Tetravalent half-arc-transitive graphs of order $8p$

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Abstract

A graph is *half-arc-transitive* if its automorphism group acts transitively on its vertex set, edge set, but not arc set. Let p be a prime. It is known that there exist no tetravalent half-arc-transitive graphs of order p or $2p$. Feng et al. (J Algebraic Combin 26:431–451, 2007) gave the classification of tetravalent half-arc-transitive graphs of order $4p$. In this paper, a classification is given of all tetravalent half-arc-transitive graphs of order $8p$.

Keywords Cayley graph · Half-arc-transitive graph · Tightly attached half-arc-transitive

Mathematics Subject Classification 05C25 · 20B25

1 Introduction

Throughout this paper all graphs are finite simple, connected and undirected unless specified otherwise. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* (*symmetric*) if $\text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. The graph X is said to be *half-arc-transitive* provided that it is vertex- and edge- but not arc-transitive. More generally, by a *half-arc-transitive* action of a subgroup G of $\text{Aut}(X)$ on X we shall mean a vertex- and edge-, but not arc-transitive action of G on X . In this case, we say that the graph X is *G-half-arc-transitive*.

The investigation of half-arc-transitive graphs was initiated by Tutte [22] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970, Bouwer [2] constructed the first family of half-arc-transitive graph. Following these two classical articles, half-arc-transitive graphs have been extensively studied from different perspectives over decades by many authors. See, for example [6, 8, 9, 14–17, 21, 23, 25, 26, 29, 30].

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In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying the tetravalent half-arc-transitive graphs is currently one of active topics in algebraic graph theory (for example, see [1, 4, 10–13, 19, 20, 24]). Let p be a prime. By Chao [3], there are no tetravalent half-arc-transitive graphs of order p , and by Cheng and Oxley [5], there are no tetravalent half-arc-transitive graphs of order $2p$. In 2007, Feng et al. [13] classified the tetravalent half-arc-transitive graphs of order $4p$. In this paper, we classify the tetravalent half-arc-transitive graphs of order $8p$.

2 Preliminary results

Let X be a tetravalent G -half-arc-transitive graph for a subgroup G of $\text{Aut}(X)$. The fact that no automorphism of the half-arc-transitive graph X interchanges a pair of adjacent vertices also implies that the action of $\text{Aut}(X)$ gives rise to two (paired) natural orientations of the edges of X . For quartic graph X , then fixing one of these two orientations we obtain the so-called alternating cycles of X . Half of the length of these cycles is called the radius of X and the size of the intersection of two non-disjoint alternating cycles is called the attachment number of X [18]. In the case that the attachment number of X coincides with its radius we say that X is tightly attached.

Let $m \geq 3$ be an integer, $n \geq 3$ an odd integer and let $r \in \mathbb{Z}_n^*$ satisfy $r^m = \pm 1$. The graph $X(r; m, n)$ is defined to have vertex set $V = \{u_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and edge set $E = \{\{u_i^j, u_{i+1}^{j \pm r^i}\} \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$. Define the maps: $\rho : u_i^j \mapsto u_i^{j+1}$, $\sigma : u_i^j \mapsto u_{i+1}^{rj}$, $\tau : u_i^j \mapsto u_i^{-j}$ for $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$. Then $\rho, \sigma, \tau \in \text{Aut}(X(r; m, n))$. Furthermore, it is easy to check that $\langle \rho \rangle \triangleleft \langle \rho, \sigma, \tau \rangle$. For the purpose of this paper, we introduce a result due to Marušič.

Proposition 2.1 [18, Theorem 3.4, Proposition 4.1, Corollary 4.3] *A connected tetravalent graph X is a tightly attached half-arc-transitive graph of odd radius n if and only if $X \cong X(r; m, n)$, where $m \geq 3$, and $r \in \mathbb{Z}_n^*$ satisfying $r^m = \pm 1$, and moreover none of the following conditions is fulfilled:*

- (1) $r^2 = \pm 1$;
- (2) $(r; m, n) = (2; 3, 7)$;
- (3) $(r; m, n) = (r; 6, 7k)$, where $k \geq 1$ is odd, $(7, k) = 1$, $r^6 = 1$, and there exists a unique solution $q \in \{r, -r, r^{-1}, -r^{-1}\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q - 1) = 0$ and $q \equiv 5 \pmod{7}$.

Furthermore, $\text{Aut}(X(r; m, n)) = \langle \rho, \sigma, \tau \rangle$ and $X(-s; t, r) \cong X(s; t, r) \cong X(s^{-1}; t, r) \cong X(-s^{-1}; t, r)$.

By [27], we have the following proposition.

Proposition 2.2 *Let $G = \langle a, b \mid a^m = b^n = 1, a^b = a^s \rangle$ where $s^n = 1 \pmod{m}$. For $k \in \mathbb{Z}_n^*$, set $C^k := \text{Cay}(G, \{b^k, b^{-k}, b^k a, (b^k a)^{-1}\})$. Then $C^k \cong X(s^k; n, m)$.*

The following proposition is straightforward (see [13]).

Proposition 2.3 [13, Propositions 2.1 and 2.2] *Let $X = \text{Cay}(G, S)$ be half-arc-transitive. Then S contains no involutions, and there is no $\alpha \in \text{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian group.*

A graph X is called a bi-Cayley if it admits a group H of automorphisms acting semiregularly on $V(X)$ with two orbits. By [7], we have the following proposition.

Proposition 2.4 *Every connected bi-Cayley graph X over an abelian group H is vertex-transitive. Moreover, if X is half-arc-transitive, the valency of X is at least 6.*

To end this section, we give a simple proposition.

Proposition 2.5 *Let G be a finite group. If $M \triangleleft A$, $N \triangleleft A$ and $M \cap N = 1$, then $mn = nm$ for any $m \in M$ and $n \in N$.*

3 Classification

In this section, we determine the classification of tetravalent half-arc-transitive graphs of order $8p$. First, we prove the following lemma.

Lemma 3.1 *Let $p \geq 17$ be a prime, $r \in \mathbb{Z}_p^*$ and $r^8 = \pm 1 \pmod{p}$. Then, up to isomorphism, if $8 \mid p-1$ and $16 \nmid p-1$, there is only one tightly attached half-arc-transitive graph $X(r; 8, p)$ where $r^4 = -1 \pmod{p}$, if $16 \mid p-1$, there are three tightly attached half-arc-transitive graphs $X(r; 8, p)$, $X(r^2; 8, p)$, $X(r^3; 8, p)$ where $r^8 = -1 \pmod{p}$.*

Proof Assume that $8 \mid p-1$ and $16 \nmid p-1$. Let r be an element of $r \in \mathbb{Z}_p^*$ with $r^4 = -1 \pmod{p}$. Then all elements of order 8 in $\mathbb{Z}_p^*$ are r, r^3, r^5 and r^7 , furthermore, $r^{-1} = r^7, (r^3)^{-1} = r^5$ and $-r = r^5$. By Proposition 2.1, $X(r^7; 8, p) \cong X(r; 8, p) \cong X(r^5; 8, p) \cong X(r^3; 8, p)$ is half-arc-transitive.

Assume that $16 \mid p-1$. In this case, we assume that $r^8 = -1$ for some $r \in \mathbb{Z}_p^*$. Then all elements of order 16 in $\mathbb{Z}_p^*$ are $r, r^3, r^5, r^7, r^9, r^{11}, r^{13}$ and r^{15} , all elements of order 8 in $\mathbb{Z}_p^*$ are r^2, r^6, r^{10} and r^{14} . Furthermore, $r^{-1} = r^{15}, -r = r^9, (r^7)^{-1} = r^9$, and $(r^3)^{-1} = r^{13}, -r^3 = r^{11}, (r^5)^{-1} = r^{11}$, and $(r^2)^{-1} = r^{14}, (r^6)^{-1} = r^{10}, -r^2 = r^{10}$. By Proposition 2.1, $X(r^{15}; 8, p) \cong X(r; 8, p) \cong X(r^9; 8, p) \cong X(r^7; 8, p)$, $X(r^5; 8, p) \cong X(r^{11}; 8, p) \cong X(r^3; 8, p) \cong X(r^{13}; 8, p)$, $X(r^{14}; 8, p) \cong X(r^2; 8, p) \cong X(r^{10}; 8, p) \cong X(r^6; 8, p)$ are half-arc-transitive.

To complete the proof, it suffices to show that $X(r; 8, p), X(r^2; 8, p), X(r^3; 8, p)$ are pair-wise non-isomorphic where $r^8 = -1$. Suppose on the contrary that $X(r^m; 8, p) \cong X(r^n; 8, p)$ where $m, n \in \{1, 2, 3\}$ and $m \neq n$. It is easy to know that $r^{m \pm n} \neq 1$ and $r^{8+m \pm n} \neq 1$.

Let $V_i = \{v_i^j \mid j \in \mathbb{Z}_p\}$ for $i \in \mathbb{Z}_8$. Then $V(X(r^m; 8, p)) = V(X(r^n; 8, p)) = \bigcup_{i \in \mathbb{Z}_8} V_i$. Recall that $E(X(r^m; 8, p)) = \{v_i^j, v_{i+1}^{j \pm (r^m)^i}\} \mid i \in \mathbb{Z}_8, j \in \mathbb{Z}_p\}$. Note that

$X(r^n; 8, p)$ has an automorphism which fixes v_0^0 and interchanges v_1^1 and v_1^{-1} , and $v_7^{r^n}$ and $v_7^{-r^n}$. Since $X(r^m; 8, p) \cong X(r^n; 8, p)$, there is an isomorphism α from $X(r^m; 8, p)$ to $X(r^n; 8, p)$ such that $(v_0^0)^\alpha = v_0^0$ and $(v_1^1)^\alpha = v_1^1$ or $(v_1^1)^\alpha = v_7^{-r^n}$. By Proposition 2.1, $\text{Aut}(X(r; m, n)) = \langle \rho, \sigma, \tau \rangle$. Recall that $\rho : v_i^j \mapsto v_i^{j+1}$, $\sigma : v_i^j \mapsto v_{i+1}^j$, $\tau : v_i^j \mapsto v_i^{-j}$ for $i \in \mathbb{Z}_8$, $j \in \mathbb{Z}_p$. It follows that $\langle \rho \rangle$ is a normal Sylow p -subgroup of $\langle \rho, \sigma, \tau \rangle$. Then V_i ($i \in \mathbb{Z}_8$) are orbits of the unique normal Sylow p -subgroup of $\text{Aut}(X(r^m; 8, p))$ and $\text{Aut}(X(r^n; 8, p))$, respectively. This implies that α maps each V_i to some V_j . Thus, $V_0^\alpha = V_0$ and $V_1^\alpha = V_1$ or V_7 .

Suppose that $V_1^\alpha = V_1$. Then $(v_1^1)^\alpha = v_1^1$ and $V_k^\alpha = V_k$ for any $k \in \mathbb{Z}_8$. Since the induced subgraph $[V_0 \cup V_1]$ by $V_0 \cup V_1$ in $X(r^m; 8, p)$ and in $X(r^n; 8, p)$ are cycles of length $2p$, it is easy to see that $(v_0^k)^\alpha = v_0^k$ and $(v_1^k)^\alpha = v_1^k$ for any $k \in \mathbb{Z}_p$. Similarly, since the induced subgraph $[V_1 \cup V_2]$ by $V_1 \cup V_2$ in $X(r^m; 8, p)$ and in $X(r^n; 8, p)$ are cycles of length $2p$, one has $(v_2^{r^m})^\alpha = v_2^{r^n}$ or $v_2^{-r^n}$ because $(v_1^0)^\alpha = v_1^0$. If $(v_2^{r^m})^\alpha = v_2^{r^n}$ then $(v_1^{2r^m})^\alpha = v_1^{2r^n}$. Note that $(v_1^{2r^m})^\alpha = v_1^{2r^m}$. Thus, $2r^m = 2r^n$ in $\mathbb{Z}_p^*$, that is $r^{m-n} = 1$ in $\mathbb{Z}_p^*$, a contradiction. Similarly, if $(v_2^{r^m})^\alpha = v_2^{-r^n}$ then $(v_1^{2r^m})^\alpha = v_1^{-2r^n}$. Thus, $2r^m = -2r^n$ in $\mathbb{Z}_p^*$, that is $r^{8+m-n} = 1$ in $\mathbb{Z}_p^*$, a contradiction.

Now let $V_1^\alpha = V_7$. Then $(v_1^1)^\alpha = v_7^{r^{-n}}$ and $V_k^\alpha = V_{8-k}$ for any $k \in \mathbb{Z}_8$. Since the induced subgraph $[V_0 \cup V_7]$ in $X(r^m; 8, p)$ and in $X(r^n; 8, p)$ are cycles of length $2p$, one has $(v_0^j)^\alpha = v_0^{jr^{-n}}$ and $(v_1^j)^\alpha = v_7^{jr^{-n}}$ for any $j \in \mathbb{Z}_p$. In particular, $(v_1^0)^\alpha = v_7^0$ and $(v_1^{2r^m})^\alpha = v_7^{2r^{m-n}}$. It follows that $(v_2^{r^m})^\alpha = v_6^{r^{-2n}}$ or $v_6^{-r^{-2n}}$. If $(v_2^{r^m})^\alpha = v_6^{-r^{-2n}}$ then $(v_1^{2r^m})^\alpha = v_7^{2r^{-2n}}$; thus $v_7^{2r^{m-n}} = v_7^{2r^{-2n}}$, implying $r^{m+n} = 1$ in $\mathbb{Z}_p^*$, a contradiction. One may assume that $(v_2^{r^m})^\alpha = v_6^{-r^{-2n}}$ and hence $(v_1^{2r^m})^\alpha = v_7^{-2r^{-2n}}$; thus $v_7^{2r^{m-n}} = v_7^{-2r^{-2n}}$, implying $r^{8+m+n} = 1$, a contradiction. $\square$

Lemma 3.2 *Let p be an odd prime, and let X be a tetravalent half-arc-transitive graph of order $8p$. Then $\text{Aut}(X)$ has a normal Sylow p -subgroup.*

Proof Let X be a connected tetravalent half-arc-transitive graph of order $8p$ and $u \in V(X)$. Set $A = \text{Aut}(X)$. Note that A_u is a 2-group. Then we may assume that $|A| = 2^{4+z}p$ with some integer $z \geq 0$. Thus, A is solvable. Let P be a Sylow p -subgroup of A . Then $|P| = p$. By [28], we have that $p \geq 17$ (In fact, $p = 17$ or 41 if $8p < 252$).

Suppose to the contrary that A has no normal Sylow p -subgroups. Then A has a normal 2-subgroup since A is solvable. Assume that N is a maximal normal 2-subgroup of A . Let X_N be the quotient graph of X corresponding to the orbits of N , and let K be the kernel of A acting on $V(X_N)$. The maximality of N forces that $N = K$. By half-arc-transitivity of X , the subgraph of X induced by each orbit of N has no edges. Then X_N has valency 2 or 4. Assume that $K_u = 1$. Note that $|X| = 8p$. Then $|N| = |K| \mid 8$. Since N is a maximal normal 2-subgroup of A , we have that A/N has no normal 2-subgroup. It follows that A/N has normal Sylow p -subgroup PN/N , that is $PN \triangleleft A$. It is easy to know that P is characteristic in PN , implying that $P \triangleleft A$, a contradiction.

Thus, we may assume that $N_u = K_u > 1$. Then X_N has valency 2, we may assume that $X_N = (B_0, B_1, \dots, B_{k-1})$, where B_i is an orbit of N and B_i is adjacent to B_{i+1} for each $i \in \mathbb{Z}_k$ such that $k \mid B_i| = 8p$. By the edge-transitivity of X , the induced subgraph

Fig. 1 The induced subgraph $[B_0 \cup B_1]$

$[B_i \cup B_{i+1}]$ in X has valency 2. Since N is transitive on each B_i and $N_u > 1$, all edges in the induced subgraph $[B_i \cup B_{i+1}]$ have the same direction either from B_i to B_{i+1} or from B_{i+1} to B_i in the oriented graph $D_A(X)$. This implies that $A/N \cong \mathbb{Z}_k$. Since N is a maximal normal 2-subgroup of A , we have that $A/N \cong \mathbb{Z}_p$. Thus, $|B_i| = 8$ and $k = p$. By the half-arc-transitivity of X , if an automorphism of X maps B_0 to B_1 then it maps B_i to B_{i+1} for each $i \in \mathbb{Z}_p$. Let $\beta \in \text{Aut}(X)$ be of order p . Clearly, the action of β on $\{B_0, B_1, \dots, B_{p-1}\}$ is a p -cycle. Thus, there exists an integer s such that $B_0^{\beta^s} = B_1$. Set $\alpha = \beta^s$. It follows that α has order p and $B_i^\alpha = B_{i+1}$ for each $i \in \mathbb{Z}_p$. Since the induced subgraph $[B_i \cup B_{i+1}]$ has valency 2, the transitivity of N on each B_i implies that $[B_i \cup B_{i+1}]$ is either an 16-cycle, an union of two 8-cycles, or an union of four 4-cycles for each $i \in \mathbb{Z}_p$. Suppose that $[B_i \cup B_{i+1}]$ is an 16-cycle. Then N acts faithfully on each B_i and $N_u \neq 1$ implies $N_u \cong \mathbb{Z}_2$. It follows that $|N| = 16$. As $A/N \cong \mathbb{Z}_p$, one has $|A| = 16p$. Recall that $p \geq 17$. Then P is normal in A , a contradiction.

Suppose that $[B_i \cup B_{i+1}]$ is an union of two 8-cycles. Remember that $\alpha \in \text{Aut}(X)$ has order p and $B_i^\alpha = B_{i+1}$ for each $i \in \mathbb{Z}_p$. Let $B_i = \{v_i^j \mid j \in \mathbb{Z}_8\}$ such that $(v_i^j)^\alpha = v_{i+1}^j$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Consider a 8-cycle C in the induced subgraph $[B_0 \cup B_1]$ and let n be the number of edges of C which are contained in some orbit of α . Then $n = 0, 1, 2, 3$, or 4 and consequently, $[B_0 \cup B_1]$ is one of the following five cases in Fig. 1. $\square$

It is easy to see that for the Case I, X is disconnected, a contradiction.

Noting that $\alpha \in \text{Aut}(X)$, for Case II, we have that

$$E(X) = \{(v_i^0, v_{i+1}^0), (v_i^0, v_{i+1}^3), (v_i^1, v_{i+1}^1), (v_i^1, v_{i+1}^0), (v_i^2, v_{i+1}^2), (v_i^2, v_{i+1}^1), \\ (v_i^3, v_{i+1}^4), (v_i^3, v_{i+1}^5), (v_i^4, v_{i+1}^3), (v_i^4, v_{i+1}^2), (v_i^5, v_{i+1}^5), (v_i^5, v_{i+1}^6), \\ (v_i^6, v_{i+1}^6), (v_i^6, v_{i+1}^7), (v_i^7, v_{i+1}^4), (v_i^7, v_{i+1}^7) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{p-i+1}^2$, $(v_i^1)^\rho = v_{p-i+1}^1$, $(v_i^2)^\rho = v_{p-i+1}^0$, $(v_i^3)^\rho = v_{p-i+1}^4$, $(v_i^4)^\rho = v_{p-i+1}^3$, $(v_i^5)^\rho = v_{p-i+1}^7$, $(v_i^6)^\rho = v_{p-i+1}^6$, $(v_i^7)^\rho = v_{p-i+1}^5$. One may easily show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$, furthermore, ρ interchanges v_0^1 and v_1^1 . Note that $(v_0^1, v_1^1) \in E(X)$. Clearly, a half-arc-transitive graph has no automorphisms which interchange the two arcs of any given edge, a contradiction.

For Case III, we have that

$$E(X) = \{(v_i^0, v_{i+1}^0), (v_i^0, v_{i+1}^1), (v_i^1, v_{i+1}^1), (v_i^1, v_{i+1}^2), (v_i^2, v_{i+1}^3), (v_i^2, v_{i+1}^7), (v_i^3, v_{i+1}^3), (v_i^3, v_{i+1}^4), (v_i^4, v_{i+1}^3), (v_i^4, v_{i+1}^5), (v_i^5, v_{i+1}^4), (v_i^5, v_{i+1}^0), (v_i^6, v_{i+1}^5), (v_i^6, v_{i+1}^6), (v_i^7, v_{i+1}^6), (v_i^7, v_{i+1}^7) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{p-i+1}^1$, $(v_i^1)^\rho = v_{p-i+1}^0$, $(v_i^2)^\rho = v_{p-i+1}^5$, $(v_i^3)^\rho = v_{p-i+1}^4$, $(v_i^4)^\rho = v_{p-i+1}^3$, $(v_i^5)^\rho = v_{p-i+1}^2$, $(v_i^6)^\rho = v_{p-i+1}^7$, $(v_i^7)^\rho = v_{p-i+1}^6$ ($i \in \mathbb{Z}_p$). One may show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$. Note that $(v_0^0, v_1^1) \in E(X)$, ρ interchanges v_0^0 and v_1^1 . It is a contradiction.

For Case IV, we have that

$$E(X) = \{(v_i^0, v_{i+1}^0), (v_i^0, v_{i+1}^1), (v_i^1, v_{i+1}^2), (v_i^1, v_{i+1}^7), (v_i^2, v_{i+1}^1), (v_i^2, v_{i+1}^3), (v_i^3, v_{i+1}^2), (v_i^3, v_{i+1}^4), (v_i^4, v_{i+1}^3), (v_i^4, v_{i+1}^5), (v_i^5, v_{i+1}^4), (v_i^5, v_{i+1}^6), (v_i^6, v_{i+1}^0), (v_i^6, v_{i+1}^5), (v_i^7, v_{i+1}^6), (v_i^7, v_{i+1}^7) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{p-i+1}^0$, $(v_i^1)^\rho = v_{p-i+1}^6$, $(v_i^2)^\rho = v_{p-i+1}^5$, $(v_i^3)^\rho = v_{p-i+1}^4$, $(v_i^4)^\rho = v_{p-i+1}^3$, $(v_i^5)^\rho = v_{p-i+1}^2$, $(v_i^6)^\rho = v_{p-i+1}^1$, $(v_i^7)^\rho = v_{p-i+1}^7$ ($i \in \mathbb{Z}_p$). One may easily show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$. Note that $(v_0^0, v_1^0) \in E(X)$, ρ interchanges v_0^0 and v_1^0 . It is a contradiction.

For Case V, we have that

$$E(X) = \{(v_i^0, v_{i+1}^1), (v_i^0, v_{i+1}^7), (v_i^1, v_{i+1}^0), (v_i^1, v_{i+1}^2), (v_i^2, v_{i+1}^1), (v_i^2, v_{i+1}^3), (v_i^3, v_{i+1}^2), (v_i^3, v_{i+1}^4), (v_i^4, v_{i+1}^3), (v_i^4, v_{i+1}^5), (v_i^5, v_{i+1}^4), (v_i^5, v_{i+1}^6), (v_i^6, v_{i+1}^5), (v_i^6, v_{i+1}^7), (v_i^7, v_{i+1}^0), (v_i^7, v_{i+1}^6) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{p-i+1}^1$, $(v_i^1)^\rho = v_{p-i+1}^0$, $(v_i^2)^\rho = v_{p-i+1}^7$, $(v_i^3)^\rho = v_{p-i+1}^6$, $(v_i^4)^\rho = v_{p-i+1}^5$, $(v_i^5)^\rho = v_{p-i+1}^4$, $(v_i^6)^\rho = v_{p-i+1}^3$, $(v_i^7)^\rho = v_{p-i+1}^2$ ($i \in \mathbb{Z}_p$). One may easily show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$. Note that $(v_0^0, v_1^1) \in E(X)$, ρ interchanges v_0^0 and v_1^1 . It is a contradiction.

Fig. 2 The induced subgraph $[B_0 \cup B_1]$

Suppose that $[B_i \cup B_{i+1}]$ is a union of four 4-cycles. Remember that $\alpha \in \text{Aut}(X)$, $B_i = \{v_i^j \mid j \in \mathbb{Z}_8\}$ and $(v_i^j)^\alpha = v_{i+1}^j$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Consider a 4-cycle C in the induced subgraph $[B_0 \cup B_1]$ and let n be the number of edges of C which are on some orbit of α . Then $n = 0, 1$, or 2 and consequently, $[B_0 \cup B_1]$ is one of the following five cases in Fig. 2.

It is easy to see that for Case I, II and III, X is disconnected.

For Case IV, we have that

$$E(X) = \{(v_i^0, v_{i+1}^0), (v_i^0, v_{i+1}^1), (v_i^1, v_{i+1}^6), (v_i^1, v_{i+1}^7), \\ (v_i^2, v_{i+1}^2), (v_i^2, v_{i+1}^3), (v_i^3, v_{i+1}^0), (v_i^3, v_{i+1}^1), \\ (v_i^4, v_{i+1}^4), (v_i^4, v_{i+1}^5), (v_i^5, v_{i+1}^2), (v_i^5, v_{i+1}^3), (v_i^6, v_{i+1}^6), \\ (v_i^6, v_{i+1}^7), (v_i^7, v_{i+1}^4), (v_i^7, v_{i+1}^5) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{p-i+1}^0$, $(v_i^1)^\rho = v_{p-i+1}^3$, $(v_i^2)^\rho = v_{p-i+1}^6$, $(v_i^3)^\rho = v_{p-i+1}^1$, $(v_i^4)^\rho = v_{p-i+1}^4$, $(v_i^5)^\rho = v_{p-i+1}^7$, $(v_i^6)^\rho = v_{p-i+1}^2$, $(v_i^7)^\rho = v_{p-i+1}^5$ ($i \in \mathbb{Z}_p$). One may show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$. Note that $(v_0^0, v_1^0) \in E(X)$, ρ interchanges v_0^0 and v_1^0 . It is a contradiction.

For Case V, we have that

$$E(X) = \{(v_i^0, v_{i+1}^1), (v_i^0, v_{i+1}^7), (v_i^1, v_{i+1}^0), (v_i^1, v_{i+1}^2), \\ (v_i^2, v_{i+1}^1), (v_i^2, v_{i+1}^7), (v_i^3, v_{i+1}^0), (v_i^3, v_{i+1}^2), \\ (v_i^4, v_{i+1}^3), (v_i^4, v_{i+1}^5), (v_i^5, v_{i+1}^4), (v_i^5, v_{i+1}^6), (v_i^6, v_{i+1}^3), \\ (v_i^6, v_{i+1}^5), (v_i^7, v_{i+1}^4), (v_i^7, v_{i+1}^6) \mid i \in \mathbb{Z}_p\}.$$

Define a permutation ρ on $V(X)$ by $(v_i^0)^\rho = v_{-i}^0$, $(v_i^1)^\rho = v_{-i}^1$, $(v_i^2)^\rho = v_{-i}^2$, $(v_i^3)^\rho = v_{-i}^7$, $(v_i^4)^\rho = v_{-i}^6$, $(v_i^5)^\rho = v_{-i}^5$, $(v_i^6)^\rho = v_{-i}^4$, $(v_i^7)^\rho = v_{-i}^3$ ($i \in \mathbb{Z}_p$). One may show that $N((v_i^j)^\rho) = (N(v_i^j))^\rho$ for $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_8$. Thus, $\rho \in \text{Aut}(X)$. Note that ρ fixes v_0^1 and interchanges v_{-1}^0 and v_1^0 , a contradiction.

Now, we give the main theorem.

Theorem 3.3 *Let p be a prime and let X be a connected tetravalent vertex-transitive and edge-transitive graph of order $8p$. Then X is half-arc-transitive if and only if $X \cong X(r; 8, p)$ where $r^4 = -1 \pmod{p}$ and $16 \nmid p-1$, or $X \cong X(r; 8, p)$, $X(r^2; 8, p)$ or $X(r^3; 8, p)$ where $r^8 = -1 \pmod{p}$.*

Proof By Lemma 3.1, the sufficiency is true. To prove the necessity, let X be a connected tetravalent half-arc-transitive graph of order $8p$ and $u \in V(X)$. By [28], we have that $p \geq 17$. Set $A = \text{Aut}(X)$ and P is a Sylow p -subgroup of A . By Lemma 3.2, $P \triangleleft A$. Let X_P be the quotient graph of X corresponding to the orbits of P , and K the kernel of A acting on $V(X_P)$. Then P has eight orbits of length p , say $V_i = \{v_i^j \mid j \in \mathbb{Z}_p\}$ for $i \in \mathbb{Z}_8$. Let $P = \langle \delta \rangle$ and $(v_i^j)^\delta = v_i^{j+1}$ for $i \in \mathbb{Z}_8$ and $j \in \mathbb{Z}_p$. Since X is edge-transitive, X_P has valency 2 or 4. First, we prove the following claim.

Claim $K_u \neq 1$. Furthermore, X_P has valency 2.

Suppose that $K_u = 1$. Then $K = P$, furthermore X_P has valency 4 or X_P is isomorphic to an 8-cycle. It means that X_P is A/P -half-arc-transitive or $A/P \cong D_{16}$, implying that A/P is non-abelian. Set $C = C_A(P)$. Then $A/C \leq \mathbb{Z}_{p-1}$. Suppose that $C = P$. Then $A/C = A/P \leq \mathbb{Z}_{p-1}$ is abelian, a contradiction. Thus, $C > P$. Let N be a Sylow 2-subgroup of C . Then $C = P \times N$ and N is characteristic in C , implying that $N \triangleleft A$. Let $O_2(A)$ be a maximal normal 2-subgroup of A . Then $O_2(A) \cap P = 1$, by Proposition 2.5, we have that $O_2(A) \leq C$. Since N is a Sylow 2-subgroup of C and $N \triangleleft A$, we have $N = O_2(A)$, that is, N is the maximal normal 2-subgroup of A . Suppose that $|N| = 2$. Then $|N| = |C/P| = 2$ implies that $C/P \leq Z(A/P)$. Since $(A/P)/(C/P) \cong A/C \leq \mathbb{Z}_{p-1}$, we have that A/P is abelian, it is a contradiction.

Thus, $4 \mid |N|$. Let X_N be the quotient graph of X corresponding to the orbits of N , and let L be the kernel of A acting on $V(X_N)$. Note that N is a maximal normal 2-subgroup of A . Thus, we have $L = N$ and A/N has no normal 2-subgroup. Suppose that $N_u > 1$. By the proof of Lemma 3.2 (Note that B_i is an orbit of N and B_i is adjacent to B_{i+1} for each $i \in \mathbb{Z}_p$), we have that $[B_i \cup B_{i+1}]$ is an 16-cycle. It means that $N \cong D_{16}$ and $A/N \cong \mathbb{Z}_p$. Note that $P \triangleleft A$. Then $A = P \times D_{16}$, furthermore, $A_u \cong \mathbb{Z}_2$. Let $B = P \times \mathbb{Z}_8 \cong \mathbb{Z}_{8p}$. Clearly, B is a normal subgroup of A . Then $B_u \leq A_u \cong \mathbb{Z}_2$. Suppose that $B_u = \mathbb{Z}_2$. Since $B \cong \mathbb{Z}_{8p}$, we have that B_u is characteristic in B . Since $B \triangleleft A$, we have that $B_u \triangleleft A$. It follows that $B_u = 1$, a contradiction. Thus, $B_u = 1$ and B is regular on $V(X)$. It means that X is a Cayley graph on B . By Proposition 2.3, it is a contradiction.

Thus, $N_u = 1$. Note that $4 \mid |N|$. Then $|N| = 4$ or 8. Suppose that $|N| = 4$. Then N is abelian, implying that $C = P \times N$ is abelian. Then C has two orbits on $V(X)$ and C is regular on each orbit. Thus, X is a bi-Cayley graph on group C . By Proposition 2.4, it is impossible. Thus, $|N| = 8$, and $C = P \times N$ is regular on X , implying that X is a Cayley graph on C . By Proposition 2.3, we have that N is non-abelian, implying that $N \cong D_8$ or Q_8 . Suppose that $N \cong D_8$. Now, we assume

that $X = \text{Cay}(C, S)$ where $C = P \times \langle a, b \rangle = \langle \rho \rangle \times \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$ and $S = \{x, x^{-1}, y, y^{-1}\}$. Note that $C \triangleleft A$. Thus, X is a connected normal half-arc-transitive graph. Then $o(x) = o(y)$ and $C = \langle x, y \rangle$. Up to graph isomorphism, we may assume that $S \cong \{\rho b, \rho^{-1}b, \rho ab, \rho^{-1}ab\}$. Define a map $\alpha : \rho \mapsto \rho^{-1}, a \mapsto a, b \mapsto b$. Then $\alpha \in \text{Aut}(C)$ and $(\rho b)^\alpha = \rho^{-1}b = (\rho b)^{-1}$. By Proposition 2.3, it is impossible. Suppose that $N \cong Q_8$. Similarly, we may assume that $C = P \times \langle a, b \rangle = \langle \rho \rangle \times \langle a, b \mid a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle$ and $S = \{\rho a, \rho^{-1}a^{-1}, \rho b, \rho^{-1}b^{-1}\}$. Define a map $\alpha : \rho \mapsto \rho^{-1}, a \mapsto a^{-1}, b \mapsto b^{-1}$. Then $\alpha \in \text{Aut}(C)$ and $(\rho b)^\alpha = \rho^{-1}b^{-1} = (\rho b)^{-1}$. By Proposition 2.3 again, it is impossible.

Thus, $K_u \neq 1$ and X_P has valency 2. It means that $X_P = (V_0, V_1, \dots, V_7)$ is an 8-cycle, furthermore, all edges in the induced subgraph $[V_i \cup V_{i+1}]$ have the same direction either from V_i to V_{i+1} or from V_{i+1} to V_i in the oriented graph $D_A(X)$. It follows that the attachment number of X is equal to its radius. Thus, X is a tetravalent tightly attached half-arc-transitive graph of odd radius p . By Proposition 2.1 and Lemma 3.1, $X \cong X(r; 8, p)$ where $r^4 = -1 \pmod{p}$, $X \cong X(r; 8, p)$, $X \cong X(r^2; 8, p)$ or $X \cong X(r^3; 8, p)$ where $r^8 = -1 \pmod{p}$. $\square$

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