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Seven combinatorial problems around isolated quasihomogeneous singularities

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Abstract
This paper proposes seven combinatorial problems around formulas for the characteristic polynomial and the spectral numbers of an isolated quasihomogeneous hypersurface singularity. One of them is a new conjecture on the characteristic polynomial. It is an amendment to an old conjecture of Orlik on the integral monodromy of an isolated quasihomogeneous singularity. The search for a combinatorial proof of the new conjecture led us to the seven purely combinatorial problems.

Keywords Isolated quasihomogeneous singularity · Weight system · Monodromy · Characteristic polynomial · Combinatorial problems · Orlik blocks

Mathematics Subject Classification 32S40 · 12Y05 · 05C22 · 05C25

1 Introduction

This paper proposes seven combinatorial problems around formulas for the characteristic polynomial and the spectral numbers of a quasihomogeneous singularity. One of them is a new conjecture on the characteristic polynomial. It is an amendment to an old conjecture of Orlik on the integral monodromy of an isolated quasihomogeneous singularity. The search for a combinatorial proof of the new conjecture led us to the seven purely combinatorial problems.

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We start with a result on $\mathbb{Z}$-lattices with automorphisms. Then we describe Orlik's conjecture and our new conjecture. Finally, we give a rough outline of the seven problems.

Definition 1.1 Let $M \subseteq \mathbb{N} = \{1, 2, 3, \ldots\}$ be a finite nonempty subset. Its *Orlik block* is a pair (H_M, h_M) with H_M a $\mathbb{Z}$-lattice of rank $\sum_{m\in M} \varphi(m)$ and $h_M : H_M \to H_M$ an automorphism with characteristic polynomial $\prod_{m\in M} \Phi_m$ (Φ_m is the m-th cyclotomic polynomial) and with a cyclic generator $e_1 \in H_M$, i.e.,

$$H_M = \bigoplus_{j=1}^{rkM} \mathbb{Z} \cdot h_M^{j-1}(e_1). \tag{1.1}$$

(H_M, h_M) is unique up to isomorphism. $\mathrm{Aut}_{S^1}(H_M, h_M)$ denotes the group of all automorphisms of H_M which commute with h_M and which have all eigenvalues in S^1.

Definition 6.1 enriches the set M to a directed graph $\mathcal{G}(M)$. An edge goes from $m_1 \in M$ to $m_2 \in M$ if $\frac{m_1}{m_2}$ is a power of a prime number p and if no $m_3 \in M\backslash\{m_1, m_2\}$ with $m_2|m_3|m_1$ exists. Then it is called a p-edge. The main result in [5] is cited precisely in Theorem 6.2. Roughly it is as follows.

Theorem 1.2 [5, Theorem 1.2] *Let (H_M, h_M) be the Orlik block of a finite nonempty subset $M \subseteq \mathbb{N}$. Then $\mathrm{Aut}_{S^1}(H_M, h_M) = \{\pm h_M^k \mid k \in \mathbb{Z}\}$ if and only if condition (I) or condition (II) in Theorem 6.2 is satisfied. They are conditions on the graph $\mathcal{G}(M)$.*

A weight system $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_i \in \mathbb{Q}_{>0}$ equips any monomial $\mathbf{x}^{\mathbf{j}} = x_1^{j_1} \ldots x_n^{j_n}$ with a weighted degree $\deg_{\mathbf{w}} \mathbf{x}^{\mathbf{j}} := \sum_{i=1}^n w_i j_i$. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is called an *isolated quasihomogeneous singularity* if for some weight system $\mathbf{w}$ with $w_i \in \mathbb{Q} \cap (0, 1)$ each monomial in f has weighted degree 1 and if the functions $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ vanish simultaneously only at $0 \in \mathbb{C}^n$. Then the *Milnor lattice* $H_{\mathrm{Milnor}} := H_{n-1}(f^{-1}(1), \mathbb{Z})$ is a $\mathbb{Z}$-lattice of some rank $\mu \in \mathbb{N}$ [12], which is called *Milnor number*. It comes equipped with a natural automorphism $h_{\mathrm{mon}} : H_{\mathrm{Milnor}} \to H_{\mathrm{Milnor}}$ of finite order, the *monodromy*. Thus its characteristic polynomial has the form

$$p_{ch,h_{\mathrm{mon}}} = \prod_{m\in M_1} \Phi_m^{\nu(m)}$$

for a finite subset $M_1 \subseteq \mathbb{N}$ and a function $\nu : M_1 \to \mathbb{N}$. Denote $\nu_{\max} := \max(\nu(m) \mid m \in \mathbb{N})$ and for $j = 1, \ldots, \nu_{\max}$

$$M_j := \{m \in M_1 \mid \nu(m) \geq j\}, \quad g_j := \prod_{m\in M_j} \Phi_m.$$

Then

$$M_1 \supset M_2 \supset ... \supset M_{\nu_{\max}} \neq \emptyset$$
$$\text{and} \qquad p_{ch,h_{\text{mon}}} = \prod_{j=1}^{\nu_{\max}} g_j.$$

The polynomials $g_1, \ldots, g_{\nu_{\max}}$ are called *elementary divisors* of $p_{ch,h_{\text{mon}}}$.

Conjecture 1.3 (Orlik's conjecture, [14, Conjecture 3.1]) *For any isolated quasihomogeneous singularity, there is an isomorphism*

$$(H_{\text{Milnor}}, h_{\text{mon}}) \cong \bigoplus_{j=1}^{\nu_{\max}} (H_{M_j}, h_{M_j}).$$

The conjecture is known to be true for curve singularities [11] and a few other cases, but it is still (after 45 years) open in general. The following conjecture is independent of Orlik's conjecture, but it is motivated by Orlik's conjecture, as it concerns the sets $M_1, \ldots, M_{\nu_{\max}}$.

Conjecture 1.4 *For any isolated quasihomogeneous singularity, each of the sets $M_1,\ldots, M_{\nu_{\max}}$ satisfies condition (I) in Theorem 6.2.*

If this is true for some singularity, then Theorem 1.2 gives for these sets $\text{Aut}_{S^1}(H_{M_j}, h_{M_j}) \cong \{\pm h^k_{M_j} \mid k \in \mathbb{Z}\}$. If also Orlik's conjecture holds, then this is helpful in determining the automorphisms of the Milnor lattice which respect the monodromy (and intersection form or Seifert form).

Examples 1.5 (i) Fix $\mu \in \mathbb{Z}$. The singularity A_μ, $f(x_1) = x_1^{\mu+1}$, is quasihomogeneous with $w_1 = \frac{1}{\mu+1}$. Here

$$p_{ch,h_{\text{mon}}}(t) = \frac{t^{\mu+1} - 1}{t - 1} = \prod_{m \in M} \Phi_m(t),$$
$$\text{with} \quad M = \{a \in \mathbb{N} \mid a \neq 1 \text{ and } a \text{ divides } \mu + 1\}.$$

Here $(H_{\text{Milnor}}, h_{\text{mon}}) \cong (H_M, h_M)$ is well known, so Orlik's conjecture holds here. The set M satisfies condition (I) in Theorem 6.2 (because of Theorem 6.9), so Conjecture 1.4 holds here. Therefore $\text{Aut}_{S^1}(H_{\text{Milnor}}, h_{\text{mon}}) = \{\pm h^k_{\text{mon}} \mid k \in \mathbb{Z}\}$.

(ii) The Thom–Sebastiani sum of two singularities $f(x)$ and $g(y)$ is $f(x) + g(y)$. Then [20]

$$(H_{\text{Milnor}}, h_{\text{mon}})(f + g) \cong (H_{\text{Milnor}}, h_{\text{mon}})(f) \otimes (H_{\text{Milnor}}, h_{\text{mon}})(g)$$

But it is unknown whether Orlik's conjecture holds for $f + g$ if it holds for f and g. Example 6.5 (ii) shows that condition (I) in Theorem 6.2 does not behave well under tensor product of products of cyclotomic polynomials. This motivates problem 7.

(iii) It is not even clear for all Brieskorn–Pham singularities $\sum_{j=1}^n x_j^{a_j}$ whether Orlik's conjecture or Conjecture 1.4 hold. Though both conjectures hold if $\gcd(a_i, a_j) = 1$ for all i and j with $i \neq j$ [3, Proposition 6.3].

Orlik's Conjecture 1.3 concerns the Milnor lattice. Any proof requires to go into geometry. But we hope that our Conjecture 1.4 is amenable to a combinatorial proof. It just concerns the characteristic polynomial. Milnor and Orlik [13] proved a formula which expresses this in terms of the weight system $\mathbf{w}$ of the quasihomogeneous singularity. It says div $p_{ch,h_{\text{mon}}} = D_{\mathbf{w}}$, where $D_{\mathbf{w}}$ is defined in (3.9). See Theorem 3.9. Therefore we hope that there will be a purely combinatorial Proof of Conjecture 1.4 dealing solely with properties of $\mathbf{w}$. This is problem 6 below. For most of the other problems, we need two more data.

First, an isolated quasihomogeneous singularity comes also equipped with *exponents* $\alpha_1, \ldots, \alpha_\mu \in \mathbb{Q} \cap (0, n)$. They are slightly finer invariants than $p_{ch,h_{\text{mon}}}$. They satisfy

$$\text{div } p_{ch,h_{\text{mon}}} = \sum_{j=1}^{\mu} [e^{2\pi i \alpha_j}]$$

and, for any $d \in \mathbb{N}$ with $v_i := d \cdot w_i \in \mathbb{N}$ for all $i \in \{1, \ldots, n\}$,

$$\sum_{j=1}^{\mu} t^{d \cdot \alpha_j} = \rho_{(\mathbf{v},d)}$$

where $\rho_{(\mathbf{v},d)}$ is defined in (3.8). See Theorem 3.9.

Second, the weight systems $\mathbf{w}$ for which isolated quasihomogeneous singularities exist, can be characterized by a combinatorial condition $(C1)$ (and equivalent combinatorial conditions $(C1)'$ and $(C2)$, see Lemma 3.3). This is cited in Theorem 3.5. It was proved first by Kouchnirenko [8, Remarque 1.13 (i)]. The necessity of $(C1)$ had already been seen by K. Saito [17], the sufficiency not. A weaker combinatorial property $\overline{(C1)}$ is equivalent to $\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]$ (3.16).

The seven problems are given in detail in the later chapters. Roughly, they are as follows.

Problem 1: (Remark 3.8) Let $(\mathbf{v}, d) = (v_1, \ldots, v_n, d) \in \mathbb{N}^{n+1}$ with $d > \max_i v_i$ be given which satisfies $\overline{(C1)}$. Write $\rho_{(\mathbf{v},d)} = \sum_{\alpha \in \frac{1}{d}\mathbb{Z}} \sigma(\alpha) \cdot t^{d \cdot \alpha} \in \mathbb{Z}[t]$. Is $D_{\mathbf{w}} = \sum_\alpha \sigma(\alpha) \cdot [e^{2\pi i \alpha}]$?

Problem 2 : (Remark 3.11 (ii)) Let $(\mathbf{v}, d) = (v_1, \ldots, v_n, d) \in \mathbb{N}^{n+1}$ with $d > \max_i v_i$ be given which satisfies $(C1)$. Give combinatorial proofs of the formulas in Theorem 3.9 which connect $D_{\mathbf{w}}$ and $\rho_{(\mathbf{v},d)}$ with the exponents and with one another.

Problem 3: (Remark 3.11 (iii)) Make some good use of the conditions for J with $|J| \geq 2$ in $(C1)$.

Problem 4: (Remarks 5.2) Find examples different from Ivlev's example for weight systems $\mathbf{w}$ which satisfy $\overline{(C1)}$, but not $(C1)$.

Problem 5: (Remark 5.6) Prove or disprove K. Saito's Conjecture 5.4 that $d_{\mathbf{w}} \in M_1$ or $\frac{d_{\mathbf{w}}}{2} \in M_1$ for $\mathbf{w}$ with $(C1)$. Here $d_{\mathbf{w}} := \mathrm{lcm}(\text{denominator of } w_i \mid i \in \{1, \ldots, n\})$.

Problem 6: (Remark 6.4 (i)) Prove (or disprove) combinatorially Conjecture 1.4.

Problem 7: (Remark 6.4 (iii)) Find a natural condition on products f of cyclotomic polynomials which implies for any elementary divisor of f condition (I) in Theorem 6.2 and which is stable under tensor product. Prove that $D_{\mathbf{w}}$ for $\mathbf{w}$ with $(C1)$ satisfies it (this would prove Conjecture 1.4).

Some comments: The problems 1, 2, 3 and 7 are motivated by problem 6, i.e., the wish to prove *combinatorially* Conjecture 1.4. If Conjecture 1.4 and Orlik's conjecture hold, we understand the Milnor lattice with monodromy of an isolated quasihomogeneous singularity much better, and we can determine $\mathrm{Aut}_{S^1}(H_{\mathrm{Milnor}}, h_{\mathrm{mon}})$ much easier. This, in turn, will be useful for many problems on singularities, for example, for period maps and Torelli problems. The problems 1 and 2 are closely related. A positive solution to one of them will probably also give a positive solution to the other one. And both will probably be useful for positive solutions of the problems 6 and 7. [6] made good use of the conditions for $|J| = 1$ in $(C1)$. They give rise to a graph. But the problems here probably require to involve also the conditions for $|J| \geq 2$. Problem 3 is vague, but fundamental. The condition $(C1)$ is because of Theorem 3.5 central for the classification of (weight systems of) isolated quasihomogeneous singularities. We need to be able to work with the condition $(C1)$. Problem 3 addresses this. It looks surprisingly difficult to find solutions for the very concrete problem 4. Though the problems 4 and 5 are less important than the other problems. Problem 5 is motivated by the (more important) problems 6 and 7. They are closely related. A positive solution of problem 6 goes probably via a positive solution of problem 7.

The paper is structured as follows. Section 2 gives notations and basic facts around cyclotomic polynomials. Section 3 introduces for abstract weight systems $\mathbf{w}$ and $(\mathbf{v}, d)$ the objects $D_{\mathbf{w}}$ and $\rho_{(\mathbf{v},d)}$ and the conditions $\overline{(C1)}$ and $(C1)$, and it states elementary facts as well as the formulas and facts which hold for the weight systems $\mathbf{w}$ of isolated quasihomogeneous singularities. This is all classical. Section 4 gives more explicit formulas in the cases of the quasihomogeneous singularities of cycle type and chain type. This builds on Sect. 3 and on [6] and is elementary. Section 5 presents examples. Especially, it gives counter-examples to the part of K. Saito's Conjecture 5.4 which says that $d_{\mathbf{w}} \in M_1$ in the case of a weight system $\mathbf{w}$ with all $w_i < \frac{1}{2}$. These counter-examples are interesting also in Sect. 6. Section 6 formulates in Theorem 6.2 the main result from [5] on automorphisms of Orlik blocks. It discusses Conjecture 1.4, it gives examples, and it proves Conjecture 1.4 in special cases, which include the cycle type, the chain type, the cases with $n = 2$ and many of the cases with $n = 3$ (Theorem 6.9).

2 Notations around cyclotomic polynomials

This section fixes some notations and recalls some well-known formulas around products of cyclotomic polynomials.

In this paper $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Whenever a number $n \in \mathbb{N}$ is fixed then $N := \{1, \ldots, n\}$.

Denote by $\mu(\mathbb{C}) \subseteq S^1$ the group of all unit roots. Denote by $\mathbb{Q}[\mu(\mathbb{C})]$ and $\mathbb{Z}[\mu(\mathbb{C})]$ the group rings with elements $\sum_{j=1}^{l} b_j[\zeta_j]$ where $b_j \in \mathbb{Q}$, respectively, $b_j \in \mathbb{Z}$ and where $\zeta_j \in \mu(\mathbb{C})$, with multiplication $[\zeta_1] \cdot [\zeta_2] = [\zeta_1 \cdot \zeta_2]$. The unit element is $[1]$. The trace of an element $\sum_{j=1}^{l} b_j[\zeta_j]$ is

$$\mathrm{tr}\left(\sum_{j=1}^{l} b_j[\zeta_j]\right) := \sum_{j=1}^{l} b_j \cdot \zeta_j \in \mathbb{C}. \tag{2.1}$$

The degree of it is

$$\deg\left(\sum_{j=1}^{l} b_j[\zeta_j]\right) := \sum_{j=1}^{l} b_j \in \mathbb{Q}. \tag{2.2}$$

The trace map $\mathrm{tr} : \mathbb{Q}[\mu(\mathbb{C})] \to \mathbb{C}$ and the degree map $\deg : \mathbb{Q}[\mu(\mathbb{C})] \to \mathbb{Q}$ are ring homomorphisms.

The divisor of a unitary polynomial $f = (t - \lambda_1) \cdot \ldots \cdot (t - \lambda_l) \in \mathbb{C}[t]$ with $\lambda_j \in \mu(\mathbb{C})$ is

$$\mathrm{div}\, f := [\lambda_1] + \cdots + [\lambda_l]. \tag{2.3}$$

Of course $\mathrm{tr}(\mathrm{div}\, f) = \lambda_1 + \cdots + \lambda_l$ and $\deg(\mathrm{div}\, f) = \deg f$.

For two polynomials, f as above and $g = (t - \kappa_1) \cdot \ldots \cdot (t - \kappa_k)$ with $\kappa_j \in \mu(\mathbb{C})$, define the new polynomial $f \otimes g \in \mathbb{C}[t]$ with zeros in $\mu(\mathbb{C})$ by

$$(f \otimes g)(t) := \prod_{i=1}^{k} \prod_{j=1}^{l} (t - \kappa_i \lambda_j). \tag{2.4}$$

Then

$$\mathrm{div}\,(f \otimes g) = (\mathrm{div}\, f) \cdot (\mathrm{div}\, g), \tag{2.5}$$
$$\mathrm{tr}(\mathrm{div}\,(f \otimes g)) = (\mathrm{tr}(\mathrm{div}\, f)) \cdot (\mathrm{tr}(\mathrm{div}\, g)), \tag{2.6}$$
$$\deg(f \otimes g) = (\deg f) \cdot (\deg g). \tag{2.7}$$

The order $\mathrm{ord}\,(\zeta) \in \mathbb{N}$ of a unit root $\zeta \in \mu(\mathbb{C})$ is the minimal number $m \in \mathbb{N}$ with $\zeta^m = 1$. For $m \in \mathbb{N}$, the m-th cyclotomic polynomial is

$$\Phi_m(t) = \prod_{\zeta :\, \mathrm{ord}\,(\zeta)=m} (t - \zeta) \in \mathbb{C}[t]. \tag{2.8}$$

It is in $\mathbb{Z}[t]$, it has degree $\varphi(m)$, and it is irreducible in $\mathbb{Z}[t]$ and $\mathbb{Q}[t]$. Denote

$$\Lambda_m := \operatorname{div}(t^m - 1), \quad \Psi_m := \operatorname{div}\Phi_m, \quad E_m := \frac{1}{m}\Lambda_m. \tag{2.9}$$

Then $\Lambda_1 = [1]$. Of course

$$t^n - 1 = \prod_{m|n} \Phi_m(t), \qquad \Lambda_n = \sum_{m|n} \Psi_m, \tag{2.10}$$

$$\Phi_n = \prod_{m|n} (t^m - 1)^{\mu_{\text{Moeb}}(\frac{n}{m})}, \quad \Psi_n = \sum_{m|n} \mu_{\text{Moeb}}\left(\frac{n}{m}\right) \cdot \Lambda_m. \tag{2.11}$$

Here μ_{Moeb} is the Möbius function [1]

$$\mu_{\text{Moeb}} : \mathbb{N} \to \{0, 1, -1\},$$
$$m \mapsto \begin{cases} (-1)^r & \text{if } m = p_1 \cdot \ldots \cdot p_r \text{ with } p_1, \ldots, p_r \\ & \text{different prime numbers}, \\ 0 & \text{else} \end{cases} \tag{2.12}$$

(here $r = 0$ is allowed, so $\mu_{\text{Moeb}}(1) = 1$). The traces of Λ_m and Ψ_m are

$$\operatorname{tr}\Lambda_m = \begin{cases} 1 \text{ if } m = 1 \\ 0 \text{ if } m \geq 2, \end{cases} \tag{2.13}$$

$$\operatorname{tr}\Psi_m = \mu_{\text{Moeb}}(m). \tag{2.14}$$

It is easy to see that

$$\Lambda_a \cdot \Lambda_b = \gcd(a, b) \cdot \Lambda_{\text{lcm}(a,b)}, \quad E_a \cdot E_b = E_{\text{lcm}(a,b)}, \tag{2.15}$$

$$[\zeta] \cdot \Lambda_b = \Lambda_b \qquad \text{if ord}(\zeta)|b, \tag{2.16}$$

$$\operatorname{div}(f) \cdot \Lambda_b = \deg f \cdot \Lambda_b \qquad \text{if } f|(t^b - 1). \tag{2.17}$$

Especially

$$\Lambda_a \cdot \Lambda_b = a \cdot \Lambda_b \qquad \text{if } a|b. \tag{2.18}$$

It is more difficult to write down formulas for $\Psi_a \cdot \Psi_b$. They can be cooked up from the following special cases.

$$\Psi_a \cdot \Psi_b = \Psi_{a\cdot b} \qquad \text{if } \gcd(a, b) = 1, \tag{2.19}$$

$$\Psi_{p^a} \cdot \Psi_{p^b} = \varphi(p^b) \cdot \Psi_{p^a} \qquad \text{if } p \text{ is a prime number and } a > b \geq 0, \tag{2.20}$$

$$\Psi_{p^a} \cdot \Psi_{p^a} = \varphi(p^a) \cdot \sum_{j=0}^{a} \Psi_{p^j} - p^{a-1} \cdot \Psi_{p^a} \qquad \text{if } p \text{ is a prime number and } a > 0. \tag{2.21}$$

Especially

$$\Psi_{2^a} \cdot \Psi_{2^a} = 2^{a-1} \cdot \sum_{j=0}^{a-1} \Psi_{p^j}. \tag{2.22}$$

Fix a finite set $M \subseteq \mathbb{N}$ and a map $\nu : \mathbb{N} \to \mathbb{N}_0$ with support M (so $M = \{m \in \mathbb{N} \,|\, \nu(m) \neq 0\}$) and define the unitary polynomial

$$\Delta := \prod_{m \in M} \Phi_m^{\nu(m)} \in \mathbb{Z}[t] \tag{2.23}$$

(if $M = \emptyset$, then $\Delta = 1$). Of course, then

$$\operatorname{div} \Delta = \sum_{m \in M} \nu(m) \cdot \Psi_m. \tag{2.24}$$

Define also

$$d_M := \operatorname{lcm}(m \in M\}. \tag{2.25}$$

Then $M \subseteq \{m \in \mathbb{N} \,|\, m \text{ divides } d_M\}$. (2.11) and (2.24) give a unique function $\chi : \mathbb{N} \to \mathbb{Z}$ with finite support

$$\operatorname{supp}(\chi) \subseteq \{n \in \mathbb{N} \,|\, \exists\, m \in M \text{ with } n|m\} \subseteq \{n \in \mathbb{N} \,|\, n \text{ divides } d_M\} \tag{2.26}$$

and

$$\operatorname{div} \Delta = \sum_{n \in \mathbb{N}} \chi(n) \cdot \Lambda_n, \tag{2.27}$$

$$\nu(m) = \sum_{n:\, m|n} \chi(n), \tag{2.28}$$

$$\chi(n) = \sum_{m:\, n|m} \nu(m) \cdot \mu_{\text{Moeb}}\left(\frac{m}{n}\right). \tag{2.29}$$

ν and χ and the following third function $L : \mathbb{N} \to \mathbb{Z}$ determine each other. L does not have finite support. The numbers $L(k) \in \mathbb{Z}$ are the *Lefschetz numbers* of Δ. They are defined by

$$L(k) := \sum_{j=1}^{\deg \Delta} \lambda_j^k \qquad \text{if } \Delta = \prod_{j=1}^{\deg \Delta} (t - \lambda_j). \tag{2.30}$$

Especially $L(1) = \mathrm{tr}(\mathrm{div}\,\Delta)$. Observe

$$\sum_{a=0}^{m-1}[e^{2\pi ia/m}] = \Lambda_m,$$
$$\sum_{a=0}^{m-1}[e^{2\pi ika/m}] = \gcd(k,m)\cdot\Lambda_{m/\gcd(k,m)} \tag{2.31}$$

Thus

$$L(k) = \sum_{m\in M}\chi(m)\cdot\mathrm{tr}\Big(\gcd(k,m)\Lambda_{m/\gcd(k,m)}\Big)$$
$$= \sum_{m:\,m|k} m\cdot\chi(m). \tag{2.32}$$

Möbius inversion [1] gives

$$m\cdot\chi(m) = \sum_{k|m}\mu_{\mathrm{Moeb}}(\frac{m}{k})\cdot L(k). \tag{2.33}$$

L does not have finite support, but the following extended periodicity property:

$$L(k) = L(\gcd(k, d_M)). \tag{2.34}$$

Therefore L is determined by its values on $\{m\in\mathbb{N}\,|\,m|d_M\}$. (2.34) implies the periodicity

$$L(k) = L(k + d_M), \tag{2.35}$$

but it is stronger. In fact, (2.34) is equivalent to $\mathrm{supp}\,(\chi)\subseteq\{m\in\mathbb{N}\,|\,m|d_M\}$ and to $M\subseteq\{m\in\mathbb{N}\,|\,m|d_M\}$.

All the formulas (2.24)–(2.35) make also sense if $\mathrm{div}\,\Delta$ is replaced by any element of $\mathbb{Q}[\mu(\mathbb{C})]$. Then ν, χ and L have values in $\mathbb{Q}$.

3 Weight system and characteristic polynomial of an isolated quasihomogeneous singularity

Fix a number $n\in\mathbb{N}$, and denote $N := \{1, 2, \ldots, n\}$ and $e_i := (0, .., 0, 1, 0, .., 0)\in\mathbb{N}_0^n$ (with 1 at the i-th place) for $i\in N$.

Definition 3.1 A *weight system* is a tuple $(v_1,\ldots,v_n,d)\subseteq(\mathbb{Q}_{>0})^{n+1}$ with $v_i < d$. Another weight system is *equivalent* to it, if the second one has the form $q\cdot(v_1,\ldots,v_n,d)$ for some $q\in\mathbb{Q}_{>0}$. A weight system is *integer* if $(v_1,\ldots,v_n,d)\in\mathbb{N}^{n+1}$. It is *reduced* if it is integer and it is minimal with this property, i.e., $\gcd(v_1,\ldots,v_n,d)=1$. It is normalized if $d=1$.

Any equivalence class contains a unique reduced weight system and a unique normalized weight system. From now on, the letters $(v_1, \ldots, v_n, d)$ will be reserved for integer weight systems, and $(w_1, \ldots, w_n, 1)$ will be the equivalent normalized weight system, i.e., $w_i = \frac{v_i}{d}$.

Let $(v_1, \ldots, v_n, d)$ be an integer weight system (not necessarily reduced, it does not matter here). For $J \subseteq N$ and $k \in \mathbb{N}_0$ denote

$$\mathbb{Z}^J := \{\alpha \in \mathbb{Z}^n \mid \alpha_i = 0 \text{ for } i \notin J\}, \qquad \mathbb{N}_0^J := \mathbb{Z}^J \cap \mathbb{N}_0^n,$$
$$(\mathbb{Z}^n)_k := \{\alpha \in \mathbb{Z}^n \mid \sum_i \alpha_i \cdot v_i = k\}, \qquad (\mathbb{N}_0^n)_k := (\mathbb{Z}^n)_k \cap \mathbb{N}_0^n,$$
$$(\mathbb{Z}^J)_k := \mathbb{Z}^J \cap (\mathbb{Z}^n)_k, \qquad (\mathbb{N}_0^J)_k := (\mathbb{Z}^J)_k \cap \mathbb{N}_0^n = \mathbb{N}_0^J \cap (\mathbb{N}_0^n)_k.$$

So, $\mathbb{Z}^J$ is a coordinate plane in $\mathbb{Z}^n$ where some coordinates are 0, $(\mathbb{Z}^J)_k$ is an affine hyperplane in $\mathbb{Z}^J$, and $\mathbb{N}_0^J$ and $(\mathbb{N}_0^J)_k$ are the intersections of $\mathbb{Z}^J$ and $(\mathbb{Z}^J)_k$ with the *quadrant* $\mathbb{N}_0^n$. Of course, $\mathbb{Z}^N = \mathbb{Z}^n$ and $\mathbb{Z}^\emptyset = \{(0, \ldots, 0)\}$.

Remark 3.2 For $J \subseteq N$ with $J \neq \emptyset$ define the semigroup

$$SG(J) := \sum_{j \in J} \mathbb{N}_0 \cdot v_j \subseteq \mathbb{N}_0 \tag{3.1}$$

and observe

$$\sum_{j \in J} \mathbb{Z} \cdot v_j = \mathbb{Z} \cdot \gcd(v_j \mid j \in J). \tag{3.2}$$

Then

$$(\mathbb{N}_0^J)_k \neq \emptyset \iff k \in SG(J), \tag{3.3}$$
$$(\mathbb{Z}^J)_k \neq \emptyset \iff \gcd(v_j \mid j \in J) | k. \tag{3.4}$$

The following combinatorial lemma is a specialization of [6, Lemma 2.1]. It will be useful in Theorem 3.5. (The conditions $(C2)'$ and $(C3)$ in [6, Lemma 2.1] are less important.)

Lemma 3.3 *Fix an integer weight system $(v_1, \ldots, v_n, d)$. The following three conditions $(C1)$, $(C1)'$ and $(C2)$ are equivalent.*

(C1): $\forall\, J \subseteq N$ *with* $J \neq \emptyset$ $\;(\mathbb{N}_0^J)_d \neq \emptyset$ *or* $\exists\, K \subseteq N \backslash J$ *with* $|K| = |J|$ *and* $\forall\, k \in K$ $(\mathbb{N}_0^J)_{d-v_k} \neq \emptyset$.

(C1)': *As (C1), but only for* J *with* $|J| \leq \frac{n+1}{2}$.

(C2): $\forall\, J \subseteq N$ *with* $J \neq \emptyset$ $\;\exists\, K \subseteq N$ *with* $|K| = |J|$ *and* $\forall\, k \in K$ $(\mathbb{N}_0^J)_{d-v_k} \neq \emptyset$.

Proof $(C1) \Rightarrow (C1)'$ is trivial. $(C1)' \Rightarrow (C1)$ and $(C2) \Rightarrow (C1)$ are easy. See [6] for details. The least easy implication is $(C1) \Rightarrow (C2)$. In [6] it was proved via the condition $(C3)$ there. A more direct proof will be given now.

Suppose that $(C1)$ holds. Fix $J \subseteq N$ with $J \neq \emptyset$. We want to find a $K \subseteq N$ such that J and K satisfy $(C2)$. Define the support of J by

$$\text{supp}\,(J) := \{j \in J \mid \exists\, \alpha \in (\mathbb{N}_0^J)_d \text{ with } \alpha_j \neq 0\} \subseteq J.$$

Consider $J_1 := J \backslash \text{supp}\,(J)$.

1st case, $J_1 = \emptyset$: Then J and $K := J$ satisfy $(C2)$.

2nd case, $J_1 \neq \emptyset$: The definition of J_1 implies $(\mathbb{N}_0^{J_1})_d = \emptyset$. Therefore $(C1)$ gives the existence of a set $K_1 \subseteq N \backslash J_1$ with $|K_1| = |J_1|$ and $\forall\, k \in K_1 \quad (\mathbb{N}_0^{J_1})_{d-v_k} \neq \emptyset$.

Because of $v - d_k > 0$, any element $\beta \in (\mathbb{N}_0^{J_1})_{d-v_k}$ satisfies $\beta_j \neq 0$ for some $j \in J_1$.

If some $k \in K_1$ would be in $J \backslash J_1$, then for any element $\beta \in (\mathbb{N}_0^{J_1})_{d-v_k}$ the element $\alpha := \beta + e_k$ would contradict $J_1 \cap \text{supp}\,(J) = \emptyset$. Thus $K_1 \subseteq N \backslash J$. Now J and $K := K_1 \cup \text{supp}\,(J)$ satisfy $(C2)$. □

Remarks 3.4 (i) Denote with a bar the analogous conditions $\overline{(C1)}$, $\overline{(C1)'}$, $\overline{(C2)}$ where $\mathbb{N}_0$ is replaced by $\mathbb{Z}$. Also these conditions are equivalent to one another. The proof is the same as above.

(ii) Recall that a polynomial

$$f = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \cdot x^\alpha \in \mathbb{C}[x_1, \ldots, x_n] \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

is quasihomogeneous with respect to a weight system $(v_1, \ldots, v_n, d)$ if

$$\sum_{i=1}^n \alpha_i \cdot v_i = d \quad \text{for all } \alpha \text{ with } a_\alpha \neq 0.$$

Recall that a quasihomogeneous polynomial has an isolated singularity at 0 if the functions $\frac{\partial f}{\partial x_i}$ vanish simultaneously precisely at 0. The set of all quasihomogeneous polynomials with a given weight system $(v_1, \ldots, v_n, d)$ is the (finite dimensional) space $\{\sum_{\alpha \in (\mathbb{N}_0^n)_d} a_\alpha \cdot x^\alpha \mid a_\alpha \in \mathbb{C}\}$. A generic quasihomogeneous polynomial with the given weight system is one in a Zariski open subset of this space.

(iii) Let f be a quasihomogeneous polynomial with weight system $(v_1, \ldots, v_n, d)$ and an isolated singularity at 0. Then all fibers $f^{-1}(\tau) \subseteq \mathbb{C}^n$ for $\tau \in \mathbb{C}^*$ are smooth and diffeomorphic. The Milnor lattice $H_{\text{Milnor}} := H_{n-1}^{(red)}(f^{-1}(1), \mathbb{Z})$ (reduced homology if and only if $n = 1$) is a $\mathbb{Z}$-lattice of rank $\mu = \prod_{j=1}^n (\frac{d}{v_j} - 1) \in \mathbb{N}$. It comes equipped with a natural automorphism h_{mon} of finite order, called *monodromy*. The pair $(H_{\text{Milnor}}, h_{\text{mon}})$ depends up to isomorphism only upon the weight system. Theorem 3.9 expresses $p_{ch,h_{\text{mon}}}$ in terms of the weight system. Orlik's conjecture predicts the isomorphism class of the pair $(H_{\text{Milnor}}, h_{\text{mon}})$ and uses only the weight system for this.

The following theorem is cited from [6, Theorem 2.2]. It was first proved by Kouchnirenko [8, Remarque 1.13 (i)]. See [6, Remarks 2.3] for its history and contributions in [8–10,15,17,21,22].

Theorem 3.5 *Let $(v_1, \dots, v_n, d) \in \mathbb{N}^{n+1}$ be an integer weight system. The following three conditions are equivalent.*

(IS3): There exists a quasihomogeneous polynomial f with the weight system $(v_1, \dots, v_n, d)$ and an isolated singularity at 0.
(IS3)': A generic quasihomogeneous polynomial with the weight system $(v_1, \dots, v_n, d)$ has an isolated singularity at 0.
(C1) to (C2): The weight system $(v_1, \dots, v_n, d)$ satisfies one of the equivalent conditions $(C1)$, $(C1)'$, $(C2)$.

In Definition 3.6, some objects will be associated to any weight system. Before studying them in the case of weight systems of isolated quasihomogeneous singularities, their shape under weaker conditions will be discussed in Lemma 3.7.

Definition 3.6 Let $(\mathbf{v}, d) = (v_1, \dots, v_n, d) \in \mathbb{N}^{n+1}$ be an integer weight system.

(a) Define unique numbers $s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{N}$ by

$$\frac{v_i}{d} = \frac{s_i}{t_i} \quad \text{and} \quad \gcd(s_i, t_i) = 1,$$

so by reducing the fractions v_i/d. The numbers depend only on the normalized weight system $\mathbf{w} = (w_1, \dots w_n) = (\frac{v_1}{d}, \dots, \frac{v_n}{d})$.

(b) Define

$$d_{\mathbf{w}} := \operatorname{lcm}(t_j \mid j \in N). \tag{3.5}$$

Of course $d_{\mathbf{w}}|d$. If $(\mathbf{v}, d)$ is reduced and $\gcd(v_1, \dots, v_n)|d$ (which holds, for example, if $\overline{(C2)}$ holds), then $\gcd(v_1, \dots, v_n) = 1$ and then $d_{\mathbf{w}} = d$.

(c) For $k \in \mathbb{N}$ define

$$M(k) := \{j \in N \mid t_j \text{ divides } k\}, \tag{3.6}$$

$$\text{and } \mu(k) := \prod_{j \in M(k)} \left(\frac{1}{w_j} - 1 \right) = \prod_{j \in M(k)} \frac{d - v_j}{v_j} \in \mathbb{Q}_{>0} \tag{3.7}$$

(the empty product is by definition 1).

(d) Define a quotient of polynomials

$$\rho_{(\mathbf{v},d)}(t) := t^{v_1 + \dots + v_n} \cdot \prod_{j=1}^{n} \frac{t^{d-v_j} - 1}{t^{v_j} - 1} \in \mathbb{Q}(t) \tag{3.8}$$

and an element of $\mathbb{Q}[\mu(\mathbb{C})]$

$$D_{\mathbf{w}} := \prod_{j=1}^{n} \left(\frac{1}{s_j} \Lambda_{t_j} - \Lambda_1 \right) \in \mathbb{Q}[\mu(\mathbb{C})]. \tag{3.9}$$

Lemma 3.7 *Let* $(\mathbf{v}, d)$ *be an integer weight system.*

(a) Then

$$M(k) = M(\gcd(k, d_{\mathbf{w}})) \tag{3.10}$$
$$= \left\{ j \in N \mid \frac{d}{\gcd(k, d_{\mathbf{w}})} \text{ divides } v_j \right\}, \tag{3.11}$$
$$\mu(k) = \mu(\gcd(k, d_{\mathbf{w}})). \tag{3.12}$$

(b) The Lefschetz numbers $L(k)$ *of the element* $D_{\mathbf{w}}$ *are*

$$L(k) = L(\gcd(k, d_{\mathbf{w}})) \tag{3.13}$$
$$= (-1)^{n-|M(k)|} \cdot \mu(k) \in \mathbb{Q}^*. \tag{3.14}$$

(c)

$$(\mathbf{v}, d) \text{ satisfies } \overline{(C2)} \quad \Rightarrow \quad \mu(k) \in \mathbb{N} \quad \text{for all } k \in \mathbb{N}. \tag{3.15}$$
$$(\mathbf{v}, d) \text{ satisfies } \overline{(C2)} \iff \rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]. \tag{3.16}$$

Proof (a) As all t_j divide $d_{\mathbf{w}}$ by definition of $d_{\mathbf{w}}$, $t_j|k$ is equivalent to $t_j|\gcd(k, d_{\mathbf{w}})$. This shows (3.10) and (3.12). Now suppose $k|d_{\mathbf{w}}$ (just for simplicity of notations). By definition $t_j = d/\gcd(v_j, d)$. Thus for any $j \in J$

$$t_j|k \iff \frac{d}{\gcd(v_j, d)}|k \iff \frac{d}{k}|\gcd(v_j, d) \iff \frac{d}{k}|v_j.$$

This shows (3.11).

(b) The following calculation gives (3.14).

$$\begin{aligned} L(k) &= \operatorname{tr}\left(\prod_{j=1}^{n}\left(\frac{\gcd(k, t_j)}{s_j}\Lambda_{t_j/\gcd(k,t_j)} - \Lambda_1\right)\right) \\ &= \prod_{j=1}^{n}\left(\frac{\gcd(k, t_j)}{s_j} \cdot \operatorname{tr}\left(\Lambda_{t_j/\gcd(k,t_j)}\right) - 1\right) \\ &= \prod_{j\in M(k)}\left(\frac{t_j}{s_j} - 1\right) \cdot \prod_{j\notin M(k)}(-1) \\ &= (-1)^{n-|M(k)|} \cdot \mu(k). \end{aligned}$$

The equality (3.13) $L(k) = L(\gcd(k, d_{\mathbf{w}}))$ is a consequence of the analogous properties (3.10) of $M(k)$ and (3.12) of $\mu(k)$ and of (3.14).

(c) Recall from Remark 3.2 that $(\mathbb{Z}^J)_{d-v_l} \neq \emptyset \iff \gcd(v_j \mid j \in J)|(d - v_l)$. Therefore $\overline{(C2)}$ says

(GCD) $\forall\, J \subseteq N$ the $\gcd(v_j \mid j \in J)$ divides at least $|J|$ of the numbers $d - v_l$ for $l \in N$.

For any $k \in \mathbb{N}$ with $k|d$, obviously the analogous condition with $M(k)$ instead of N holds then, too. It is easy to derive from this directly $\mu(k) \in \mathbb{N}$ for this k. But it will also follow from the consideration below of $\rho_{(\mathbf{v},d)}$. Then $\mu(k) \in \mathbb{N}$ for all $k \in \mathbb{N}$ follows with (3.12).

$\rho_{(\mathbf{v},d)}$ is a quotient of cyclotomic polynomials. The condition (GCD) says that any cyclotomic polynomial in the denominator turns up with at least the same multiplicity in the numerator, especially the cyclotomic polynomials $\Phi_{\gcd(v_j \mid j \in J)}$ for some $J \subseteq N$. Thus $\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]$ is equivalent to $\overline{(C2)}$.

Now suppose that $\overline{(C2)}$ holds. Then (GCD) holds for N and also for any $M(k)$ instead of N. The argument above for $\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]$ applies also to the partial product $\prod_{j \in M(k)}(\ldots)$ for any $k \in \mathbb{N}$ and shows that it is in $\mathbb{Z}[t]$. Dividing out all factors $(t-1)$, one can insert $t = 1$ and obtains for the partial product

$$\mu_k = \prod_{j \in M(k)} \frac{d - v_j}{v_j} \in \mathbb{Z} \cap \mathbb{Q}_{>0} = \mathbb{N}.$$

□

Remark 3.8 Let $(\mathbf{v}, d)$ be an integer weight system with $\overline{(C2)}$. Then (3.16) gives $\rho_{(\mathbf{v},d)} = \sum_{\alpha \in \frac{1}{d}\mathbb{Z}} \sigma(\alpha) \cdot t^{d \cdot \alpha}$ with a function $\sigma : \frac{1}{d}\mathbb{Z} \to \mathbb{Z}$ with finite support. And (3.15) gives $L(k) \in \mathbb{Z}$.

Open problem 1:

(a) Is $D_{\mathbf{w}} \in \mathbb{Z}[\mu(\mathbb{C})]$? Equivalent: Are the $\chi(m)$ which are determined by the $L(k)$ and (2.33) in $\mathbb{Z}$?
(b) Is $D_{\mathbf{w}} = \sum_{\alpha \in \frac{1}{d}\mathbb{Z}} \sigma(\alpha) \cdot [e^{2\pi i \alpha}]$?

Yes for problem 1 (b) would imply *Yes* for problem 1 (a).

The following theorem is classical, see the Remarks 3.10 for its origins.

Theorem 3.9 *Let $(\mathbf{v}, d)$ be an integer weight system with $(C2)$, i.e., a weight system of isolated quasihomogeneous singularities.*

Then the divisor of the characteristic polynomial of its monodromy is $D_{\mathbf{w}}$, so here $D_{\mathbf{w}} \in \mathbb{N}_0[t]$, and

$$\mathrm{tr}((monodromy)^k) = (-1)^{n-|M(k)|} \cdot \mu(k). \tag{3.17}$$

Also $\rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t]$, thus

$$\rho_{(\mathbf{v},d)} = \sum_{j=1}^{\mu} t^{d \cdot \alpha_j} \quad \text{for certain } \alpha_j \in \mathbb{Q}. \tag{3.18}$$

These numbers $(\alpha_1, \ldots, \alpha_\mu)$ *are the* exponents *of the singularity, and* $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_\mu}$ *are the eigenvalues of the monodromy, i.e., the zeros of the characteristic polynomial, so here*

$$D_{\mathbf{w}} = \sum_{j=1}^{\mu}[e^{2\pi i\alpha_j}] \in \mathbb{N}_0[\mu(\mathbb{C})]. \tag{3.19}$$

Remarks 3.10 (i) Formula (3.17) was shown by Milnor in [12, §9.6]. Of course, the trace of the k-th power of the monodromy is precisely the k-th Lefschetz number of the characteristic polynomial of the monodromy. Therefore (3.17) together with (3.14) and the equivalence of the data L, χ, ν, Δ in Sect. 2 implies that the characteristic polynomial of the monodromy has the divisor $D_{\mathbf{w}}$. This was first seen in [13].

(ii) The polynomial $\rho_{(\mathbf{v},d)}$ is the generating function of the exponents, which are up to the shift $v_1 + \cdots + v_n$ the weighted degrees of the Jacobi algebra

$$\frac{\mathbb{C}\{x_1, \ldots, x_n\}}{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)} \overset{f \text{ q.h.}}{\cong} \frac{\mathbb{C}[x_1, \ldots, x_n]}{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}.$$

This was (re)discovered by many people. Therefore $\rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t]$.

(iii) Let $(\mathbf{v}, d)$ be an integer weight system with $n \leq 3$. Theorem 3 in [18] says

$$\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t] \iff (IS3).$$

With (3.16) and Theorem 3.5, this is equivalent to $\overline{(C1)} \iff (C1)$ for $n \leq 3$. This equivalence $\overline{(C1)} \iff (C1)$ for $n \leq 3$ is lemma 2.5 in [6]. It has a short combinatorial proof.

Remarks 3.11 (i) Theorem 3.9 implies that for an integer weight system $(\mathbf{v}, d)$ with $(C2)$ the answer to the problem 1 (a)+(b) is *Yes*. But the proof is not combinatorial. Theorem 3.9 gives also the following three implications:

$$(C2) \Rightarrow D_{\mathbf{w}} \in \mathbb{N}_0[\mu(\mathbb{C})], \tag{3.20}$$

$$(C2) \Rightarrow \rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t], \tag{3.21}$$

$$(C2) \Rightarrow D_{\mathbf{w}} = \sum_{j=1}^{\mu}[e^{2\pi i\alpha_j}]. \tag{3.22}$$

(3.22) is the positive answer to problem 1 (b) in the case $(C2)$. The known proofs of (3.20), (3.21) and (3.22) are not combinatorial.

(ii) Let $(\mathbf{v}, d)$ be an integer weight system with $(C2)$.

Open problem 2:

(a) Give a combinatorial proof of (3.20).
(b) Give a combinatorial proof of (3.21).
(c) Give a combinatorial proof of (3.22).

(iii) One can separate in $(C2)$ and $(C1)$ the conditions for J of different values of $|J| \in N$. The conditions for J with $|J| = 1$ lead to the graphs and types of a quasihomogeneous singularity which are discussed in [6, ch. 3]. Sections 4 and 6 in [6] make extensive and successful use of the conditions for $|J| = 1$. Below in Sect. 4, we will extend formulas in [6] for parts of the Milnor number μ to formulas for parts of $D_{\mathbf{w}}$.

But it is irritatingly difficult to make use of the conditions for J with $|J| \geq 2$ in $(C2)$ or $(C1)$. Though they must be used in solutions of the problems in (iii), and probably also in a positive solution of the Conjecture 6.3 in Sect. 6, if that has a positive solution.

Open problem 3: Make some good use of the conditions for J with $|J| \geq 2$ in $(C2)$ or $(C1)$.

The last point in this section is a discussion of a well-known fact on the order of the monodromy of a quasihomogeneous singularity. That order is

$$d_{\text{mon}} := \text{lcm}(m \in \mathbb{N} \,|\, \nu(m) > 0)$$

where the numbers $\nu(m)$ are determined by $D_{\mathbf{w}} = \sum_{m \in \mathbb{N}} \nu(m) \cdot \Psi_m$.

Lemma 3.12 *In the case of an isolated quasihomogeneous singularity with $w_j \leq \frac{1}{2}$ for all $j \in N$, $d_{\text{mon}} = d_{\mathbf{w}}$ or $d_{\text{mon}} = \frac{d_{\mathbf{w}}}{2}$. If all $w_j < \frac{1}{2}$, then $d_{\text{mon}} = d_{\mathbf{w}}$.*

Proof Because of the Definition (3.9) of $D_{\mathbf{w}}$, d_{mon} is a divisor of $d_{\mathbf{w}}$. The equalities

$$\prod_{j=1}^{n} \left(\frac{1}{w_j} - 1 \right) = \mu = \text{trid} = \text{tr}(Mon)^{d_{\text{mon}}} = L(d_{\text{mon}})$$
$$= \pm \mu(d_{\text{mon}}) = \pm \prod_{j \,:\, t_j | d_{\text{mon}}} \left(\frac{1}{w_j} - 1 \right)$$

show that the second product can miss only indices j with $w_j = \frac{1}{2}$. Therefore $\text{lcm}(t_j \mid w_j < \frac{1}{2})$ divides d_{mon}. If all $w_j < \frac{1}{2}$, then $\text{lcm}(t_j \mid w_j < \frac{1}{2}) = d_{\mathbf{w}}$. If some $w_j = \frac{1}{2}$ then $\text{lcm}(t_j \mid w_j < \frac{1}{2}) = d_{\mathbf{w}}$ or $\frac{d_{\mathbf{w}}}{2}$. □

4 Formulas for isolated quasihomogeneous singularities of cycle type and of chain type

The formulas in this section concern the isolated quasihomogeneous singularities of *cycle type*,

$$x_n x_1^{a_1} + x_1 x_2^{a_2} + x_2 x_3^{a_3} + \cdots + x_{n-1} x_n^{a_n}, \qquad \text{where } n \in \mathbb{N},\ a_1, \ldots, a_n \in \mathbb{N},$$
$$\text{and in the case } n \text{ even neither } a_j = 1 \text{ for all even } j \text{ nor } a_j = 1 \text{ for all odd } j,$$

and of *chain type*,

$$x_1^{a_1+1} + x_1 x_2^{a_2} + x_2 x_3^{a_3} + \cdots + x_{n-1} x_n^{a_n} \qquad \text{where } n \in \mathbb{N},\ a_1, \ldots, a_n \in \mathbb{N}.$$

The formulas start with normalized weight systems $(\mathbf{w}, 1) = (w_1, \ldots, w_n, 1)$ which satisfy a part of the conditions $(C1)$ and $(C2)$ in Lemma 3.3, namely that part which concerns subsets $J \subseteq N$ with $|J| = 1$. That part leads to graphs and types of weight systems, see Sect. 3 in [6]. As already said in Remark 3.10 (iv), it is difficult to make use of the conditions in $(C1)$ and $(C2)$ for J with $|J| \geq 2$. The formulas here do not make use of these higher conditions.

The formulas extend formulas in [6] for parts of the weight system and parts of the Milnor number to formulas for parts of $D_{\mathbf{w}}$. Some calculations already made in [6] will be reproduced here for better readability. We start with the cycle type, then consider a generalization of the chain type and finally specialize that to the chain type. The formulas for the generalization of the chain type will allow to glue its root to another graph.

Define the function

$$\begin{aligned} &\rho : \bigcup_{k=0}^{\infty} \mathbb{Z}^k \to \mathbb{Z}, \\ &\rho(x_1, \ldots, x_k) := x_1 \ldots x_k - x_2 \ldots x_k + \cdots + (-1)^{k-1} x_k + (-1)^k \end{aligned} \tag{4.1}$$

(the case $k = 0$ is $\rho(\emptyset) = 1$).

Lemma 4.1 (Partly [6, Lemma 3.4 and (4.6)]) *Fix $n \in \mathbb{N}$ and n numbers $a_1, \ldots, a_n \in \mathbb{N}$ such that, if n is even, neither $a_j = 1$ for all even j nor $a_j = 1$ for all odd j.*

Then there is a unique normalized weight system $(\mathbf{w}, 1) = (w_1, \ldots, w_n, 1)$ with $a_j w_j + w_{j-1} = 1$ for all $j \in N$, where $w_0 := w_n$. It is

$$\begin{aligned} w_j &= \frac{v_j}{d} \quad \textit{where} \\ v_j &:= \rho(a_{j-1}, a_{j-2}, \ldots, a_2, a_1, a_n, a_{n-1}, \ldots, a_{j+1}) \in \mathbb{N}, \\ d &:= \prod_{j=1}^{n} a_j - (-1)^n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

Define

$$\gamma := \gcd(v_1, d).$$

Then the unique numbers $s_j, t_j \in \mathbb{N}$ with $\gcd(s_j, t_j) = 1$ and $w_j = \frac{s_j}{t_j}$ from Definition 3.6 are

$$\begin{aligned} s_j &= \frac{v_j}{\gamma}, \\ t_1 = \ldots = t_n &= \frac{d}{\gamma}. \end{aligned} \tag{4.3}$$

Especially $\gamma = \gcd(v_j, d)$ for any $j \in N$.

The Milnor number is

$$\mu = \prod_{j=1}^{n} a_j. \tag{4.4}$$

The divisor $D_{\mathbf{w}}$ from Definition 3.6 is

$$D_{\mathbf{w}} = \gamma \cdot \Lambda_{d/\gamma} + (-1)^n \cdot \Lambda_1. \tag{4.5}$$

Proof The matrix of the system $a_j w_j + w_{j-1} = 1$ of linear equations has the determinant

$$\det \begin{pmatrix} a_1 & & & 1 \\ 1 & a_2 & & \\ & \ddots & \ddots & \\ & & 1 & a_n \end{pmatrix} = \prod_{j=1}^{n} a_j - (-1)^n = d,$$

here $d > 0$ by hypothesis. Therefore it has a unique solution. It is easy to see that this solution is given by (4.2). The conditions that in the case n even neither $a_j = 1$ for all even j nor $a_j = 1$ for all odd j make sure that the numbers v_j and d and the weights w_j are not zero, but positive. The equation $a_j w_j + w_{j-1} = 1$ implies $w_j < 1$.

By definition $t_1 = d/\gamma$. The identities (where $w_0 = w_n, s_0 = s_n, t_0 = t_n$)

$$\left. \begin{array}{r} \frac{s_j}{t_j} = w_j = \frac{1-w_{j-1}}{a_j} = \frac{t_{j-1}-s_{j-1}}{t_{j-1}\cdot a_j} \\ \gcd(t_{j-1}, t_{j-1} - s_{j-1}) = 1 \end{array} \right\} \Rightarrow t_j = t_{j-1} \cdot \frac{a_j}{\gcd(a_j, t_{j-1} - s_{j-1})} \tag{4.6}$$

show $t_{j-1} | t_j$. As we have a cycle here, $t_j = d/\gamma$ and $\gcd(v_j, d) = \gamma$ for any $j \in N$.

The Milnor number is calculated by (with $v_0 = v_n$)

$$\mu = \prod_{j=1}^{n} \frac{d - v_{j-1}}{v_{j-1}} = \prod_{j=1}^{n} \frac{a_j \cdot v_j}{v_{j-1}} = a_1 \cdot \ldots \cdot a_n.$$

The divisor $D_{\mathbf{w}}$ is defined in (3.9). Because of (2.15) it has only the two summands $\Lambda_{d/\gamma}$ and Λ_1, and the coefficient of Λ_1 is obviously (from (3.9)) $\chi(1) = (-1)^n$. As $\mu = \deg D_{\mathbf{w}} = \chi(d/\gamma) \cdot d/\gamma + \chi(1)$, the coefficient $\chi(d/\gamma)$ of $\Lambda_{d/\gamma}$ is $\chi(d/\gamma) = \gamma$, so (4.5) holds. □

Lemma 4.2 is a slight generalization of the chain type. Corollary 4.3 specializes it to the chain type.

Lemma 4.2 (Partly [6, (4.10)]) *Fix $n \in \mathbb{N}$, n numbers $a_1, \ldots, a_n \in \mathbb{N}$, two numbers $s_0, t_0 \in \mathbb{N}$ with $s_0 < t_0$ and $\gcd(s_0, t_0) = 1$, and define $w_0 := \frac{s_0}{t_0}$.*

Then there are unique weights $w_1, \ldots, w_n \in \mathbb{Q} \cap (0,1)$ *with* $a_j w_j + w_{j-1} = 1$ *for* $j = 1, \ldots, n$. *Write* $w_j = \frac{s_j}{t_j}$ *with* $s_j, t_j \in \mathbb{N}$ *and* $\gcd(s_j, t_j) = 1$ *and* $\beta_j := \gcd(t_{j-1} - s_{j-1}, a_j) \in \mathbb{N}$ *and* $\alpha_j := \frac{a_j}{\beta_j} \in \mathbb{N}$. *Then*

$$s_j = \frac{\rho(a_{j-1}, \ldots, a_1) \cdot t_0 + (-1)^j s_0}{\beta_j \cdot \ldots \cdot \beta_1}, \tag{4.7}$$

$$t_j = \alpha_j \cdot t_{j-1} = \alpha_j \cdot \ldots \cdot \alpha_1 \cdot t_0. \tag{4.8}$$

The partial divisor and the partial Milnor number associated to $(w_1, \ldots, w_n)$ *are*

$$\prod_{j=1}^{n} \left(\frac{1}{s_j} \Lambda_{t_j} - \Lambda_1 \right) = (-1)^n \Lambda_1 + \sum_{j=1}^{n} (-1)^{n-j} \frac{\beta_j \ldots \beta_1}{t_0 - s_0} \cdot \Lambda_{t_j} \tag{4.9}$$

$$= (-1)^n E_1 + \sum_{j=1}^{n} (-1)^{n-j} \frac{a_j \ldots a_1}{1 - w_0} \cdot E_{t_j}, \tag{4.10}$$

$$\prod_{j=1}^{n} \left(\frac{1}{w_j} - 1 \right) = \frac{\rho(a_n, a_{n-1}, \ldots, a_1) + (-1)^{n-1} w_0}{1 - w_0}. \tag{4.11}$$

Proof The weights are unique and in $\mathbb{Q} \cap (0,1)$ because they are determined inductively by the equations $a_j w_j + w_{j-1} = 1$, i.e.,

$$w_j = \frac{1 - w_{j-1}}{a_j} = \frac{t_{j-1} - s_{j-1}}{a_j \cdot t_{j-1}} = \frac{(t_{j-1} - s_{j-1})/\beta_j}{\alpha_j \cdot t_{j-1}}.$$

As $1 = \gcd(s_{j-1}, t_{j-1}) = \gcd(t_{j-1} - s_{j-1}, t_{j-1})$, this shows (4.8). For $j = 1$ (4.7) is clear. For $j \geq 2$ the additional calculation

$$\begin{aligned} \beta_j \cdot s_j &= t_{j-1} - s_{j-1} \\ &= \frac{a_{j-1} \cdot \ldots \cdot a_1 \cdot t_0}{\beta_{j-1} \cdot \ldots \cdot \beta_1} - \frac{\rho(a_{j-2}, \ldots, a_1) \cdot t_0 + (-1)^{j-1} s_0}{\beta_{j-1} \cdot \ldots \cdot \beta_1} \\ &= \frac{\rho(a_{j-1}, \ldots, a_1) \cdot t_0 + (-1)^j s_0}{\beta_{j-1} \cdot \ldots \cdot \beta_1} \end{aligned}$$

shows (4.7). Now the partial divisor is also calculated inductively. The induction uses the partial divisor and the partial Milnor number for $n - 1$. Also $t_j | t_n$ and (2.18) ($\Lambda_a \Lambda_b = a \Lambda_b$ for $a|b$) are used.

$$\left(\prod_{j=1}^{n-1}\left(\frac{1}{s_j}\Lambda_{t_j}-\Lambda_1\right)\right)\cdot\left(\frac{1}{s_n}\Lambda_{t_n}-\Lambda_1\right)$$
$$=\left(\prod_{j=1}^{n-1}\left(\frac{1}{w_j}-1\right)\right)\cdot\frac{1}{s_n}\Lambda_{t_n}-\prod_{j=1}^{n-1}\left(\frac{1}{s_j}\Lambda_{t_j}-\Lambda_1\right)$$
$$=\frac{\beta_n...\beta_1}{t_0-s_0}\cdot\Lambda_{t_n}+(-1)^n+\sum_{j=1}^{n-1}(-1)^{n-j}\frac{\beta_j...\beta_1}{t_0-s_0}\cdot\Lambda_{t_j}.$$

This shows (4.9) and (4.10). The partial Milnor number is the degree of the partial divisor. □

Corollary 4.3 *In the situation of Lemma 4.2, suppose $w_0 = w_1$. Then*

$$s_0 = s_1 = 1, t_0 = t_1 = a_1 + 1, \beta_1 = a_1, \alpha_1 = 1. \tag{4.12}$$

Define

$$b_k := (a_1 + 1)\cdot a_2\cdot ...\cdot a_k \quad \textit{for } k = 1,\dots,n, \quad b_0 := 1, \tag{4.13}$$
$$\mu_k := \rho(a_k,\dots,a_2,a_1+1), \quad \textit{for } k = 1,\dots,n, \quad \mu_0 := 1. \tag{4.14}$$

Then

$$s_j = \frac{\mu_{j-1}}{\beta_j\cdot...\cdot\beta_2}, \tag{4.15}$$
$$t_j = \alpha_j\cdot t_{j-1} = \alpha_j\cdot...\cdot\alpha_2\cdot(a_1+1) = \frac{b_j}{\beta_j\cdot...\cdot\beta_2}, \tag{4.16}$$
$$D_{\mathbf{w}} = \prod_{j=1}^{n}\left(\frac{1}{s_j}\Lambda_{t_j}-\Lambda_1\right)$$
$$= (-1)^n + \sum_{j=1}^{n}(-1)^{n-j}\beta_j...\beta_2\cdot\Lambda_{t_j} \tag{4.17}$$
$$= (-1)^n + \sum_{j=1}^{n}(-1)^{n-j}b_j\cdot E_{t_j}, \tag{4.18}$$
$$\mu_k = \prod_{j=1}^{k}\left(\frac{1}{w_j}-1\right) = b_k - \mu_{k-1}. \tag{4.19}$$

Furthermore, define

$$\sum_{j=0}^{n}(-1)^{n-j}\Lambda_{b_j} =: \sum_{i=1}^{\mu}[\lambda_i]. \tag{4.20}$$

The definition (4.20) *makes sense, as obviously for the divisor on the left hand side,*

$$\nu(m) = \begin{cases} 1 \text{ if for some even } k\ m|b_{n-k}, m \not| b_{n-k-1}, \\ 0 \text{ else.} \end{cases} \tag{4.21}$$

Then

$$\sum_{j=0}^{n}(-1)^{n-j}\Lambda_{b_j} = \prod_{j=1}^{n}\left(\frac{1}{\mu_{j-1}}\Lambda_{b_j} - \Lambda_1\right) \tag{4.22}$$

$$D_{\mathbf{w}} = \sum_{i=1}^{\mu}[\lambda_i^{\mu}]. \tag{4.23}$$

Proof Formula (4.12) is trivial. The formulas (4.15) to (4.19) are immediate consequences of the formulas in Lemma 4.2. (4.22) is proved inductively by a similar calculation as (4.9),

$$\begin{aligned} &\left(\prod_{j=1}^{n}\left(\frac{1}{\mu_{j-1}}\Lambda_{b_j} - \Lambda_1\right)\right) \cdot \left(\frac{1}{\mu_{n-1}}\Lambda_{b_n} - \Lambda_1\right) \\ &= \left(\prod_{j=1}^{n-1}\left(\frac{1}{w_j} - 1\right)\right) \cdot \frac{1}{\mu_{n-1}}\Lambda_{b_n} - \prod_{j=1}^{n-1}\left(\frac{1}{\mu_{j-1}}\Lambda_{b_j} - \Lambda_1\right) \\ &= \Lambda_{b_n} + \sum_{j=0}^{n-1}(-1)^{n-j}\Lambda_{b_j}. \end{aligned}$$

For the final formula (4.23), it is in view of (2.31) and (4.18) enough to show

$$\frac{b_j}{\gcd(b_j,\mu)} = t_j \qquad \text{for } j \geq 1. \tag{4.24}$$

But $\mu_k = b_k - \mu_{k-1}$ (for $k \geq 1$) and $b_k = a_k b_{k-1}$ (for $k \geq 2$) show

$$\gcd(b_j,\mu) = \gcd(b_j,\mu_{n-1}) = \ldots = \gcd(b_j,\mu_j) = \gcd(b_j,\mu_{j-1}).$$

As $w_j = \frac{s_j}{t_j} = \frac{\mu_{j-1}}{b_j}$,

$$t_j = \frac{b_j}{\gcd(b_j,\mu_{j-1})} = \frac{b_j}{\gcd(b_j,\mu)}.$$

□

Remark 4.4 The last formula (4.23) in corollary 4.3 fits to a result of Orlik and Randell [16, (2.11) theorem]. They showed that the integral monodromy is the μ-th power of a cyclic automorphism of the Milnor lattice, whose eigenvalues are given by the divisor in (4.20). Formula (4.23) just confirms that the divisor $D_{\mathbf{w}}$ has the eigenvalues which fit to this theorem.

We made this calculation mainly to see how it works and to get some inspiration for good guesses for other types of weight systems of isolated quasihomogeneous singularities.

5 Examples and counter-examples

This section offers examples. Some of them are counter-examples to conjectures or hopes.

Example 5.1 Recall that for weight systems with $n = 3$ $\overline{(C1)} \iff (C1)$ by Remark 3.10 (iii). The first known (and the only documented) example of a weight system $(v_1, \ldots, v_n, d)$ which satisfies $\overline{(C1)}$, but not $(C1)$, is an example of Ivlev [2, 12.3] with $n = 4$. It is the integer weight system

$$(\mathbf{v}, d) = (1, 24, 33, 58, 265). \tag{5.1}$$

By Lemma 3.7 (c) $\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]$. Ivlev (and we, too) calculated that $\rho_{(\mathbf{v},d)}$ is even in $\mathbb{N}_0[t]$. By Theorem 3.5 any quasihomogeneous polynomial with this weight system has a non-isolated singularity at 0.

Now we show that $(\mathbf{v}, d)$ satisfies $\overline{(C1)}$, but not $(C1)$. Observe

$$(\mathbf{w}, 1) = \left(\frac{1}{265}, \frac{24}{265}, \frac{33}{265}, \frac{58}{265}, 1\right), \quad w_j = \frac{v_j}{d} = \frac{s_j}{t_j}, \text{ with } s_j = v_j, t_j = d,$$

and

$$\begin{aligned}
&265 = 5 \cdot 53, \quad 264 = 3 \cdot 8 \cdot 11 = 8 \cdot 33 = 11 \cdot 24, \\
&265 - 33 = 232 = 4 \cdot 58, \\
&\gcd(24, 33) = 3, 265 - 58 = 207 = 3 \cdot 69, \\
&\text{but } 207 \notin SG(24, 33) := \mathbb{N}_0 \cdot 24 + \mathbb{N}_0 \cdot 33.
\end{aligned}$$

The following table lists the sets J with $|J| \leq 2$ which satisfy alone or with a suitable set $K \subseteq N \backslash J$ the condition $(C1)$.

J	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1, 2\}$	$\{1, 3\}$	$\{1, 4\}$	$\{2, 4\}$	$\{3, 4\}$
K		$\{1\}$	$\{1\}$	$\{3\}$				$\{1, 3\}$	

The set $J = \{2, 3\}$ satisfies with $K = \{1, 4\}$ $\overline{(C1)}$, but not $(C1)$. In the notation of [6, Example 3.2 (iii)], the weight system is of type XII (but with a different numbering). The sets $M(k)$ are

$$M(k) = \begin{cases} M(265) = N = \{1, 2, 3, 4\} & \text{if } 265|k, \\ M(1) = \emptyset & \text{if } 265 \not| k. \end{cases}$$

Therefore only the values of $L(k) = (-1)^{n-|M((k)|} \cdot \mu(k)$ and $\chi(k)$ for $k \in \{1, 265\}$ are interesting.

$$(L(265), L(1)) = (66516, 1) = (\mu, 1),$$
$$(\chi(265), \chi(1)) = (251, 1),$$
$$D_{\mathbf{w}} = 251 \cdot \Lambda_{265} + 1 \cdot \Lambda_1 = \frac{\mu - 1}{265} \cdot \Lambda_{265} + \Lambda_1.$$

Remarks 5.2 **Open problem 4:** Find other examples of integer weight systems $(\mathbf{v}, d)$ which satisfy $\overline{(C1)}$, but not $(C1)$. Both cases, $\rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t]$ and $\rho_{(\mathbf{v},d)} \in \mathbb{Z}[t]\backslash\mathbb{N}_0[t]$, are interesting. Because of Remark 3.10 (iii), all such examples satisfy $n \geq 4$. Find examples with $n = 4$ of other types as Ivlev's example, which is of type XII in the notation of [6, Example 3.2 (iii)].

Examples 5.3 Here some examples of weight systems of isolated quasihomogeneous singularities are given, together with the values of ν, χ and L from Sect. 2.

(i) $n = 3, \; N = \{1, 2, 3\}$,

$$(w_1, w_2, w_3, 1) = \left(\frac{1}{4}, \frac{1}{6}, \frac{5}{12}, 1\right).$$

One singularity with this weight system is $x_1^4+x_2^6+x_2x_3^2$. The monomials $x_1^4, x_2^6, x_2x_3^2$ give the type II in [6, example 3.2 (ii)]. The following table lists all sets $M(k)$ and suitable values of k.

N	$\{1\}$	$\{2\}$	$\emptyset$
12	4	6	1

Therefore only the values of $L(k) = (-1)^{n-|M(k)|} \cdot \mu(k)$ and $\chi(k)$ for $k \in \{12, 4, 6, 1\}$ are interesting.

$$(L(12), L(4), L(6), L(1)) = (21, 3, 5, -1),$$
$$(\chi(12), \chi(4), \chi(6), \chi(1)) = (1, 1, 1, -1),$$
$$D_{\mathbf{w}} = \Lambda_{12} + \Lambda_4 + \Lambda_6 - \Lambda_1$$
$$= \Lambda_{12} + (\Psi_6 + \Psi_4 + \Psi_3 + \Psi_2 + \Psi_1) + \Psi_2.$$

(ii) $n = 4, \; N = \{1, 2, 3, 4\}$,

$$(w_1, w_2, w_3, w_4, 1) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{6}, \frac{5}{12}, 1\right).$$

One singularity with this weight system is $x_1^5 + x_1x_2^2 + x_3^6 + x_3x_4^2$. The monomials $x_1^5, x_1x_2^2, x_3^6, x_3x_4^2$ give the type XIII in [6, example 3.2 (iii)]. The following table lists all sets $M(k)$ and suitable values of k.

N	$\{1, 2, 3\}$	$\{3, 4\}$	$\{1, 2\}$	$\{3\}$	$\emptyset$
60	30	12	5	6	1

Therefore only the values of $L(k) = (-1)^{n-|M(k)|} \cdot \mu(k)$ and $\chi(k)$ for $k \in \{60, 30, 12, 5, 6, 1\}$ are interesting.

$$\begin{aligned}
&(L(60), L(30), L(12), L(5), L(6), L(1)) = (42, -30, 7, 6, -5, 1),\\
&(\chi(60), \chi(30), \chi(12), \chi(5), \chi(6), \chi(1)) = (1, -1, 1, 1, -1, 1),\\
&D_{\mathbf{w}} = \Lambda_{60} - \Lambda_{30} + \Lambda_{12} + \Lambda_5 - \Lambda_6 + \Lambda_1\\
&\quad = (\Psi_{60} + \Psi_{20} + \Psi_{12} + \Psi_5 + \Psi_4 + \Psi_1) + (\Psi_{12} + \Psi_4 + \Psi_1).
\end{aligned}$$

(iii) The curve singularity D_{2q}, $x_1^{2q-1} + x_1x_2^2$:

$$n = 2, N = \{1, 2\}, \mu = 2q, (w_1, w_2, 1) = \left(\frac{1}{2q-1}, \frac{q-1}{2q-1}, 1\right).$$

The monomials $x_1^{2q-1}, x_1x_2^2$ give the type II in [6, example 3.2 (i)]. The following table lists all sets $M(k)$ and suitable values of k.

N	$\emptyset$
$2q-1$	1

Therefore only the values of $L(k) = (-1)^{n-|M(k)|} \cdot \mu(k)$ and $\chi(k)$ for $k \in \{2q-1, 1\}$ are interesting.

$$\begin{aligned}
(L(2q-1), L(1)) &= (2q, 1),\\
(\chi(2q-1), \chi(1)) &= (1, 1),\\
D_{\mathbf{w}} &= \Lambda_{2q-1} + \Lambda_1.
\end{aligned}$$

(iv) The curve singularity D_{2q+1}, $x_1^{2q} + x_1x_2^2$:

$$n = 2, N = \{1, 2\}, \mu = 2q+1, (w_1, w_2, 1) = \left(\frac{1}{2q}, \frac{2q-1}{4q}, 1\right).$$

The monomials $x_1^{2q}, x_1x_2^2$ give the type II in [6, example 3.2 (i)]. The following table lists all sets $M(k)$ and suitable values of k.

N	$\{1\}$	$\emptyset$
$4q$	$2q$	1

Therefore only the values of $L(k) = (-1)^{n-|M(k)|} \cdot \mu(k)$ and $\chi(k)$ for $k \in \{4q, 2q, 1\}$ are interesting.

$$\begin{aligned} (L(4q), L(2q), L(1)) &= (2q+1, -(2q-1), 1), \\ (\chi(4q), \chi(2q), \chi(1)) &= (1, -1, 1), \\ D_{\mathbf{w}} &= \Lambda_{4q} - \Lambda_{2q} + \Lambda_1. \end{aligned} \tag{5.2}$$

K. Saito proposed the following conjecture.

Conjecture 5.4 [19, (3.13) and (4.2)] *Let $(w_1, \dots, w_n, 1)$ be a normalized weight system such that $\rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t]$ or (in general stronger) such that $(IS3)$ (from Theorem 3.5) holds for the reduced weight system. Then $D_{\mathbf{w}} = \sum_{m\in\mathbb{N}} \nu(m) \cdot \Psi_m$ satisfies*

$$\nu(d_{\mathbf{w}}) > 0 \text{ or } \nu\left(\frac{d_{\mathbf{w}}}{2}\right) > 0, \tag{5.3}$$

$$\nu(d_{\mathbf{w}}) > 0 \text{ if all } w_j < \frac{1}{2}, \tag{5.4}$$

i.e., in the case $(IS3)$ the monodromy has eigenvalues of order $d_{\mathbf{w}}$ or of order $\frac{d_{\mathbf{w}}}{2}$, and if all $w_j < \frac{1}{2}$ it has eigenvalues of order $d_{\mathbf{w}}$.

Saito was not aware of the part of Theorem 3.5 saying that the condition $(C1)$ is sufficient for $(IS3)$ (necessity is proved in [17]). Probably therefore he gave in the conjecture in [19, (3.13] the characterization $\rho_{(\mathbf{v},d)} \in \mathbb{N}_0[t]$, which is in the cases $n \le 3$ sufficient and necessary for $(IS3)$ ([2,18] [6, lemma 2.4]). In [19, (4.2)] he gave the condition $(IS3)$.

He proved in [19] a result which implies the conjecture for $n = 3$. He also stated that it is true for $n = 2$.

The following examples disprove the part (5.4) of the conjecture for $n = 4$. They can be extended easily to $n \ge 5$.

Examples 5.5 Consider two curve singularities $D_{2^k q_1+1}$ and $D_{2^k q_2+1}$ with $k, q_1, q_2 \in \mathbb{N}$ with q_1 and q_2 odd and $\mathrm{lcm}(q_1, q_2) > \max(q_1, q_2)$. Then their Thom–Sebastiani sum $D_{2^k q_1+1} \otimes D_{2^k q_2+1}$ is a quasihomogeneous singularity in $n = 4$ variables with normalized weights

$$\left(\frac{1}{2^k q_1}, \frac{2^k q_1 - 1}{2^{k+1} q_1}, \frac{1}{2^k q_2}, \frac{2^k q_2 - 1}{2^{k+1} q_2}\right)$$

and $d_{\mathbf{w}} = 2^{k+1}\mathrm{lcm}(q_1, q_2)$. The divisor of the characteristic polynomial is because of (5.2)

$$\begin{aligned} D_{\mathbf{w}} &= (\Lambda_{2^{k+1}q_1} - \Lambda_{2^k q_1} + \Lambda_1) \cdot (\Lambda_{2^{k+1}q_2} - \Lambda_{2^k q_2} + \Lambda_1) \\ &= (2^{k+1} - 2^k - 2^k) \gcd(q_1, q_2) \Lambda_{2^{k+1}\mathrm{lcm}(q_1,q_2)} \\ &\quad + 2^k \gcd(q_1, q_2) \Lambda_{2^k \mathrm{lcm}(q_1,q_2)} + \Lambda_{2^{k+1}q_1} + \Lambda_{2^{k+1}q_2} \end{aligned}$$

$$
\begin{aligned}
&-\Lambda_{2^k q_1} - \Lambda_{2^k q_2} + \Lambda_1. \\
= &\ 2^k \gcd(q_1, q_2)\Lambda_{2^k \mathrm{lcm}(q_1,q_2)} + \Lambda_{2^{k+1} q_1} + \Lambda_{2^{k+1} q_2} \\
&-\Lambda_{2^k q_1} - \Lambda_{2^k q_2} + \Lambda_1.
\end{aligned}
$$

Part (5.4) of Conjecture 5.3 does not hold here.

Remarks 5.6 In Examples 5.5 the part (5.3) of the conjecture does hold. That part of the conjecture is still open.

We checked the tables of weight systems of isolated quasihomogeneous singularities in $n = 4$ variables in [7] up to $\mu = 500$ for all weight systems for which (5.4) does not hold. There are 25 cases, and they are precisely those Thom–Sebastiani sums $D_{2^k q_1+1} \otimes D_{2^k q_2+1}$ in Examples 5.4 which satisfy $\mu \leq 500$. In 23 cases $k = 1$, in 2 cases $k = 2$.

This indicates that for $n = 4$ their might be no counter-examples to (5.3) and only the counter-examples in example 5.5 to (5.4).

Open problem 5:

(a) Prove or disprove the part (5.3) of Conjecture 5.4.
(b) Settle whether in the case $n = 4$ the only counter-examples to (5.4) are those in example 5.5.

6 A conjecture on the orders of the eigenvalues of the monodromy of an isolated quasihomogeneous singularity

Recall the Definition 1.1 of the Orlik block (H_M, h_M) and of the group $\mathrm{Aut}_{S^1}(H_M, h_M)$ for a finite nonempty set $M \subseteq \mathbb{N}$. The main result in [5] characterizes those sets M for which $\mathrm{Aut}_{S^1}(H_M, h_M)$ is as small as possible in terms of conditions on the set M. It is recalled below in Theorem 6.2. The following definitions are needed.

Definition 6.1 Let $M \subseteq \mathbb{N}$ be a finite set of positive integers.

(a) A graph $\mathcal{G}(M) = (M, E(M))$ is associated to it as follows. M itself is the set of vertices. The edges in $E(M)$ are directed. The set $E(m)$ is defined as follows. From a vertex $m_1 \in M$ to a vertex $m_2 \in M$ there is no edge if at least one of the following two conditions holds:

(i) m_1/m_2 is not a power of a prime number.
(ii) An $m_3 \in M \backslash \{m_1, m_2\}$ with $m_2|m_3|m_1$ exists.

If m_1/m_2 is a power p^k with $k \in \mathbb{N}$ of a prime number p and if no $m_3 \in M\backslash\{m_1, m_2\}$ with $m_2|m_3|m_1$ exists, then there is a directed edge from m_1 to m_2, which is additionally labeled with p. It is called a p-edge. Together such edges form the set $E(M)$ of all edges.

(b) For any prime number p the components of the graph $(M, E(M)\backslash\{p\text{-edges}\})$ which is obtained by deleting all p-edges, are called the p-planes of the graph. A p-plane is called a highest p-plane if no p-edge ends at a vertex of the p-plane. A p-edge from m_1 to m_2 is called a highest p-edge if no p-edge ends at m_1.

(c) A property (T_p) for a prime number p and a property (S_2) for the prime number 2:

$$(T_p): \text{The graph } \mathcal{G}(M) \text{ has only one highest } p\text{-plane.} \tag{6.1}$$

$$(S_2): \text{The graph } (M, E(M)\backslash\{\text{highest 2-edges}\}) \text{ has only 1 or 2 components.} \tag{6.2}$$

(d) The least common multiple of the numbers in M is denoted $\text{lcm}(M) \in \mathbb{N}$. For any prime number p denote

$$l(m, p) := \max(l \in \mathbb{N}_0 \mid p^l \text{ divides } m) \quad \text{for any } m \in \mathbb{N},$$
$$l(M, p) := \max(l(m, p) \mid m \in M) = l(\text{lcm}(M), p).$$

Then $m = \prod_{p \text{ prime number}} p^{l(m,p)}$.

Theorem 6.2 [5, Theorem 1.2] *Let $M \subseteq \mathbb{N}$ be a finite set of positive integers, and let (H_M, h_M) be its Orlik block. Then*

$$\text{Aut}_{S^1}(H_M, h_M) = \{\pm h_M^k \mid k \in \mathbb{Z}\} \tag{6.3}$$

holds if and only if the graph $\mathcal{G}(M)$ satisfies one of the following two properties.

(I) $\mathcal{G}(M)$ is connected. It satisfies (S_2). It satisfies (T_p) for any prime number $p \geq 3$.
(II) $\mathcal{G}(M)$ has two components M_1 and M_2. The graphs $\mathcal{G}(M_1)$ and $\mathcal{G}(M_2)$ are 2-planes of $\mathcal{G}(M)$ and satisfy (T_p) for any prime number $p \geq 3$. Furthermore

$$\gcd(\text{lcm}(M_1), \text{lcm}(M_2)) \in \{1; 2\}, \tag{6.4}$$

$$l(M_1, 2) > l(M_2, 2) \in \{0; 1\}. \tag{6.5}$$

Motivated by Orlik's Conjecture 1.3, Theorem 6.2, and a search in the lists of weight systems and associated divisors $D_{\mathbf{w}}$ in [7], here we propose the following conjecture.

Conjecture 6.3 (= Conjecture 1.4) *For any isolated quasihomogeneous singularity, each of the sets $M_1, \ldots, M_{\nu_{\max}}$ satisfies condition (I) in Theorem 6.2.*

Remarks 6.4 (i) **Open problem 6:** Prove Conjecture 6.3 combinatorially (or disprove it by a counter-example).

(ii) The conjecture is hard to deal with, because it requires to split the characteristic polynomial into its elementary divisors (as also Orlik's conjecture). It is not easy to extract from the formula for $D_{\mathbf{w}}$, which is by the result of Milnor and Orlik the divisor of the characteristic polynomial, information about these elementary divisors. This formula is rather nice in terms of the Λ_m (though as a product, not a sum), but the elementary divisors require to consider the Ψ_m.

(iii) Example 6.5 (i) shows that the conditions (I) and (II) together in Theorem 6.2 do not behave well under tensor product. Example 6.5 (ii) shows that condition (I) alone does not behave well under tensor product. This leads to the open problem 7. It generalizes Conjecture 6.3. A solution of problem 7 (a)+(b) would imply a positive solution of problem 6.

(iv) **Open problem 7:**

(a) Find a natural condition for products f of cyclotomic polynomials which implies for any elementary divisor of f condition (I) in Theorem 6.2, and which is stable under tensor product.
(b) Prove that the characteristic polynomial of any quasihomogeneous singularity satisfies this condition.

(iv) It would be desirable to have other ways to express condition (I) in Theorem 6.2, e.g., in terms of the $\chi(m)$ of the divisor of a characteristic polynomial. But it is not clear how they could look like.

(v) Conjecture 6.3 is therefore proved only in a few cases, in Theorem 6.9. The proofs use Lemma 4.1 and Corollary 4.3.

Examples 6.5 (i) Consider $f_1 := \Phi_{12}\Phi_6^2\Phi_4^2\Phi_2$ and $f_2 := \Phi_5\Phi_1$. Then

$$f_1 \otimes f_2 = \Phi_{60}\Phi_{30}^2\Phi_{20}^2\Phi_{12}\Phi_{10}\Phi_6^2\Phi_4^2\Phi_2$$

by (2.19)–(2.20). Denote by $f_1 = g_{1,1} \cdot g_{1,2}$ and $f_2 = g_2$ and $f_1 \otimes f_2 = g_{\otimes,1} \cdot g_{\otimes,2}$ the decompositions into elementary divisors and by $M_{1,1}, M_{1,2}, M_2, M_{\otimes,1} M_{\otimes,2} \subseteq \mathbb{N}$ the corresponding sets (Fig. 1). Then

$M_{1,1} = \{12, 6, 4, 2\}$	satisfies condition (I),
$M_{1,2} = \{6, 4\}$	satisfies condition (II),
$M_2 = \{5, 1\}$	satisfies condition (I),
$M_{\otimes,1} = \{60, 30, 20, 12, 10, 6, 4, 2\}$	satisfies condition (I),
$M_{\otimes,2} = \{30, 20, 6, 4\}$	satisfies neither (I) nor (II).

(ii) Consider $f_1 := \Phi_7^2\Phi_3\Phi_1$ and $f_2 := \Phi_5^2\Phi_3\Phi_1$. Then

$$f_1 \otimes f_2 = \Phi_{35}^4\Phi_{21}^2\Phi_{15}^2\Phi_7^2\Phi_5^2\Phi_3^3\Phi_1^3$$

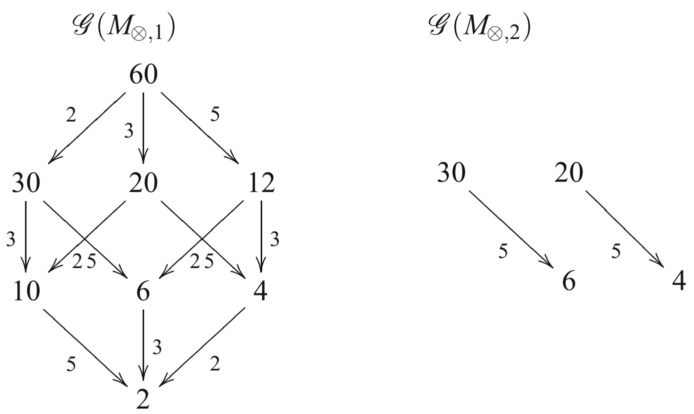

Fig. 1 For example 6.5 (i)

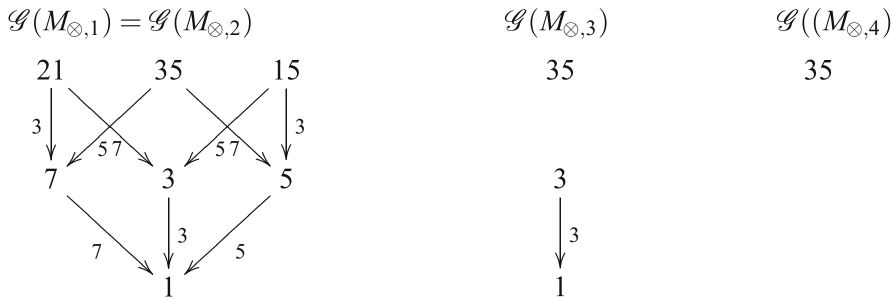

Fig. 2 For example 6.5 (ii)

by (2.19)–(2.21). Denote by $f_1 = g_{1,1} \cdot g_{1,2}$ and $f_2 = g_{2,1} \cdot g_{2,2}$ and $f_1 \otimes f_2 = g_{\otimes,1} \cdot g_{\otimes,2} \cdot g_{\otimes,3} \cdot g_{\otimes,4}$ the decompositions into elementary divisors and by $M_{i,j}$ and $M_{\otimes,j} \subseteq \mathbb{N}$ the corresponding sets (Fig. 2). Then

$M_{1,1} = \{7, 3, 1\}$,	$M_{1,2} = \{7\}$,
$M_{2,1} = \{5, 3, 1\}$,	$M_{2,2} = \{5\}$,
$M_{\otimes,1} = M_{\otimes,2} = \{35, 21, 15, 7, 5, 3, 1\}$	and $M_{\otimes,4} = \{35\}$
	satisfy all condition (I), but
$M_{\otimes,3} = \{35, 3, 1\}$	satisfies neither (I) nor (II).

Remark 6.6 (i) Lemma 8.2 in [4] gives the sufficient condition in part (ii) for $\mathrm{Aut}_{S^1}(H_M, h_M) = \{\pm h_M^k \,|\, k \in \mathbb{Z}\}$. It is a special case of condition (I) in Theorem 6.2. It holds for many elementary divisors of characteristic polynomials of isolated quasihomogeneous singularities. But Examples 6.7 (i)–(iii) give quasihomogeneous singularities where it does not hold for all elementary divisors of the characteristic polynomial.

(ii) A special case of condition (I) [4, Lemma 8.2]: M contains a largest number m_1 such that $\mathcal{G}(M)$ is a directed graph with root m_1. This implies (T_p) for any p. Additionally, a chain of 2-edges exists which connects all 2-planes. This implies (S_2).

Examples 6.7 (i) The weight system $(\mathbf{w}, 1) = (\frac{1}{6}, \frac{1}{10}, \frac{1}{15}, 1)$ satisfies $(C1)$ and $(IS3)$. It is of type I (=*Fermat type*) in the notation of [6, Example 3.2 (ii)]. The Brieskorn–Pham singularity $x_1^6 + x_2^{10} + x_3^{15}$ has this weight system. Here

$$\begin{aligned} D_{\mathbf{w}} &= (\Lambda_6 - \Lambda_1)(\Lambda_{10} - \Lambda_1)(\Lambda_{15} - \Lambda_1) \\ &= (2\Lambda_{30} - \Lambda_6 - \Lambda_{10} + \Lambda_1)(\Lambda_{15} - \Lambda_1) \\ &= 20\Lambda_{30} + \Lambda_6 + \Lambda_{10} + \Lambda_{15} - \Lambda_1 = \sum_{j=1}^{22} \mathrm{div}\, g_j, \end{aligned}$$

with the elementary divisors g_j with

$$\mathrm{div}\, g_j = \Lambda_{30} \quad \text{for } 1 \leq j \leq 20,$$

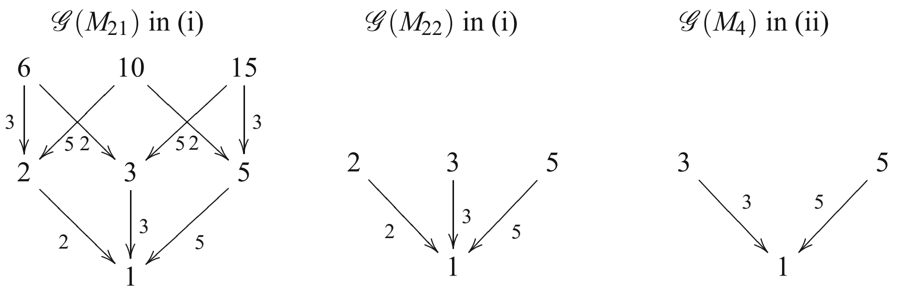

Fig. 3 For Examples 6.7 (i) and (ii)

$$\begin{aligned}\operatorname{div} g_{21} &= \Lambda_6 + \Lambda_{10} + \Lambda_{15} - \Lambda_2 - \Lambda_3 - \Lambda_5 + \Lambda_1 \\ &= \Psi_6 + \Psi_{10} + \Psi_{15} + \Psi_2 + \Psi_3 + \Psi_5 + \Psi_1, \\ \operatorname{div} g_{22} &= \Lambda_2 + \Lambda_3 + \Lambda_5 - 2\Lambda_1 = \Psi_2 + \Psi_3 + \Psi_5 + \Psi_1.\end{aligned}$$

The sets $M_{21} = \{6, 10, 15, 2, 3, 5, 1\}$ for g_{21} and $M_{22} = \{2, 3, 5, 1\}$ for g_{22} satisfy condition (I) in Theorem 6.2, but not the stronger conditions in Remark 6.6 (ii) (Fig. 3).

(ii) The weight system $(\mathbf{w}, 1) = (\frac{2}{15}, \frac{1}{5}, \frac{1}{3}, 1)$ satisfies $(C1)$ and $(IS3)$. It is of type II in the notation of [6, Example 3.2 (ii)]. One singularity with this weight system is $x_1^5x_2 + x_2^5 + x_3^3$. Here

$$\begin{aligned}D_{\mathbf{w}} &= \left(\frac{1}{2}\Lambda_{15} - \Lambda_1\right)(\Lambda_5 - \Lambda_1)(\Lambda_3 - \Lambda_1) = (2\Lambda_{15} - \Lambda_5 + \Lambda_1)(\Lambda_3 - \Lambda_1) \\ &= 3\Lambda_{15} + \Lambda_5 + \Lambda_3 - \Lambda_1 \\ &= 3\Psi_{15} + 4\Psi_5 + 4\Psi_3 + 4\Psi_1 = \sum_{j=1}^{4} \operatorname{div} g_j,\end{aligned}$$

with the elementary divisors g_j with

$$\begin{aligned}\operatorname{div} g_j &= \Lambda_{15} \quad \text{for } 1 \le j \le 3, \\ \operatorname{div} g_4 &= \Psi_5 + \Psi_3 + \Psi_1.\end{aligned}$$

The set $M_4 = \{5, 3, 1\}$ for g_4 satisfies condition (I) in Theorem 6.2, but not the stronger conditions in Remark 6.6 (ii).

(iii) The first of Examples 5.4 is $D_7 \otimes D_{11}$ with $(k, q_1, q_2) = (1, 3, 5)$ and $(\mathbf{w}, 1) = (\frac{1}{6}, \frac{5}{12}, \frac{1}{10}, \frac{9}{20}, 1)$. It satisfies $(C1)$ and is of type IV in the notation of [6, Example 3.2 (iii)]. Here

$$D_{\mathbf{w}} = 2\Lambda_{30} + \Lambda_{12} + \Lambda_{20} - \Lambda_6 - \Lambda_{10} + \Lambda_1 = \sum_{j=1}^{3} \operatorname{div} g_j,$$

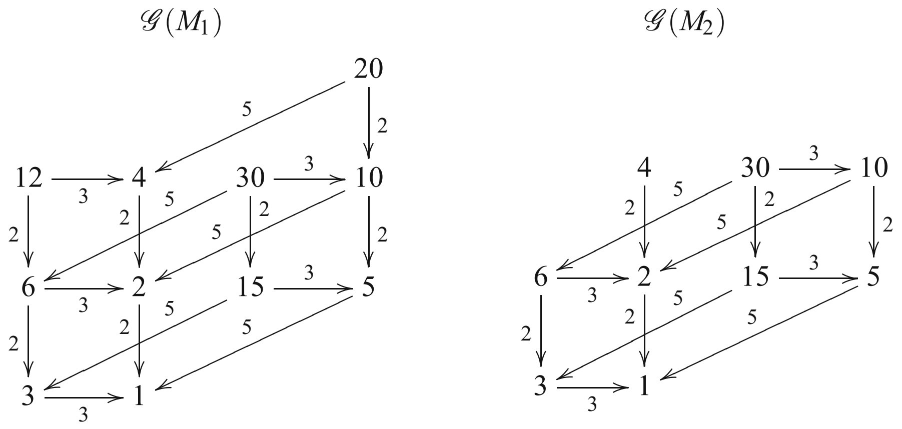

Fig. 4 For example 6.7 (iii)

with the elementary divisors g_j with div $g_j = \sum_{m \in M_j} \Psi_m$ and

$$M_1 = \{30, 20, 15, 12, 10, 6, 5, 4, 3, 2, 1\},$$
$$M_2 = \{30, 15, 10, 6, 5, 4, 3, 2, 1\},$$
$$M_3 = \{1\}.$$

The sets M_1 and M_2 satisfy condition (I) in Theorem 6.2, but not the stronger conditions in Remark 6.6 (ii) (Fig. 4).

Lemma 6.8 *Suppose that numbers $k_1, \ldots, k_l \in \mathbb{N}$ with $k_j | k_{j-1}$ for $j = 2, \ldots, l$ are given. Then the set $M \subseteq \mathbb{N}$ which is defined by $\Lambda_{k_1} - \Lambda_{k_2} + \cdots + (-1)^{l-1}\Lambda_{k_l} = \sum_{m \in M} \Psi_m$ is either empty or satisfies the conditions in Remark 6.6(ii).*

Proof We suppose that the set M is not empty. If $k_j = k_{j-1}$ for some $j \in \{2, 3, \ldots, l\}$, we can drop k_j and k_{j-1}. Therefore we can suppose $k_j < k_{j-1}$ for $j \in \{2, \ldots, l\}$. We have to prove the following two claims.

Claim 1 *The graph $\mathscr{G}(M)$ is a directed graph with root k_1.*

Claim 2 *In $\mathscr{G}(M)$ a chain of 2-edges exists which connects all 2-planes.*

Proof of claim 1: The cases $l \in \{1, 2\}$ are trivial. Suppose $l \geq 3$. The proof uses induction over l.

Define the sets M_1 and M_2 by $\Lambda_{k_1} - \Lambda_{k_2} = \sum_{m \in M_1} \Psi_m$ and $\sum_{j=3}^{l} (-1)^{j-1} \Lambda_{k_j} = \sum_{m \in M_2} \Psi_m$, so that $M = M_1 \dot{\cup} M_2$. The graph $\mathscr{G}(M_1)$ is obviously a directed graph with root k_1. The graph $\mathscr{G}(M_2)$ is by induction hypothesis a directed graph with root k_3. For the proof of the claim it is sufficient to show that the graph $\mathscr{G}(M)$ contains a directed edge from a vertex in M_1 to k_3. As $k_2 < k_1$, a prime number q with $l(k_2, q) < l(k_1, q)$ exists. Then the number

$$m := q^{l(k_1,q)} \cdot \prod_{p \text{ prime number}, p\neq q} p^{l(k_3,p)}$$

is in M_1, and there is a directed edge from m to k_3. □

Useful for the proof of claim 2 will be

Claim 3 *For any prime number p and any $r \in \mathbb{N}_0$, the set $M(p,r) := \{m \in M \mid l(m,p) = r\}$ is either empty or a single $p-$ plane. In the second case its graph is a directed graph with a root.*

Proof of claim 3: $M(p,r) = \{p^r \cdot m \mid m \in \widetilde{M}(p,r)\}$ where $\widetilde{M}(p,r)$ is the support of the divisor

$$\sum_{j:l(k_j,p)\geq r} (-1)^{j-1}\Lambda_{\widetilde{k_j}} \quad \text{with } \widetilde{k_j} := p^{-l(k_j,p)} \cdot k_j.$$

If this divisor is not 0, claim 1 applies and gives claim 3.

Proof of claim 2: Two cases will be distinguished.

1st case, for any odd $j \in \{1,\ldots,l-1\}$ $\frac{k_j}{k_{j+1}} = 2^{l(k_j,2)-l(k_{j+1},2)}$: Then

$$M(2,r) = \begin{cases} \emptyset & \text{if } r > l(k_1,2), \\ & \text{or if } l(k_j,2) \geq r > l(k_{j+1},2) \text{ for an even } j, \\ & \text{or if } l(k_l,2) \geq r \text{ and } l \text{ is even.} \\ \{2^r \cdot m \mid m|\widetilde{k_j}\} \text{ where } \widetilde{k_j} := 2^{-l(k_j,2)} \cdot k_j & \\ & \text{if } l(k_j,2) \geq r > l(k_{j+1},2) \text{ for an odd } j, \\ & \text{or if } l(k_l,2) \geq r \text{ and } j = l \text{ is odd.} \end{cases}$$

Define

$$\widetilde{k_{\min}} := \begin{cases} \widetilde{k_l} & \text{if } l \text{ is odd,} \\ \widetilde{k_{l-1}} & \text{if } l \text{ is even.} \end{cases}$$

Then the set $\{2^r \cdot \widetilde{k_{\min}} \mid M(2,r) \neq \emptyset\}$ is the set of vertices in M of a chain of 2-edges which connects all 2-planes.

2nd case, a minimal odd $j \in \{1,2,\ldots,l-1\}$ with $\frac{k_j}{k_{j+1}} \neq 2^{l(k_j,2)-l(k_{j+1},2)}$ exists: Then a prime number $p \geq 3$ with $l(k_j,p) > l(k_{j+1},p)$ exists. And then $\{2^{r-l(k_j,2)} \cdot k_j \mid 0 \leq r \leq l(k_j,2)\} \subseteq M$. Therefore for $0 \leq r \leq l(k_j,2)$ the set $M(2,r)$ is not empty. For $r > l(k_j,2)$ the set $M(2,r)$ is as in the 1st case. Thus the set $\{2^{r-l(k_j,2)} \cdot k_j \mid M(2,r) \neq \emptyset\}$ is the set of vertices in M of a chain of 2-edges which connects all 2-planes. □

Theorem 6.9 *Conjecture 6.3 holds for the weight systems of isolated quasihomogeneous singularities of cycle type and of chain type. It holds for all isolated* *quasihomogeneous singularities in $n = 2$ variables. It holds for the isolated quasihomogeneous singularities in $n = 3$ variables which are of the types III, IV, V, VI and VII in example 3.2 (ii) in* [6] *(see Remark 6.10 for the types I and II).*

In fact, in all these cases the set M of each elementary divisor satisfies even the stronger conditions in Remark 6.6 (ii).

Proof First consider the cycle type. Recall Lemma 4.1, and especially formula (4.5) for $D_{\mathbf{w}}$. It implies that all elementary divisors except one have the divisor $\Lambda_{d/\gamma}$, and the last one has the divisor $\Lambda_{d/\gamma} - \Lambda_1$ if n is odd, and it has the divisor Λ_1 if n is even. These divisors satisfy by Lemma 6.8 the conditions in Remark 6.6 (ii).

Next consider the chain type. Recall corollary 4.3 and especially formula (4.17) for $D_{\mathbf{w}}$. It implies that any elementary divisor satisfies the conditions in Lemma 6.8. Therefore it satisfies the conditions in Remark 6.6 (ii).

Now consider the case $n = 2$. By example 3.2 (i) in [6], there are three types. Type III is a cycle type. Type II is a chain type. They are treated above. Type I is the tensor product of two A-type singularities, it is called Fermat type. In general, the tensor product is difficult to deal with, but this case is fairly easy. Here the weights are $(w_1, w_2) = (\frac{1}{t_1}, \frac{1}{t_2})$, and $D_{\mathbf{w}}$ is

$$\begin{aligned} D_{\mathbf{w}} &= (\Lambda_{t_1} - \Lambda_1)(\Lambda_{t_2} - \Lambda_1) \\ &= \gcd(t_1, t_2)\Lambda_{\mathrm{lcm}(t_1,t_2)} - \Lambda_{t_1} - \Lambda_{t_2} + \Lambda_1. \end{aligned}$$

The elementary divisors are as follows.

$$\begin{aligned} \text{For } k \le \gcd(t_1, t_2) - 2: &\ \operatorname{div} g_k = \Lambda_{\mathrm{lcm}(t_1,t_2)}, \\ \text{for } k = \gcd(t_1, t_2) - 1: &\ \operatorname{div} g_k = \Lambda_{\mathrm{lcm}(t_1,t_2)} - \Lambda_{\gcd(t_1,t_2)} + \Lambda_1, \\ \text{for } k = \gcd(t_1, t_2): &\ \operatorname{div} g_k = \Lambda_{\mathrm{lcm}(t_1,t_2)} - \Lambda_{t_1} - \Lambda_{t_2} + \Lambda_{\gcd(t_1,t_2)}. \end{aligned}$$

The divisors in the first two cases satisfy the conditions in Lemma 6.8 and therefore the conditions in Remark 6.6 (ii).

Consider the divisor $\operatorname{div} g_k$ in the third case. Suppose that $t_1 \nmid t_2$ and $t_2 \nmid t_1$, because else $\operatorname{div} g_k = 0$. The set $M \subseteq \mathbb{N}$ with $\operatorname{div} g_k = \sum_{m \in M} \Psi_m$ is

$$M = \{m \in \mathbb{N} \,|\, m | \mathrm{lcm}(t_1, t_2), m \nmid t_1, m \nmid t_2\}.$$

Obviously, its graph is a directed graph with root $\mathrm{lcm}(t_1, t_2)$. This gives the first condition in Remark 6.6 (ii). For the second condition, we distinguish the following two cases. Write $\widetilde{t_j} = 2^{-m(t_j,2)} \cdot t_j$, so that $t_j = 2^{m(t_j,2)} \cdot \widetilde{t_j}$. Suppose $m(t_1, 2) \ge m(t_2, 2)$. Then $\widetilde{t_2} \nmid \widetilde{t_1}$ and $\mathrm{lcm}(t_1, t_2) = 2^{m(t_1,2)} \cdot \mathrm{lcm}(\widetilde{t_1}, \widetilde{t_2})$.

1st case, $\widetilde{t_1} \nmid \widetilde{t_2}$: Then the set $\{2^r \cdot \mathrm{lcm}(\widetilde{t_1}, \widetilde{t_2}) \,|\, 0 \le r \le m(t_1, 2)\}$ is a subset of M and is a chain of 2-edges which connects all 2-planes.

2nd case, $\widetilde{t_1} | \widetilde{t_2}$: Then $m(t_1, 2) > m(t_2, 2)$. Then the set $\{2^r \cdot \widetilde{t_2} \,|\, m(t_2, 2) + 1 \le r \le m(t_1, 2)\}$ is a subset of M and is a chain of 2-edges which connects all 2-planes.

Now consider the case $n = 3$. By example 3.2 (ii) in [6], there are seven types. Type V is a chain type, and type VII is a cycle type. They are treated above. The types III, IV and VI will be treated in a similar way as the type I for $n = 2$.

Type III for $n = 3$: weights $\mathbf{w} = (\frac{1}{t_1}, \frac{s_2}{t_2}, \frac{s_3}{t_3})$ with

$$w_j = \frac{1 - w_1}{a_j}, \quad t_j = t_1 \cdot \alpha_j \text{ with } \alpha_j = \frac{a_j}{\gcd(a_j, t_1 - 1)} \quad \text{for } j = 2, 3,$$

for some $a_2, a_3 \in \mathbb{N}$. Write $\widetilde{\alpha} := \operatorname{lcm}(\alpha_2, \alpha_3)$. Then

$$\begin{aligned}
D_{\mathbf{w}} &= \left(\frac{1}{s_1}\Lambda_{t_1} - \Lambda_1\right)\left(\frac{1}{s_2}\Lambda_{t_2} - \Lambda_1\right)\left(\frac{1}{s_3}\Lambda_{t_3} - \Lambda_1\right) \\
&= (\Lambda_{t_1} - \Lambda_1)\left(\frac{t_1 \cdot \gcd(\alpha_2, \alpha_3)}{s_2 \cdot s_3}\Lambda_{t_1\widetilde{\alpha}} - \frac{1}{s_2}\Lambda_{t_2} - \frac{1}{s_3}\Lambda_{t_3} + \Lambda_1\right) \\
&= r_1\Lambda_{t_1\widetilde{\alpha}} - r_2\Lambda_{t_2} - r_3\Lambda_{t_3} + \Lambda_{t_1} - \Lambda_1 \\
\text{with } r_1 &= \frac{t_1(t_1 - 1)\gcd(\alpha_2, \alpha_3)}{s_2 s_3}, \quad r_2 = \frac{t_1 - 1}{s_2}, \quad r_3 = \frac{t_1 - 1}{s_3}.
\end{aligned}$$

Suppose (without loss of generality) that $r_2 \leq r_3$. The elementary divisors g_k are as follows:

$$\begin{aligned}
\text{For } 1 \leq k \leq r_1 - r_2 - r_3 &: \ \operatorname{div} g_k = \Lambda_{t_1\widetilde{\alpha}}, \\
\text{for } k = r_1 - r_2 - r_3 + 1 &: \ \operatorname{div} g_k = \Lambda_{t_1\widetilde{\alpha}} - \Lambda_{t_1 \gcd(\alpha_2,\alpha_3)} \\
&\qquad\qquad\qquad + \Lambda_{t_1} - \Lambda_1, \\
\text{for } r_1 - r_2 - r_3 + 2 \leq k \leq r_1 - r_3 &: \ \operatorname{div} g_k = \Lambda_{t_1\widetilde{\alpha}} - \Lambda_{t_1 \gcd(\alpha_2,\alpha_3)}, \\
\text{for } r_1 - r_3 + 1 \leq k \leq r_1 - r_2 &: \ \operatorname{div} g_k = \Lambda_{t_1\widetilde{\alpha}} - \Lambda_{t_3}, \\
\text{for } r_1 - r_2 + 1 \leq k \leq r_1 &: \ \operatorname{div} g_k = \Lambda_{t_1\widetilde{\alpha}} - \Lambda_{t_2} - \Lambda_{t_3} \\
&\qquad\qquad\qquad + \Lambda_{t_1 \gcd(\alpha_2,\alpha_3)}.
\end{aligned}$$

The divisors in the first four cases satisfy the conditions in Lemma 6.8 and therefore the conditions in Remark 6.6 (ii). The divisors in the fifth case are of the same type as the divisor in the third case in type I for $n = 2$.

Type IV for $n = 3$ is a sum of a 1 variable Fermat type and a 2 variable cycle type. The weights are $\mathbf{w} = (\frac{1}{t_1}, \frac{s_2}{t_2}, \frac{s_3}{t_3})$ with

$$\begin{aligned}
\gamma &= \gcd(a_2 - 1, a_2a_3 - 1) = \gcd(a_3 - 1, a_2a_3 - 1), \\
t_2 = t_3 &= \frac{a_2a_3 - 1}{\gamma}, \quad s_2 = \frac{a_3 - 1}{\gamma}, \quad s_3 = \frac{a_2 - 1}{\gamma}
\end{aligned}$$

for some $a_2, a_3 \in \mathbb{N}_{\geq 2}$, by Lemma 4.1. Write $\widetilde{t} := \gcd(t_1, t_2)$. Again by Lemma 4.1, $D_{\mathbf{w}}$ is the product

$$\begin{aligned}
D_{\mathbf{w}} &= (\Lambda_{t_1} - \Lambda_1)(\gamma\Lambda_{t_2} + \Lambda_1) \\
&= \gamma\widetilde{t}\Lambda_{\operatorname{lcm}(t_1,t_2)} - \gamma\Lambda_{t_2} + \Lambda_{t_1} - \Lambda_1.
\end{aligned}$$

The elementary divisors g_k are as follows:

$$\begin{aligned}
\text{For } 1 \leq k \leq \gamma(\widetilde{t}-1): \ & \operatorname{div} g_k = \Lambda_{\operatorname{lcm}(t_1,t_2)},\\
\text{for } k = \gamma(\widetilde{t}-1)+1: \ & \operatorname{div} g_k = \Lambda_{\operatorname{lcm}(t_1,t_2)} - \Lambda_{t_2} + \Lambda_{\widetilde{t}} - \Lambda_1,\\
\text{for } \gamma(\widetilde{t}-1)+2 \leq k \leq \gamma\widetilde{t}: \ & \operatorname{div} g_k = \Lambda_{\operatorname{lcm}(t_1,t_2)} - \Lambda_{t_2},\\
\text{for } k = \gamma\widetilde{t}+1: \ & \operatorname{div} g_k = \Lambda_{t_1} - \Lambda_{\widetilde{t}}.
\end{aligned}$$

All these divisors satisfy the conditions in Lemma 6.8 and therefore the conditions in Remark 6.6 (ii).

Type VI for $n = 3$ consists of a cycle such that one of its vertices is the root of a 2 variable chain. The weights are $\mathbf{w} = (\frac{s_1}{t_1}, \frac{s_2}{t_2}, \frac{s_3}{t_3})$ with

$$\begin{aligned}
&\gamma = \gcd(a_2-1, a_1a_2-1) = \gcd(a_1-1, a_1a_2-1),\ t_1 = t_2 = \frac{a_1a_2-1}{\gamma},\\
&t_3 = t_1 \cdot \alpha \text{ for some } \alpha \in \mathbb{N},\ s_1 = \frac{a_2-1}{\gamma},\ s_2 = \frac{a_1-1}{\gamma},
\end{aligned}$$

for some $a_1, a_2 \in \mathbb{N}_{\geq 2}$. By Lemma 4.1, $D_{\mathbf{w}}$ is the product

$$\begin{aligned}
D_{\mathbf{w}} &= (\gamma\Lambda_{t_1} + \Lambda_1)\left(\frac{1}{s_3}\Lambda_{t_3} - \Lambda_1\right)\\
&= r\Lambda_{t_3} - \gamma\Lambda_{t_1} - \Lambda_1 \quad \text{with} \quad r = \frac{\gamma t_1 + 1}{s_3}.
\end{aligned}$$

Observe $r \geq \gamma + 1$, because $r - \gamma - 1$ is the coefficient of $[1]$ in $D_{\mathbf{w}}$. The elementary divisors g_k are as follows:

$$\begin{aligned}
\text{For } 1 \leq k \leq r-\gamma-1: \ & \operatorname{div} g_k = \Lambda_{t_3},\\
\text{for } k = r-\gamma: \ & \operatorname{div} g_k = \Lambda_{t_3} - \Lambda_1,\\
\text{for } r-\gamma+1 \leq k \leq r: \ & \operatorname{div} g_k = \Lambda_{t_3} - \Lambda_{t_1}.
\end{aligned}$$

All these divisors satisfy the conditions in Lemma 6.8 and therefore the conditions in Remark 6.6 (ii). □

Remark 6.10 In example 3.2 (ii) in [6], i.e., for $n = 3$, type I is the Fermat type with $\mathbf{w} = (\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3})$, and type II is the sum of a 1 variable Fermat type and 2 variable chain type, so $\mathbf{w} = (\frac{1}{t_1}, \frac{1}{t_2}, \frac{s_3}{t_3})$ with $\frac{s_3}{t_3} = \frac{1-w_2}{a_3}$ and $a_3 \geq 2$. In both cases, a similar ansatz as in the Proof of Theorem 6.9 leads to an unpleasant multitude of different subcases. Examples 6.7 (i)+(ii) show that in special cases of both types, some elementary divisor does not satisfy the conditions in Remark 6.6 (ii). It does not seem worth to try to prove Conjecture 6.3 in this way.

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