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A monoid of Kostka–Foulkes polynomials

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Abstract
We introduce the monoid of the admissible KF polynomials. These polynomials are invariant under uniform translation of partitions. Moreover, each Kostka–Foulkes polynomial turns out to be a linear combination of admissible KF polynomials with coefficients -1 or 1. Elementary manipulations of triangular matrices provide identities on Kostka–Foulkes polynomials which are not obvious a priori.

Keywords Kostka–Foulkes polynomials · Kostant partition function · Raising operators · Triangular matrices

Mathematics Subject Classification 05E05 · 05E10 · 06A11

1 Introduction

The Kostka–Foulkes polynomials are the transition coefficients $K_{\lambda\mu}(q)$ in the expansion of the Schur functions s_λ in terms of the (classical) Hall-Littlewood symmetric functions $P_\mu(q)$, that is:

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu}(q)\ P_\mu(q).$$

The sum is extended to all the (integer) partitions $\mu = (\mu_1, \mu_2, \ldots)$ that are dominated by the given partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, that is to those μ satisfying

$$\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \text{ for all } i.$$

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When $q = 1$, $K_{\lambda\mu}(q)$ reduces to the Kostka number $K_{\lambda\mu}$ which counts the semi-standard Young tableaux of shape λ and content μ [10]. It is known since the seventies [9] that $K_{\lambda\mu}(q)$ is a polynomial with non-negative integer coefficients and that the coefficient of q^k in $K_{\lambda\mu}(q)$ counts the semi-standard Young tableaux of shape λ, content μ and charge k. The combinatorics of this subject is very rich and intriguing [5,7,11]. Here, we introduce a subset of Kostka–Foulkes polynomials closed under multiplication for which the computation benefits of nice combinatorial properties.

We consider the following set of pairs of partitions,

$$\mathfrak{P} = \left\{(\lambda, \mu) \,\middle|\, 0 \leq (\lambda_1 + \cdots + \lambda_i) - (\mu_1 + \cdots + \mu_i) \leq \lambda_i - \lambda_{i+1} \text{ for all } i\right\},$$

and we name *admissible pair* each $(\lambda, \mu) \in \mathfrak{P}$. The main subject of this paper is set $\mathfrak{K}$ of all *admissible KF polynomials* $K_{\lambda\mu}(q)$, that is

$$\mathfrak{K} := \left\{K_{\lambda\mu}(q) \,\middle|\, (\lambda, \mu) \in \mathfrak{P}\right\}.$$

The set $\mathfrak{K}$ turns out to be a sub-monoid [see identity (24)] of the Liskova semigroup introduced in [7]. These polynomials are invariant under uniform translation of partitions (Corollary 3.3). Each Kostka–Foulkes polynomial turns out to be a linear combination of admissible KF polynomials with coefficients in $\{-1, 1\}$ (Corollary 3.5). A simple combinatorial description of admissible KF polynomials is achieved by means of *class polynomials* [identity (17)] arising from a q-analogue of the Kostant partition function. Basic identities on class polynomials are obtained via elementary manipulations of triangular matrices. Class polynomials enumerate certain triangular matrices by trace [identity (31)], and satisfy a recursive formula [identity (35)]. When such identities are interpreted in terms of (admissible) Kostka–Foulkes polynomials the resulting scenario turns out to be not obvious a priori. For instance, we state a reducibility criterion for Kostka–Foulkes polynomials [Corollary (25)]. Moreover, we recover Kostka–Foulkes polynomials satisfying

$$K_{\lambda\mu}(q) = q^{\binom{n+1}{2}}[n]_q! \text{ and } K_{\lambda\mu} = q(1+q)^{n-1},$$

where $[n]_q!$ is the q-factorial. Finally, we obtain the explicit expression of the minimum integer k such that q^k occurs in the admissible KF polynomial $K_{\lambda\mu}(q)$ [identity (37)] which in general is not an easy task.

2 Preliminary notions

Throughout this paper, by an integer partition of length at most n we mean a vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Given $\alpha \in \mathbb{Z}^n$, we set $\tilde{\alpha}_i := \alpha_1 + \alpha_2 + \cdots + \alpha_i$ for $1 \leq i \leq n$, then $\tilde{\alpha} := (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$. Let

$$\mathcal{P}_n := \left\{\alpha \in \mathbb{Z}^n \,\middle|\, \tilde{\alpha}_n = 0\right\} \text{ and } \mathcal{P}_n^+ := \left\{\alpha \in \mathcal{P}_n \,\middle|\, \tilde{\alpha} \in \mathbb{N}^n\right\},$$

and define the *dominance ordering* on $\mathbb{Z}^n$ by

$$\alpha \leq \beta \text{ if and only if } \beta - \alpha \in \mathcal{P}_n^+.$$

For each $\alpha \in \mathcal{P}_n^+$ we define the *raising operator* $R_\alpha : \mathbb{Z}^n \to \mathbb{Z}^n$ by

$$R_\alpha(\beta) := \beta + \alpha.$$

Hence, we set $R_{ij} := R_{e_i - e_j}$, for $1 \leq i < j \leq n$, to denote the raising operator associated with the difference $e_i - e_j$ of vectors chosen from the standard basis $e_1, e_2, \ldots, e_n$ of $\mathbb{Z}^n$ [10]. The *Kostant partition function* $P : \mathcal{P}_n^+ \to \mathbb{N}$ associates each $\alpha \in \mathcal{P}_n^+$ with the number $P(\alpha)$ of ways α can be written as a linear combination with non-negative integer coefficients of the differences $e_i - e_j$ [3]. Equivalently, $P(\alpha)$ counts the number of ways R_α can be written as a product of the maps R_{ij}. An interesting q-analogue $P(\alpha, q)$ of $P(\alpha)$ is obtained by setting

$$\sum_{\alpha \in \mathcal{P}_n^+} P(\alpha, q)\, R_\alpha := \prod_{1 \leq i < j \leq n} (1 - q\, R_{ij})^{-1}. \tag{1}$$

We have $P(\alpha) = P(\alpha, 1)$, and the coefficient of q^k in $P(\alpha, q)$ counts the number of ways R_α can be written as a product of k maps R_{ij}, counting repetitions. The power series (1) provides a rather simple definition of the Kostka–Foulkes polynomials $K_{\lambda\mu}(q)$. In detail, let R_α map the Schur function s_λ to $R_\alpha(s_\lambda) := s_{\lambda+\alpha}$, and define the transformed Hall-Littlewood symmetric function $H_\mu(q)$ [4] (denoted $Q'_\mu(q)$ in [10]) by

$$H_\mu(q) := \prod_{1 \leq i < j \leq n} (1 - q\, R_{ij})^{-1}\, s_\mu. \tag{2}$$

Recently, (2) has been generalized in [2] via the chip-firing game providing an analogue of the transformed Hall-Littlewood symmetric functions for any connected simple graph. The Kostka–Foulkes polynomial $K_{\lambda\mu}(q)$ turns out to be the coefficient of s_λ in the expansion of $H_\mu(q)$:

$$H_\mu(q) = \sum_{\lambda \geq \mu} K_{\lambda\mu}(q) s_\lambda. \tag{3}$$

In order to apply (2), some care is needed. First, we formally expand the right-hand side via (1) to obtain

$$H_\mu(q) = \sum_{\alpha \in \mathcal{P}_n^+} P(\alpha, q)\, s_{\mu+\alpha} = \sum_{\beta \geq \mu} P(\beta - \mu, q)\, s_\beta. \tag{4}$$

Second, we have to take into account that for all $\beta \geq \mu$, it is either $s_\beta = 0$ or $s_\beta = \varepsilon(w)\, s_\lambda$ for a unique permutation w of sign $\varepsilon(w)$ in the symmetric group $\mathfrak{S}_n$, and a unique partition $\lambda \geq \mu$. More precisely, the triple (β, w, λ) satisfies $w(\lambda+\rho) - \rho = \beta$, where $\rho := (n-1, n-2, \ldots, 0)$. Hence, by combining with (3) and (4), we recover the following expansion of $K_{\lambda\mu}(q)$ which is a special case of a more general formula conjectured by Lusztig [8] and stated by Kato [6]:

$$K_{\lambda\mu}(q) = \sum_{w \in \mathfrak{S}_n} \varepsilon(w) P(w(\lambda + \rho) - (\mu + \rho), q), \tag{5}$$

where $P(\alpha, q) := 0$ is assumed whenever $\alpha \notin \mathcal{P}_n^+$. In the following section we will give an explicit characterization of those pairs (λ, μ) of partitions satisfying $w(\lambda+\rho)-(\mu+\rho) \in \mathcal{P}_n^+$ only if $w = e$ (the identity permutation). The corresponding Kostka–Foulkes polynomials $K_{\lambda\mu}(q)$ satisfy

$$K_{\lambda\mu}(q) = P(\lambda - \mu, q) \tag{6}$$

and further properties of interest in their own right.

3 Admissible KF polynomials

Given two partitions λ, μ of length at most n, we set

$$\mathfrak{S}_n(\lambda, \mu) := \{w \in \mathfrak{S}_n \mid w(\lambda + \rho) - (\mu + \rho) \in \mathcal{P}_n^+\},$$

and define

$$\mathfrak{P}_n := \big\{(\lambda, \mu) \,\big|\, \mathfrak{S}_n(\lambda, \mu) = \{e\}\big\} \text{ and } \mathfrak{P} := \bigcup_{n \geq 1} \mathfrak{P}_n.$$

We name *admissible pair* any pair $(\lambda, \mu) \in \mathfrak{P}$. Note that, if (λ, μ) is an admissible pair, then $e(\lambda + \rho) - (\mu + \rho) = \lambda - \mu \in \mathcal{P}_n^+$, hence $\mu \leq \lambda$. This means that the following subset of Kostka–Foulkes polynomials is non-empty:

$$\mathfrak{K} := \big\{K_{\lambda\mu}(q) \,\big|\, (\lambda, \mu) \in \mathfrak{P}\big\}. \tag{7}$$

Henceforth, we call the polynomials in $\mathfrak{K}$ *admissible KF polynomials*. The following theorem provides an explicit description of admissible pairs.

Theorem 3.1 *Let λ, μ be partitions of length at most n. The pair (λ, μ) is admissible if and only if we have*

$$0 \leq \tilde{\lambda}_i - \tilde{\mu}_i \leq \lambda_i - \lambda_{i+1} \textit{ for } 1 \leq i \leq n-1. \tag{8}$$

In order to prove Theorem 3.1, we need some basic facts concerning permutations. As it is well known, the symmetric group $\mathfrak{S}_n$ is generated by the adjacent transpositions $s_1, s_2, \ldots, s_{n-1}$, defined by $s_i := (i, i+1)$ for $1 \leq i \leq n-1$. A decomposition of $w \in \mathfrak{S}_n$ is an expression $w = s_{i_1} s_{i_2} \cdots s_{i_d}$ of w as a product of adjacent transpositions. The decompositions of w of minimal length are said reduced and the number $\ell(w)$

of adjacent transpositions involved is said to be the length of w. If $\mathrm{inv}(w)$ denote the number of inversions on w, then we have $\mathrm{inv}(w) = \ell(w)$ for all $w \in \mathfrak{S}_n$ [1]. Now, consider any $w \in \mathfrak{S}_n$ and observe that

$$\mathrm{inv}(ws_i) = \mathrm{inv}(w) + 1 \text{ if and only if } w(i) < w(i+1). \tag{9}$$

Lemma 3.2 *Let $w, s_i \in \mathfrak{S}_n$ satisfy $\ell(ws_i) = \ell(w) + 1$. If $w \notin \mathfrak{S}(\lambda, \mu)$ then $ws_i \notin \mathfrak{S}(\lambda, \mu)$.*

Proof Set $\alpha := ws_i(\lambda + \rho)$, $\beta := w(\lambda + \rho)$, $\gamma := \mu + \rho$. We have

$$\tilde{\alpha} = \tilde{\beta} - (\lambda_{w(i)} - \lambda_{w(i+1)} - w(i) + w(i+1))e_i. \tag{10}$$

Since λ is a partition, and being $\mathrm{inv}(ws_i) = \mathrm{inv}(w) + 1$, by (9) we have

$$w(i+1) - w(i) > 0 \text{ and } \lambda_{w(i)} - \lambda_{w(i+1)} \geq 0,$$

then $\tilde{\alpha}_i - \tilde{\gamma}_i < \tilde{\beta}_i - \tilde{\gamma}_i$, and finally

$$\tilde{\alpha}_j - \tilde{\gamma}_j \leq \tilde{\beta}_j - \tilde{\gamma}_j \text{ for } 1 \leq j \leq n.$$

Whit this established, if $w \notin \mathfrak{S}(\lambda, \mu)$ then there is j such that $\tilde{\alpha}_j - \tilde{\gamma}_j \leq \tilde{\beta}_j - \tilde{\gamma}_j < 0$, and this says $ws_i \notin \mathfrak{S}(\lambda, \mu)$. □

We now have all the necessary tools to prove Theorem 3.1.

Proof of Theorem 3.1 Let (λ, μ) be an admissible pair. As $e \in \mathfrak{S}(\lambda, \mu)$, we have $\lambda - \mu \in \mathcal{P}_n^+$, that is $0 \leq \tilde{\lambda}_i - \tilde{\mu}_i$ for $1 \leq i \leq n$ ($\tilde{\lambda}_n - \tilde{\mu}_n = 0$). Moreover, since $s_i \notin \mathfrak{S}(\lambda, \mu)$ for $1 \leq i \leq n-1$, we deduce

$$s_i(\lambda + \rho) - (\mu + \rho) \notin \mathcal{P}_n^+ \text{ for } 1 \leq i \leq n-1, \tag{11}$$

hence

$$\tilde{\lambda}_i - (\lambda_i - \lambda_{i+1}) + \tilde{\rho}_i - 1 - \tilde{\mu}_i - \tilde{\rho}_i < 0 \text{ for } 1 \leq i \leq n-1, \tag{12}$$

that is

$$\tilde{\lambda}_i - \tilde{\mu}_i \leq \lambda_i - \lambda_{i+1} \text{ for } 1 \leq i \leq n.$$

and (8) is obtained. Conversely, assume that λ and μ satisfies (8). From $0 \leq \tilde{\lambda}_i - \tilde{\mu}_i$ for $1 \leq i \leq n$ it follows that $\lambda - \mu \in \mathcal{P}_n^+$ and then $e \in \mathfrak{S}(\lambda, \mu)$. Now, let $w \neq e$, choose any reduced decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ of w, then set $w_j := s_{i_1} s_{i_2} \cdots s_{i_j}$ for $1 \leq j \leq k$. From (8) we get (12), then (11), and finally $s_i \notin \mathfrak{S}(\lambda, \mu)$ for $1 \leq i \leq n-1$. In particular, this means $w_1 \notin \mathfrak{S}(\lambda, \mu)$. Moreover, since $s_{i_1} s_{i_2} \cdots s_{i_j}$ is reduced, we have $\mathrm{inv}(w_j s_{i_{j+1}}) = \mathrm{inv}(w_j) + 1$ for $1 \leq j \leq k$. By means of Lemma 3.2 we have $w_j \notin \mathfrak{S}(\lambda, \mu)$ for $1 \leq j \leq k$, so that $w = w_k \notin \mathfrak{S}(\lambda, \mu)$. □

Theorem 3.1 and identity (5) provide a first nice property of admissible Kostka–Foulkes polynomials.

Corollary 3.3 *Let $K_{\lambda\mu}(q) \in \mathfrak{K}$ and let $\alpha \in \mathbb{Z}^n$ be such that both $\lambda + \alpha$ and $\mu + \alpha$ are partitions. We have*

$$K_{\lambda\mu}(q) = K_{\lambda+\alpha\,\mu+\alpha}(q).$$

Proof Set $\beta := \lambda + \alpha$ and $\gamma := \mu + \alpha$ and observe that $\tilde{\beta}_i - \tilde{\gamma}_i = \tilde{\lambda}_i - \tilde{\mu}_i$ for $1 \le i \le n$. This means $K_{\lambda+\alpha\,\mu+\alpha}(q) \in \mathfrak{K}$ and (6) provides

$$K_{\lambda+\alpha\,\mu+\alpha}(q) = P(\lambda - \mu, q) = K_{\lambda\mu}(q).$$

□

The following theorem states that any polynomial $P(\alpha, q)$ actually is an admissible KF polynomial, so that (1) can be seen as a generating function of the admissible KF polynomials.

Theorem 3.4 *For all $\alpha \in \mathcal{P}_n^+$ there exists $(\lambda, \mu) \in \mathfrak{P}_n$ such that $\lambda - \mu = \alpha$.*

Proof Given $\alpha \in \mathcal{P}_n^+$, define $\lambda \in \mathbb{Z}^n$ recursively by

$$\lambda_i := \lambda_{i+1} + \max\{\tilde{\alpha}_i, \alpha_i - \alpha_{i+1}\} \text{ for } 1 \le i \le n \;\; (\lambda_{n+1} := 0),$$

then set $\mu := \lambda - \alpha$. Since

$$\lambda_i - \lambda_{i+1} = \max\{\tilde{\alpha}_i, \alpha_i - \alpha_{i+1}\} \ge \tilde{\alpha}_i \ge 0 \text{ for } 1 \le i \le n,$$

λ is a partition. Analogously, as we have

$$\mu_i - \mu_{i+1} = \max\{\tilde{\alpha}_i, \alpha_i - \alpha_{i+1}\} - (\alpha_i - \alpha_{i+1}) \ge 0 \text{ for } 1 \le i \le n,$$

we deduce that μ is a partition. Finally, we have

$$0 \le \tilde{\alpha}_i = \tilde{\lambda}_i - \tilde{\mu}_i \le \lambda_i - \lambda_{i+1} \text{ for } 1 \le i \le n - 1,$$

and (λ, μ) is an admissible pair by virtue of Theorem 3.1. □

Theorem 3.4 combined with identity (5) says us that each Kostka–Foulkes polynomial is a signed sum of the admissible KF polynomials.

Corollary 3.5 *There exist maps $f_{\lambda\mu} : \mathfrak{P} \to \{-1, 0, 1\}$ satisfying*

$$K_{\lambda\mu}(q) = \sum_{(\alpha,\beta)\in\mathfrak{P}} f_{\lambda\mu}(\alpha, \beta) K_{\alpha\beta}(q). \tag{13}$$

Proof Let $\lambda, \mu \in \mathbb{Z}^n$ be partitions such that $\mu \leq \lambda$. For each $w \in \mathfrak{S}(\lambda, \mu)$ let (α_w, β_w) denote any admissible pair satisfying $\alpha_w - \beta_w = w(\lambda + \rho) - (\mu + \rho)$, then define

$$f_{\lambda\mu}(\alpha, \beta) = \begin{cases} \varepsilon(w), & \text{if } (\alpha, \beta) = (\alpha_w, \beta_w); \\ 0, & \text{otherwise.} \end{cases}$$

Hence, (13) comes from (5). □

4 Counting triangular matrices by sum and class

The identification of the polynomials $P(\alpha, q)$ with the admissible KF polynomials leads to a combinatorial description of $\mathfrak{K}$ in terms of enumeration of triangular matrices of non-negative integers. With this goal, we let $\mathbf{T}_n$ denote the set of all lower triangular matrices of non-negative integers and we display each element of $\mathbf{T}_n$ as a staircase tableau: we write

$$T = \begin{matrix} 1 & & & \\ 0 & 0 & & \\ 2 & 0 & 3 & \\ 1 & 2 & 1 & 4 \end{matrix} \quad \text{instead of } T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 1 & 2 & 1 & 4 \end{bmatrix}. \tag{14}$$

Now, for all $T = (t_{ij})$ in $\mathbf{T}_n$ we set

$$\operatorname{sum}(T) := \sum_{1 \leq j \leq i \leq n} t_{ij}, \tag{15}$$

and call $\operatorname{sum}(T)$ the *sum* of T. Furthermore, we define

$$\operatorname{class}_k(T) := \sum_{1 \leq j \leq k \leq i \leq n} t_{ij}, \quad \text{for } 1 \leq k \leq n, \tag{16}$$

and call $(\operatorname{class}_1(T), \operatorname{class}_2(T), \ldots, \operatorname{class}_n(T))$ the *class* of T.

Example 4.1 The sum of the tableau T in (14) is $\operatorname{sum}(T) = 14$. Its class $(4, 5, 9, 8)$ is obtained by summing up all the entries in each of the following rectangles:

$$\begin{array}{|c|ccc} \hline 1 & & & \\ 0 & 0 & & \\ 2 & 0 & 3 & \\ 1 & 2 & 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|cc|cc} 1 & & & \\ \hline 0 & 0 & & \\ 2 & 0 & 3 & \\ 1 & 2 & 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|ccc|c} 1 & & & \\ 0 & 0 & & \\ \hline 2 & 0 & 3 & \\ 1 & 2 & 1 & 4 \\ \hline \end{array} \qquad \begin{array}{|cccc|} 1 & & & \\ 0 & 0 & & \\ 2 & 0 & 3 & \\ \hline 1 & 2 & 1 & 4 \\ \hline \end{array}.$$

Given $\boldsymbol{c} \in \mathbb{N}^n$, we let $\mathbf{T}_{\boldsymbol{c}}$ denote the set of all tableaux of class $\boldsymbol{c}$, and write $\mathbf{T}_{\boldsymbol{c},k}$ to denote the set of all tableaux of class $\boldsymbol{c}$ and of sum k. Therefore, we define the $\boldsymbol{c}$*-class polynomial* $P_{\boldsymbol{c}}(q)$ by

$$P_{\boldsymbol{c}}(q) := \sum_{T \in \mathbf{T}_{\boldsymbol{c}}} q^{\operatorname{sum}(T)} = \sum_{k \geq 0} \left|\mathbf{T}_{\boldsymbol{c},k}\right| q^k. \tag{17}$$

The following proposition is at root of our combinatorial description.

Proposition 4.1 *If* $\alpha \in \mathcal{P}_{n+1}^{+}$ *and* $\boldsymbol{c} \in \mathbb{N}^n$ *satisfy* $\tilde{\alpha} = (c_1, c_2, \ldots, c_n, 0)$*, then we have*

$$P(\alpha, q) = P_{\boldsymbol{c}}(q). \tag{18}$$

Proof By virtue of (1), for all $\alpha \in \mathcal{P}_{n+1}^{+}$ we have

$$P(\alpha, q) = \sum_{A \in \mathbf{SUT}_\alpha} q^{\sum a_{ij}}, \tag{19}$$

where $A = (a_{ij})$ ranges over the set $\mathbf{SUT}_\alpha$ of all strictly upper triangular $(n+1) \times (n+1)$ matrices of non-negative integers satisfying

$$\alpha = \sum_{1 \le i < j \le n+1} a_{ij}\,(e_i - e_j).$$

On the other hand, from

$$\sum_{1 \le i < j \le n+1} a_{ij}\,(e_i - e_j) = \alpha = \sum_{i=1}^{n+1} \alpha_i\, e_i = \sum_{k=1}^{n} \tilde{\alpha}_k\,(e_k - e_{k+1}),$$

we obtain

$$\tilde{\alpha}_k = \sum_{1 \le i \le k < j \le n+1} a_{ij}, \text{ for } 1 \le k \le n. \tag{20}$$

By comparing (20) and (16), one easily sees that $\mathbf{SUT}_\alpha$ bijectively corresponds to $\mathbf{T}_{\boldsymbol{c}}$ via the map $(a_{ij}) \mapsto (t_{ij})$ defined by $t_{ij} := a_{j\,i+1}$. Finally, the polynomials (17) and (19) agree, provided that $\tilde{\alpha} = (c_1, c_2, \ldots, c_n, 0)$. □

Basic identities on class polynomials follow from (17) via elementary manipulations of triangular matrices. On the other hand, once such identities are written in terms of (admissible) KF polynomials they provide results not obvious a priori. Let us give some preliminary examples. For all $T = (t_{ij}) \in \mathbf{T}_n$, let T' denote the tableau obtained by reflecting T along its SW-NE diagonal, that is

$$T' := (t_{n-j+1\,n-i+1}).$$

So, for T in (14) we obtain

$$T' = \begin{matrix} 4 & & & \\ 1 & 3 & & \\ 2 & 0 & 0 & \\ 1 & 2 & 0 & 1 \end{matrix} \quad .$$

The map $T \mapsto T'$ is bijective from $\mathbf{T}_n$ to itself; hence, it immediately provides the following result.

Proposition 4.2 *For all* $\boldsymbol{c} = (c_1, c_2, \dots, c_n) \in \mathbb{N}^n$ *we have*

$$P_{\boldsymbol{c}}(q) = P_{\boldsymbol{c}'}(q), \tag{21}$$

where $\boldsymbol{c}' := (c_n, c_{n-1}, \dots, c_1)$. *Equivalently, if* $(\lambda, \mu), (\eta, \nu) \in \mathfrak{P}$ *satisfy* $\tilde{\eta} - \tilde{\nu} = (\tilde{\lambda} - \tilde{\mu})'$, *then we have*

$$K_{\lambda\mu}(q) = K_{\eta\nu}(q). \tag{22}$$

Proof Identity (21) comes obviously from $T \mapsto T'$, while identity (22) comes from identity (21) via Theorem 3.4. □

For all $S \in \mathbf{T}_n$ and $T \in \mathbf{T}_m$, let $S \oplus T$ denote the tableau in $\mathbf{T}_{n+m+1}$ obtained by putting S and T on the top and on the right, respectively, of a $(n+1) \times (m+1)$ rectangular tableau whose entries all are equal to zero. Formally, if $S = (s_{ij})$ and $T = (t_{ij})$ then we set $S \oplus T = (u_{ij})$ where

$$u_{ij} := \begin{cases} s_{ij}, & \text{for } 1 \le i, j \le n, \\ 0, & \text{for } n+1 \le i \le n+m+1 \text{ and } 1 \le j \le n+1, \\ t_{i-n-1, j-n-1}, & \text{for } n+2 \le i \le n+m+1 \text{ and } n+2 \le j \le n+m+1. \end{cases}$$

For instance, we have

$$\begin{array}{lll} 2 & & \\ 1 & 0 & \\ 2 & 0 & 8 \end{array} \;\oplus\; \begin{array}{ll} 3 & \\ 2 & 6 \end{array} \;=\; \begin{array}{llll|ll} 2 & & & & & \\ 1 & 0 & & & & \\ 2 & 0 & 8 & & & \\ \hline 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 3 & \\ 0 & 0 & 0 & 0 & 2 & 6 \end{array}.$$

The map $(S, T) \mapsto S \oplus T$ is bijective from $\mathbf{T}_n \times \mathbf{T}_m$ to the subset $\mathbf{T}_{n,m}$ of $\mathbf{T}_{n+m+1}$ of all tableaux T satisfying $\text{class}_{n+1}(T) = 0$. As a consequence, we obtain the following multiplication rule for class polynomials.

Proposition 4.3 *For all* $\boldsymbol{a} = (a_1, a_2, \dots, a_n) \in \mathbb{N}^n$ *and* $\boldsymbol{b} = (b_1, b_2, \dots, b_m) \in \mathbb{N}^m$, *we have*

$$P_{\boldsymbol{a}}(q) P_{\boldsymbol{b}}(q) = P_{\boldsymbol{a}0\boldsymbol{b}}(q), \tag{23}$$

where

$$\boldsymbol{a}0\boldsymbol{b} := (a_1, a_2, \dots, a_n, 0, b_1, b_2, \dots, b_m).$$

Equivalently, for all $(\lambda, \mu), (\eta, \nu) \in \mathfrak{P}$ *there exists* $(\sigma, \tau) \in \mathfrak{P}$ *such that*

$$K_{\lambda\mu}(q) K_{\eta\nu}(q) = K_{\sigma\tau}(q). \tag{24}$$

Proof Identity (23) comes via the bijection $(T, S) \mapsto T \oplus S$, while identity (24) comes from identity(23) via identities (18) and (6) and Theorem 3.4. □

Identity (24) has a couple of particularly nice direct consequences.

Corollary 4.4 *Let* $(\lambda, \mu) \in \mathfrak{P}$ *satisfy* $\tilde{\lambda} - \tilde{\mu} = \boldsymbol{a}0\boldsymbol{b}$ *for suitable vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$. *Then, we have*

$$K_{\lambda\mu}(q) = K_{\eta\nu}(q)\, K_{\sigma\tau}(q), \tag{25}$$

for suitable $(\eta, \nu), (\sigma, \tau) \in \mathfrak{P}$ *such that* $(\lambda, \mu) = (\mu \cup \sigma, \nu \cup \tau)$.

Proof Let

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \quad \boldsymbol{b} = (b_1, b_2, \ldots, b_m),$$
$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+m+1}), \quad \mu = (\mu_1, \mu_2, \ldots, \mu_{n+m+1}).$$

If $\tilde{\lambda} - \tilde{\mu} = \boldsymbol{a}0\boldsymbol{b}$ then we set

$$\eta := (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \nu := (\mu_1, \mu_2, \ldots, \mu_n),$$
$$\sigma := (\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+m+1}), \quad \tau := (\mu_{n+1}, \mu_{n+2}, \ldots, \mu_{n+m+1})$$

to obtain $\lambda = \eta \cup \sigma$, $\mu = \nu \cup \tau$, $\tilde{\eta} - \tilde{\nu} = \boldsymbol{a}$ and $\tilde{\sigma} - \tilde{\tau} = \boldsymbol{b}$. Finally, we apply (6), (18) and (23) to recover

$$K_{\lambda\mu}(q) = P_{\boldsymbol{a}0\boldsymbol{b}}(q) = P_{\boldsymbol{a}}(q)P_{\boldsymbol{b}}(q) = K_{\eta\nu}(q)K_{\sigma\tau}(q).$$

□

Theorem 4.5 *The set* $\mathfrak{K}$ *of all admissible KF polynomials is a monoid with respect to multiplication.*

Proof $K_{\lambda\lambda}(q) = P_0(q) = 1 \in \mathfrak{K}$ and, by virtue of (24), $\mathfrak{K}$ is closed under multiplication. □

We close this section with an application to the q-factorial. Let $(a, b) \in \mathbb{N}^2$ and assume $a \leq b$. Consider the following generic tableau in $\mathbf{T}_2$:

$$T = \begin{matrix} y & \\ x & z \end{matrix}.$$

Once that x is chosen such that $0 \leq x \leq a$, in order to obtain $T \in \mathbf{T}_{(a,b)}$ we set $y = a - x$ and $z = b - x$. This provides

$$P_{ab}(q) = \sum_{0 \leq x \leq a} q^{a+b-x} = q^b(1 + q + \cdots + q^a) = q^b\,[a+1]_q,$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ is the n-th q-integer. In the case $b \leq a$, by means of (21) we recover

$$P_{ab}(q) = P_{(b,a)}(q) = q^a\,[b+1]_q,$$

and this explicitly determines all the class polynomials indexed in $\mathbb{N}^2$:

$$P_{ab}(q) = q^{\max\{a,b\}}\,[\min\{a,b\}+1]_q.$$

Note that, for each positive integer i, thanks to (18) and (6), we obtain

$$K_{3i\,2i-1\;2i\,2i\,i-1}(q) = P_{i\,i-1}(q) = q^i\,[i]_q.$$

Set $\boldsymbol{i} := (i-1, i)$ and $\boldsymbol{n}! := \mathbf{1020}\cdots 0\boldsymbol{n} = (0,1,0,1,2,\ldots,0,n-1,n)$, then apply (23) to obtain

$$\begin{aligned} P_{\boldsymbol{n}!}(q) &= \prod_{i=1}^{n} P_{\boldsymbol{i}}(q) = \prod_{i=1}^{n} q^i[i]_q = q^{\binom{n+1}{2}}[n]_q! \\ &= \frac{q^{\binom{n+1}{2}}}{(1-q)^n}(1-q)(1-q^2)\cdots(1-q^n), \end{aligned}$$

where $[n]_q! := [1]_q[2]_q\cdots[n]_q$ is the q-factorial. On the other hand, thanks to Theorem 3.4, we known there exists $(\lambda,\mu)\in\mathfrak{P}$ such that $\tilde{\lambda}-\tilde{\mu} = \boldsymbol{n}!0$, and then

$$K_{\lambda\mu}(q) = q^{\binom{n+1}{2}}[n]_q!. \tag{26}$$

It is well known that the coefficient of q^k in $[n]_q!$ equals the number of permutations $\pi\in\mathfrak{S}_n$ with exactly $i(\pi)=k$ inversions [12]. Hence, one may look for a bijection $T\mapsto\pi(T)$ from $\mathbf{T}_{\boldsymbol{n}!}$ to $\mathfrak{S}_n$ satisfying $\mathrm{sum}(T) = i(\pi(T)) + \binom{n+1}{2}$. Such a bijection is obtained as follows. First, note that each tableau

$$\begin{matrix} y & \\ x & z \end{matrix} \in \mathbf{T}_{(i-1,i)}$$

is uniquely determined by the value taken by y: choose y such that $0\le y\le i-1$, then set $x = i-1-y$ and $z = i-x$ (that is $x+z=i$). Hence, the tableaux $T_1\oplus T_2\oplus\cdots\oplus T_n$ in $\mathbf{T}_{\boldsymbol{n}!}$ bijectively correspond to the vectors $(y_1,y_2,\ldots,y_n)$ of integers such that $0\le y_i\le i-1$. On the other hand, such vectors exactly are the inversion tables of the permutations in $\mathfrak{S}_n$ [12]. Finally, given $T = T_1\oplus T_2\oplus\cdots\oplus T_n$, let $\pi(T)$ denote the unique permutation of $\mathfrak{S}_n$ with inversion table $(y_1,y_2,\ldots,y_n)$ to obtain

$$\mathrm{sum}(T) = \sum_{i=1}^{n}(y_i+x_i+z_i) = \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} i = i(\pi(T)) + \binom{n+1}{2}.$$

5 A recursive formula for the class polynomials

In the previous section, we have shown that every identity on class polynomials can be interpreted in terms of Kostka–Foulkes polynomials. Henceforth, we adopt the notation $P_{\boldsymbol{c}}(q)$ that we believe more effective. Given $T = (t_{ij}) \in \mathbf{T}_n$, set $T_n := T$ and, for $1 \le i \le n-1$, let $T_{n-i} \in \mathbf{T}_{n-i}$ denote the tableau obtained from T_{n-i+1} by removing its bottom row. Therefore, set

$$c_{h,k}(T) := \text{class}_k(T_h) = \sum_{1 \le j \le k \le i \le h} t_{ij}, \text{ for } 1 \le k \le h \le n. \tag{27}$$

The triangular matrix $C(T) = (c_{h,k}(T)) \in \mathbf{T}_n$ will be named the *class tableau* of T. For instance, starting from T in (14) we obtain

$$T_4 = \begin{matrix} 1 & & & \\ 0 & 0 & & \\ 2 & 0 & 3 & \\ 1 & 2 & 1 & 4 \end{matrix} \qquad T_3 = \begin{matrix} 1 & & \\ 0 & 0 & \\ 2 & 0 & 3 \end{matrix} \qquad T_2 = \begin{matrix} 1 & \\ 0 & 0 \end{matrix} \qquad T_1 = 1\,,$$

then

$$\text{class}(T_4) = (4, 5, 9, 8), \quad \text{class}(T_3) = (3, 2, 5), \quad \text{class}(T_2) = (1, 0),$$
$$\text{class}(T_1) = (1),$$

and finally

$$C(T) = \begin{matrix} 1 & & & \\ 1 & 0 & & \\ 3 & 2 & 5 & \\ 4 & 5 & 9 & 8 \end{matrix} \quad . \tag{28}$$

Starting from (27) one recovers the explicit expression of the entries of each tableau T in terms of the entries of the associated class tableau: we have

$$t_{ij} = \big(c_{ij}(T) - c_{i,j-1}(T)\big) - \big(c_{i-1,j}(T) - c_{i-1,j-1}(T)\big) \tag{29}$$

for $1 \le i \le j \le n$, where $c_{ij}(T) := 0$ if $i = 0$ or $j = 0$. Identities (27) and (29) say that the map $T \mapsto C(T)$ is bijective from $\mathbf{T}_n$ to the subset $\mathbf{CT}_n \subset \mathbf{T}_n$ of all tableaux $C = (c_{ij})$ satisfying

$$c_{ij} - c_{i,j-1} \ge c_{i-1,j} - c_{i-1,j-1}, \tag{30}$$

where as above we assume $c_{ij} := 0$ if $i = 0$ or $j = 0$. Henceforth, we will name *class tableau* any of such tableaux. Moreover, given $\boldsymbol{c} \in \mathbb{N}^n$, we name $\boldsymbol{c}$*-class tableau* any class tableau $C = (c_{ij})$ such that $\boldsymbol{c} = (c_{n,1}, c_{n,2}, \dots, c_{n,n})$, that is any class tableau whose bottom row agrees with a. Hence, we let $\mathbf{CT}_{\boldsymbol{c}}$ denote the set of all $\boldsymbol{c}$-class tableaux or, equivalently, the image of $\mathbf{T}_{\boldsymbol{c}}$ under the map $T \mapsto C(T)$. Recalling that $\mathbf{T}_n$ is the set of all lower triangular $n \times n$ matrices of non-negative integers, for all $T \in \mathbf{T}_n$ we set

$$\text{trace}\, T = t_{1,1} + t_{2,2} + \cdots + t_{n,n}.$$

The expansion of the class polynomial $P_{\boldsymbol{c}}(q)$ in terms of $\boldsymbol{c}$-class tableaux follows.

Theorem 5.1 *For all* $\boldsymbol{c} \in \mathbb{N}^n$ *we have*

$$P_{\boldsymbol{c}}(q) = \sum_{C \in \mathbf{CT}_{\boldsymbol{c}}} q^{\operatorname{trace} C}. \tag{31}$$

Proof Given $T \in \mathbf{T}_n$, from (27) we have

$$c_{k,k}(T) = t_{k,1} + t_{k,2} + \cdots + t_{k,k},$$

for $1 \leq k \leq n$, and then

$$\operatorname{trace} C(T) = \operatorname{sum}(T). \tag{32}$$

Now, (31) follows from (17) and (32) via the map $T \mapsto C(T)$. □

Thanks to (31) we are able to determine the degree of any class polynomial.

Corollary 5.2 *For all* $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n$ *the class polynomial* $P_{\boldsymbol{c}}(q)$ *is monic of degree*

$$\deg\ P_{\boldsymbol{c}}(q) = c_1 + c_2 + \cdots + c_n.$$

Proof From (27), it easily seen that class tableaux have non-decreasing column (from top to bottom). This means that there exists a unique $\boldsymbol{c}$-class tableau of maximal trace, such a tableau being

$$C = \begin{pmatrix} c_1 & & & \\ c_1 & c_2 & & \\ \vdots & \vdots & & \\ c_1 & c_2 & \ldots & c_n \end{pmatrix}. \tag{33}$$

□

A further application of (31) is a recursive formula for the class polynomials. To recover such a formula, for all $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{N}^n$, write $\boldsymbol{b} \prec \boldsymbol{c}$ to mean $\boldsymbol{b} = (b_1, b_2, \ldots, b_{n-1}) \in \mathbb{N}^{n-1}$ and

$$c_j - c_{j-1} \geq b_j - b_{j-1}, \tag{34}$$

for $1 \leq j \leq n$ (assume $b_n = c_0 = b_0 = 0$). By comparing (30) and (34), one sees that $C \in \mathbf{T}_n$ is a $\boldsymbol{c}$-class tableau if and only if its rows $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n$ satisfy $\boldsymbol{c}_1 \prec \boldsymbol{c}_2 \prec \ldots \prec \boldsymbol{c}_n$. For example, the class tableau (28) can be identified with the sequence

$$(1) \prec (1, 0) \prec (3, 2, 5) \prec (4, 5, 9, 8).$$

Proposition 5.3 *For all $\boldsymbol{c} \in \mathbb{N}^n$ we have*

$$P_{\boldsymbol{c}}(q) = q^{c_n} \sum_{\boldsymbol{b} \prec \boldsymbol{c}} P_{\boldsymbol{b}}(q). \tag{35}$$

Proof From (31) we have

$$\begin{aligned} P_{\boldsymbol{c}}(q) &= \sum_{\boldsymbol{c}_1 \prec \boldsymbol{c}_2 \prec \ldots \prec \boldsymbol{c}_n = a} q^{c_{1,1}+c_{2,2}+\cdots+c_{n,n}} \\ &= q^{c_n} \sum_{\boldsymbol{b} \prec \boldsymbol{c}} \left(\sum_{\boldsymbol{c}_1 \prec \boldsymbol{c}_2 \prec \ldots \prec \boldsymbol{c}_{n-1} = \boldsymbol{b}} q^{c_{1,1}+c_{2,2}+\cdots+c_{n-1,n-1}} \right) \\ &= q^{c_n} \sum_{\boldsymbol{b} \prec \boldsymbol{c}} P_{\boldsymbol{b}}(q). \end{aligned}$$

□

By virtue of the recursive rule (35), we are able to recover the minimum power of q occurring in any class polynomial.

Proposition 5.4 *The minimum integer k such that q^k occurs in $P_{\boldsymbol{c}}(q)$ is*

$$a(\boldsymbol{c}) := \sum_{i=1}^{n} \max\{0, c_i - c_{i+1}\} = \sum_{\substack{1 \le i \le n \\ c_i \ge c_{i+1}}} (c_i - c_{i+1}), \tag{36}$$

where we assume $c_{n+1} = 0$ and $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$.

Proof Let $\tilde{a}(\boldsymbol{c})$ denote the minimum integer k such that q^k occurs in $P_{\boldsymbol{c}}(q)$. We use induction on $n \ge 1$ to prove that $\tilde{a}(\boldsymbol{c}) = a(\boldsymbol{c})$ for all $\boldsymbol{c} \in \mathbb{N}^n$. If $n = 1$ and $\boldsymbol{c} = (c)$, then we have $P_{\boldsymbol{c}}(q) = q^c$ and $\tilde{a}(\boldsymbol{c}) = c = a(\boldsymbol{c})$. Now, let $n \ge 2$ and assume $\tilde{a}(\boldsymbol{b}) = a(\boldsymbol{b})$ for all $\boldsymbol{b} \in \mathbb{N}^{n-1}$. By means of (35) we have

$$\tilde{a}(\boldsymbol{c}) = c_n + \min\{a(\boldsymbol{b}) \mid \boldsymbol{b} \prec \boldsymbol{c}\}.$$

From (34), if $\boldsymbol{b} \prec \boldsymbol{c}$ then we have $b_i - b_{i+1} \ge c_i - c_{i+1}$ for $1 \le i \le n-1$, hence

$$a(\boldsymbol{b}) = \sum_{i=1}^{n-1} \max\{0, b_i - b_{i+1}\} \ge \sum_{i=1}^{n-1} \max\{0, c_i - c_{i+1}\} = a(\boldsymbol{c}) - c_n.$$

Now, set $c_n^\star := 0$ and consider the integral vector $\boldsymbol{c}^\star = (c_1^\star, c_2^\star, \ldots, c_{n-1}^\star)$ defined recursively by

$$c_{n-i}^\star := \max\{0, c_{n-i} - c_{n-i+1} + c_{n-i+1}^\star\},$$

for $1 \leq i \leq n-1$. We have

$$c_i^\star = \begin{cases} 0, & \text{if } c_i - c_{i+1} + c_{i+1}^\star < 0; \\ c_i - c_{i+1} + c_{i+1}^\star, & \text{if } c_i - c_{i+1} + c_{i+1}^\star \geq 0; \end{cases}$$

then $c_i^\star - c_{i+1}^\star \geq c_i - c_{i+1}$ for $1 \leq i \leq n-1$, which says that $\boldsymbol{c}^\star \prec \boldsymbol{c}$. Moreover, we also have $c_i^\star - c_{i+1}^\star = c_i - c_{i+1}$ if $c_i - c_{i+1} \geq 0$, and $c_i^\star - c_{i+1}^\star \leq 0$ if $c_i - c_{i+1} < 0$, which gives

$$a(\boldsymbol{c}^\star) = \sum_{\substack{1 \leq i \leq n-1 \\ c_i^\star \geq c_{i+1}^\star}} (c_i^\star - c_{i+1}^\star) = \sum_{\substack{1 \leq i \leq n-1 \\ c_i \geq c_{i+1}}} (c_i - c_{i+1}) = a(\boldsymbol{c}) - c_n.$$

Finally, we conclude that $a(\boldsymbol{c}^\star) = \min\{a(\boldsymbol{b}) \mid \boldsymbol{b} \prec \boldsymbol{c}\}$, then we have

$$\tilde{a}(\boldsymbol{c}) = c_n + \min\{a(\boldsymbol{b}) \mid \boldsymbol{b} \prec \boldsymbol{c}\} = c_n + a(\boldsymbol{c}^\star) = a(\boldsymbol{c}).$$

□

Remark 5.1 For all $K_{\lambda\mu}(q) \in \mathfrak{K}$, via (6), (18) and (36) we obtain the following explicit formula for the minimum integer k such that q^k occurs in $K_{\lambda\mu}(q)$:

$$a(\lambda, \mu) := a(\tilde{\lambda} - \tilde{\mu}) = \sum_{i=1}^{n} \max\left\{0, (\tilde{\lambda}_i - \tilde{\lambda}_{i+1}) - (\tilde{\mu}_i - \tilde{\mu}_{i+1})\right\}. \tag{37}$$

In closing, we give the explicit formula for the class polynomial $P_{\mathbf{1}^n}(q)$, where $\mathbf{1}^n := (1, 1, \ldots, 1) \in \mathbb{N}^n$. Note that $\boldsymbol{b} \prec \mathbf{1}^n$ is possible if and only if $\boldsymbol{b} = \mathbf{1}^i\mathbf{0}^{n-1-i}$, this expression denoting a vector whose first i entries are equal to 1 and the remaining $(n-1-i)$ entries are equal to 0. As we have $P_{\mathbf{1}^i\mathbf{0}^{n-1-i}}(q) = P_{\mathbf{1}^i}(q)$, by means of (35), for $n \geq 2$ we can write

$$\begin{aligned} P_{\mathbf{1}^n}(q) &= q \sum_{0 \leq i \leq n-1} P_{\mathbf{1}^i}(q) \\ &= q \sum_{0 \leq i \leq n-2} P_{\mathbf{1}^i}(q) + q P_{\mathbf{1}^{n-1}}(q) \\ &= P_{\mathbf{1}^{n-1}}(q) + q P_{\mathbf{1}^{n-1}}(q) \\ &= (1+q) P_{\mathbf{1}^{n-1}}(q), \end{aligned}$$

and thus

$$P_{\mathbf{1}^n}(q) = (1+q)^{n-1} q. \tag{38}$$

Again, observe that (38) states the existence of an admissible KF polynomial $K_{\lambda\mu}(q)$ satisfying

$$K_{\lambda\mu}(q) = (1+q)^{n-1} q. \tag{39}$$

Consequence of (38) is that the number $\left|\mathbf{T}_{\mathbf{1}^{n+1},k+1}\right|$ of tableaux of class $\mathbf{1}^{n+1}$ and sum $k+1$ equals the binomial coefficient $\binom{n}{k}$. This suggests to look for a bijection $T \mapsto I(T)$ mapping each tableaux T in $\mathbf{T}_{\mathbf{1}^{n+1}}$ to a subset $I(T)$ of $\{1, 2, \ldots, n\}$, such that $\text{sum}(T) = |I(T)| + 1$. To get the desired bijection, if $I = \{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$ and if $i_1 < i_2 < \ldots < i_k$, define $T(I) = (t_{ij}) \in \mathbf{T}_{\mathbf{1}^{n+1}}$ by $t_{i_1,1} = t_{i_2,i_1+1} = \cdots = t_{i_k,i_{k-1}+1} = t_{n+1,i_k+1} = 1$ ($t_{ij} = 0$ otherwise). The map $I \mapsto T(I)$ is bijective and satisfies $\text{sum}(T(I)) = |I| + 1$. Finally, let $T \mapsto I(T)$ denote the inverse of $I \mapsto T(I)$ and obtain $\text{sum}(T) = |I(T)| + 1$, as desired.

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