we have

$$K^*(P_\ell W_{n,k}) = \Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}[y, y^{-1}]/\langle b \rangle$$

where $w_0, \dots, w_{k-2}$ and y are of degree one each, $z_i := y^{l_i} - 1$ for $1 \leq i \leq k$ and $b := \gcd(S_{n-k+1}^k(Z), \dots, S_{n-1}^k(Z), S_n^k(Z))$ with $Z = (z_1, \dots, z_k)$ and for $i = n - k + 1, \dots, n - 1$, $S_i^k(Z) := \sum_{1 \leq j_1 \leq \dots \leq j_i \leq k} z_{j_1} z_{j_2} \cdots z_{j_i}$. In particular,

$$K^*(PW_{n,k}) = \Lambda_{\mathbb{Z}}^*(w_0, \dots, w_{k-2}) \otimes_{\mathbb{Z}} \mathbb{Z}[z]/\langle b \rangle$$

where $b := \gcd(\binom{n}{i} z^i \mid n - k < i \leq n)$.

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