Journal of Algebraic Combinatorics (2020) 51:195–235
https://doi.org/10.1007/s10801-019-00872-z

Polynomial super representations of $U_q^{\mathrm{res}}(\mathfrak{gl}_{m|n})$ at roots of unity

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Received: 26 May 2018 / Accepted: 16 January 2019 / Published online: 2 February 2019

Abstract
As a homomorphic image of the hyperalgebra $U_{q,R}(m|n)$ associated with the quantum linear supergroup $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$, we first give a presentation for the q-Schur superalgebra $S_{q,R}(m|n,r)$ over a commutative ring R. We then develop a criterion for polynomial supermodules of $U_{q,F}(m|n)$ over a field F and use this to determine a classification of polynomial irreducible supermodules at roots of unity. This also gives classifications of irreducible $S_{q,F}(m|n,r)$-supermodules for all r. As an application when $m = n \geq r$ and motivated by the beautiful work (Brundan and Kujawa in J Algebraic Combin 18:13–39, 2003) in the classical (non-quantum) case, we provide a new proof for the Mullineux conjecture related to the irreducible modules over the Hecke algebra $H_{q^2,F}(\mathfrak{S}_r)$; see Brundan (Proc Lond Math Soc 77:551–581, 1998) for a proof without using the super theory.

Keywords Quantum linear supergroup · Quantum hyperalgebra · Polynomial representation · q-Schur superalgebra · Hecke algebra · Mullineux conjecture

Mathematics Subject Classification 17B35 · 17B37 · 17B70 · 20C08 · 20G43

The work was supported by a 2017 UNSW Science Goldstar Grant, the Natural Science Foundation of China (#11871404), and Jiangsu Provincial Department of Education. The authors would like to thank UNSW and Xiamen University for their hospitality during the writing of the paper.

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1 Introduction

The Mullineux conjecture [22] refers to a combinatorial algorithmic map $\lambda \mapsto \mathtt{M}(\lambda)$ on p-regular partitions such that if D^λ is an irreducible p-modular representation of the symmetric group $\mathfrak{S}_r$, then $D^{\mathtt{M}(\lambda)} \cong D^\lambda \otimes \mathrm{sgn}$, where sgn is the sign representation. Building on his work on modular branching rules, Kleshchev [18] developed an alternative algorithm to describe the partition associated with $D^\lambda \otimes \mathrm{sgn}$. With some technical combinatorics, Ford and Kleshchev [16] then proved that Kleshchev's algorithm is equivalent to the Mullineux map and thereby proved the Mullineux conjecture. See [1] for a shorter proof for the equivalence. The Hecke algebra version of this conjecture was proved by Brundan [2]. Like the p-modular case, quantum branching rules play a decisive role in the proof.

In 2003, Brundan and Kujawa [4] discovered an excellent new proof for the original conjecture without using branching rules. Instead, they used representations of the general linear Lie supergroup. This proof involves a different algorithm introduced by Xu [25] for the Mullineux map and the Serganova algorithm for computing the highest weights of w-twisted irreducible supermodules. The latter relies on the highest weight theory developed in [4, §4] associated with a representative w of an $\mathfrak{S}_m \times \mathfrak{S}_n$-coset. However, this theory does not seem to have a quantum analogue. Thus, generalising the work in [4] to the quantum case requires some new ideas.

In this paper, we will use the polynomial super representation theory of the (super) quantum hyperalgebra associated with the linear Lie superalgebra $\mathfrak{gl}_{m|n}$ to give a new proof of the quantum Mullineux conjecture. Here are the main ideas to tackle the two algorithms used in [4]. First, we directly link the map j_l used in Xu's algorithm to a non-vanishing condition of certain products of Gaussian polynomials which naturally occur in root vector actions on a maximal vector; see Lemmas 6.1 and 6.3. This results in a classification of polynomial irreducible supermodules. Second, we realise the Serganova algorithm through a sequence of root vector actions on a highest weight vector. It is worth noting that the graph automorphism σ, available only when $m = n$, and a pair of Schur functors play crucial roles in the final stage of the proof.

We organise the paper as follows: We first discuss in Sect. 2 the Lusztig $\mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-form $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$ of the quantum supergroup $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$ over $\mathbb{Q}(\boldsymbol{v})$ and their base change $U_{q,R}(m|n)$ to any commutative ring R via $\boldsymbol{v} \mapsto q$, the quantum (super) hyperalgebras. We also display the commutation formulas of root vectors which are used throughout the paper. In Sect. 3, we introduce the q-Schur superalgebra $S_{q,R}(m|n,r)$ not only as an endomorphism algebra of a module of the Hecke algebra $H_{q^2,R}(r)$ but also as a homomorphic image of $U_{q,R}(m|n)$. By working out a presentation for $S_{q,R}(m|n,r)$ in Sect. 4, we develop a criterion which tests when a finite-dimensional weight $U_{q,F}(m|n)$-supermodule is polynomial in Sect. 5. A classification of irreducible weight $U_{q,F}(m|n)$-supermodule is also given as an extension of its non-super counterpart [19].

In Sect. 6, we classify all polynomial irreducible $U_{q,F}(m|n)$-supermodules (Theorem 6.4) which are indexed by the sets used in [4]. Notably, the method here is very different from those used in [4]. As a simple application, a classification of irreducible $S_{q,F}(m|n,r)$-supermodules is given in Sect. 7. Unlike the classification given in [11,12], which is independent of quantum supergroups, this classification is constructive. We further investigate the structure of q-Schur superalgebras through a certain filtration of ideals and Weyl supermodules. The last two sections are devoted to prove the quantum Mullineux conjecture. The combinatorics of the Mullineux map, largely following [4], and the quantum Serganova algorithm (Proposition 8.2, Theorem 8.5) are discussed in Sect. 8. In the last section, we introduce two Schur functors and compare their images on supermodules (Proposition 9.3). The conjecture is proved in Theorem 9.5.

Throughout the paper, we assume that R is a commutative ring with 1 of characteristic $\neq 2$. Let $q \in R$ be an invertible element. From Sect. 5 onwards, we assume that $R = F$ is a field and q is a primitive l'th root of unity. To include the non-roots of unity case, we set $l' = \infty$ if q is not a unit of unity.

For fixed non-negative integers m, n with $m + n > 0$ and $i \in [1, m + n] := \{1, 2, \ldots, m + n\}$, define the parity function $i \mapsto \bar{i}$ by

$$\bar{i} = \begin{cases} \bar{0}, & \text{if } 1 \leq i \leq m; \\ \bar{1}, & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

For the standard basis $\{\epsilon_1, \ldots, \epsilon_{m+n}\}$ for $\mathbb{Z}^{m+n}$, define the "super dot product" by $(\epsilon_i, \epsilon_j) = (\epsilon_i, \epsilon_j)_s = (-1)^{\bar{i}}\delta_{ij}$, and call $\alpha_i = \epsilon_i - \epsilon_{i+1}, i \in [1, m+n) := [1, m+n]\backslash\{m+n\}$ simple roots. We have positive root system $\Phi^+ = \{\alpha_{i,j} = \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m+n\}$ and negative root system $\Phi^- = -\Phi^+$. Define $\bar{\alpha}_{i,j} = \bar{i} + \bar{j}$, and call $\alpha_{i,j}$ an even (resp. odd) root if $\bar{\alpha}_{i,j} = \bar{0}$ (resp., $\bar{1}$). Note that α_m is the only odd simple root. Let $\Phi = \Phi^+ \cup \Phi^-$.

2 The quantum hyperalgebra $U_{q,R}(m|n)$

Let $\mathbb{Q}(\boldsymbol{v})$ be the field of rational functions in indeterminate $\boldsymbol{v}$ and let

$$\boldsymbol{v}_a = \boldsymbol{v}^{(-1)^{\bar{a}}} \quad (1 \leq a \leq m+n).$$

Define the super (or graded) commutator on the homogeneous elements X, Y for an (associative) superalgebra by

$$[X, Y] = [X, Y]_s = XY - (-1)^{\bar{X}\bar{Y}} YX.$$

The following quantum enveloping superalgebra $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$ is defined in [6,26].

Definition 2.1 The quantum enveloping superalgebra $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$ over $\mathbb{Q}(\boldsymbol{v})$ is generated by the homogeneous elements

$$E_1, \ldots, E_{m+n-1}, F_1, \ldots, F_{m+n-1}, K_1^{\pm 1}, \ldots, K_{m+n}^{\pm 1},$$

with a $\mathbb{Z}_2$-grading given by setting $\overline{E_m} = \overline{F_m} = \bar{1}$, $\overline{E_a} = \overline{F_a} = \bar{0}$ for $a \neq m$, and $\overline{K_a^{\pm 1}} = \bar{0}$. These elements are subject to the following relations:

(QG1) $K_aK_b = K_bK_a,\ K_aK_a^{-1} = K_a^{-1}K_a = 1$;
(QG2) $K_aE_b = \boldsymbol{v}^{(\varepsilon_a,\alpha_b)}E_bK_a,\ K_aF_b = \boldsymbol{v}^{(\varepsilon_a,-\alpha_b)}F_bK_a$;
(QG3) $[E_a, F_b] = \delta_{a,b}\frac{K_aK_{a+1}^{-1}-K_a^{-1}K_{a+1}}{\boldsymbol{v}_a-\boldsymbol{v}_a^{-1}}$;
(QG4) $E_aE_b = E_bE_a,\ F_aF_b = F_bF_a$, if $|a-b| > 1$;
(QG5) For $a \neq m$ and $|a-b| = 1$,

$$E_a^2E_b - (\boldsymbol{v}_a + \boldsymbol{v}_a^{-1})E_aE_bE_a + E_bE_a^2 = 0,$$
$$F_a^2F_b - (\boldsymbol{v}_a + \boldsymbol{v}_a^{-1})F_aF_bF_a + F_bF_a^2 = 0;$$

(QG6) $E_m^2 = F_m^2 = [E_m, E_{m-1,m+2}] = [F_m, E_{m+2,m-1}] = 0$, where

$$\begin{aligned} E_{m-1,m+2} &= E_{m-1}E_mE_{m+1} - \boldsymbol{v}E_{m-1}E_{m+1}E_m - \boldsymbol{v}^{-1}E_mE_{m+1}E_{m-1} \\ &\quad + E_{m+1}E_mE_{m-1}, \\ E_{m+2,m-1} &= F_{m+1}F_mF_{m-1} - \boldsymbol{v}^{-1}F_mF_{m+1}F_{m-1} - \boldsymbol{v}F_{m-1}F_{m+1}F_m \\ &\quad + F_{m-1}F_mF_{m+1}. \end{aligned}$$

Note that, if $a = b = m$ in (QG3), then $E_mF_m + F_mE_m = \frac{K_mK_{m+1}^{-1}-K_m^{-1}K_{m+1}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.

By directly checking the relations, it is clear that there is a $\mathbb{Q}(\boldsymbol{v})$-algebra automorphism (of order 4); cf. [8, Lem. 6.5(1)]:

$$\varpi : U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n}) \longrightarrow U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n}), \quad E_a \mapsto (-1)^{\overline{a}+\overline{a+1}}F_a,\ F_a \mapsto E_a,\ K_j^{\pm 1} \mapsto K_j^{\mp 1}, \tag{2.1}$$

and a ring anti-automorphism of order 2

$$\Upsilon : U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n}) \longrightarrow U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n}), \quad E_a \mapsto F_a,\ F_a \mapsto E_a,\ K_j^{\pm 1} \mapsto K_j^{\mp 1},\ \boldsymbol{v} \mapsto \boldsymbol{v}^{-1}. \tag{2.2}$$

When $m = n$, we have the following $\mathbb{Q}(\boldsymbol{v})$-algebra automorphism induced from a "graph automorphism"

$$\sigma : U_{\boldsymbol{v}}(\mathfrak{gl}_{n|n}) \longrightarrow U_{\boldsymbol{v}}(\mathfrak{gl}_{n|n}), \quad E_a \to F_{2n-a},\ F_a \to E_{2n-a},\ K_j^{\pm 1} \to K_{2n+1-j}^{\mp 1}. \tag{2.3}$$

We now introduce Lusztig's $\mathcal{Z}$-form[1] of $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$, where $\mathcal{Z} := \mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$. Let $[i] = \frac{\boldsymbol{v}^i-\boldsymbol{v}^{-i}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$ and $[i]! = [1][2]\cdots[i]$. For any integers $t \in \mathbb{N}$, $s \in \mathbb{Z}$, define (symmetric) Gaussian polynomials by

[1] This form in the non-super case, first introduced in [19], is also known as the restricted integral form in [5, §9.3]; compare with the non-restricted form in [5, §9.2]. See also their respective representation theories of their specialisations in [5, §§11.1–2].

$$\begin{bmatrix} s \\ t \end{bmatrix} = \frac{[s]!}{[t]![s-t]!} = \prod_{i=1}^{t} \frac{\boldsymbol{v}^{s-i+1} - \boldsymbol{v}^{-s+i-1}}{\boldsymbol{v}^{i} - \boldsymbol{v}^{-i}}.$$

Note that, by the evaluation map from $\mathcal{Z}$ to R via $\boldsymbol{v} \mapsto q$, the evaluation of the polynomial $[{s \atop t}]$ at q is denoted $[{s \atop t}]_q$. Note also that if q_a is the value of $\boldsymbol{v}_a$ at q then

$$\begin{bmatrix} s \\ t \end{bmatrix}_{q_a} = \begin{bmatrix} s \\ t \end{bmatrix}_q.$$

For $c \in \mathbb{Z}$, $t \in \mathbb{N}$, set $\left[{K_i;c \atop 0}\right] = 1$ and, for $t > 0$,

$$\begin{bmatrix} K_i;c \\ t \end{bmatrix} = \begin{bmatrix} K_i;c \\ t \end{bmatrix}_i = \prod_{s=1}^{t} \frac{K_i \boldsymbol{v}_i^{c-s+1} - K_i^{-1} \boldsymbol{v}_i^{-c+s-1}}{\boldsymbol{v}_i^{s} - \boldsymbol{v}_i^{-s}}. \tag{2.4}$$

Here, we sometimes use the subscript i to indicate the use of $\boldsymbol{v}_i$. Let $E_i^{(M)} = \frac{E_i^M}{[M]^!}$, $F_i^{(M)} = \frac{F_i^M}{[M]^!}$, and $\left[{K_i \atop t}\right] = \left[{K_i;0 \atop t}\right]$. Then $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$ has the Lusztig $\mathcal{Z}$-form $U_{\boldsymbol{v},\mathcal{Z}} := U_{\boldsymbol{v},\mathcal{Z}}(m|n)$. This is the $\mathcal{Z}$-subsuperalgebra generated by

$$\left\{ E_i^{(M)}, F_i^{(M)}, K_j^{\pm 1}, \begin{bmatrix} K_j \\ t \end{bmatrix} \,\middle|\, t, M \in \mathbb{N}, 1 \le i < m+n,\ 1 \le j \le m+n \right\}.$$

It is clear to see that the automorphisms ϖ, Υ stabilise the $\mathcal{Z}$-form $U_{\boldsymbol{v},\mathcal{Z}}$. Likewise, the graph automorphism σ restricts to an $\mathcal{Z}$-algebra automorphism

$$\sigma : U_{\boldsymbol{v},\mathcal{Z}}(n|n) \longrightarrow U_{\boldsymbol{v},\mathcal{Z}}(n|n). \tag{2.5}$$

We need quantum root vectors to describe PBW-type bases for $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$ below. For a root $\alpha = \epsilon_a - \epsilon_b$, define recursively the *root vectors* $E_\alpha = E_{a,b}$ as follows:

$$E_{a,a+1} = E_a, \quad E_{a+1,a} = F_a \text{ and, for } |a-b| > 1,$$
$$E_{a,b} = \begin{cases} E_{a,c}E_{c,b} - \boldsymbol{v}_c E_{c,b} E_{a,c}, & \text{if } a > b; \\ E_{a,c}E_{c,b} - \boldsymbol{v}_c^{-1} E_{c,b} E_{a,c}, & \text{if } a < b. \end{cases}$$

Here c can be any number strictly between a and b. Note that the definition is independent of the choice of c. Observe that

$$\Upsilon(E_{a,b}) = E_{b,a}, \quad \varpi(E_{a,b}) = \pm(-q_a)^{f_{a,b}} E_{b,a} \text{ for some } f_{a,b} \in \mathbb{N}. \tag{2.6}$$

The following three sets of commutation formulas, for divided powers of root vectors $E_{a,b}^{(M)} := \frac{E_{a,b}^M}{[M]^!}$ ($a \ne b$, $M \ge 1$) in $U_{\boldsymbol{v},\mathcal{Z}}$, are given in [15]. They continue to hold in the specialisation to an arbitrary commutative ring R via $\boldsymbol{v} \mapsto q \in R$:

$$U_{q,R} = U_{q,R}(m|n) = U_{\boldsymbol{v},\mathcal{Z}} \otimes_{\mathcal{Z}} R. \tag{2.7}$$

Following [2, §3], we call $U_{q,R}$ the *quantum (super) hyperalgebra* associated with $U_{\boldsymbol{v}}(\mathfrak{gl}_{m|n})$, which is also denoted by $U_q^{\mathrm{res}}(\mathfrak{gl}_{m|n})$ in [5, §9.3]. For notational simplicity, we write $X = X\otimes 1$ for all $X \in U_{\boldsymbol{v},\mathbb{Z}}$. We also set $\varpi_R = \varpi\otimes \mathrm{id}_R$, Υ_R and σ_R to denote the corresponding automorphisms. For example, $\sigma_R : U_{q,R}(n|n) \longrightarrow U_{q,R}(n|n)$ satisfies

$$\left(E_i^{(M)}, F_i^{(M)}, K_j^{\pm 1}, \begin{bmatrix} K_j \\ t \end{bmatrix}\right) \longmapsto \left(F_{2n-i}^{(M)}, E_{2n-i}^{(M)}, K_{2n+1-j}^{\mp 1}, \begin{bmatrix} K_{2n+1-j}^{-1} \\ t \end{bmatrix}\right). \tag{2.8}$$

Note that applying ϖ_R, Υ_R to the commutation relations below may produce other commutation relations in $U_{q,R}$.

Proposition 2.2 *For any $1 \le a, b, c \le m+n$ with $b \ne c$, we have, in $U_{q,R}$,*

$$K_a E_{b,c} = \begin{cases} q_a^{\delta_{a,b}-\delta_{a,c}} E_{b,c} K_a, & \text{if } a = b \text{ or } c; \\ E_{b,c} K_a, & \text{if } a \ne b, c. \end{cases}$$

Proposition 2.3 ([15, (27)&Proposition 3.8.1]) *Let $E_{a,b}$ and $E_{c,d}$ be two root vectors with $a < b$ and $c < d$, and let $M, N \ge 1$. We then have the following commutation formulas in $U_{q,R}$.*

(0) $E_{a,b}^2 = 0$ *for all odd root* $\alpha = \epsilon_a - \epsilon_b$;

(1) If $b < c$ or $c < a < b < d$, then

$$E_{a,b}^{(M)} E_{c,d}^{(N)} = \begin{cases} (-1)^{\overline{E}_{a,b}\overline{E}_{c,d}} E_{c,d} E_{a,b}, & \text{if } M = N = 1; \\ E_{c,d}^{(N)} E_{a,b}^{(M)}, & \text{otherwise.} \end{cases}$$

(2) If $a = c < b < d$ or $a < c < b = d$, then

$$E_{a,b}^{(M)} E_{c,d}^{(N)} = \begin{cases} (-1)^{\overline{E}_{a,b}\overline{E}_{c,d}} q_b E_{c,d} E_{a,b}, & \text{if } M = N = 1, b = d; \\ (-1)^{\overline{E}_{a,b}\overline{E}_{c,d}} q_a E_{c,d} E_{a,b}, & \text{if } M = N = 1, a = c; \\ q_b^{MN} E_{c,d}^{(N)} E_{a,b}^{(M)}, & \text{otherwise.} \end{cases}$$

(3) If $a < b = c < d$, then

$$E_{a,b}^{(M)} E_{c,d}^{(N)} = \begin{cases} E_{a,d} + q_c^{-1} E_{c,d} E_{a,b}, & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} q_b^{-(N-t)(M-t)} E_{c,d}^{(N-t)} E_{a,d}^{(t)} E_{a,b}^{(M-t)}, & \text{otherwise.} \end{cases}$$

(4) If $a < c < b < d$, then

$$E_{a,b}^{(M)} E_{c,d}^{(N)} \\ = \begin{cases} (-1)^{\overline{E}_{a,b}\overline{E}_{c,d}} E_{c,d} E_{a,b} + (q_b - q_b^{-1}) E_{a,d} E_{c,b}, & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} q_b^{\frac{t(t-1)}{2}} (q_b - q_b^{-1})^t [t]_q! E_{c,b}^{(t)} E_{c,d}^{(N-t)} E_{a,b}^{(M-t)} E_{a,d}^{(t)}, & \text{otherwise.} \end{cases}$$

For $\alpha = \epsilon_i - \epsilon_j \in \Phi$, let $K_\alpha = K_{i,j} = K_i K_j^{-1}$ and define $\left[{K_{i,j};c \atop t}\right] = \left[{K_{i,j};c \atop t}\right]_i$ as in (2.4), replacing K_i there by $K_{i,j}$.

Proposition 2.4 ([15, (29)&Proposition 3.9.1]) *Let $E_{a,b}$ and $E_{d,c}$ be two root vectors with $a < b$ and $c < d$, and let $M, N \geq 1$. We then have the following commutation formulas in $U_{q,R}$.*

(1) *If $b \leq c$ or $c < a < b < d$, then*

$$E_{a,b}^{(M)} E_{d,c}^{(N)} = \begin{cases} (-1)^{\bar{E}_{a,b}\bar{E}_{d,c}} E_{d,c} E_{a,b}, & \text{if } M = N = 1; \\ E_{d,c}^{(N)} E_{a,b}^{(M)}, & \text{otherwise.} \end{cases}$$

(2) *If $a < c < b = d$*

$$E_{a,b}^{(M)} E_{d,c}^{(N)} = \begin{cases} (-1)^{\bar{E}_{a,b}\bar{E}_{d,c}} E_{d,c} E_{a,b} + K_{c,d} E_{a,c}, & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} q_b^{-t(N-t)} E_{d,c}^{(N-t)} K_{c,d}^t E_{a,b}^{(M-t)} E_{a,c}^{(t)}, & \text{otherwise.} \end{cases}$$

(3) *If $a = c < b < d$, then*

$$E_{a,b}^{(M)} E_{d,c}^{(N)} = \begin{cases} (-1)^{\bar{E}_{a,b}\bar{E}_{d,c}} E_{d,c} E_{a,b} - (-1)^{\bar{E}_{a,b}\bar{E}_{d,c}} K_{a,b} E_{d,b}, & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} (-1)^t q_b^{-t(M-1-t)} E_{d,b}^{(t)} E_{d,c}^{(N-t)} K_{a,b}^t E_{a,b}^{(M-t)}, & \text{otherwise.} \end{cases}$$

(4) *If $a < b$, then*

$$E_{a,b}^{(M)} E_{b,a}^{(N)} = \begin{cases} (-1)^{\bar{E}_{a,b}\bar{E}_{b,a}} E_{b,a} E_{a,b} + (q_a - q_a^{-1})^{-1}(K_{a,b} - K_{a,b}^{-1}), & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} E_{b,a}^{(N-t)} \left[{K_{a,b}; 2t - M - N \atop t}\right] E_{a,b}^{(M-t)}, & \text{otherwise.} \end{cases}$$

(5) *If $a < c < b < d$, then*

$$E_{a,b}^{(M)} E_{d,c}^{(N)} = \begin{cases} (-1)^{\bar{E}_{a,b}\bar{E}_{d,c}} E_{d,c} E_{a,b} - (q_b - q_b^{-1})^{-1} K_{c,b} E_{a,c} E_{d,b}, & \text{if } M = N = 1; \\ \sum_{t=0}^{\min(M,N)} (-1)^t q_b^{\frac{-t(2N-3t-1)}{2}} (q_b - q_b^{-1})^t [t]_q! E_{d,c}^{(N-t)} E_{d,b}^{(t)} K_{c,b}^t E_{a,b}^{(M-t)} E_{a,c}^{(t)}, & \text{o.w.} \end{cases}$$

The commutation formulas can easily be used to obtained the so-called PBW bases. Let

$$P(m|n) = \{A = (A_\alpha)_{\alpha \in \Phi} \mid A_\alpha \in \mathbb{N} \text{ if } \bar{\alpha} = \bar{0} \text{ and } A_\alpha \in \{0, 1\} \text{ if } \bar{\alpha} = \bar{1}\}. \quad (2.9)$$

For $A \in P(m|n)$ and any fixed ordering on Φ^+ and Φ^-, let

$$E_A = \prod_{\alpha\in\Phi^+} E_\alpha^{(A_\alpha)}, \ \ F_A = \prod_{\beta\in\Phi^-} E_\beta^{(A_\beta)}. \tag{2.10}$$

Then $U_{q,R}$ has an (integral) R-basis (see [15, §3.10])

$$\left\{ E_A \prod_{a=1}^{m+n} \left(K_a^{\sigma_a} \begin{bmatrix} K_a \\ \mu_a \end{bmatrix} \right) F_A \,\middle|\, A \in P(m|n), \sigma_a \in \{0,1\}, \mu \in \mathbb{N}^{m+n} \right\}. \tag{2.11}$$

We define analogously positive part, negative part and zero part as in the non-super case: $U^+_{q,R}, U^-_{q,R}, U^0_{q,R}$. Denote $U^{\geq 0}_{q,R} = U^+_{q,R}U^0_{q,R}$.

Remark 2.5 In [20, §2.3, Thm 4.5], Lusztig gave a presentation for the $\mathcal{Z}$-form $U_{\mathcal{Z}}$ of a quantum group associated with a symmetric Cartan matrix. It should not be hard to generalise this work to get a presentation for $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$ and for $U_{\boldsymbol{v},R}(m|n)$.

3 The q-Schur superalgebras $S_{q,R}(m|n,r)$

We first review the definition of q-Schur superalgebras in terms of an endomorphism algebra of a q-permutation module over the Hecke algebra $H_{q^2,R}$ associated with the symmetric group $\mathfrak{S}_r$ on r letters. Let $S = \{s_i = (i, i+1)\}$ be the generating set of basic transpositions.

The Hecke algebra $H_{q^2,R} = H_{q^2,R}(r)$ is the R-algebra with generators T_i, $1 \leq i \leq r-1$, which subject to the relations

$$T_iT_j = T_jT_i, |i-j| > 1; \quad T_iT_jT_i = T_jT_iT_j, |i-j| = 1; \quad T_i^2 = (q^2-1)T_i + q^2.$$

By setting $T_{s_i} = T_i$ and $T_w = T_{i_1} \cdots T_{i_l}$ if $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression, $H_{q^2,R}$ is a free R-module with basis $\{T_w \mid w \in \mathfrak{S}_r\}$ and the multiplication satisfies the rules: for $s \in S$,

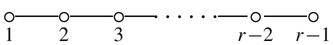
$$T_wT_s = \begin{cases} T_{ws}, & \text{if } \ell(ws) > \ell(w); \\ (q^2-1)T_w + q^2T_{ws}, & \text{if } \ell(ws) < \ell(w). \end{cases} \tag{3.1}$$

Since q^2 is invertible, it follows that T_i^{-1} exists and every basis element T_w is invertible.

The Hecke algebra $H_{q^2,R}$ admits the following R-algebra automorphism

$$(-)^\sharp : H_{q^2,R} \longrightarrow H_{q^2,R}, \quad T_i \longmapsto (q^2-1) - T_i. \tag{3.2}$$

Since the symmetric group $\mathfrak{S}_r$ is the Coxeter group associated with Coxeter graph



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the graph automorphism $(-)^\dagger$ sending i to $r-i$ induces a group automorphism and an R-algebra automorphism

$$\begin{aligned}(-)^\dagger &: \mathfrak{S}_r \longrightarrow \mathfrak{S}_r, \quad s_i \longmapsto s_{r-i};\\ (-)^\dagger &: H_{q^2,R} \longrightarrow H_{q^2,R}, \quad T_i \longmapsto T_{r-i}.\end{aligned} \tag{3.3}$$

For a composition λ of r, i.e. λ is an element of the set

$$\Lambda(N,r) = \left\{(\lambda_1,\ldots,\lambda_N) \in \mathbb{N}^N \mid \sum_{i=1}^N \lambda_i = r\right\}, \text{ for some } N,$$

let $\mathfrak{S}_\lambda$ be the associated parabolic (or standard Young) subgroup and let $\mathcal{D}_\lambda := \mathcal{D}_{\mathfrak{S}_\lambda}$ be the set of all shortest coset representatives of the right cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$. Let $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ be the set of the shortest $\mathfrak{S}_\lambda$-$\mathfrak{S}_\mu$ double coset representatives.

For $\lambda, \mu \in \Lambda(N,r)$ and $d \in \mathcal{D}_{\lambda\mu}$, the subgroup

$$\mathfrak{S}_{\lambda d\cap\mu} := \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$$

is a parabolic subgroup associated with the composition $\lambda d \cap \mu$ which can be easily described in terms of the following $N \times N$-matrix $A = (a_{i,j})$, where $a_{i,j} = |R_i^\lambda \cap d(R_j^\mu)|$: if $\nu^{(j)} = (a_{1,j}, a_{2,j}, \ldots, a_{N,j})$ denotes the jth column of A, then

$$\lambda d \cap \mu = (\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(N)}). \tag{3.4}$$

Putting $\jmath(\lambda, d, \mu) = \big(|R_i^\lambda \cap d(R_j^\mu)|\big)_{i,j}$, we obtain a bijection

$$\jmath : \bigcup_{\lambda,\mu\in\Lambda(N,r)} \{(\lambda, d, \mu) \mid d \in \mathcal{D}_{\lambda\mu}\} \longrightarrow \mathcal{M}(N,r), \tag{3.5}$$

where $\mathcal{M}(N,r)$ is the subset of the $N \times N$ matrix ring $M_N(\mathbb{N})$ over $\mathbb{N}$ consisting of matrices $A = (a_{i,j})$ whose entries sum to r, i.e. $|A| := \sum_{i,j} a_{i,j} = r$. Note that, if $\jmath(\lambda, d, \mu) = A$, then

$$\begin{aligned}\lambda = \mathrm{ro}(A) &:= \left(\sum_{j=1}^N a_{1,j}, \ldots, \sum_{j=1}^N a_{N,j}\right),\\ \mu = \mathrm{co}(A) &:= \left(\sum_{i=1}^N a_{i,1}, \ldots, \sum_{i=1}^N a_{i,N}\right).\end{aligned} \tag{3.6}$$

For $A = (a_{i,j}) \in \mathcal{M}(N,r)$, let $A^\dagger = (a^\dagger_{i,j})$, where $a^\dagger_{i,j} = a_{N-j+1,N-i+1}$. So $A^\dagger$ is obtained by two transposes along diagonal and anti-diagonal, respectively. We thus have a bijection

$$(-)^{\dagger}: \mathcal{M}(N,r) \longrightarrow \mathcal{M}(N,r), \quad A \longmapsto A^{\dagger},$$

and $\jmath(\lambda^{\dagger}, d^{\dagger}, \mu^{\dagger}) = A^{\dagger}$, where $\nu^{\dagger}$ denotes the composition obtained by reversing the sequences ν, i.e.

$$\nu^{\dagger} = (\nu_N, \ldots, \nu_2, \nu_1), \text{ if } \nu = (\nu_1, \nu_2, \ldots, \nu_N).$$

For the description of a superstructure, we consider two non-negative integers m, n. Thus, a composition λ of $m+n$ parts will be written as

$$\lambda = (\lambda^{(0)}|\lambda^{(1)}) = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_m^{(0)}|\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_n^{(1)})$$

to indicate the "even" and "odd" parts of λ. Let

$$\Lambda(m|n,r) := \Lambda(m+n,r) = \bigcup_{r_1+r_2=r} (\Lambda(m,r_1) \times \Lambda(n,r_2)),$$
$$\Lambda(m|n) := \bigcup_{r \geq 0} \Lambda(m|n,r) = \mathbb{N}^{m+n}.$$

For $\lambda = (\lambda^{(0)} \mid \lambda^{(1)}) \in \Lambda(m|n,r)$, we also write

$$\mathfrak{S}_{\lambda} = \mathfrak{S}_{\lambda^{(0)}} \mathfrak{S}_{\lambda^{(1)}} \cong \mathfrak{S}_{\lambda^{(0)}} \times \mathfrak{S}_{\lambda^{(1)}}, \tag{3.7}$$

where $\mathfrak{S}_{\lambda^{(0)}} \leq \mathfrak{S}_{\{1,2,\ldots,|\lambda^{(0)}|\}}$ and $\mathfrak{S}_{\lambda^{(1)}} \leq \mathfrak{S}_{\{|\lambda^{(0)}|+1,\ldots,r\}}$ are the even and odd parts of $\mathfrak{S}_{\lambda}$, respectively. Define

$$\lfloor xy \rfloor_{\lambda} := x_{\lambda^{(0)}} y_{\lambda^{(1)}}, \quad \lfloor yx \rfloor_{\lambda} = y_{\lambda^{(0)}} x_{\lambda^{(1)}}, \tag{3.8}$$

where $x_{\lambda^{(i)}} = \sum_{w \in \mathfrak{S}_{\lambda^{(i)}}} T_w, \quad y_{\lambda^{(i)}} = \sum_{w \in \mathfrak{S}_{\lambda^{(i)}}} (-q^2)^{-\ell(w)} T_w.$

The endomorphism algebra

$$S_{q,R} = S_{q,R}(m|n,r) := \mathrm{End}_{H_{q^2,R}(r)} \Big(\bigoplus_{\lambda \in \Lambda(m|n,r)} \lfloor xy \rfloor_{\lambda} H_{q^2,R}(r) \Big) \tag{3.9}$$

is called the *q-Schur superalgebra* of degree $(m|n,r)$.

By definition, for $\lambda = (\lambda^{(0)}, \lambda^{(1)})$, $\lambda^{\dagger} = (\lambda^{(1)\dagger}, \lambda^{(0)\dagger})$. Let $\lambda^{+} = (\lambda^{(0)\dagger}, \lambda^{(1)\dagger})$. Then

$$(\lfloor xy \rfloor_{\lambda})^{\dagger} = (x_{\lambda^{(0)}})^{\dagger} (y_{\lambda^{(1)}})^{\dagger} = y_{\lambda^{(1)\dagger}} x_{\lambda^{(0)\dagger}} = \lfloor yx \rfloor_{\lambda^{\dagger}}.$$

Since $\mathfrak{S}_{\lambda^{\dagger}}$ and $\mathfrak{S}_{\lambda^{+}}$ are conjugate, there exists $d \in \mathfrak{S}_r$ such that $\lfloor yx \rfloor_{\lambda^{\dagger}} T_d = T_d \lfloor xy \rfloor_{\lambda^{+}}$. Hence, $\lfloor yx \rfloor_{\lambda^{\dagger}} H_{q^2,R} = T_d \lfloor xy \rfloor_{\lambda^{+}} H_{q^2,R} \cong \lfloor xy \rfloor_{\lambda^{+}} H_{q^2,R}$. Now, we see the following easily.

Lemma 3.1 *For $m = n$, we may identify $S_{q,R}(n|n, r)$ with the endomorphism algebra* $\mathrm{End}_{H_{q^2,R}}\Big(\bigoplus_{\lambda\in\Lambda(n|n,r)} {}_{[}yx_{]_\lambda} H_{q^2,R}(r)\Big)$. *In particular, the isomorphism $(\ )^\dagger$ in (3.3) induces an isomorphism of right $H_{q^2,R}$-modules*

$$f:\bigoplus_{\lambda\in\Lambda(n|m,r)} {}_{[}xy_{]_\lambda} H_{q^2,R}\longrightarrow \bigoplus_{\lambda\in\Lambda(m|n,r)} {}_{[}yx_{]_{\lambda^\dagger}} H_{q^2,R},\quad m\longmapsto m^\dagger,$$

which further results in an R-algebra automorphism

$$(\ )^\dagger: S_{q,R}(n|n,r)\longmapsto S_{q,R}(n|n,r), \phi\longmapsto f\phi f^{-1}. \tag{3.10}$$

Following [14, (5.3.2)], for $\lambda,\mu\in\Lambda(m|n,r)$, define

$$\mathcal{D}^\circ_{\lambda\mu}=\{d\in\mathcal{D}_{\lambda\mu}\mid \mathfrak{S}^d_{\lambda^{(i)}}\cap\mathfrak{S}_{\mu^{(j)}}=1\ \forall \bar{i}+\bar{j}=1\}.$$

Then all $\jmath(\lambda,d,\mu)$ with $\lambda,\mu\in\Lambda(m|n,r)$, $d\in\mathcal{D}^\circ_{\lambda\mu}$ form the matrix set

$$\begin{aligned}\mathcal{M}(m|n,r)&=\{A=(a_{ij})\in M_{m+n}(\mathbb{N})\mid a_{i,j}\in\{0,1\}\ \forall\bar{i}+\bar{j}=1, |A|=r\},\\ \mathcal{M}(m|n)&=\bigcup_{r\geq 0}\mathcal{M}(m|n,r).\end{aligned} \tag{3.11}$$

We may interpret an element $(A_\alpha)_{\alpha\in\Phi}\in P(m|n)$ in (2.9) as a matrix $A=(A_{i,j})\in\mathcal{M}(m|n)$, where $A_{i,j}=A_\alpha$ if $\alpha=\epsilon_i-\epsilon_j$ and $A_{i,i}=0$ for all i.

For $A=\jmath(\lambda,d,\mu)$, putting

$$T_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu}:=\sum_{\substack{w_0\in\mathfrak{S}_{\mu^{(0)}},w_1\in\mathfrak{S}_{\mu^{(1)}}\\ w_0w_1\in\mathfrak{S}_\mu\cap\mathcal{D}_{\lambda d\cap\mu}}}(-\boldsymbol{v}^2)^{-\ell(w_1)}x_{\lambda^{(0)}}y_{\lambda^{(1)}}T_dT_{w_0}T_{w_1},$$

there exists an $\mathcal{H}$-homomorphism $\phi_A:=\phi^d_{\lambda\mu}$ defined by

$$\phi^d_{\lambda\mu}(x_{\alpha^{(0)}}y_{\alpha^{(1)}}h)=\delta_{\mu,\alpha}T_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu}h, \forall\alpha\in\Lambda(m|n,r), h\in\mathcal{H}.$$

Let $d^{(0)}$ (resp. $d^{(1)}$) to be the longest element in the double coset $\mathfrak{S}_{\lambda^{(0)}}d\mathfrak{S}_{\mu^{(0)}}$(resp. $\mathfrak{S}_{\lambda^{(1)}}d\mathfrak{S}_{\mu^{(1)}}$). Following [14, (6.0.2)], let $\mathcal{T}_A=\boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)}T_{\mathfrak{S}_\lambda d\mathfrak{S}_\mu}$. Then

$$[A]=\boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)+l(w_{0,\mu^{(0)}})-l(w_{0,\mu^{(1)}})}\phi_A,$$

where $w_{0,\lambda}$ denotes the longest element in $\mathfrak{S}_\lambda$, is the map $\mathcal{T}_{\mathfrak{S}_\mu}\mapsto\mathcal{T}_A$. The first assertion of the following result is given in [14, 5.8].

Lemma 3.2 *The set $\{[A]\mid A\in\mathcal{M}(m|n,r)\}$ forms a R-basis for $S_{q,R}(m|n,r)$. Moreover, for $m=n$, we have $[A]^\dagger=[A^\dagger]$.*

Proof It suffices to prove the last statement for $R = \mathcal{Z}$. Let $A = \jmath(\lambda, d, \mu)$. We have

$$\begin{aligned}[A]^\dagger(\mathcal{T}_{\mathfrak{S}_{\mu^\dagger}}) &= \boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)}(\phi_A([xy]_\mu))^\dagger = \boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)}(T_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu})^\dagger \\ &= \frac{\boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)}}{P_\nu(\boldsymbol{v}^2)}([xy]_\lambda T_d [xy]_\mu)^\dagger \quad (\nu = \lambda d \cap \mu) \\ &= \frac{\boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)}}{P_{\nu^\dagger}(\boldsymbol{v}^2)}([yx]_{\lambda^\dagger} T_{d^\dagger} [yx]_{\mu^\dagger}) \\ &= \boldsymbol{v}^{-l(d^{(0)})+l(d^{(1)})-l(d)} T_{\mathfrak{S}_{\lambda^\dagger} d^\dagger \mathfrak{S}_{\mu^\dagger}} = \mathcal{T}_{\mathfrak{S}_{\lambda^\dagger} d^\dagger \mathfrak{S}_{\mu^\dagger}},\end{aligned}$$

where $P_\nu(\boldsymbol{v}^2) = \sum_{w_0 \in \mathfrak{S}_{\nu^{(0)}}, w_1 \in \mathfrak{S}_{\nu^{(1)}}} \boldsymbol{v}^{2l(w_0)}(\boldsymbol{v}^{-1})^{2l(w_1)} = P_{\nu^\dagger}(\boldsymbol{v}^2)$ and the last equality is seen from the fact that $\ell(d^{(i)}) = \ell(d^{(i)\dagger})$ for $i = 0, 1$. □

El Turkey and Kujawa ([15, Thm 3.3.1]) gave a presentation of the $\boldsymbol{v}$-Schur superalgebra $S_{\boldsymbol{v}}(m|n, r)$ over $\mathbb{Q}(\boldsymbol{v})$. They proved that $S_{\boldsymbol{v}}(m|n, r)$ is generated by the similar generators and defining relations for $U_{\boldsymbol{v}}(m|n)$ over $\mathbb{Q}(\boldsymbol{v})$ along with relations:

$$K_1 \ldots K_m K_{m+1}^{-1} \ldots K_{m+n}^{-1} - \boldsymbol{v}^r = 0, \quad (K_a - 1)(K_a - \boldsymbol{v}_a) \ldots (K_a - \boldsymbol{v}_a^r) = 0. \tag{3.12}$$

Thus, if I_r denotes the ideal of $U_{\boldsymbol{v}}(m|n)$ generated by $K_1 \ldots K_m K_{m+1}^{-1} \ldots K_{m+n}^{-1} - \boldsymbol{v}^r$ and $(K_a - 1)(K_a - \boldsymbol{v}_a) \ldots (K_a - \boldsymbol{v}_a^r)$, $1 \le a \le m+n$, then $U_{\boldsymbol{v}}(m|n)/I_r \cong S_{\boldsymbol{v}}(m|n, r)$. So we have an algebra epimorphism (see [15, (20)] or [10, Cor. 6.4]):

$$\eta_r : U_{\boldsymbol{v}}(m|n) \longrightarrow S_{\boldsymbol{v}}(m|n, r). \tag{3.13}$$

In particular, $S_{\boldsymbol{v}}(m|n, r)$ has generators:

$$\mathtt{e}_a = \eta_r(E_a), \ \mathtt{f}_a = \eta_r(F_a), \ \mathtt{k}_j^{\pm 1} = \eta_r(K_j^{\pm 1}).$$

Put $\mathtt{e}_a^{(M)} = \eta_r(E_a^{(M)})$, $\left[\begin{smallmatrix}\mathtt{k}_j\\ t\end{smallmatrix}\right] = \eta_r(\left[\begin{smallmatrix}K_j\\ t\end{smallmatrix}\right])$, etc., and let $S_{\boldsymbol{v},\mathcal{Z}} = S_{\boldsymbol{v},\mathcal{Z}}(m|n, r)$ be the $\mathcal{Z}$-subalgebra of $S_{\boldsymbol{v}}(m|n, r)$ generated by

$$\left\{ \mathtt{e}_a^{(M)}, \mathtt{f}_a^{(M)}, \mathtt{k}_j^{\pm 1}, \begin{bmatrix}\mathtt{k}_j \\ t\end{bmatrix} \,\middle|\, t, M \in \mathbb{N}, 1 \le a < m+n, \ 1 \le j \le m+n \right\}. \tag{3.14}$$

Then $S_{\boldsymbol{v},\mathcal{Z}}$ has a $\mathcal{Z}$-basis of (see [15, Thm 3.12.1])

$$\mathbf{Y} = \bigcup \{\mathtt{e}_A 1_\lambda \mathtt{f}_A \mid A \in P(m|n), \lambda \in \Lambda(m|n, r), \chi(E_A F_A) \preceq \lambda\}, \tag{3.15}$$

where χ is the content function defined in [15, 3.11][2] and $\mathtt{e}_A, \mathtt{f}_A$ are images of the elements E_A, F_A defined in (2.10). (Here $\mu \preceq \lambda$ mean $\mu_i \le \lambda_i$ for all i.)

[2] If we identify A with a matrix $(m_{i,j})$, then the hth component $\chi(E_A F_A)_h = \sum_{i<h}(m_{i,h} + m_{h,i})$.

For any commutative ring R and any invertible element $q \in R$, base change via the specialisation $\mathcal{Z} \to R$, $\boldsymbol{v} \mapsto q$ results in R-algebra

$$S_{q,R} = S_{q,R}(m|n,r) \cong S_{\boldsymbol{v},\mathcal{Z}}(m|n,r) \otimes_{\mathcal{Z}} R.$$

Moreover, by restriction and specialisation, the map η_r in (3.13) induces an R-algebra epimorphism (see [10, Cor. 8.4]):

$$\eta_{r,R} := \eta_r \otimes 1 : U_{q,R}(m|n) \longrightarrow S_{q,R}(m|n,r). \tag{3.16}$$

Like in Sect. 2, we will also abuse X as $X \otimes 1$ for simplicity. Thus, $S_{q,R}(m|n,r)$ is generated by the elements in (3.14).

4 Presenting $S_{q,R}(m|n,r)$ over a commutative ring R

For any $\mu \in \Lambda(m|n)$, let $\left[{K \atop \mu}\right] = \prod_{a=1}^{m+n}\left[{K_a \atop \mu_a}\right]$. Let $J_r = J_{r,R}$ be the ideal of $U_{q,R} = U_{q,R}(m|n)$ generated by

$$1 - \sum_{\lambda\in\Lambda(m|n,r)}\begin{bmatrix}K\\ \lambda\end{bmatrix},\quad K_a^{\pm1}\begin{bmatrix}K\\ \lambda\end{bmatrix} - q_a^{\pm\lambda_a}\begin{bmatrix}K\\ \lambda\end{bmatrix},\quad \begin{bmatrix}K_a;c\\ t\end{bmatrix}\begin{bmatrix}K\\ \lambda\end{bmatrix} - \begin{bmatrix}\lambda_a+c\\ t\end{bmatrix}_q\begin{bmatrix}K\\ \lambda\end{bmatrix}, \tag{4.1}$$

where $1 \le a \le m+n$, $t \in \mathbb{N}$, $c \in \mathbb{Z}$, $\lambda \in \Lambda(m|n,r)$. Let $\pi_{r,R} : U_{q,R} \to \overline{U}_{q,R} := U_{q,R}/J_r$ be the natural homomorphism and put

$$\mathrm{E}_{a,b}^{(M)} = \pi_{r,R}\left(E_{a,b}^{(M)}\right),\ \mathrm{K}_a^{\pm1} = \pi_{r,R}\left(K_a^{\pm1}\right),$$
$$\begin{bmatrix}\mathrm{K}_a\\ t\end{bmatrix} = \pi_{r,R}\left(\begin{bmatrix}K_a\\ t\end{bmatrix}\right),\ \begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix} = \pi_{r,R}\left(\begin{bmatrix}K\\ \lambda\end{bmatrix}\right).$$

Lemma 4.1 *For any $\lambda \in \Lambda(m|n,r)$, let $1_\lambda := \left[{\mathrm{K} \atop \lambda}\right]$. Then the following hold in $\overline{U}_{q,R}$:*

(1) $\sum_{\lambda\in\Lambda(m|n,r)} 1_\lambda = 1$;

(2) $\mathrm{K}_a^{\pm1}1_\lambda = q_a^{\pm\lambda_a}1_\lambda$, $\left[{\mathrm{K}_a;c \atop t}\right]1_\lambda = \left[{\lambda_a+c \atop t}\right]_q 1_\lambda$, *for all* $1 \le a \le m+n,\ t \in \mathbb{N}, c \in \mathbb{Z}$.

(3) $\mathrm{K}_1 \dots \mathrm{K}_m \mathrm{K}_{m+1}^{-1} \dots \mathrm{K}_{m+n}^{-1} = q^r$.

(4) $(\mathrm{K}_a - 1)(\mathrm{K}_a - q_a) \dots (\mathrm{K}_a - q_a^r) = 0,\ 1 \le a \le m+n$.

(5) $\left[{\mathrm{K} \atop \mu}\right]1_\lambda = \left[{\lambda \atop \mu}\right]_q 1_\lambda$ *for all* $\mu \in \Lambda(m|n)$, *where* $\left[{\lambda \atop \mu}\right]_q = \prod_{a=1}^{m+n}\left[{\lambda_a \atop \mu_a}\right]_q$. *Hence,* $1_\lambda 1_\mu = \delta_{\mu,\lambda}1_\lambda$. *Moreover,* $\left[{\mathrm{K} \atop \mu}\right] = \sum_{\lambda\in\Lambda(m|n,r)}\left[{\lambda \atop \mu}\right]_q\left[{\mathrm{K} \atop \lambda}\right]$ *and* $\left[{\mathrm{K} \atop \mu}\right] = 0$, *if* $|\mu| > r$.

(6) For each $\alpha \in \Phi$, $\mathrm{E}_\alpha^{(M)}1_\lambda = \begin{cases} 1_{\lambda+M\alpha}\mathrm{E}_\alpha^{(M)}, & \text{if } \lambda + M\alpha \in \Lambda(m|n,r),\\ 0, & \text{otherwise,}\end{cases}$ *and* $1_\lambda E_\alpha^{(M)} = 0$ *if* $\lambda - M\alpha \notin \Lambda(m|n,r)$.

Proof The relations (1) and (2) are clear from the definition, while (3) and (4) follow from (1) and (2) as

$$\mathrm{K}_1\dots\mathrm{K}_m\mathrm{K}_{m+1}^{-1}\dots\mathrm{K}_{m+n}^{-1}=\sum_{\lambda\in\Lambda(m|n,r)}\mathrm{K}_1\dots\mathrm{K}_m\mathrm{K}_{m+1}^{-1}\dots\mathrm{K}_{m+n}^{-1}\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}$$

$$=\sum_{\lambda\in\Lambda(m|n,r)}q_1^{\lambda_1}\dots q_m^{\lambda_m}q_{m+1}^{-\lambda_{m+1}}\dots q_{m+n}^{-\lambda_{m+n}}\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=q^r\sum_{\lambda\in\Lambda(m|n,r)}\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=q^r,$$

$$(\mathrm{K}_a-1)(\mathrm{K}_a-q_a)\dots(\mathrm{K}_a-q_a^r)=\sum_{\lambda\in\Lambda(m|n,r)}(\mathrm{K}_a-1)(\mathrm{K}_a-q_a)\dots(\mathrm{K}_a-q_a^r)\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}$$

$$=\sum_{\lambda\in\Lambda(m|n,r)}(q_a^{\lambda_a}-1)(q_a^{\lambda_a}-q_a)\dots(q_a^{\lambda_a}-q_a^r)\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=0.$$

Similarly, (5) is seen as follows:

$$\begin{bmatrix}\mathrm{K}\\ \mu\end{bmatrix}\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=\prod_{a=1}^{m+n}\begin{bmatrix}\mathrm{K}_a\\ \mu_a\end{bmatrix}\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=\prod_{a=1}^{m+n}\begin{bmatrix}\lambda_a\\ \mu_a\end{bmatrix}_q\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix}=\begin{bmatrix}\lambda\\ \mu\end{bmatrix}_q\begin{bmatrix}\mathrm{K}\\ \lambda\end{bmatrix},$$

and $\left[\begin{smallmatrix}\mathrm{K}\\ \mu\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathrm{K}\\ \mu\end{smallmatrix}\right](\sum_{\lambda\in\Lambda(m|n,r)}\left[\begin{smallmatrix}\mathrm{K}\\ \lambda\end{smallmatrix}\right])=\sum_{\lambda\in\Lambda(m|n,r)}\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]_q\left[\begin{smallmatrix}\mathrm{K}\\ \lambda\end{smallmatrix}\right]$, which is 0 if $|\mu|>r$, as in this case there is i such that $\lambda_i<\mu_i$ and so $\left[\begin{smallmatrix}\lambda_i\\ \mu_i\end{smallmatrix}\right]_q=0$. Consequently, $1_\mu 1_\lambda=\delta_{\mu,\lambda}1_\lambda$ if $|\mu|=|\lambda|=r$.

It remains to prove (6). Recall the relations in $U_{\boldsymbol{v},\mathcal{Z}}$: $K_bE_{b,c}=\boldsymbol{v}_bE_{b,c}K_b$, $K_cE_{b,c}=\boldsymbol{v}_c^{-1}E_{b,c}K_c$ and

$$E_{b,c}\begin{bmatrix}K_b\\ \lambda_b\end{bmatrix}=\begin{bmatrix}K_b;-1\\ \lambda_b\end{bmatrix}E_{b,c};\quad E_{b,c}\begin{bmatrix}K_c\\ \lambda_c\end{bmatrix}=\begin{bmatrix}K_c;1\\ \lambda_c\end{bmatrix}E_{b,c},$$

(see, Proposition 2.2 or [15, p.306]). Thus, by induction, we have, for $M>0$,

$$E_{b,c}^{(M)}\begin{bmatrix}K_b\\ \lambda_b\end{bmatrix}=\begin{bmatrix}K_b;-M\\ \lambda_b\end{bmatrix}E_{b,c}^{(M)};\quad E_{b,c}^{(M)}\begin{bmatrix}K_c\\ \lambda_c\end{bmatrix}=\begin{bmatrix}K_c;M\\ \lambda_c\end{bmatrix}E_{b,c}^{(M)}.\tag{4.2}$$

Hence, for $\lambda\in\Lambda(m|n,r)$, $b\neq c$,

$$E_{b,c}^{(M)}\begin{bmatrix}K\\ \lambda\end{bmatrix}=E_{b,c}^{(M)}\prod_{a=1}^{m+n}\begin{bmatrix}K_a\\ \lambda_a\end{bmatrix}=\begin{bmatrix}K_b;-M\\ \lambda_b\end{bmatrix}\begin{bmatrix}K_c;M\\ \lambda_c\end{bmatrix}\prod_{a\neq b,c}\begin{bmatrix}K_a\\ \lambda_a\end{bmatrix}E_{b,c}^{(M)}.$$

Multiplying both sides on the left by $\left[\begin{smallmatrix}K_b;0\\ \lambda_b+M\end{smallmatrix}\right]$ and applying (4.2) yield in $U_{\boldsymbol{v},\mathcal{Z}}$:

$$E_{b,c}^{(M)}\begin{bmatrix}K_b;M\\ \lambda_b+M\end{bmatrix}\begin{bmatrix}K\\ \lambda\end{bmatrix}=\begin{bmatrix}K_b;0\\ \lambda_b+M\end{bmatrix}\begin{bmatrix}K_b;-M\\ \lambda_b\end{bmatrix}\begin{bmatrix}K_c;M\\ \lambda_c\end{bmatrix}\prod_{a\neq b,c}\begin{bmatrix}K_a\\ \lambda_a\end{bmatrix}E_{b,c}^{(M)}.$$

We now compute the images of both sides in the quotient algebra $\overline{U}_{q,R}$:

$$\begin{aligned}\text{LHS} &= \mathrm{E}_{b,c}^{(M)}\begin{bmatrix}\mathrm{K}_b;\, M\\ \lambda_b+M\end{bmatrix}1_\lambda\\ &= \mathrm{E}_{b,c}^{(M)}\begin{bmatrix}\lambda_b+M\\ \lambda_b+M\end{bmatrix}_q 1_\lambda = \mathrm{E}_{b,c}^{(M)}1_\lambda\end{aligned}$$

by (2), and

$$\begin{aligned}\text{RHS} &= \begin{bmatrix}\mathrm{K}_b;\, 0\\ \lambda_b+M\end{bmatrix}\begin{bmatrix}\mathrm{K}_b;\, -M\\ \lambda_b\end{bmatrix}\begin{bmatrix}\mathrm{K}_c;\, M\\ \lambda_c\end{bmatrix}\prod_{a\neq b,c}\begin{bmatrix}\mathrm{K}_a\\ \lambda_a\end{bmatrix}\mathrm{E}_{b,c}^{(M)}\\ &= \sum_{\mu\in\Lambda(m|n,r)}\left(\begin{bmatrix}\mathrm{K}_b;\, 0\\ \lambda_b+M\end{bmatrix}\begin{bmatrix}\mathrm{K}_b;\, -M\\ \lambda_b\end{bmatrix}\begin{bmatrix}\mathrm{K}_c;\, M\\ \lambda_c\end{bmatrix}\prod_{a\neq b,c}\begin{bmatrix}\mathrm{K}_a\\ \lambda_a\end{bmatrix}1_\mu\right)\mathrm{E}_{b,c}^{(M)}\\ &= \sum_{\mu\in\Lambda(m|n,r)}\left(\begin{bmatrix}\mu_b\\ \lambda_b+M\end{bmatrix}_q\begin{bmatrix}\mu_b-M\\ \lambda_b\end{bmatrix}_q\begin{bmatrix}\mu_c+M\\ \lambda_c\end{bmatrix}_q\prod_{a\neq b,c}\begin{bmatrix}\mu_a\\ \lambda_a\end{bmatrix}_q 1_\mu\right)\mathrm{E}_{b,c}^{(M)}.\end{aligned}$$

Since, for $\lambda, \mu \in \Lambda(m|n, r)$,

$$\begin{aligned}&\begin{bmatrix}\mu_b\\ \lambda_b+M\end{bmatrix}_q\begin{bmatrix}\mu_b-M\\ \lambda_b\end{bmatrix}_q\begin{bmatrix}\mu_c+M\\ \lambda_c\end{bmatrix}_q\prod_{a\neq b,c}\begin{bmatrix}\mu_a\\ \lambda_a\end{bmatrix}_q\neq 0\\ &\iff \mu_b\geq\lambda_b+M,\ \mu_c+M\geq\lambda_c,\ \mu_a\geq\lambda_a,\quad \text{for } a\neq b,c\\ &\iff \mu=\lambda+M\alpha,\end{aligned}$$

it follows that

$$\text{RHS} = \begin{cases}1_{\lambda+M\alpha}\mathrm{E}_{b,c}^{(M)}, & \text{if } \lambda+M\alpha\in\Lambda(m|n,r),\\ 0, & \text{otherwise}.\end{cases}$$

as desired. The other case can be done similarly. □

Remark 4.2 The proof above is a modification of that of [15, Proposition 3.7.1]. It works now over an arbitrary commutative ring and parameter $q \in R$.

We are now ready to give a presentation for $S_{q,R}(m|n,r)$; compare the presentation over $\mathbb{Q}(\boldsymbol{v})$ in [15]. Recall the map $\eta_{r,R}$ in (3.16) and Remark 2.5 for a presentation of $U_{q,R}(m|n)$.

Theorem 4.3 *For any commutative ring R, the kernel of $\eta_{r,R}$ is the ideal $J_{r,R}$ generated by the elements in* (4.1)*. In particular, the q-Schur superalgebra $S_{q,R}(m|n,r)$ can be presented by the generators as given in* (3.14) *and relations for $U_{q,R}(m|n)$ together with* (1)–(2) *in Lemma 4.1.*

Proof Recall the ideal I_r of $U_{\boldsymbol{v}}(m|n)$ (over $\mathbb{Q}(\boldsymbol{v})$) generated by the elements in (3.12). Let $J_{r,\mathcal{Z}}$ be the ideal of $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$ when $R=\mathcal{Z}$. Base change to $\mathbb{Q}(\boldsymbol{v})$ gives an ideal $J_{r,\mathbb{Q}(\boldsymbol{v})}$ of $U_{\boldsymbol{v}}(m|n)$. Lemma 4.1 (3)&(4) shows that $I_r \subseteq J_{r,\mathbb{Q}(\boldsymbol{v})}$. On the other hand, by [15, Propositions 3.6.1–2] these elements in (4.1), when regarded as elements in $U_{\boldsymbol{v},\mathcal{Z}}$, are all in I_r. Hence, $I_r = J_{r,\mathbb{Q}(\boldsymbol{v})}$. In particular, $J_{r,\mathcal{Z}} \subseteq I_r \cap U_{\boldsymbol{v},\mathcal{Z}}$. Thus, there is an algebra epimorphism $\bar{\pi} : \overline{U}_{\boldsymbol{v},\mathcal{Z}} = U_{\boldsymbol{v},\mathcal{Z}}(m|n)/J_{r,\mathcal{Z}} \to U_{\boldsymbol{v},\mathcal{Z}}/(I_r \cap U_{\boldsymbol{v},\mathcal{Z}})$ with $\bar{\pi}(\mathtt{E}_i) = \mathtt{e}_i$ etc.. The latter is isomorphic to the image of $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$ in $\overline{U}_{\boldsymbol{v},\mathbb{Q}(\boldsymbol{v})} = U_{\boldsymbol{v}}(m|n)/I_r$.

Now the proof of in [15, Theorem 3.12.1], especially that given in [9, Proposition 9.1], shows that the set $\mathbf{Y}$ in (3.15) forms a spanning set for $U_{\boldsymbol{v},\mathcal{Z}}/(I_r \cap U_{\boldsymbol{v},\mathcal{Z}})$. Similarly, by replacing $\mathtt{e}_i$, $\mathtt{f}_i$ etc. by $\mathtt{E}_i$, $\mathtt{F}_i$ etc., one constructs by Lemma 4.1 a spanning set $\widetilde{\mathbf{Y}}$ for $\overline{U}_{\boldsymbol{v},\mathcal{Z}}$. Since $\overline{U}_{\boldsymbol{v},\mathcal{Z}} \otimes R \cong \overline{U}_{\boldsymbol{v},R}$, it follows that $\widetilde{\mathbf{Y}}_R$ spans $\overline{U}_{q,R}$.

On the other hand, since the elements in (4.1) are all in the kernel of $\eta_{r,\mathcal{Z}}$, it follows that $J_{r,R} \subseteq \ker \eta_{r,R}$, where $\eta_{r,R}$ is the epimorphism given in (3.16). Consequently, $\eta_{r,R}$ induces an epimorphism $\bar{\eta}_{r,R} : \overline{U}_{q,R} = U_{q,R}/J_{r,R} \to S_{q,R}$. Hence, the image $\bar{\eta}_{r,R}(\widetilde{\mathbf{Y}})$ spans $S_{q,R}$. Since $S_{q,R}$ is R-free of rank $|\widetilde{\mathbf{Y}}|$, the transition matrix from $\bar{\eta}_{r,R}(\widetilde{\mathbf{Y}}_R)$ to a basis for $S_{q,R}$ must be invertible. This forces $\bar{\eta}_{r,R}(\widetilde{\mathbf{Y}}_R)$ is linearly independent. Therefore, $\bar{\eta}_{r,R}$ must be an isomorphism. □

Remark 4.4 Both proofs in [15] and [9] for the fact that $\mathbf{Y}$ spans $U_{\boldsymbol{v},\mathcal{Z}}/I_r \cap U_{\boldsymbol{v},\mathcal{Z}}$ and $\widetilde{\mathbf{Y}}$ spans $\overline{U}_{\boldsymbol{v},\mathcal{Z}}$ use a PBW-type basis involving all root vectors. Thus, almost all the commutation formulas in Propositions 2.3 and 2.4 have to be used in a lengthy case-by-case argument. However, if we use a monomial basis in the divided powers of generators as given in [13], the number of cases can be reduced significantly and a complete proof can be seen easily.

The following result is a super version of [13, Thm 9.3].

Corollary 4.5 *Assume $m, n \geq r$ and let $\omega \in \Lambda(m|n, r)$ be of the form*

$$\omega = (0^a, 1^r, 0^{m-a-r}|\boldsymbol{0}) \text{ or } \omega = (\boldsymbol{0}|0^a, 1^r, 0^{n-a-r}).$$

Then the elements $\mathtt{C}_i = 1_\omega \mathtt{e}_i \mathtt{f}_i 1_\omega$ in $S_{q,R}(m|n,r)$ for $\omega_i = 1$ satisfy the relations

$$\mathtt{C}_i^2 = (q^{-1}+q)\mathtt{C}_i, \quad \mathtt{C}_i\mathtt{C}_j = \mathtt{C}_j\mathtt{C}_i (|i-j|>1),$$
$$\mathtt{C}_i\mathtt{C}_{i+1}\mathtt{C}_i - \mathtt{C}_i = \mathtt{C}_{i+1}\mathtt{C}_i\mathtt{C}_{i+1} - \mathtt{C}_{i+1}.$$

In particular, there is an R-algebra isomorphism $1_\omega S_{q,R}(m|n,r)1_\omega \cong H_{q^2,R}(r)$.

Proof We may simply modify the proof of [13, Thm 9.3] to prove these relations. To see the last assertion, let $t_i = q\mathtt{C}_i - 1_\omega$ for all i with $\omega_i = 1$. Then $\{t_i \mid i \in [1, m+n], \omega_i = 1\}$ generate a subalgebra isomorphic to $H_{q^2,R}(r)$ under the map $t_{a+i} \mapsto T_i$ (cf. the proof for [13, Thm 9.3]).[3] □

[3] If we put $t_i' = q^{-1}\mathtt{C}_i - 1$, then the relations for $\mathtt{C}_i$ are equivalent to $(t_i')^2 = (q^{-2}-1)t_i' + q^{-2}$, $t_i't_{i'}' = t_{i'}'t_i'$, $t_j't_{j'}'t_j' = t_{j'}'t_j't_{j'}'$, where $|i-i'|>1$ and $|j-j'|=1$. Then t_i' generate a subalgebra isomorphic to $H_{q^{-2},R}(r)$.

Lemma 4.6 *The isomorphism σ considered in (2.3) and (2.5) induces an R-algebra isomorphism $\sigma_R : U_{q,R}(n|n) \longrightarrow U_{q,R}(n|n)$, which further induces an algebra automorphism, by abuse of notation,*

$$\sigma_R : S_{q,R}(n|n,r) \longrightarrow S_{q,R}(n|n,r).$$

Moreover, if $m = n \geq r$ and $\omega = (1^r, 0^{n-r}|0^n)$, $\omega' = (0^n|0^{n-r}, 1^r) \in \Lambda(n|n,r)$, then the automorphism σ restricts to an R-algebra isomorphism

$$\bar{\sigma} : 1_\omega S_{q,R}(n|n,r)1_\omega \longrightarrow 1_{\omega'} S_{q,R}(n|n,r)1_{\omega'}, \quad t_i \mapsto t_{2n-i} \quad (1 \leq i \leq r-1). \tag{4.3}$$

Proof The first automorphism is induced from (2.5), while the second is clear since $\sigma(\ker \eta_{r,R}) = \ker \eta_{r,R}$ by Theorem 4.3. The last assertion follows easily from the fact that $\sigma(1_\omega) = 1_{\omega'}$ and $1_\omega \mathtt{e}_i \mathtt{f}_i 1_\omega = 1_\omega \mathtt{f}_i \mathtt{e}_i 1_\omega$ since $\frac{\mathtt{k}_i \mathtt{k}_{i+1}^{-1} - \mathtt{k}_i^{-1}\mathtt{k}_{i+1}}{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}} 1_\omega = 0$. □

This result shows that the automorphism given in (3.10) agrees with the automorphism σ above.

5 Finite-dimensional weight supermodules of $U_{q,F}(m|n)$

From now on, we will assume $R = F$ is a field of characteristic $\neq 2$ and $q \in F$ is an l'th primitive root of unity with $l' \geq 3$. By setting $l' = \infty$, we may also include the case where q is not a root of unity. We first describe a classification of the irreducible weight supermodules of $U_{q,F} = U_{q,F}(m|n)$ by their highest weights. We then use the result in the previous section to give a criterion for polynomial weight $U_{q,F}$-supermodules.

For a $U_{q,F}$-supermodule V and $\lambda \in \mathbb{Z}^{m+n}$, define its (nonzero) λ-weight space (of type **1**) by

$$V_\lambda = \left\{ v \in V \mid K_a.v = q_a^{\lambda_a} v,\ \begin{bmatrix} K_a; c \\ t \end{bmatrix}.v = \begin{bmatrix} \lambda_a + c \\ t \end{bmatrix}_q v,\ \forall t \in \mathbb{N}, c \in \mathbb{Z} \right\}. \tag{5.1}$$

Note that, by using $\left[\begin{smallmatrix} K_b^{-1}; c \\ t \end{smallmatrix}\right] = (-1)^t \left[\begin{smallmatrix} K_b; -(c+1)+t \\ t \end{smallmatrix}\right]$ and [8, Lemma 14.18], we deduce from (5.1) that, for $v \in V_\lambda$,

$$\begin{bmatrix} K_a^{-1}; c \\ t \end{bmatrix}.v = \begin{bmatrix} -\lambda_a + c \\ t \end{bmatrix}_q v, \quad \begin{bmatrix} K_{a,b}; c \\ t \end{bmatrix}.v = \begin{bmatrix} \lambda_a - (-1)^{\bar{a}+\bar{b}}\lambda_b + c \\ t \end{bmatrix}_q v. \tag{5.2}$$

For example, as a quotient of $U_{q,F}$, $V = S_{q,F}$ is a $U_{q,F}$-supermodule. By Lemma 4.1(2), the λ-weight space $V_\lambda = 1_\lambda S_{q,F}$.

Define the partial ordering $\leq$ on $\mathbb{Z}^{m+n}$ by setting $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a non-negative sum of simple roots α_i. Denote $\mathrm{wt}(v) = \lambda$ for $v \in V_\lambda$ and let $\mathrm{wt}(v)_h$ be the hth component of $\mathrm{wt}(v)$.

Lemma 5.1 *For a $U_{q,F}$-supermodule V, $\lambda \in \mathbb{Z}^{m+n}$, and $\alpha \in \Phi$, we have*

$$E_{\alpha}^{(M)} V_{\lambda} \subseteq V_{\lambda+M\alpha}. \tag{5.3}$$

In particular, for $0 \neq v \in V_{\lambda}$ and $h \in [1, m+n)$, if $E_{b,a}^{(M)}.v \neq 0$ for some $a < b$ and $M > 0$, then

$$\mathrm{wt}(v) > \mathrm{wt}(E_{b,a}^{(M)}.v), \quad \mathrm{wt}(v)_h \leq \mathrm{wt}(E_{b,a}^{(M)}.v)_h \ \ (\forall h \neq a). \tag{5.4}$$

Proof The first assertion follows easily from (4.2). Suppose now $E_{b,a}^{(M)}.v \neq 0$. By (5.3), $\mathrm{wt}(E_{b,a}^{(M)}.v) = \mathrm{wt}(v) - M(\epsilon_a - \epsilon_b) < \mathrm{wt}(v)$, and $\mathrm{wt}(E_{b,a}^{(M)}.v)_h \geq \mathrm{wt}(v)_h$ whenever $h \neq a$. □

Call $\lambda = \sum_{i=1}^{m+n} \lambda_i \epsilon_i$ ($\lambda_i \in \mathbb{Z}$) a *weight* of V if $V_{\lambda} \neq 0$ and denote by $\pi(V) = \{\mu \in \mathbb{Z}^{m+n} \mid V_{\mu} \neq 0\}$ the set of weights of V. By (5.3), $\bigoplus_{\lambda \in \pi(V)} V_{\lambda}$ is a submodule of V. If $V = \bigoplus_{\lambda \in \pi(V)} V_{\lambda}$, we call V a *weight supermodule* (of type **1**). For example, the natural supermodule $V(m|n)$ and its tensor product $V(m|n)^{\otimes r}$ are weight supermodules with $\pi(V(m|n)^{\otimes r}) = \Lambda(m|n, r)$.

Let $U_{q,F}$-**mod** be the category of finite-dimensional weight $U_{q,F}(m|n)$-supermodules with *enriched* morphism sets in the sense of [3, Remark 2.1].

For every weight supermodule $V = \bigoplus_{\lambda \in \pi(V)} V_{\lambda}$, we may change its superspace structure (i.e. the parity) to get a standard one V^{s} associated with V, where

$$V^{\mathsf{s}} = V \text{ as a } U_{q,F}\text{-module, but } (V^{\mathsf{s}})_i = \bigoplus_{\overline{|\mu^{(1)}|} \equiv i} V_{\mu} \ \ (i \in \mathbb{Z}_2).$$

Clearly, V^{s} is a weight supermodule.

Lemma 5.2 *For every weight $U_{q,F}$-supermodule V, there is a supermodule isomorphism $V \cong V^{\mathsf{s}}$.*

Proof Since the parity function $\delta_V : v \mapsto (-1)^{\bar{v}} v$ on the superspace V stabilises every weight space V_{μ}, it follows that $V_{\mu} = (V_{\mu})_{\bar{0}} \oplus (V_{\mu})_{\bar{1}}$. Putting $\pi(V)_i = \{\mu \in \pi(V) \mid \overline{|\mu^{(1)}|} = i\}$, we have

$$V = \bigoplus_{\mu \in \pi(V)} \big((V_{\mu})_{\bar{0}} \oplus (V_{\mu})_{\bar{1}}\big) = V_+ \oplus V_-,$$

where $V_+ = \bigoplus_{\mu \in \pi(V)_{\bar{0}}} (V_{\mu})_{\bar{0}} \oplus \bigoplus_{\mu \in \pi(V)_{\bar{1}}} (V_{\mu})_{\bar{1}}$ and $V_- = \bigoplus_{\mu \in \pi(V)_{\bar{0}}} (V_{\mu})_{\bar{1}} \oplus \bigoplus_{\mu \in \pi(V)_{\bar{1}}} (V_{\mu})_{\bar{0}}$. Clearly, both V_+ and V_- are $U_{q,F}$-subsupermodules and $V_+ \cong (V_+)^{\mathsf{s}}$ and $V_- \cong \Pi(V_-)^{\mathsf{s}}$, where Π is the parity functor. Thus, $V \cong V_+ \oplus \Pi(V_-) \cong (V_+)^{\mathsf{s}} \oplus (V_-)^{\mathsf{s}} = V^{\mathsf{s}}$. □

We remark that this result will not be necessarily used later on. However, with this result, it is convenient to assume that every weight $U_{q,F}$-supermodule can have the standard parity.

We call a nonzero weight vector $\mathfrak{m}_\lambda$ a *maximal vector* if it satisfies

$$E_i^{(M)}.\mathfrak{m}_\lambda = 0, \quad \text{for } 1 \le i \le m+n-1 \text{ and } M > 0.$$

Call V a *highest weight supermodule* if it is generated by a maximal vector.

Let[4]

$$\begin{aligned}\mathbb{Z}_{++}^{m|n} &= \{\lambda \in \mathbb{Z}^{m+n} \mid \lambda_1 \ge \cdots \ge \lambda_m, \lambda_{m+1} \ge \cdots \ge \lambda_{m+n}\},\\ \Lambda^{++}(m|n) &= \Lambda(m|n) \cap \mathbb{Z}_{++}^{m|n}, \quad \Lambda^{++}(m|n,r) = \Lambda^{++}(m|n) \cap \Lambda(m|n,r).\end{aligned} \tag{5.5}$$

For $\lambda \in \mathbb{Z}_{++}^{m|n}$, let F_λ be a one-dimensional $U_{q,F}^0$-module of weight λ. By inflating F_λ to a $U_{q,F}^{\ge 0}$-supermodule and then inducing to $U_{q,F}$, we obtain the induced supermodule or Verma supermodule $Y(\lambda) = U_{q,F} \otimes_{U_{q,F}^{\ge 0}} F_\lambda$, where the highest weight space has the parity $i \equiv |\lambda^{(1)}| (\text{mod } 2)$. Thus, the parity of $Y(\lambda)$ is standard.

Similar to the non-super quantum case (see [19, §§6.1,6.2]) or the non-quantum super case [3, 2.4], $Y(\lambda)$ is a highest weight supermodule with highest weight λ, and $\dim Y(\lambda)_\lambda = 1$. Thus, every proper submodule of $Y(\lambda)$ is contained in the subspace $\bigoplus_{\mu<\lambda} Y(\lambda)_\mu$. Hence, $Y(\lambda)$ has a unique maximal subsupermodule and hence a unique irreducible quotient $L(\lambda)$.

Note that the irreducible supermodule $L(\lambda)$ can also be constructed through Kac modules, cf. [26, Section III][5] and [4, Thm 4.5]. The proof of the following result is standard; see, e.g. [3, Thm 2.4].

Proposition 5.3 *The set $\{L(\lambda) \mid \lambda \in \mathbb{Z}_{++}^{m|n}\}$ forms a complete set of irreducible objects in the category $U_{q,F}$-***mod**.

From now on, unless otherwise stated, we assume every weight supermodule is finite dimensional. Let $V = \bigoplus_{\lambda \in \mathbb{Z}^{m+n}} V_\lambda$ be a finite-dimensional weight $U_{q,F}$-supermodule. Then, for any $r \in \mathbb{Z}$, $V_r = \oplus_{|\mu|=r} V_\mu$ is a subsupermodule by (5.3). If $V = V_r$, then V is called a $U_{q,F}$-supermodule of *degree* r. We call V a *polynomial supermodule* of $U_{q,F}$ if V is a weight module with $\pi(V) \subseteq \Lambda(m|n)$. Since, for a polynomial supermodule V we have $V = \oplus_{r \ge 0} V_r$, we need only consider V_r, which is called a polynomial supermodule of degree r. Unlike the non-super case, we will see in the next section that only a subset of $\Lambda^{++}(m|n)$ labels all polynomial irreducible $U_{q,F}$-supermodules.

We now describe a vanishing ideal for all polynomial $U_{q,F}$-supermodules of degree r. Recall the algebra epimorphism $\eta_{r,F}$ in (3.16) and its kernel in Theorem 4.3.

Proposition 5.4 *Let V a polynomial $U_{q,F}$-supermodule and $r > 0$. Then $V = V_r$ if and only if* $\ker(\eta_{r,F}).V = 0$. *Hence, every polynomial $U_{q,F}(m|n)$-supermodule of degree r is an inflation of a $S_{q,F}(m|n,r)$-supermodule.*

[4] The set $\mathbb{Z}_{++}^{m|n}$ is denoted by $X^+(T)$ in [4, p.23]. Also, the notation $\Lambda^{++}(m|n,r)$ there has a different meaning; see footnote 6.

[5] Note that the quantum supergroup at a root of unity in [26] is not the quantum hyperalgebra $U_{q,F}(m|n)$ here. Compare [5, §§11.1,11.2].

Proof The sufficiency is clear since every $S_{q,F}$-supermodule has its weights in $\Lambda(m|n,r)$. Suppose now V is a polynomial $U_{q,F}$-supermodule of degree r. To prove $\ker(\eta_{r,F}).V = 0$, by Theorem 4.3, it is sufficient to verify every element in (4.1) vanishes V.

Choose $0 \neq m_\mu \in V_\mu$ with $\mu \in \Lambda(m|n,r)$. Then, for $a \in [1, m+n]$,

$$(1) \quad \left(1 - \sum_{\lambda\in\Lambda(m|n,r)} \begin{bmatrix} K \\ \lambda \end{bmatrix}\right).m_\mu = \left(1 - \sum_{\lambda\in\Lambda(m|n,r)} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q\right) m_\mu$$

$$\left(\text{Recall} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q = \prod_{i=1}^{m+n} \begin{bmatrix} \mu_i \\ \lambda_i \end{bmatrix}_q.\right)$$

$$= \left(1 - \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}_q\right) m_\mu = 0 \qquad \left(\text{since} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q = \delta_{\mu\lambda}\right).$$

$$(2) \quad \left(K_a^{\pm 1}\begin{bmatrix} K \\ \lambda \end{bmatrix} - q_a^{\pm\lambda_a}\begin{bmatrix} K \\ \lambda \end{bmatrix}\right).m_\mu = \left(K_a^{\pm 1} - q_a^{\pm\lambda_a}\right)\begin{bmatrix} K \\ \lambda \end{bmatrix}.m_\mu$$

$$= \left(K_a^{\pm 1} - q_a^{\pm\lambda_a}\right)\begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q.m_\mu$$

$$= \left(q_a^{\pm\mu_a} - q_a^{\pm\lambda_a}\right)\begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q m_\mu = 0 \qquad \left(\text{since} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_q = \delta_{\mu\lambda}\right).$$

The proof of $(\left[\begin{smallmatrix} K_a;c \\ t \end{smallmatrix}\right]\left[\begin{smallmatrix} K \\ \lambda \end{smallmatrix}\right] - \left[\begin{smallmatrix} \lambda_a+c \\ t \end{smallmatrix}\right]_q\left[\begin{smallmatrix} K \\ \lambda \end{smallmatrix}\right]).m_\mu = 0$ is similar to that of (2). □

Remark 5.5 By this proposition, the full subcategory of finite-dimensional polynomial $U_{q,F}(m|n)$-supermodules of degree r is equivalent to the category of finite-dimensional supermodules over the q-Schur superalgebra $S_{q,F}(m|n,r)$.

Rui and the first author ([14, Thm 9.8]) showed that $S_{q,F}(m|n,r) \cong A_F(m|n,r)^*$, where $A_F(m|n,r)$ is the rth homogenous component of the quantum matrix superalgebra. Hence, one may also follow Green's original definition in [17] to define polynomial $U_{q,F}(m|n)$-supermodules through $A_F(m|n,r)$-cosupermodules.

6 Polynomial irreducible $U_{q,F}(m|n)$-supermodules

Throughout the section, F denotes a field of characteristic $\neq 2$ and $q \in F$. If q is an l'th primitive root of unity in F, let

$$l = \begin{cases} l', & \text{if } l' \text{is odd;} \\ \frac{l'}{2}, & \text{if } l' \text{is even.} \end{cases}$$

In this case, q^2 is an l-th primitive root of unity. If q is not a root of unity, then we set $l = \infty$. As before, we use the abbreviation $U_{q,F}$, $S_{q,F}$ for $U_{q,F}(m|n)$, $S_{q,F}(m|n,r)$, respectively.

Let Λ^+ be the set of all partitions. Following [4, Lem. 6.2] or [25, Lem. 1], we first define the map

$$j_l : \Lambda^+ \to \mathbb{N}. \tag{6.1}$$

For $\lambda \in \Lambda^+$ of length $\ell(\lambda) = d$ (i.e. $\lambda_{d+1} = 0$ and $\lambda_d \neq 0$), let $x_{d+1} = x_{d+2} = \cdots = 0$ and define $x_d, x_{d-1}, \ldots x_1 \in \{0, 1\}$ recursively by setting

$$x_i = \begin{cases} 1, & \text{if } \lambda_i + x_{i+1} + x_{i+2} + \cdots \not\equiv 0 \pmod l; \\ 0, & \text{if } \lambda_i + x_{i+1} + x_{i+2} + \cdots \equiv 0 \pmod l. \end{cases} \tag{6.2}$$

Let

$$j_l(\lambda) = x_1 + x_2 + \cdots = \sum_{i=1}^{d} x_i.$$

Then $j_l(\lambda) \leq d$. It is clear from the definition that, if $l = \infty$ (i.e. q is not a root of unity), then $j_\infty(\lambda) = d$. In general, there exists a subsequence $1 \leq i_1 < \cdots < i_t \leq d$, where $t = j_l(\lambda)$, such that $x_{i_1} + \cdots + x_{i_t} = t$ and $j_l(\lambda_{i_1}, \ldots, \lambda_{i_t}) = t$.

The map j_l is closely related to Xu's algorithm for computing the Mullineux map; see Sect. 8, [4, §6], and Remark 8.1(2).

Lemma 6.1 *For $\lambda \in \Lambda^+$, if $1 \leq i_1 < i_2 \cdots < i_t \leq \ell(\lambda)$, then $j_l(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t}) \leq j_l(\lambda)$. In particular, $j_l(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t}) = t$ if and only if*

$$\prod_{s=1}^{t} \begin{bmatrix} \lambda_{i_s} + t - s \\ 1 \end{bmatrix}_q \neq 0.$$

Proof Let $\mu = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t})$. By the observation above, we may assume that $j_l(\mu) = t$ and prove $t \leq j_l(\lambda)$. Define $y_1, y_2, \ldots, y_t, \ldots \in \{0, 1\}$ for μ, similar to the x_i for λ, as above. Since $j_l(\mu) = t$, we have $y_t = y_{t-1} = \cdots = y_1 = 1$ and $y_i = 0$ for all $i > t$. Thus, by definition, we have, for $s = t, t-1, \ldots, 1$,

$$\lambda_{i_s} + t - s = \mu_s + t - s \not\equiv 0 \pmod l.$$

We claim that there exist $1 \leq i'_1 < \cdots < i'_t \leq d = \ell(\lambda)$ such that $x_{i'_s} = 1$. Indeed, let $i'_t \in [1, d]$ be the maximal index such that $\lambda_{i'_t} \not\equiv 0 (\operatorname{mod} l)$. Then $i'_t \in [i_t, d]$ and $x_{i'_t} = 1$. For every $s = t-1, t-2, \ldots 1$, the above congruence relations guarantee that there exists $i'_s \in [i_s, i'_{s+1})$, maximal in the interval, such that $\lambda_{i'_s} + t - s \not\equiv 0 (\operatorname{mod} l)$. Thus, by the selection of i'_s, we have $x_{i'_s} = 1$ for all $s \in [1, t]$, proving the claim and, hence, the first assertion.

Since $\left[{\lambda_{i_s}+t-s \atop 1}\right]_q = [\lambda_{i_s} + t - s]_q \neq 0$ if and only if $\lambda_{i_s} + t - s \not\equiv 0 (\operatorname{mod} l)$ (noting q^2 is a primitive lth root of unity), the last assertion is clear. □

For $\lambda \in \Lambda^{++}(m|n,r)$, define the "modulo l" subset

$$\Lambda_l^{++}(m|n,r) = \{\lambda \in \Lambda^{++}(m|n,r) \mid j_l(\lambda^{(1)}) \le \lambda_m\}. \tag{6.3}$$

If $l = \infty$, then $\lambda^{(0)}$ concatenating with the dual of $\lambda^{(1)}$ is a well-defined partition of r and the set $\Lambda_\infty^{++}(m|n,r)$ is identified with

$$\Lambda^+(r)_{m|n} := \{\lambda \in \Lambda^+ \mid \lambda_{m+1} \le n, |\lambda| = r\} \tag{6.4}$$

(see [14, (4.0.2)]). This set is used in [14] to label irreducible $S_{\boldsymbol{v},\mathbb{Q}(\boldsymbol{v})}(m|n,r)$-modules.

Remark 6.2 If $l = p$ is a prime, $\Lambda_p^{++}(m|n,r)$ is used in [4][6] to parametrise the irreducible supermodules of the Schur superalgebra $S(m|n,r)$ in positive characteristic p. In the theorem below, we will generalise this result to the quantum Schur superalgebras at every primitive l'-th root of unity q.

Lemma 6.3 *For $\lambda \in \Lambda^{++}(m|n,r)$, let $\mathfrak{m}_\lambda$ be a maximal vector of weight λ for a weight $U_{q,F}$-supermodule. Consider the sequences $1 \le h < i_1 < i_2 < \cdots < i_s \le m+n$ and $(a_1,\ldots,a_s) \in \mathbb{Z}_{>0}^s$, where $a_t = 1$ whenever α_{h,i_t} is an odd root.*

(1) If $1 \le t \le s$ and $i > i_t$, then $E_{h,i}^{(b)}.(E_{i_t,h}^{(a_t)} \ldots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda) = 0, \forall b > 0$.

(2) $(E_{h,i_1}^{(a_1)} \ldots E_{h,i_s}^{(a_s)}).(E_{i_s,h}^{(a_s)} \ldots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda)$
$$= \prod_{t=1}^{s} \begin{bmatrix} \lambda_h - (-1)^{\bar{h}+\bar{i}_t}\lambda_{i_t} - a_{t-1} \ldots - a_1 \\ a_t \end{bmatrix}_q \mathfrak{m}_\lambda.$$

Proof Applying the anti-automorphism Υ defined in (2.2) to the formulas in Proposition 2.4(3) yields the following commutation formulas in $U_{q,F}$:

$$E_{h,i}^{(b)} E_{i_t,h}^{(a_t)} = \begin{cases} (-1)^{\bar{E}_{h,i}\bar{E}_{i_t,h}} E_{i_t,h}E_{h,i} - (-1)^{\bar{E}_{h,i}\bar{E}_{i_t,h}} E_{i_t,i}K_{i_t,h}, & \text{if } b = a_t = 1; \\ \displaystyle\sum_{k=0}^{\min(b,a_t)} (-1)^k q_{i_t}^{k(a_t-1-k)} E_{i_t,h}^{(a_t-k)} K_{h,i_t}^{-k} E_{h,i}^{(b-k)} E_{i_t,i}^{(k)}, & \text{otherwise.} \end{cases}$$

Since either $b-k > 0$ or $k > 0$ and $\mathfrak{m}_\lambda$ is a maximal vector, assertion (1) is clear if $t = 1$. The general case follows from induction.

We now prove (2). By Proposition 2.4(4) and assertion (1), if $a_s > 1$,

$$E_{h,i_s}^{(a_s)}.(E_{i_s,h}^{(a_s)} \ldots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda)$$
$$= \left(\sum_{t=0}^{a_s} E_{i_s,h}^{(a_s-t)} \begin{bmatrix} K_{h,i_s}; 2t - a_s - a_s \\ t \end{bmatrix} E_{h,i_s}^{(a_s-t)}\right)(E_{i_{s-1},h}^{(a_{s-1})} \ldots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda)$$

[6] This set is denoted by $\Lambda^{++}(m|n,r)$ in [4, Thm 6.5], where $\lambda^{(1)}$ is called the tail $t(\lambda)$ of λ.

$$= \begin{bmatrix} K_{h,i_s}; 0 \\ a_s \end{bmatrix} (E^{(a_{s-1})}_{i_{s-1},h} \dots E^{(a_1)}_{i_1,h}.\mathfrak{m}_\lambda) \quad \text{(by (1))}$$
$$= \begin{bmatrix} \lambda_h - a_{s-1} \dots - a_1 - (-1)^{\bar{h}+\bar{i}_s}\lambda_{i_s} \\ a_s \end{bmatrix}_q (E^{(a_{s-1})}_{i_{s-1},h} \dots E^{(a_1)}_{i_1,h}.\mathfrak{m}_\lambda),$$

by (5.3) and (5.2). The $a_s = 1$ case is similar. Induction on s proves (2). □

Theorem 6.4 *For $r > 0$ and $\lambda \in \Lambda^{++}(m|n,r)$, the irreducible $U_{q,F}$-supermodule $L(\lambda)$ is polynomial if and only if $\lambda \in \Lambda_l^{++}(m|n,r)$. In particular, the set $\{L(\lambda) \mid \lambda \in \Lambda_l^{++}(m|n,r)\}$ forms a complete set of all pairwise non-isomorphic polynomial irreducible $U_{q,F}$-supermodules of degree r in $U_{q,F}$-***mod**.

Proof Choose a maximal vector $0 \neq \mathfrak{m}_\lambda \in L(\lambda)_\lambda$ then

$$L(\lambda) = U_F.\mathfrak{m}_\lambda = U_F^-.\mathfrak{m}_\lambda.$$

First, we prove that, for $\lambda \in \Lambda^{++}(m|n,r)$, if $j_l(\lambda^{(1)}) > \lambda_m$, then $L(\lambda)$ is not a polynomial supermodule. Suppose that $\lambda \notin \Lambda_l^{++}(m|n,r)$ and so $s := j(\lambda^{(1)}) \geq \lambda_m + 1 \geq 1$. For partition $\lambda^{(1)} = (\lambda_{m+1}, \dots, \lambda_{m+n})$, the sequence $x_1, x_2, \dots, x_d$ defined in (6.2) satisfies $\sum_{i\geq 1} x_i = s$. Let $i_1 < i_2 < \cdots < i_{\lambda_m+1}$ be the indices of the last $\lambda_m + 1$ nonzero terms in the sequence. Then $x_{i_t} = 1$ for all $t \in [1, \lambda_m + 1]$ and

$$j_l(\lambda_{m+i_1}, \dots, \lambda_{m+i_{\lambda_m+1}}) = \lambda_m + 1.$$

By Lemma 6.1

$$\prod_{t=1}^{\lambda_m+1} \begin{bmatrix} \lambda_{m+i_t} + \lambda_m + 1 - t \\ 1 \end{bmatrix}_q \neq 0.$$

Thus, Lemma 6.3 implies

$$\begin{aligned} &(E_{m,m+i_1} \dots E_{m,m+i_{\lambda_m+1}})(E_{m+i_{\lambda_m+1},m} \dots E_{m+i_1,m}).\mathfrak{m}_\lambda \\ &= \prod_{t=1}^{\lambda_m+1} \begin{bmatrix} \lambda_m + \lambda_{m+i_t} - t + 1 \\ 1 \end{bmatrix}_q \mathfrak{m}_\lambda \neq 0. \end{aligned} \tag{6.5}$$

Hence, $E_{m+i_{\lambda_m+1},m} \dots E_{m+i_1,m}.\mathfrak{m}_\lambda \neq 0$, forcing $L(\lambda)_{\lambda-\alpha_{m,m+i_1}-\cdots-\alpha_{m,m+i_{\lambda_m+1}}} \neq 0$. Since

$$\lambda - \alpha_{m,m+i_1} - \cdots - \alpha_{m,m+i_{\lambda_m+1}} = \lambda - (\lambda_m + 1)\epsilon_m + \epsilon_{m+i_1} + \cdots + \epsilon_{m+i_{\lambda_m+1}},$$

whose mth component is -1, it follows that $\pi(L(\lambda)) \not\subseteq \Lambda(m|n,r)$. Hence, $L(\lambda)$ is not a polynomial supermodule.

We now prove the converse. Suppose $L(\lambda)$ is not a polynomial supermodule of degree $r = |\lambda|$. Then there exists $\nu \in \mathbb{Z}^{m+n}$ such that $|\nu| = r$, $L(\lambda)_\nu \neq 0$, and $\nu_h < 0$

for some $h \in [1, m+n]$. In fact, we may assume that $h < m+n$. This can be seen from the fact that $L(\lambda) = U_F^-.\mathfrak{m}_\lambda$ is spanned by vectors $\prod_{a<b} E_{b,a}^{(A_{b,a})}.\mathfrak{m}_\lambda$ whose weights are of the form $\lambda - \sum_{a<b} A_{b,a}(\epsilon_a - \epsilon_b)$. We need to prove that $\lambda \notin \Lambda_l^{++}(m|n, r)$.

Let

$$\mu = \max\{\nu \in \pi(L(\lambda)) \mid \nu_h < 0\}.$$

Claim 1 The weight space $L(\lambda)_\mu$ is spanned by the vectors

$$\left\{ \prod_{h+1 \le b \le m+n} E_{b,h}^{(A_{b,h})}.\mathfrak{m}_\lambda \mid A_{b,h} \in \mathbb{N} \text{ and } \mu = \lambda - A_{h+1,h}\alpha_{h,h+1} - \cdots - A_{m+n,h}\alpha_{h,m+n} \right\}.$$

Proof of Claim 1 Fix an ordering on Φ^+ such that the sequence ends with the $m+n-h$ positive roots: $\alpha_{h,m+n}, \alpha_{h,m+n-1}, \ldots, \alpha_{h,h+1}$. Then

$$L(\lambda) = U_F^-.\mathfrak{m}_\lambda = \operatorname{span}\left\{ \prod_{\epsilon_a - \epsilon_b \in \Phi^+, a \ne h} E_{b,a}^{(A_{b,a})} \prod_{h+1 \le b \le m+n} E_{b,h}^{(A_{b,h})}.\mathfrak{m}_\lambda \,\middle|\, A \in \mathcal{M}(m|n)^- \right\}.$$

Here $\mathcal{M}(m|n)^-$ consists of matrices in $\mathcal{M}(m|n)$ whose diagonal and upper triangular parts are zeros.

For every nonzero spanning vector of the form,

$$w = \prod_{\substack{\epsilon_a - \epsilon_b \in \Phi^+ \\ a \ne h}} E_{b,a}^{(A_{b,a})} \left(\prod_{h+1 \le b \le m+n} E_{b,h}^{(A_{b,h})}.\mathfrak{m}_\lambda \right)$$

satisfying $\mathrm{wt}(w)_h < 0$, if $\prod_{\substack{\epsilon_a - \epsilon_b \in \Phi^+ \\ a \ne h}} E_{b,a}^{(A_{b,a})} \ne 1$, repeatedly applying (5.4) yields

$$\mathrm{wt}\left(\prod_{h+1 \le b \le m+n} E_{b,h}^{(A_{b,h})}.\mathfrak{m}_\lambda \right) > \mathrm{wt}(w).$$

and

$$\left(\mathrm{wt}\left(\prod_{h+1 \le b \le m+n} E_{b,h}^{(A_{b,h})}.\mathfrak{m}_\lambda \right) \right)_h \le \mathrm{wt}(w)_h < 0$$

Thus, if $w \in L(\lambda)_\mu$, then the maximality of μ forces $\prod_{\substack{\epsilon_a - \epsilon_b \in \Phi^+ \\ a \ne h}} E_{b,a}^{(A_{b,a})} = 1$, proving Claim 1.

By Claim 1, we choose a nonzero vector $v \in L(\lambda)_\mu$ of the form:

$$v = E_{i_s,h}^{(a_s)} E_{i_{s-1},h}^{(a_{s-1})} \dots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda \neq 0, \tag{6.6}$$

for some sequences $h < i_1 < i_2 < \cdots < i_s \leq m+n$ and $(a_1, \dots, a_s) \in (\mathbb{Z}_{>0})^s$ where $a_t = 1$ whenever α_{h,i_t} is an odd root. Then $\mu = \mathrm{wt}(v) = \lambda + \sum_{t=1}^{s} a_t(\epsilon_{i_t} - \epsilon_h)$.

Since $\mathrm{wt}(E_{i_{s-1},h}^{(a_{s-1})} \dots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda) > \mathrm{wt}(v) = \mu$, by the selection of μ, we must have $\mathrm{wt}(E_{i_{s-1},h}^{(a_{s-1})} \dots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda)_h \geq 0$. In other words, we have

$$a_1 + \cdots a_{s-1} \leq \lambda_h < a_1 + \cdots a_s. \tag{6.7}$$

If, for some $a < b$, $a \neq h$, and $M \in \mathbb{Z}_{>0}$, $u = E_{a,b}^{(M)}.v \neq 0$, then by (5.3), $\mathrm{wt}(u) = \mu + M(\epsilon_a - \epsilon_b) > \mathrm{wt}(v) = \mu$ and $\mathrm{wt}(u)_h \leq \mathrm{wt}(v)_h < 0$, contrary to the selection of μ. Thus, we have

$$E_{a,b}^{(M)}.v = 0 \text{ for all } 1 \leq a < b \leq m+n, a \neq h, M \in \mathbb{Z}_{>0}. \tag{6.8}$$

Claim 2 For the selected v as in (6.6), we have

$$\prod_{t=1}^{s} \begin{bmatrix} \lambda_h - (-1)^{\bar{h}+\bar{i}_t}\lambda_{i_t} - a_{t-1} \cdots - a_1 \\ a_t \end{bmatrix}_q \neq 0. \tag{6.9}$$

Proof of Claim 2 Since λ is the highest weight of $L(\lambda) = U_{q,F}.v$ and $\mathfrak{m}_\lambda \in U_{q,F}.v = U_{q,F}^- U_{q,F}^0 U_{q,F}^+.v$, no vectors with weight λ can occur in the set $I_F^- U_F^0 U_F^+.v$, where I_F^- is the ideal spanned by all monomials of positive degree. Hence, we must have

$$\mathfrak{m}_\lambda \in U_{q,F}.v = U_{q,F}^+.E_{i_s,h}^{(a_s)} \dots E_{i_1,h}^{(a_1)}\mathfrak{m}_\lambda.$$

By using a PBW-type basis for $U_{q,F}^+$ over an ordering on positive roots, beginning with $\alpha_{h,m+n}, \alpha_{h,m+n-1}, \dots, \alpha_{h,h+1}$, (6.8) implies

$$\begin{aligned} U_F^+.v &= \mathrm{span}\left\{ \prod_{h+1\leq b\leq m+n} E_{h,b}^{(A_{h,b})} \prod_{\substack{\epsilon_a-\epsilon_b\in\Phi^+ \\ a\neq h}} E_{a,b}^{(A_{a,b})}.v \;\middle|\; A \in P(m|n) \right\} \\ &= \mathrm{span}\left\{ \left(\prod_{h+1\leq b\leq m+n} E_{h,b}^{(A_{h,b})} \right) v \;\middle|\; A_{h,b} \in \mathbb{N} \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} (U_F^+.v)_\lambda &= \mathrm{span}\left\{ \prod_{h+1\leq b\leq m+n} E_{h,b}^{(A_{h,b})}.v \;\middle|\; \sum_{b=h+1}^{m+n} A_{h,b}(\epsilon_h - \epsilon_b) = \sum_{t=1}^{s} a_t(\epsilon_h - \epsilon_{i_t}) \right\} \\ &= \mathrm{span}\left\{ (E_{h,i_1}^{(a_1)} \dots E_{h,i_s}^{(a_s)}).(E_{i_s,h}^{(a_s)} \dots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda) \right\}. \end{aligned}$$

However, by Lemma 6.3(2),

$$\left(E_{h,i_1}^{(a_1)} \dots E_{h,i_s}^{(a_s)}\right) \cdot \left(E_{i_s,h}^{(a_s)} \dots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda\right) = \prod_{t=1}^{s} \begin{bmatrix} \lambda_h - (-1)^{\bar{h}+\bar{i}_t}\lambda_{i_t} - a_{t-1} \cdots - a_1 \\ a_t \end{bmatrix}_q \mathfrak{m}_\lambda.$$

We must have $\prod_{t=1}^{s} \begin{bmatrix} \lambda_h - (-1)^{\bar{h}+\bar{i}_t}\lambda_{i_t} - a_{t-1} \cdots - a_1 \\ a_t \end{bmatrix}_q \neq 0$, proving Claim 2.

Now, by claim (6.9), we see $\begin{bmatrix} \lambda_h - (-1)^{\bar{h}+\bar{i}_s}\lambda_{i_s} - a_{s-1} \cdots - a_1 \\ a_s \end{bmatrix}_q \neq 0$. This implies

$$\lambda_h - (-1)^{\bar{h}+\bar{i}_s}\lambda_{i_s} - a_{s-1} \cdots - a_1 \geq a_s,$$

or $\lambda_h - (-1)^{\bar{h}+\bar{i}_s}\lambda_{i_s} \geq a_s + a_{s-1} \cdots + a_1$. Thus, the second inequality in (6.7) forces $\bar{h} + \bar{i}_s = 1$. Since $h < i_s$, we must have $h \leq m < i_s$. Hence, α_{h,i_s} is an odd root and so $a_s = 1$. By (6.7), $\lambda_h = a_1 + \cdots + a_{s-1}$ and, consequently, $\mu_h = \mathrm{wt}(v)_h = -1$.

Finally, we are ready to prove $j_l(\lambda^{(1)}) \geq \lambda_m + 1$. Let s' be the minimal index such that $m < i_{s'}$. Then $1 \leq h < i_1 < \cdots < i_{s'-1} \leq m < i_{s'} < i_{s'+1} < \cdots < i_s$. This implies $a_{i_t} = 1$ for all $s' \leq t \leq s$ and so (6.6) becomes

$$v = E_{i_s,h} \cdots E_{i_{s'},h}\, E_{i_{s'-1},h}^{(a_{s'-1})} \cdots E_{i_1,h}^{(a_1)}.\mathfrak{m}_\lambda \quad \text{and} \quad \bar{h} + \bar{i}_t = \begin{cases} 1, & s' \leq t \leq s; \\ 0, & 1 \leq t < s'. \end{cases}$$

Since $\mathrm{wt}(v)_h = -1$, it follows that $s - s' = \lambda_h - a_{i_{s'-1}} - \cdots - a_1$. In this case, expression (6.9) has the form

$$\prod_{t=s'}^{s} \begin{bmatrix} (s - s') + \lambda_{i_t} - (t - s') \\ 1 \end{bmatrix}_q \prod_{t=1}^{s'-1} \begin{bmatrix} \lambda_h - \lambda_{i_t} - a_{t-1} \cdots - a_1 \\ a_t \end{bmatrix}_q \neq 0. \quad (6.10)$$

The factor for $t = s' - 1$ in the second product of (6.10) being nonzero implies

$$\lambda_h - \lambda_{i_{s'-1}} - a_{s'-2} - \cdots - a_1 \geq a_{s'-1}$$

or equivalently, $s - s' \geq \lambda_{i_{s'-1}}$.

On the other hand, the first product in (6.10) can be rewritten as

$$\prod_{t=s'}^{s} \begin{bmatrix} (s - s') + \lambda_{i_t} - (t - s') \\ 1 \end{bmatrix}_q = \prod_{t=1}^{s-s'+1} \begin{bmatrix} \lambda_{i_{s'+t-1}} + (s - s' + 1) - t \\ 1 \end{bmatrix}_q \neq 0,$$

which implies $j_l(\lambda_{i_{s'}}, \dots, \lambda_{i_s}) = s - s' + 1$ by Lemma 6.1. Hence, by Lemma 6.1 again,

$$j_l(\lambda^{(1)}) \geq j_l(\lambda_{i_{s'}}, \dots, \lambda_{i_s}) = s - s' + 1 \geq \lambda_{i_{s'-1}} + 1 \geq \lambda_m + 1,$$

noting $i_{s'-1} \leq m$. Hence, $\lambda \notin \Lambda_l^{++}(m|n, r)$, as required. □

7 Classification of irreducible supermodules of $S_{q,F}(m|n,r)$

We keep the assumption on F and q and assume l is the order of q^2 as in Sect. 6.

Theorem 7.1 *The set $\{L(\lambda) \mid \lambda \in \Lambda_l^{++}(m|n,r)\}$ forms a complete set of all non-isomorphic irreducible $S_{q,F}(m|n,r)$-supermodules.*

Proof By Proposition 5.4 and Theorem 6.4, the irreducible supermodules in the set are all irreducible $S_{q,F}(m|n,r)$-supermodules. Since every irreducible $S_{q,F}(m|n,r)$-supermodule L is naturally a polynomial irreducible $U_{q,F}$-supermodule of degree r by inflation, it must be of the form $L \cong L(\lambda)$ by Proposition 5.3. Now apply Theorem 6.4 to see $\lambda \in \Lambda_l^{++}(m|n,r)$. □

Remark 7.2 (1) When $m+n \geq r$, a classification is given in [11,12] without using representations of the quantum supergroup. See also a comparison of the index sets in [11, Theorem B.3] in this case. Note that the theorem above has also generalised the classification loc. cit. to the $m+n<r$ case.
(2) The theorem above is a quantum version of [4, Lemma 5.4].

Corollary 7.3 *If q is not a root of unity (i.e. if $l=\infty$), then $S_{q,F}(m|n,r)$ is semisimple with irreducible representations labelled by $\Lambda^+(r)_{m|n}$ (see (6.4)).*

We will construct irreducible $S_{q,F}(m|n,r)$-supermodules directly in the category $S_{q,F}$-**mod** of finite-dimensional $S_{q,F}$-supermodules. In this category, every module V is a weight module in the sense that $V = \oplus_{\lambda\in\Lambda(m|n,r)} 1_\lambda V$, where $1_\lambda = \eta_{r,F}(\left[{K \atop \lambda}\right]) = \phi^1_{\lambda,\lambda}$ are weight idempotents. In particular, $S_{q,F}(m|n,r)$ itself has a direct sum decomposition into projective modules

$$S_{q,F}(m|n,r) = \oplus_{\lambda\in\Lambda(m|n,r)} S_{q,F}(m|n,r)1_\lambda.$$

We define analogously the positive part, negative part and zero part $S^+_{q,F}$, $S^-_{q,F}$, $S^0_{q,F}$ for $S_{q,F}(m|n,r)$ which are generated, respectively, by $\mathrm{e}^{(M)}_{a,b}$, $a<b$, $M\geq 0$; $\mathrm{e}^{(M)}_{a,b}$, $a>b$, $M\geq 0$; $\left[{\mathrm{k}_a \atop t}\right]$, $t\geq 0$, $a\in[1,m+n]$. We may also regard them as homomorphic images of $U^+_{q,F}$, $U^-_{q,F}$, $U^0_{q,F}$, respectively. In particular, these are subsuperalgebras with identity 1.

Let I^+ denote the ideal of $S^+_{q,F}$ generated by all $\mathrm{e}^{(M)}_{a,b}$, $a<b$, $M>0$ and define, for $\lambda\in\Lambda(m|n,r)$,

$$V(\lambda) = S_{q,F}1_\lambda / S_{q,F}I^+1_\lambda.$$

We may also define the notion of highest weight module in this category. Thus, if v is a highest weight vector of an $S_{q,F}$-module, then $I^+.v=0$. Call a highest weight module V of highest weight λ to be *universal* if every highest weight module with highest weight λ is a homomorphic image of V (cf. [2, Lem. 3.15]).

Theorem 7.4 *The $S_{q,F}(m|n,r)$-supermodule $V(\lambda)$ is nonzero if and only if $\lambda \in \Lambda_l^{++}(m|n,r)$. Moreover, every such a $V(\lambda)$ is an indecomposable universal polynomial highest weight supermodule and has a unique irreducible quotient $L(\lambda)$.*

Proof If $V(\lambda) \neq 0$ then, for any highest weight supermodule V of highest weight λ, choose a maximal vector $\mathfrak{m}_\lambda \in V_\lambda$. Define a map f from the left ideal $S_{q,F}1_\lambda$ to V by the rule: $f(s1_\lambda) = (s1_\lambda).\mathfrak{m}_\lambda$. Clearly, f is a (homogeneous) supermodule homomorphism. Note that $f(1_\lambda) = 1_\lambda.\mathfrak{m}_\lambda = \mathfrak{m}_\lambda$ and, for all $s \in S_{q,F}$, we have $s.\mathfrak{m}_\lambda = s.(1_\lambda.\mathfrak{m}_\lambda) = (s1_\lambda).\mathfrak{m}_\lambda = f(s1_\lambda)$. Hence, f is a surjection. Since

$$f(S_{q,F}I^+1_\lambda) = (S_{q,F}I^+1_\lambda).\mathfrak{m}_\lambda = S_{q,F}I^+.\mathfrak{m}_\lambda = 0,$$

we see that $S_{q,F}I^+1_\lambda \subseteq \ker f$. Thus, f induces an epimorphism $\bar{f} : V(\lambda) \to V$. This proves the universal property.

If $\lambda \in \Lambda_l^{++}(m|n,r)$, the argument above for $V = L(\lambda)$ shows that $L(\lambda)$ is a homomorphic image of $V(\lambda)$. Hence, $V(\lambda) \neq 0$. Conversely, if $V(\lambda) \neq 0$, then $V(\lambda)$ has an irreducible head of highest weight λ which must be isomorphic to $L(\lambda)$. Hence, $\lambda \in \Lambda_l^{++}(m|n,r)$ by Theorem 7.1. □

Remarks 7.5 (1) The supermodules $V(\lambda)$ play the role of Weyl modules for the Schur superalgebra. It would be interesting to determine the formal character of $V(\lambda)$.

(2) If we order the set $\Lambda_l^{++}(m|n,r)$ as $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}$ such that $\lambda^{(i)} \leq \lambda^{(j)}$ implies $i > j$, then we may construct a filtration of ideals

$$0 \subseteq S_{q,F}f_1S_{q,F} \subseteq S_{q,F}f_2S_{q,F} \subseteq \cdots S_{q,F}f_NS_{q,F} \subseteq S_{q,F},$$

where $f_i = \sum_{j=1}^{i} 1_{\lambda^{(i)}}$. Note that, if $n = 0$, then the sequence is a heredity chain for the quasi-hereditary algebra $S_{q,F}(m,r)$.

Claim: $S_{q,F}f_NS_{q,F} = S_{q,F}$.

Proof It suffices to prove $S_{q,F}1_\lambda S_{q,F} \subseteq S_{q,F}f_NS_{q,F}$ for all $\lambda \in \Lambda(m|n,r)$. We apply induction on the poset structure of $\Lambda(m|n,r)$. There is nothing to prove if $\lambda \in \Lambda_l^{++}(m|n,r)$. In particular, the assertion is true for largest element $(r,0,\ldots,0|0,\ldots,0)$. Suppose $\lambda \in \Lambda(m|n,r)/\Lambda_l^{++}(m|n,r)$. Then $V(\lambda) = S_{q,F}1_\lambda/S_{q,F}I^+1_\lambda = 0$. So $1_\lambda \in S_{q,F}I^+1_\lambda$. Hence, there exist $x_i \in S_{q,F}$, $y_i \in I^+$ with $1_\lambda = \sum_i x_iy_i1_\lambda$. But, by Lemma 4.1, $x_iy_i1_\lambda = x_i1_{\lambda^{(i)}}y_i$, where $\lambda^{(i)} \in \Lambda(m|n,r)$, and $\lambda^{(i)} > \lambda$. Hence, by induction, $S_{q,F}1_\lambda S_{q,F} \subseteq \sum_i S_{q,F}1_{\lambda^{(i)}}S_{q,F} \subseteq S_{q,F}f_NS_{q,F}$. □

(3) In [11,12], a classification is done by using the defect groups of primitive idempotents. By (2), we see that the non-equivalent primitive idempotents $e_1, e_2, \ldots, e_N$ can be selected to satisfy the condition $e_i1_{\lambda^{(i)}} = e_i$ for every i. It would be interesting to know whether this condition can be used to determine the defect group of e_i.

8 The Mullineux map and Serganova's algorithm

In this section, we keep the notations F, q, l as defined at the beginning of Sect. 6.

Recall the R-algebra automorphism $(\ )^\sharp$ defined in (3.2). For any $H_{q^2,F}$-module W, define a new $H_{q^2,F}$-module $W^\sharp$ by twisting the action via $(\ )^\sharp$: $h.v := h^\sharp v$. Note that, if $q^2 = 1$, i.e. if $H_{q^2,F} = F\mathfrak{S}_r$ is the group algebra, then $W^\sharp \cong W \otimes \mathrm{sgn}$.

Let $\Lambda_l^+(r)$ be the set of all l-restricted partitions of r. Then, by [7, §§4,6], this set indexes the isomorphism classes of irreducible $H_{q^2,F}$-module. Let D_λ, $\lambda \in \Lambda_l^+(r)$ denote a representative from the class λ; see (9.1) for the definition of D_λ. Thus, $\{D_\lambda\}_{\lambda\in\Lambda_l^+(r)}$ forms a complete set of pairwise non-isomorphic irreducible $H_{q^2,F}$-modules.

Define the Mullineux conjugation map

$$\mathtt{M} : \Lambda_l^+(r) \longrightarrow \Lambda_l^+(r), \quad \lambda \longmapsto \mathtt{M}(\lambda) \tag{8.1}$$

by mimicking the definition on [4, p.32] with prime p replaced by l (see also remarks in the second paragraph on [2, p.556]). We omit the details here. Note that this is the transpose of the original definition from [22].

Remarks 8.1 (1) Irreducible p-modular representations of the symmetric group $\mathfrak{S}_r$ are indexed by p-regular partitions of r. Mullineux conjectured $(D_\lambda)^\sharp \cong D_{\mathtt{M}(\lambda)}$. This conjecture was first proved by Ford and Kleshchev [16] building on [18]. Using representations of supergroups, Brundan and Kujawa [4] gave an excellent new proof for the original conjecture. See also [24] for the ortho-symplectic super case.

The quantum version of this conjecture was first proved by Brundan [2]. The main method used there is the branching rule. However, it would be interesting to seek a proof of using quantum supergroups and q-Schur superalgebras, generalising the idea in [4] to the quantum case. In the next two sections, we will use the techniques developed in the paper to prove the quantum version of the Mullineux conjecture.

(2) There is another algorithm due to Xu [25] which is also independent of the primality of p. Thus, [4, Thm 6.1] continue to hold for all $l > 0$.

Recall the Lusztig $\mathcal{Z}$-form $U_{\boldsymbol{v},\mathcal{Z}}(m|n)$, its specialisation $U_{q,F} = U_{\boldsymbol{v},\mathcal{Z}}(m|n) \otimes_{\mathcal{Z}} F$, and the super dot product $(\ ,\ )_s$ on $\mathbb{Z}^{m+n}$ introduced at the end of the introduction. We first generalise Serganova's algorithm for the supergroup $GL(m|n)$ given in [4, Lem. 4.2, Thm 4.3]) to the quantum hyperalgebra $U_{q,F}$.

Proposition 8.2 *Let $\lambda \in \mathbb{Z}^{m|n}_{++}$ and choose a nonzero vector $\mathfrak{m}_\lambda \in L(\lambda)_\lambda$ for the irreducible $U_{q,F}$-module $L(\lambda)$. Fix the following ordering on positive odd roots:*

$$\beta_1 = \alpha_{m,m+1}, \ldots, \beta_m = \alpha_{1,m+1}, \beta_{m+1} = \alpha_{m,m+2}, \ldots,$$
$$\beta_{2m} = \alpha_{1,m+2}, \ldots, \beta_{mn} = \alpha_{1,m+n}.$$

Define recursively $\mathfrak{m}_\lambda^{(0)} = \mathfrak{m}_\lambda$ and, for $1 \le k \le m+n$,

$$\mathfrak{m}_\lambda^{(k)} = \begin{cases} \mathfrak{m}_\lambda^{(k-1)}, & \textit{if } l \mid (\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}), \beta_k)_s, \\ E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}, & \textit{if } l \nmid (\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}), \beta_k)_s. \end{cases}$$

Then we have

$$\begin{cases}(1)\quad \mathfrak{m}_\lambda^{(k)}\neq 0, \quad 0\leq k\leq m+n\\ (2)\quad E_{i,i+1}^{(M)}\,\mathfrak{m}_\lambda^{(k)}=0, \quad 1\leq i\leq m+n-1,\ i\neq m,\\ (3)\quad E_{-\beta_i}\,\mathfrak{m}_\lambda^{(k)}=0=E_{\beta_j}\,\mathfrak{m}_\lambda^{(k)}, \quad 1\leq i\leq k<j\leq mn.\end{cases} \tag{8.2}$$

Proof We apply induction on k. The case for $k=0$ is clear since $\mathfrak{m}_\lambda^{(0)}=\mathfrak{m}_\lambda$ is a highest weight vector. Assume now $k\geq 1$ and that (1)–(3) hold for $k-1$.

Case 1 Assume $l\nmid(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s$. Then $\mathfrak{m}_\lambda^{(k)}=E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}$ and $E_{\beta_k}\,\mathfrak{m}_\lambda^{(k-1)}=0$ by induction. Thus, by Proposition 2.4(4)

$$\begin{aligned}E_{\beta_k}\,\mathfrak{m}_\lambda^{(k)}&=E_{\beta_k}\,E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}=-E_{-\beta_k}E_{\beta_k}\,\mathfrak{m}_\lambda^{(k-1)}+\frac{K_{\beta_k}-K_{\beta_k}^{-1}}{q-q^{-1}}\mathfrak{m}_\lambda^{(k-1)}\\&=\frac{q^{(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s}-q^{-(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s}}{q-q^{-1}}\mathfrak{m}_\lambda^{(k-1)}\neq 0.\end{aligned} \tag{8.3}$$

Hence, $\mathfrak{m}_\lambda^{(k)}\neq 0$, proving (1).

To see (2), we assume $\beta_k=\epsilon_c-\epsilon_d$ with $1\leq c\leq m<d\leq m+n$. If $i+1\leq c$ or $c<i<i+1<d$, then Proposition 2.4(1) and induction imply (2); if $c=i<i+1<d$, then $i=c<m$ and, by Proposition 2.4(3), either $E_{i,i+1}E_{-\beta_k}=xE_{i,i+1}+yE_{-\beta_{k-1}}$ or, for $M>1$, $E_{i,i+1}^{(M)}E_{-\beta_k}=x'E_{i,i+1}^{(M)}+y'E_{i,i+1}^{(M-1)}$. So (2) follows from induction.

Finally, if $k<j$ and $\beta_k=\epsilon_a-\epsilon_b$, $\beta_j=\epsilon_c-\epsilon_d$, then either $b<d$ or $b=d$,$a>c$. For $b<d$, applying Υ to Proposition 2.4(1)(3)(5) (for $b=d$,$a>c$, using directly Proposition 2.4(2)) and induction gives $E_{\beta_j}\,\mathfrak{m}_\lambda^{(k)}=0$ in (3). To verify $E_{-\beta_i}\,\mathfrak{m}_\lambda^{(k)}=0$ for all $1\leq i\leq k$, since $E_{-\beta_k}^2=0$, it suffices to consider the commutator formulas for $E_{-\beta_i}E_{-\beta_k}$ for $1\leq i<k$. Suppose $E_{-\beta_i}=E_{b,a}, E_{-\beta_k}=E_{d,c}$, for some $1\leq a,c\leq m<b,d\leq m+n$. Then $i<k$ implies $b<d$ or $b=d,a>c$. If $b<d$, then $c<b<d$ and, by applying the automorphism ϖ defined in (2.1) to Proposition 2.3(1)(2)(4), we see that $E_{-\beta_i}E_{-\beta_k}=E_{b,a}E_{d,c}=xE_{b,a}+yE_{b,c}$. If $b=d,a>c$ then $c<a<b=d$ and, by applying Υ to Proposition 2.3 (2), we have $E_{-\beta_k}E_{-\beta_i}=E_{d,c}E_{b,a}=-q_cE_{b,a}E_{d,c}=-q_cE_{-\beta_i}E_{-\beta_k}$. In both cases, $E_{-\beta_i}\,\mathfrak{m}_\lambda^{(k)}=0$ follows from induction.

Case 2 Assume $l\mid(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s$. Then $\mathfrak{m}_\lambda^{(k)}=\mathfrak{m}_\lambda^{(k-1)}$. By induction, it remains to prove $E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}=0$. Suppose $E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}\neq 0$. Then $L(\lambda)=U_{q,F}(E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)})$ and so $\mathfrak{m}_\lambda^{(k-1)}\in U_{q,F}(E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)})$.

We claim that $\mathfrak{m}_\lambda^{(k-1)}\in\mathrm{span}\{E_{\beta_k}E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}\}$. Indeed, if $\Phi_{\bar 0}^+$ denotes the subset of even roots in Φ^+, by (2.11) and the commutation formulas of Proposition 2.3 and 2.4, we see that $U_{q,F}$ is spanned by the elements

$$\left\{\prod_{\epsilon_a-\epsilon_b\in\Phi_{\bar 0}^+}E_{b,a}^{(A_{b,a})}\prod_{i=k+1}^{mn}E_{-\beta_i}^{\sigma_i}\prod_{i=1}^{k}E_{\beta_i}^{\sigma_i}\prod_{a=1}^{m+n}\left(K_a^{\delta_a}\begin{bmatrix}K_a\\ \mu_a\end{bmatrix}\right)\prod_{i=1}^{k}E_{-\beta_i}^{\sigma_i'}\prod_{i=k+1}^{mn}E_{\beta_i}^{\sigma_i'}\prod_{\epsilon_a-\epsilon_b\in\Phi_{\bar 0}^+}E_{a,b}^{(A_{a,b})}\right\},$$

where $A \in \mathcal{M}(m|n)$, $\delta_a \in \{0, 1\}$ with $\{\sigma_i, \sigma_i'\} = \{A_{\beta_i}, A_{-\beta_i}\}$ and $\mu_a = A_{a,a}$. By the proof for (2) and (3) above, the elements $\prod_{i=1}^{k} E_{-\beta_i}^{\sigma_i'} \prod_{i=k+1}^{mn} E_{\beta_i}^{\sigma_i'} \prod_{\epsilon_a-\epsilon_b\in\Phi_0^+} E_{a,b}^{(A_{a,b})}$ vanish $E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}$. Thus, we have

$$L(\lambda) = \operatorname{span}\left\{ \prod_{\epsilon_a-\epsilon_b\in\Phi_0^+} E_{b,a}^{(A_{b,a})} \prod_{i=k+1}^{mn} E_{-\beta_i}^{\sigma_i} \prod_{i=1}^{k} E_{\beta_i}^{\sigma_i}.E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)} \right\}.$$

We now consider the weight space $L(\lambda)_\mu$ with $\mu = \mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)})$. Since a spanning vector has its weight of the form $\mu + \sum_{\epsilon_a-\epsilon_b\in\Phi_0^+} A_{b,a}(\epsilon_b - \epsilon_a) - \sum_{i=k+1}^{mn} \sigma_i\beta_i + \sum_{i=1}^{k} \sigma_i\beta_i - \beta_k$, such a vector in $L(\lambda)_\mu$ forces

$$\sum_{\epsilon_a-\epsilon_b\in\Phi_0^+} A_{b,a}(\epsilon_b - \epsilon_a) = \sum_{j=k+1}^{mn} \sigma_j\beta_j - \sum_{i=1}^{k}\sigma_i\beta_i + \beta_k =:(\nu^{(0)}, \nu^{(1)}) \in \mathbb{Z}^{m+n}. \tag{8.4}$$

Note the left-hand side implies that $|\nu^{(0)}| = |\nu^{(1)}| = 0$. This forces $\#X = \#Y$, where $X = \{i \mid 1 \le i \le k, \sigma_i \ne 0\}$ and $Y = \{j \mid k < j \le mn, \sigma_j \ne 0\} \cup \{k\}$. Suppose $\beta_i = \epsilon_{a_i} - \epsilon_{b_i}$, $\beta_{j_i} = \epsilon_{c_i} - \epsilon_{d_i}$, where $i \mapsto j_i$ is a bijection from X to Y and $1 \le a_i, c_i \le m < b_i, d_i \le m+n$. Then $\beta_{j_i} - \beta_i = (\epsilon_{c_i} - \epsilon_{a_i}) + (\epsilon_{b_i} - \epsilon_{d_i})$. Thus, $i < j_i$ forces $m < b_i \le d_i$ and so $\epsilon_{b_i} - \epsilon_{d_i} \in \Phi_0^+$. Hence, (8.4) implies

$$\sum_{1\le a<b\le m} A_{b,a}(\epsilon_b - \epsilon_a) = \nu^{(0)} = \sum_{i\in X}(\epsilon_{c_i} - \epsilon_{a_i}),$$
$$\sum_{m<a<b\le m+n} A_{b,a}(\epsilon_b - \epsilon_a) = \nu^{(1)} = \sum_{i\in X}(\epsilon_{b_i} - \epsilon_{d_i}).$$

The second equality is possible unless both sides are zero. Thus, all $b_i = d_i$, forcing $c_i \le a_i$. Hence, the first equality must be zero and so $c_i = a_i$ for all $i \in X$. Therefore, we must have all $A_{b,a} = 0$ and $\sigma_i = \delta_{k,i}$. Consequently, $L(\lambda)_\mu = \operatorname{span}\{E_{\beta_k}E_{-\beta_k}\mathfrak{m}_\lambda^{(k-1)}\}$, proving the claim.

Since $L(\lambda)_\mu \ne 0$, the claim and (8.3) force $\frac{q^{(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s} - q^{-(\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}),\beta_k)_s}}{q-q^{-1}} \ne 0$. This implies $l \nmid (\mathrm{wt}(\mathfrak{m}_\lambda^{(k-1)}), \beta_k)_s$, contrary to the assumption for Case 2. □

Proposition 8.2 gives a (bijective) map

$$\widetilde{\ }: \mathbb{Z}_{++}^{m|n} \longrightarrow \mathbb{Z}_{++}^{m|n}, \lambda \longmapsto \widetilde{\lambda} := \mathrm{wt}(\mathfrak{m}_\lambda^{(mn)}),$$

cf. [4, (4.1)]. The construction of $\tilde{\lambda}$ from λ is known as Serganova's algorithm.

Remark 8.3 The original Serganova algorithm (see [23] or [4, Lem. 4.2]) is a simple algorithm that allows us to pass between the highest weight labellings of irreducible supermodules defined by neighbouring Borel subgroups of the supergroup $GL(m|n)$. Applying a sequence of the neighbouring algorithms, by going from the upper triangular to lower triangular Borel subgroups, yields the map $\widetilde{\ }$ in [4, Thm 4.3]. Our algorithm here generalises this map to the quantum case and is presented within the module $L(\lambda)$.

Assume now $r \le m, n$. We define the following two maps as in [4, §6]:

$$\begin{aligned} x:&\quad \Lambda_l^+(r) \to \Lambda^{++}(m|n,r), \quad \lambda \to x(\lambda) = (\lambda, 0^{m-r}|0^n)); \\ y:&\quad \Lambda_l^+(r) \to \Lambda^{++}(m|n,r), \quad \lambda \to y(\lambda) = (0^m|\lambda, 0^{n-r})). \end{aligned} \tag{8.5}$$

Serganova's algorithm, together with Xu's algorithm for the Mullineux map (8.1) via the j_l map defined in (6.1), has the following relationship as revealed in [4, Lem. 6.3].

Corollary 8.4 *If the $m, n \ge r$ and $\lambda \in \Lambda_l^+(r)$, then $\widetilde{x(\lambda)} = y(\mathtt{M}(\lambda))$.*

When $m = n$, we may use this algorithm to compute the highest weight of a simple module twisted by the automorphism σ_F on $U_{q,F}(n|n)$; see (2.8).

Recall that, for any $U_{q,F}(n|n)$-supermodule V, the $U_{q,F}(n|n)$-supermodule V^σ is defined by setting $V^\sigma = V$ as a vector space with a new action defined by

$$x \centerdot v = \sigma(x)v, \quad v \in V,\ x \in U_{q,F}(n|n).$$

The map $V \mapsto V^\sigma$ defines a category isomorphism $U_{q,F}(n|n)$-mod $\cong U_{q,F}(n|n)$-mod. We now use $\tilde{\lambda}$ to determine the highest weight of the irreducible $U_{q,F}(n|n)$-supermodule $L(\lambda)^\sigma$ (cf. [4, Thm 4.5]).

Theorem 8.5 *For $\lambda \in \mathbb{Z}_{++}^{n|n}$, let $L(\lambda)$ be an irreducible $U_{q,F}(n|n)$-supermodule with a highest weight vector $\mathfrak{m}_\lambda$ and let $\widetilde{\lambda} = (\tilde{\lambda}^{(0)}|\tilde{\lambda}^{(1)}) = \mathrm{wt}(\mathfrak{m}_\lambda^{(n^2)})$. Then the $U_{q,F}(n|n)$-supermodule $L(\lambda)^\sigma$ is isomorphic to $L(\lambda^\sigma)$, where $\lambda^\sigma = (\tilde{\lambda}^{(1)}|\tilde{\lambda}^{(0)})$. Furthermore, if we assume $r \le m = n$, then, for any $\lambda \in \Lambda_l^+(r)$, we have*

$$L(x(\lambda))^\sigma \cong L(x(\mathtt{M}(\lambda))),$$

where $\mathtt{M}$ is the Mullineux map (8.1).

Proof Since $L(\lambda)^\sigma$ is an irreducible supermodule, it is enough to determine its highest weight. From the definition of the isomorphism σ, $v \in L(\lambda)^\sigma$ is a maximal vector if and only if v satisfies:

$$0 = E_i^{(M)} \centerdot v = \sigma(E_i^{(M)})\, v = F_{2n-i}^{(M)}\, v, \quad M > 0,\ 1 \le i \le 2n-1. \tag{8.6}$$

This is equivalent to say that v is a lowest weight vector of $L(\lambda)$.

By Proposition 8.2, $\mathfrak{m}_\lambda^{(n^2)}$ is a maximal vector for even subsuperalgebra $U_{q,F}(n|n)_{\bar{0}} \cong U_{q,F}(\mathfrak{gl}_n) \otimes U_{q^{-1},F}(\mathfrak{gl}_n)$. Moreover,

$$E_{-\beta_t}\, \mathfrak{m}_\lambda^{(n^2)} = 0,\ 1 \le t \le n^2. \tag{8.7}$$

Let

$$\begin{aligned}\mathfrak{m}_\lambda^\sigma =&F_1^{(\tilde\lambda^{(0)}_{n-1}-\tilde\lambda^{(0)}_n)}(F_2^{(\tilde\lambda^{(0)}_{n-2}-\tilde\lambda^{(0)}_n)}F_1^{(\tilde\lambda^{(0)}_{n-2}-\tilde\lambda^{(0)}_{n-1})})\cdots(F_{n-1}^{(\tilde\lambda^{(0)}_1-\tilde\lambda^{(0)}_n)}\cdots F_1^{(\tilde\lambda^{(0)}_1-\tilde\lambda^{(0)}_2)})\\ &\cdot F_{n+1}^{(\tilde\lambda^{(1)}_{n-1}-\tilde\lambda^{(1)}_n)}(F_{n+2}^{(\tilde\lambda^{(1)}_{n-2}-\tilde\lambda^{(1)}_n)}F_{n+1}^{(\tilde\lambda^{(1)}_{n-2}-\tilde\lambda^{(1)}_{n-1})})\cdots(F_{2n-1}^{(\tilde\lambda^{(1)}_1-\tilde\lambda^{(1)}_n)}\cdots F_{n+1}^{(\tilde\lambda^{(1)}_1-\tilde\lambda^{(1)}_2)})\mathfrak{m}_\lambda^{(n^2)}.\end{aligned}$$

Then, by Proposition A.1, we have, for all $1 \le i \le 2n-1$ and $i \ne n$, $F_i^{(M)}.\mathfrak{m}_\lambda^\sigma = 0$. To see $F_n.\mathfrak{m}_\lambda^\sigma = 0$, observe the commutation formulas in Proposition 2.3. If $E_{c,d}$ is odd and $E_{a,b}$ is even, then all the RHS of formulas (1)–(4) are sums of terms starting with an odd root vector. (Only in (4), we need (3) to swap $E_{c,b}^{(t)}E_{c,d}^{(N-t)}$.) Applying Υ produces half of the required formulas to prove $F_n\,\mathfrak{m}_\lambda^\sigma = 0$. The other half can be obtained by applying ϖ to (1)–(4) (see (2.6)), assuming $E_{a,b}$ is odd and $E_{c,d}$ is even. (In (4), we see $E_{a,d}E_{c,b} = E_{c,b}E_{a,b}$ by (1).) Repeatedly applying the eight sets of formulas, we see that $F_n\,\mathfrak{m}_\lambda^\sigma = 0$ follows from (8.7). Hence, $\mathfrak{m}_\lambda^\sigma$ is a lowest weight vector of $L(\lambda)$ or a highest weight vector of $L(\lambda)^\sigma$.

It remains to compute the weight $\mathrm{wt}_{L(\lambda)^\sigma}(\mathfrak{m}_\lambda^\sigma)$. By Proposition A.1,

$$\mathrm{wt}(\mathfrak{m}_\lambda^\sigma) = (\tilde\lambda^{(0)}_m, \ldots, \tilde\lambda^{(0)}_1 | \tilde\lambda^{(1)}_n, \ldots, \tilde\lambda^{(1)}_1) = \mu$$

in $L(\lambda)$. Since the isomorphism σ_F sends $K_i^\pm$ to $K_{2n-i+1}^\mp$ and $\left[{K_i \atop t}\right]$ to $\left[{K_{2n+1-i}^{-1} \atop t}\right]$, it follows from (5.1), (5.2) that $v \in L(\lambda)_\mu$ if and only if $v \in (L(\lambda)^\sigma)_{\mu^\dagger}$. Hence, $\mathrm{wt}_{L(\lambda)^\sigma}(\mathfrak{m}_\lambda^\sigma) = \mu^\dagger = (\tilde\lambda^{(1)}|\tilde\lambda^{(0)}) = \lambda^\sigma$ and therefore, λ^σ is the highest weight of $L(\lambda)^\sigma$.

Now, with the hypothesis $r \le m = n$, the last assertion follows from the first assertion and Corollary 8.4. □

9 Matching Schur functors and the Mullineux conjecture

Throughout this section, we assume $m, n \ge r$ and let

$$\omega = (1^r, 0^{m-r}|0^n),\ \omega' = (0^m|0^{n-r}, 1^r) \in \Lambda(m|n, r).$$

We will identify $H_{q^2,F}(r)$ with $1_\omega S_q(m|n,r)1_\omega$ under the isomorphism $t_i \mapsto T_i$, where t_i is defined in the proof of Corollary 4.5.

Consider two Schur functors $\mathfrak{f}_\omega, \mathfrak{f}_{\omega'}$ associated with the idempotents $1_\omega, 1_{\omega'}$. Thus, for every $\chi \in \{\omega, \omega'\}$,

$$\mathfrak{f}_\chi : S_{q,F}(m|n,r)\text{-}\mathsf{mod} \longrightarrow 1_\chi S_{q,F}(m|n,r)1_\chi\text{-}\mathsf{mod},$$

satisfying $\mathfrak{f}_\chi(V) = 1_\chi V$. We will make a comparison for the modules $\mathfrak{f}_\omega(V), \mathfrak{f}_{\omega'}(V)$.

Lemma 9.1 *Assume $m \ge r$. If $\lambda \in \Lambda_l^+(r)$, then $\mathfrak{f}_\omega L(x(\lambda)) \ne 0$. Hence, $\mathfrak{f}_\omega L(x(\lambda))$ is an irreducible $H_{q^2,F}(r)$-module.*

Proof This is clear since $L(x(\lambda))$ contains the irreducible module $L(x(\lambda))_{\bar{0}}$ for the even quantum subsupergroup $U_{q,F}(n|n)_{\bar{0}}$ and $1_\omega L(x(\lambda))_{\bar{0}} \neq 0$. □

This lemma guarantees that if we put (cf. [4, Thm 5.9, Rem. 5.10])

$$D_\lambda := \mathfrak{f}_\omega L(x(\lambda)), \tag{9.1}$$

then the set $\{D_\lambda\}_{\lambda \in \Lambda_l^+(r)}$ forms a complete set of irreducible $H_{q^2,F}(r)$-modules.

The following result follows from Corollary 4.5.

Lemma 9.2 *Assume $m, n \geq r$ and, for $1 \leq i \leq r-1$, let $t_i = q1_\omega \mathtt{e}_i \mathtt{f}_i 1_\omega - 1_\omega$ and $t_{m+n-r+i} = q1_{\omega'} \mathtt{e}_{m+n-r+i} \mathtt{f}_{m+n-r+i} 1_{\omega'} - 1_{\omega'}$. Then the map*

$$\tau : H_{q^2,F}(r) = 1_\omega S_{q,F}(m|n,r)1_\omega \longrightarrow 1_{\omega'} S_{q,F}(m|n,r)1_{\omega'},\ t_i \longmapsto t_{m+n-r+i}$$

defines an algebra isomorphism $1_\omega S_q(m|n,r)1_\omega \cong 1_{\omega'} S_q(m|n,r)1_{\omega'}$.

Thus, we may twist an $1_{\omega'} S_{q,F}(m|n,r)1_{\omega'}$-module V by τ to get an $H_{q,F}(r)$-module V^τ. We now establish the relationship between the two Schur functors.

Proposition 9.3 *Assume $m, n \geq r$. For any $S_{q,F}(m|n,r)$-supermodule V, there is an $H_{q^2,F}$-module isomorphism*

$$(\mathfrak{f}_\omega V)^\sharp \cong (\mathfrak{f}_{\omega'} V)^\tau.$$

Proof Recall the generators in (3.14) and let $\mathtt{e}_{a,b} = \eta_{r,F}(E_{a,b})$ (see (3.16)). Let

$$\mathsf{F} = \mathtt{e}_{m+n-r+1,1}\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}.$$

Then, by Lemma 4.1(6), $\mathsf{F}1_\omega = 1_{\omega'}\mathsf{F}$. We first claim that the map

$$g : 1_\omega V \longrightarrow 1_{\omega'} V, \quad 1_\omega v \longmapsto \mathsf{F}1_\omega v \ \ (\forall v \in V)$$

is a linear isomorphism. Indeed, applying Proposition 2.4(4) yields

$$\begin{aligned}
&\mathtt{e}_{r,m+n}\cdots\mathtt{e}_{2,m+n-r+2}\mathtt{e}_{1,m+n-r+1}(\mathsf{F}1_\omega)\\
&= \mathtt{e}_{r,m+n}\cdots\mathtt{e}_{2,m+n-r+2}(\mathtt{e}_{1,m+n-r+1}\mathtt{e}_{m+n-r+1,1})\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}1_\omega\\
&= \mathtt{e}_{r,m+n}\cdots\mathtt{e}_{2,m+n-r+2}\begin{bmatrix}\mathtt{k}_{1,m+n-r+1}\\ 1\end{bmatrix}\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}1_\omega\\
&\quad - \mathtt{e}_{r,m+n}\cdots\mathtt{e}_{2,m+n-r+2}(\mathtt{e}_{m+n-r+1,1}\mathtt{e}_{1,m+n-r+1})\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}1_\omega\\
&\overset{(*)}{=} \mathtt{e}_{r,m+n}\cdots\mathtt{e}_{2,m+n-r+2}\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}1_\omega\\
&= \cdots = \mathtt{e}_{r,m+n}\mathtt{e}_{m+n,r}1_\omega = 1_\omega.
\end{aligned}$$

Here the equality $(*)$ is seen from Lemma 4.1(6) and (5.2), since

$$\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r}1_\omega = 1_\lambda\mathtt{e}_{m+n-r+2,2}\cdots\mathtt{e}_{m+n,r},$$

where $\lambda = \omega + \alpha_{m+n-r+2,2} + \cdots + \alpha_{m+n,r} = (1, 0^{m-1}|0^{n-r+1}, 1^{r-1})$. Thus, we have $\mathsf{F}1_\omega v \neq 0 \iff 1_\omega v \neq 0$. Hence, g is injective and so $\dim 1_\omega V \leq \dim 1_{\omega'} V$. Similarly, we may use $\mathsf{F}' = \mathsf{e}_{1,m+n-r+1} \cdots \mathsf{e}_{r-1,m+n-1}\mathsf{e}_{r,m+n}$ to prove $\dim 1_{\omega'} V \leq \dim 1_\omega V$. Hence, g is a bijection.

We now show that the map g is an $H_{q^2,F}$-module isomorphism. This amounts to prove that, for any $v \in 1_\omega V$ and $1 \leq i < r$,

$$g((-t_i + (q^2 - 1)1_\omega)v) = g(t_i^\sharp v) = \tau(t_i)g(v) = t_{m+n-r+i}\, g(v). \tag{9.2}$$

We prove (9.2) by showing that in $S_{q,F}(m|n,r)$

$$-\mathsf{F}t_i 1_\omega + (q^2 - 1)\mathsf{F}1_\omega = \mathsf{F}(t_i^\sharp 1_\omega) = t_{m+n-r+i}\mathsf{F}1_\omega. \tag{9.3}$$

Let

$$n'' = m + n - r, \omega'' = \omega - \alpha_{r,n''+r} - \cdots - \alpha_{i+2,n''+i+2}$$
$$= (1^{i+1}, 0^{m-i-1}|0^a, 1^{r-i-1}).$$

Then, for $1 \leq i \leq r$,

$$\begin{aligned}&\mathsf{F} \cdot 1_\omega \mathsf{e}_i \mathsf{f}_i 1_\omega\\ &= \mathsf{e}_{m+n-r+1,1}\mathsf{e}_{m+n-r+2,2} \cdots \mathsf{e}_{m+n,r} 1_\omega \mathsf{e}_{i,i+1}\mathsf{e}_{i+1,i} 1_\omega = \mathsf{e}_{n''+1,1} \cdots \mathsf{e}_{n''+i-1,i-1}\\ &\quad \cdot \mathsf{e}_{n''+i,i}\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i,i+1}\mathsf{e}_{i+1,i} 1_{\omega''} \cdot \mathsf{e}_{n''+i+2,i+2} \cdots \mathsf{e}_{n''+r,r} 1_\omega.\end{aligned}$$

Let (a) stand for Propositions 2.4(1); (b) for Propositions 2.4(3); (c) for Lemma 4.1(6); (d) for Lemma 4.1(2). Let (e) be the formula obtained by applying Υ in (2.2) to Propositions 2.3(3). The middle part of the product above becomes

$$\begin{aligned}&\mathsf{e}_{n''+i,i}(\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i,i+1})\mathsf{e}_{i+1,i} 1_{\omega''}\\ &\overset{(a)}{=} \mathsf{e}_{n''+i,i}(\mathsf{e}_{i,i+1}\mathsf{e}_{n''+i+1,i+1})\mathsf{e}_{i+1,i} 1_{\omega''} = (\mathsf{e}_{n''+i,i}\mathsf{e}_{i,i+1})\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i} 1_{\omega''}\\ &\overset{(b)}{=} (\mathsf{e}_{i,i+1}\mathsf{e}_{n''+i,i} + \mathsf{k}_{i,i+1}\mathsf{e}_{n''+i,i+1})\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i} 1_{\omega''}\\ &\overset{(c)}{=} \mathsf{k}_{i,i+1}\mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i} 1_{\omega''} \quad (\text{as } \mathsf{e}_{n''+i,i}\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i} 1_{\omega''} = 0)\\ &\overset{(d)}{=} \mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i} 1_{\omega''} = \mathsf{e}_{n''+i,i+1}(\mathsf{e}_{n''+i+1,i+1}\mathsf{e}_{i+1,i}) 1_{\omega''}\\ &\overset{(e)}{=} \mathsf{e}_{n''+i,i+1}(\mathsf{e}_{n''+i+1,i} + q\mathsf{e}_{i+1,i}\mathsf{e}_{n''+i+1,i+1}) 1_{\omega''}\\ &\overset{(e)}{=} \mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i} 1_{\omega''} + q\mathsf{e}_{n''+i,i}\mathsf{e}_{n''+i+1,i+1} 1_{\omega''}\\ &\quad + q^2\mathsf{e}_{i+1,i}\mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i+1} 1_{\omega''}\\ &\overset{(c)}{=} \mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i} 1_{\omega''} + q\mathsf{e}_{n''+i,i}\mathsf{e}_{n''+i+1,i+1} 1_{\omega''}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathsf{F}\cdot 1_\omega \mathrm{e}_i \mathrm{f}_i 1_\omega\\&= \mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i-1,i-1}\mathrm{e}_{n''+i,i+1}\mathrm{e}_{n''+i+1,i}1_{\omega''}\mathrm{e}_{n''+i+2,i+2}\cdots \mathrm{e}_{n''+r,r}1_\omega\\&\quad + q\mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i-1,i-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\mathrm{e}_{n''+i+2,i+2}\cdots \mathrm{e}_{n''+r,r}1_\omega.\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{F}t_i 1_\omega &= \mathsf{F}(q1_\omega \mathrm{e}_{i,i+1}\mathrm{e}_{i+1,i}1_\omega - 1_\omega)\\&= q\mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i,i+1}\mathrm{e}_{n''+i+1,i}\cdots \mathrm{e}_{n''+r,r}1_\omega\\&\quad + (q^2-1)\mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}\cdots \mathrm{e}_{n''+r,r}1_\omega.\\\mathsf{F}t_i^\sharp 1_\omega &= \mathsf{F}(-t_i 1_\omega + (q^2-1)1_\omega) = -q\mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i-1,i-1}\\&\quad (\mathrm{e}_{n''+i,i+1}\mathrm{e}_{n''+i+1,i})\mathrm{e}_{n''+i+2,i+2}\cdots \mathrm{e}_{n''+r,r}1_\omega.\end{aligned} \tag{9.4}$$

Similarly,

$$\begin{aligned}&1_{\omega'}\mathrm{e}_{n''+i}\mathrm{f}_{n''+i}1_{\omega'}(\mathsf{F}1_\omega)\\&= 1_{\omega'}\mathrm{e}_{n''+i,n''+i+1}\mathrm{e}_{n''+i+1,n''+i}1_{\omega'}\mathrm{e}_{n''+1,1}\mathrm{e}_{n''+2,2}\cdots \mathrm{e}_{n''+r,r}1_\omega\\&= \mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i-1,i-1}\cdot \mathrm{e}_{n''+i,n+i+1}\mathrm{e}_{n''+i+1,n''+i}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\quad \cdot \mathrm{e}_{n''+i+2,i+2}\cdots \mathrm{e}_{n''+r,r}1_\omega.\end{aligned}$$

Let (u) be the formula obtained by applying Υ to Proposition 2.4(2) twice; (v) for Lemma 2.2; and (w) for the Υ-version of Proposition 2.3(1). Then the middle part of the product above becomes

$$\begin{aligned}&\mathrm{e}_{n''+i,n''+i+1}(\mathrm{e}_{n''+i+1,n''+i}\mathrm{e}_{n''+i,i})\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(e)}{=} \mathrm{e}_{n''+i,n''+i+1}(\mathrm{e}_{n''+i+1,i} + q^{-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,n''+i})\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(c)}{=} (\mathrm{e}_{n''+i,n''+i+1}\mathrm{e}_{n''+i+1,i})\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(u)}{=} (\mathrm{e}_{n''+i+1,i}\mathrm{e}_{n''+i,n''+i+1} + \mathrm{e}_{n''+i,i}\mathrm{k}^{-1}_{n''+i,n''+i+1})\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(v)}{=} \mathrm{e}_{n''+i+1,i}(\mathrm{e}_{n''+i,n''+i+1}\mathrm{e}_{n''+i+1,i+1})1_{\omega''} + q^{-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}\mathrm{k}^{-1}_{n''+i,n''+i+1}1_{\omega''}\\&\overset{(u)}{=} \mathrm{e}_{n''+i+1,i}(\mathrm{e}_{n''+i+1,i+1}\mathrm{e}_{n''+i,n''+i+1} + \mathrm{e}_{n''+i,i+1}\mathrm{k}^{-1}_{n''+i,n''+i+1})1_{\omega''}\\&\overset{(d)}{} + q^{-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(c,d)}{=} \mathrm{e}_{n''+i+1,i}\mathrm{e}_{n''+i,i+1}1_{\omega''} + q^{-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}1_{\omega''}\\&\overset{(w)}{=} -\mathrm{e}_{n''+i,i+1}\mathrm{e}_{n''+i+1,i}1_{\omega''} + q^{-1}\mathrm{e}_{n''+i,i}\mathrm{e}_{n''+i+1,i+1}1_{\omega''}.\end{aligned}$$

Thus,

$$\begin{aligned}1_{\omega'}\mathrm{e}_{n''+i}\mathrm{f}_{n''+i}1_{\omega'}(\mathsf{F}1_\omega) &= -\mathrm{e}_{n''+1,1}\cdots \mathrm{e}_{n''+i-1,i-1}\cdot \mathrm{e}_{n''+i,i+1}\mathrm{e}_{n''+i+1,i}\\&\quad \cdot \mathrm{e}_{n''+i+2,i+2}\cdots \mathrm{e}_{n''+r,r}1_\omega + q^{-1}\mathsf{F}1_\omega.\end{aligned}$$

Hence, by (9.4),

$$
\begin{aligned}
t_{n''+i}\mathsf{F}1_\omega &= (q1_{\omega'}\mathsf{e}_{n''+i}\mathsf{f}_{n''+i}1_{\omega'} - 1_{\omega'})(\mathsf{F}1_\omega) = -q\mathsf{e}_{n''+1,1} \\
&\quad \cdots \mathsf{e}_{n''+i-1,i-1}(\mathsf{e}_{n''+i,i+1}\mathsf{e}_{n''+i+1,i})\mathsf{e}_{n''+i+2,i+2}\cdots \mathsf{e}_{n''+r,r}1_\omega \\
&= \mathsf{F}(t_i^\sharp 1_\omega),
\end{aligned}
$$

proving (9.3), and hence, (9.2). □

When $m = n \geq r$, the automorphism σ_F on $S_{q,F}(n|n,r)$ (see Lemma 4.6) takes 1_ω to $1_{\omega'}$. So restriction induces an algebra isomorphism (see (4.3))

$$\bar{\sigma} : H_{q^2,F}(r) = 1_\omega S_{q,F}(n|n,r)1_\omega \longrightarrow 1_{\omega'}S_{q,F}(n|n,r)1_{\omega'}, \quad t_i \longmapsto t_{2n-i},$$

for $1 \leq i \leq r-1$. Thus, twisting module actions define a functor

$$\bar{\sigma} : 1_{\omega'}S_{q,F}(n|n,r)1_{\omega'}\text{-}\mathsf{mod} \longrightarrow H_{q^2,F}(r)\text{-}\mathsf{mod}.$$

Note that, if V is an $S_q(n|n,r)$-supermodule, then $(\mathfrak{f}_{\omega'}V)^{\bar{\sigma}}$ is an $H_{q^2,F}(r)$-module via the action $T_i \cdot x = \bar{\sigma}(t_i)x$ for all $x \in \mathfrak{f}_{\omega'}V$. Likewise, $\mathfrak{f}_\omega(V^\sigma)$ is an $H_{q^2,F}(r)$-module via the action $T_i \cdot y = \sigma(t_i)y$ for all $y \in \mathfrak{f}_\omega(V^\sigma)$.

Lemma 9.4 *With the notation above, the following diagram*

$$
\begin{array}{ccc}
S_{q,F}(n|n,r)\text{-}\mathsf{mod} & \xrightarrow{\ \mathfrak{f}_\omega\ } & H_{q^2,F}\text{-}\mathsf{mod} \\
\sigma\big\uparrow & & \big\uparrow\bar{\sigma} \\
S_{q,F}(n|n,r)\text{-}\mathsf{mod} & \xrightarrow{\ \mathfrak{f}_{\omega'}\ } & 1_{\omega'}S_{q,F}(n|n,r)1_{\omega'}\text{-}\mathsf{mod}
\end{array}
$$

is commutative. In other words, $(\mathfrak{f}_{\omega'}V)^{\bar{\sigma}} = \mathfrak{f}_\omega(V^\sigma)$ *for any* $S_{q,F}$*-supermodule* V*.*

Proof Since $\sigma(1_{\omega'}) = 1_\omega$, we have $1_{\omega'}V = 1_\omega(V^\sigma)$ or $\mathfrak{f}_{\omega'}(V) = \mathfrak{f}_\omega(V^\sigma)$ as vectors spaces. Now it is easy to see from the above that the $H_{q^2,F}$-module structures on both side are the same. □

We are now ready to proof the quantum version of the Mullineux conjecture.

Theorem 9.5 *For any* $\lambda \in \Lambda_l^+(r)$*, the irreducible* $H_{q^2,F}(r)$*-modules* $D_\lambda^\sharp$ *and* $D_{\mathtt{M}(\lambda)}$ *are isomorphic:* $D_\lambda^\sharp \cong D_{\mathtt{M}(\lambda)}$*.*

Proof By definition, $D_\lambda = \mathfrak{f}_\omega L(x(\lambda))$. Then by Proposition 9.3,

$$(\mathfrak{f}_{\omega'}L(x(\lambda)))^\tau \cong (\mathfrak{f}_\omega L(x(\lambda)))^\sharp = D_\lambda^\sharp.$$

By Lemma 9.4 and Theorem 8.5 then

$$(\mathfrak{f}_{\omega'}L(x(\lambda)))^{\bar{\sigma}} \cong \mathfrak{f}_\omega(L(x(\lambda))^\sigma) \cong \mathfrak{f}_\omega L(x(\mathtt{M}(\lambda))) = D_{\mathtt{M}(\lambda)}.$$

Since $\tau^{-1}\bar{\sigma}(t_i) = t_{r-i}$ is the automorphism induced by the graph automorphism for the Hecke algebra $H_{q^2,F}(r)$, we have $(D_\lambda)^{\tau^{-1}\bar{\sigma}} \cong D_\lambda$. Therefore,

$$D_\lambda^\sharp \cong (\mathfrak{f}_{\omega'} L(x(\lambda)))^\tau \cong ((\mathfrak{f}_{\omega'} L(x(\lambda)))^\tau)^{\tau^{-1}\bar{\sigma}} \cong (\mathfrak{f}_{\omega'} L(x(\lambda)))^{\bar{\sigma}} \cong D_{\mathtt{M}(\lambda)},$$

as desired. □

Acknowledgements The first author would like to thank Jonathan Brundan for his comments made at the 2016 Charlottesville and 2017 Sydney conferences. The authors also thank Weiqiang Wang and Hebing Rui for several discussions and thank the referee for some helpful comments.

Appendix A. The lowest weight of an irreducible $U_{q,F}(m|0)$-module

Let $U_{q,F}(m) = U_{q,F}(m|0)$ be the quantum hyperalgebra of $\mathfrak{gl}_m$ and, for $\lambda \in \mathbb{Z}_+^m := \mathbb{Z}_{++}^{m|0}$, let $L(\lambda)$ be the associated irreducible $U_{q,F}(m)$-module.

The following result should be the special case of a general result. For example, by the symmetries acting (or braid group actions) on the Weyl module $V(\lambda)$ ([21, Ch. 5]), [21, Lem. 39.1.2] tells exactly the result. However, for our purpose, one needs to extend these actions to the quantum hyperalgebra $U_{q,F}(m)$ and modules at roots of unity and establish a result for $L(\lambda)$ similar to [21, Lem. 39.1.2]. For completeness, we provide below a direct proof for the type A case.

Proposition A.1 *For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_+^m$, if $0 \neq \mathfrak{m}_\lambda \in L(\lambda)_\lambda$ is a highest weight vector, then*

$$F_1^{(\lambda_{m-1}-\lambda_m)}(F_2^{(\lambda_{m-2}-\lambda_m)} F_1^{(\lambda_{m-2}-\lambda_{m-1})}) \cdots (F_{m-1}^{(\lambda_1-\lambda_m)} \cdots F_2^{(\lambda_1-\lambda_3)} F_1^{(\lambda_1-\lambda_2)})\mathfrak{m}_\lambda \neq 0$$

is a lowest weight vector of $L(\lambda)$ with weight $\lambda^\dagger = (\lambda_m, \lambda_{m-1}, \dots, \lambda_1)$.

Proof Define recursively

$$\mathfrak{n}_\lambda^{(k)} = \begin{cases} \mathfrak{m}_\lambda, & \text{if } k = 0; \\ F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}, & \text{if } 1 \le k \le m-1. \end{cases}$$

We first claim that, for all $0 \le k \le m-1$

$$\begin{aligned} &(1) \quad \mathfrak{n}_\lambda^{(k)} \neq 0; \\ &(2) \quad \mathrm{wt}(\mathfrak{n}_\lambda^{(k)}) = (\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_m, \lambda_k, \lambda_{k-1}, \dots, \lambda_1), \\ &(3) \quad E_i^{(M)} \mathfrak{n}_\lambda^{(k)} = 0, \ \text{for } i < m-k, M > 0. \end{aligned} \tag{A.1}$$

Indeed, it is obvious if $k = 0$. Assume now $k > 0$ and that (1)–(3) hold for $k-1$. Then $\mathfrak{n}_\lambda^{(k-1)} \neq 0$ and, by Proposition 2.4(4)(1),

$$\begin{aligned} &E_1^{(\lambda_k-\lambda_{k+1})} \cdots E_{m-k-1}^{(\lambda_k-\lambda_{m-1})} E_{m-k}^{(\lambda_k-\lambda_m)} \mathfrak{n}_\lambda^{(k)} \\ &\quad = E_1^{(\lambda_k-\lambda_{k+1})} \cdots E_{m-k-1}^{(\lambda_k-\lambda_{m-1})} E_{m-k}^{(\lambda_k-\lambda_m)} F_{m-k}^{(\lambda_k-\lambda_m)} F_{m-k-1}^{(\lambda_k-\lambda_{m-1})} \cdots F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)} \\ &\quad = E_1^{(\lambda_k-\lambda_{k+1})} \cdots E_{m-k-1}^{(\lambda_k-\lambda_{m-1})} \begin{bmatrix} \tilde{K}_{m-k}; 0 \\ \lambda_k - \lambda_m \end{bmatrix} F_{m-k-1}^{(\lambda_k-\lambda_{m-1})} \cdots F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}. \end{aligned}$$

If $\mu = \mathrm{wt}(F_{m-k-1}^{(\lambda_k-\lambda_{m-1})} \cdots F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)})$, then $\mu_{m-k} = \lambda_{m-1} + (k - \lambda_{m-1}) = k$ and $\mu_{m-k+1} = \lambda_m$. Thus, by (5.1) and induction, we have

$$\begin{aligned} &E_1^{(\lambda_k-\lambda_{k+1})} \cdots E_{m-k-1}^{(\lambda_k-\lambda_{m-1})} E_{m-k}^{(\lambda_k-\lambda_m)} \mathfrak{n}_\lambda^{(k)} \\ &\quad = E_1^{(\lambda_k-\lambda_{k+1})} \cdots E_{m-k-1}^{(\lambda_k-\lambda_{m-1})} F_{m-k-1}^{(\lambda_k-\lambda_{m-1})} \cdots F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)} = \mathfrak{n}_\lambda^{(k-1)} \neq 0, \end{aligned}$$

proving $\mathfrak{n}_\lambda^{(k)} \neq 0$. Also, by induction

$$\begin{aligned} \mathrm{wt}(\mathfrak{n}_\lambda^{(k)}) &= \mathrm{wt}(F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}) \\ &= (\lambda_k, \lambda_{k+1}, \ldots, \lambda_m, \lambda_{k-1}, \lambda_{k-2}, \ldots, \lambda_1) - (\lambda_k - \lambda_{k+1})(\epsilon_1 - \epsilon_2) \\ &\quad - (\lambda_k - \lambda_{k+2})(\epsilon_2 - \epsilon_3) - \cdots - (\lambda_k - \lambda_m)(\epsilon_{m-k} - \epsilon_{m-k+1}) \\ &= (\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m, \lambda_k, \lambda_{k-1}, \ldots, \lambda_1), \end{aligned}$$

proving (2). It remains to prove (3).

For $i < m-k$ and $M > 0$, let $s = \min(M, (\lambda_k - \lambda_{k+i}))$. By Proposition 2.4(1)(4),

$$\begin{aligned} E_i^{(M)} \mathfrak{n}_\lambda^{(k)} &= E_i^{(M)} (F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}) \\ &= F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} (E_{i,}^{(M)} \cdot F_i^{(\lambda_k-\lambda_{k+i})}) \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)} \\ &= F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} \sum_{t=0}^{s} F_i^{(\lambda_k-\lambda_{k+i}-t)} \begin{bmatrix} K_{i,i+1}; 2t - M - (\lambda_k - \lambda_{k+i}) \\ t \end{bmatrix} \\ &\quad E_i^{(M-t)} \cdot F_{i-1}^{(\lambda_k-\lambda_{k+i-1})} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}. \end{aligned} \tag{A.2}$$

If $M > \lambda_k - \lambda_{k+i}$, then $M - t > 0$ for all $0 \leq t \leq s = \lambda_k - \lambda_{k+i}$ and, by induction,

$$\begin{aligned} E_i^{(M)} \mathfrak{n}_\lambda^{(k)} &= \cdots F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} \sum_{t=0}^{s} F_i^{(\lambda_k-\lambda_{k+i}-t)} \begin{bmatrix} K_{i,i+1}; 2t - M - (\lambda_k - \lambda_{k+i}) \\ t \end{bmatrix} \\ &\quad \cdot F_{i-1}^{(\lambda_k-\lambda_{k+i-1})} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} (E_i^{(M-t)} \mathfrak{n}_\lambda^{(k-1)}) = 0. \end{aligned}$$

If $M \leq (\lambda_k - \lambda_{k+i})$, then $s = M$ and, with a similar argument, (A.2) becomes

$$E_i^{(M)}\, \mathfrak{n}_\lambda^{(k)} = F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} F_i^{(\lambda_k-\lambda_{k+i}-M)} \begin{bmatrix} K_{i,i+1};\, M-(\lambda_k-\lambda_{k+i}) \\ M \end{bmatrix} \cdot F_{i-1}^{(\lambda_k-\lambda_{k+i-1})} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}$$
$$= F_{m-k}^{(\lambda_k-\lambda_m)} \cdots (F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} F_i^{(\lambda_k-\lambda_{k+i}-M)}) \cdot F_{i-1}^{(\lambda_k-\lambda_{k+i-1})} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} \mathfrak{n}_\lambda^{(k-1)}.$$

By applying Υ to Proposition 2.3(3), we obtain

$$F_{i+1}^{(\lambda_k-\lambda_{k+i+1})} F_i^{(\lambda_k-\lambda_{k+i}-M)} = \sum_{t=0}^{s'} q^{a_t b_t} F_i^{(\lambda_k-\lambda_{k+i}-M-t)} E_{i+2,i}^{(t)} F_{i+1}^{(\lambda_k-\lambda_{k+i+1}-t)},$$

where $s' = \min((\lambda_k - \lambda_{k+i+1}), (\lambda_k - \lambda_{k+i} - M))$, $a_t = \lambda_k - \lambda_{k+i} - M - t$, $b_t = \lambda_k - \lambda_{k+i+1} - t$. Substituting gives

$$E_i^{(M)}\, \mathfrak{n}_\lambda^{(k)} = \sum_{t=0}^{s'} q^{a_t b_t} \big(F_{m-k}^{(\lambda_k-\lambda_m)} \cdots F_i^{(\lambda_k-\lambda_{k+i}-M-t)} E_{i+2,i}^{(t)} \cdot F_{i-1}^{(\lambda_k-\lambda_{k+i-1})} \cdots F_2^{(\lambda_k-\lambda_{k+2})} F_1^{(\lambda_k-\lambda_{k+1})} F_{i+1}^{(\lambda_k-\lambda_{k+i+1}-t)} \mathfrak{n}_\lambda^{(k-1)}\big) = 0.$$

Here the last equation follows from induction and [8, Prop. 6.25]. Indeed, by restricting $L(\lambda)$ to the subalgebra $U_{q,F}(m-k+1)$ of $U_{q,F}(m)$ and induction, $\mathfrak{m}_\mu = \mathfrak{n}_\lambda^{(k-1)}$ is a maximal vector of weight $\mu = (\lambda_k, \ldots, \lambda_m)$ with $\mu_i = \lambda_{k+i-1}$. Since

$$\lambda_k - \lambda_{k+i+1} - t \geq \lambda_k - \lambda_{k+i+1} - (\lambda_k - \lambda_{k+i} - M) > \lambda_{k+i} - \lambda_{k+i+1},$$

[8, Prop. 6.25] implies $F_{i+1}^{(N)}\, \mathfrak{m}_\mu = 0$ for all $N > \mu_{i+1} - \mu_{i+2}$. This completes the proof of (3).

Finally, by the claim, $\lambda^\dagger = \mathrm{wt}(\mathfrak{n}_\lambda^{(m-1)})$ is a weight of $L(\lambda)$. Since $\lambda^\dagger$ is the lowest weight of the Weyl module $V(\lambda)$ and $L(\lambda)$ is a quotient of $V(\lambda)$, we conclude that $\lambda^\dagger$ is the lowest weight of $L(\lambda)$. □

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