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Uniform column sign-coherence and the existence of maximal green sequences

Peigen Cao[1] **· Fang Li**[1]

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Abstract

In this paper, we prove that each matrix in $M_{m\times n}(\mathbb{Z}_{\geq 0})$ is uniformly column sign-coherent (Definition 2.2 (ii)) with respect to any $n \times n$ skew-symmetrizable integer matrix (Corollary 3.3 (ii)). Using such matrices, we introduce the definition of irreducible skew-symmetrizable matrix (Definition 4.1). Based on this, the existence of maximal green sequences for skew-symmetrizable matrices is reduced to the existence of maximal green sequences for irreducible skew-symmetrizable matrices.

Keywords Cluster algebra · Sign-coherence · Maximal green sequence · Green-to-red sequence

Mathematics Subject Classification 13F60 · 05E40

1 Introduction

C-matrices (respectively, *G-matrices*) [7] are important research objects in the theory of cluster algebras. It is known that *C-matrices* (respectively, *G-matrices*) are column (respectively, row) sign-coherent (see Definition 2.2 (i)). In this paper, we consider the matrices which have the similar property with *C-matrices*. This property is called uniform column sign-coherence (see Definition 2.2 (ii)). By the definition of uniform column sign-coherence and a result in [9] (see Theorem 2.4 below), we know that I_n is uniformly column sign-coherent using the terminology in this paper.

The motivation to consider the uniform column sign-coherence comes from Proposition 3.7. This proposition indicates if some submatrix of a skew-symmetrizable

✉ Fang Li
fangli@zju.edu.cn

Peigen Cao
peigencao@126.com

1 Department of Mathematics, Zhejiang University (Yuquan Campus), Hangzhou 310027, Zhejiang, People's Republic of China

matrix B is uniformly column sign-coherent, then there is another non-trivial submatrix of B which is invariant under any particular sequence of mutations (see Proposition 3.7 for details).

It is natural to ask when a matrix is uniformly column sign-coherent. This is actually a hard question. However, we can turn our mind to the other side to think about how to produce new uniformly column sign-coherent matrices from a given one. Theorem 3.2 in this paper is an answer to this. As a corollary, matrices in $M_{m\times n}(\mathbb{Z}_{\geq 0})$ are proved to be uniformly column sign-coherent (Corollary 3.3).

Maximal green sequences are particular sequences of mutations of skew-symmetrizable matrices introduced by Keller [10]. Such particular sequences have numerous applications, including the computations of spectrums of BPS states, Donaldson–Thomas invariants, tilting of hearts in derived categories, and quantum dilogarithm identities.

A very important problem in cluster algebra theory is the existence of maximal green sequences for a given skew-symmetrizable matrix B. Most of the results on this topic have been given on a case-by-case basis, for example, quivers of finite type or acyclic quivers by Brüstle et al. [2], quivers from specific triangulations of various marked surfaces [1,4,8,11,13]. Generally, the existence of maximal green sequences is not mutation invariant (see [12]). In some special cases, many other authors proved that the existence of maximal green sequences is mutation invariant, for example, for finite-type quiver by Brüstle et al. [2], and for any quiver Q of finite mutation type by Mills [13]. The authors in [3] proved Rotation Lemma which says that if B admits a maximal green sequence, so does any skew-symmetrizable matrices along this sequence.

Now we give the question that we focus on in this paper. In [12, Theorem 9], Muller proved that if B has a maximal green sequence, so does any principal submatrix of B. Conversely, can we get the information about the existence of maximal green sequences for B from its proper submatrices? Our answer to this question is given in Theorem 4.5, whose proof depends on the uniform column sign-coherence.

Thanks to Theorem 4.5 in this paper and [12, Theorem 9], we reduce the existence of maximal green sequences for skew-symmetrizable matrices to the existence of maximal green sequences for irreducible skew-symmetrizable matrices (Definition 4.1). We also give a characterization for irreducible skew-symmetrizable matrices (Proposition 4.2).

Note that a very special case of Theorem 4.5 has been given in [8, Theorem 3.12]. In detail, the authors proved that if both quivers Q_1 and Q_2 have a maximal green sequences, then so does the quiver Q which is a "t-colored" direct sum of quivers Q_1 and Q_2. They believe that this result also holds for any direct sum of Q_1 and Q_2 ([8, Remark 3.13]) but they did not have a proof. Theorem 4.5 in this paper actually gives an affirmative answer to this.

This paper is organized as follows: In Sect. 2 some basic definitions are given. In Sect. 3 we give a method to produce uniformly column sign-coherent matrices from a given one (Theorem 3.2). Thus, we prove that each matrix in $M_{m\times n}(\mathbb{Z}_{\geq 0})$ is uniform column sign-coherent (Corollary 3.3). In Sect. 4 we give the definition of irreducible skew-symmetrizable matrices and their characterization. Then we reduce the existence of maximal green sequences for skew-symmetrizable matrices to the existence of maximal green sequences for irreducible skew-symmetrizable matrices.

2 Preliminaries

Recall that an integer matrix $B_{n\times n} = (b_{ij})$ is called **skew-symmetrizable** if there is a positive integer diagonal matrix S such that SB is skew-symmetric, where S is said to be a **skew-symmetrizer** of B. In this case, we say that B is S-skew-symmetrizable. For an $(m+n)\times n$ integer matrix $\tilde{B} = (b_{ij})$, the square submatrix $B = (b_{ij})_{1\le i,j\le n}$ is called the **principal part** of $\tilde{B}$. Abusing terminology, we say that $\tilde{B}$ itself is skew-symmetrizable or skew-symmetric if its principal part B is so.

Definition 2.1 Let $\tilde{B}_{(m+n)\times n} = (b_{ij})$ be S-skew-symmetrizable, the mutation of $\tilde{B}$ in the direction $k \in \{1, 2, \dots, n\}$ is the $(m+n)\times n$ matrix $\mu_k(\tilde{B}) = (b'_{ij})$, where

$$b'_{ij} = \begin{cases} -b_{ij}\ , & i = k \text{ or } j = k;\\ b_{ij} + \mathrm{sgn}(b_{ik})\max(b_{ik}b_{kj}, 0), & \text{otherwise.} \end{cases} \tag{1}$$

It is easy to see that $\mu_k(\tilde{B})$ is still S-skew-symmetrizable, and $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$.

Definition 2.2 (i) For $m, n > 0$, an $m\times n$ integer matrix A is called **column sign-coherent** (respectively, **row sign-coherent**) if any two nonzero entries of A in the same column (respectively, row) have the same sign.

(ii) Let B_1 be an $n\times n$ skew-symmetrizable matrix, and $B_2 \in M_{m\times n}(\mathbb{Z})$ be a column sign-coherent matrix. B_2 is called **uniformly column sign-coherent with respect to** B_1 if for any sequence of mutations $\mu_{k_s}\dots\mu_{k_2}\mu_{k_1}$, the lower $m\times n$ submatrix of $\mu_{k_s}\dots\mu_{k_2}\mu_{k_1}\begin{pmatrix}B_1\\B_2\end{pmatrix}$ is column sign-coherent.

Remark 2.3 (i) Note that the uniform column sign-coherence of B_2 is invariant up to permutation of its row vectors, by the equality (1).

(ii) Roughly, the uniform column sign-coherence means that the column sign-coherence is invariant after a sequence of mutations.

Given an S-skew-symmetrizable matrix $\tilde{B} = \begin{pmatrix}B\\I_n\end{pmatrix} \in M_{2n\times n}(\mathbb{Z})$, let $\tilde{B}_\sigma = \begin{pmatrix}B_\sigma\\C_\sigma\end{pmatrix}$ be the matrix obtained from $\tilde{B}$ by a sequence of mutations $\sigma := \mu_{k_s}\dots\mu_{k_2}\mu_{k_1}$. Recall that the lower part C_σ of $\tilde{B}_\sigma$ is called a C**-matrix** of B, see [7]. Note that the matrix $\tilde{B} = \begin{pmatrix}B\\I_n\end{pmatrix}$ is used to define cluster algebra with principal coefficients in [7], but we do not talk much about cluster algebra here.

Theorem 2.4 ([9]) *Using the above notations, each C-matrix of a skew-symmetrizable matrix B is column sign-coherent.*

Remark 2.5 By Definition 2.2, this theorem means that I_n is uniformly column sign-coherent with respect to the skew-symmetrizable matrix B.

Thanks to Theorem 2.4, one can define the sign functions on the column vectors of a *C-matrix* of a skew-symmetrizable matrix B. For a sequence of mutations $\sigma := \mu_{k_s} \dots \mu_{k_2}\mu_{k_1}$, denote by $\begin{pmatrix} B_\sigma \\ C_\sigma \end{pmatrix} := \mu_{k_s} \dots \mu_{k_2}\mu_{k_1} \begin{pmatrix} B \\ I_n \end{pmatrix}$. If the entries of j-th column of C_σ are all nonnegative (respectively, nonpositive), the sign of the j-th column of C_σ is defined as $\varepsilon_\sigma(j) = 1$ (respectively, $\varepsilon_\sigma(j) = -1$).

Definition 2.6 Let C_σ be the C-matrix of B given by a sequence of mutations σ, a column index $j \in \{1, \dots, n\}$ of C_σ is called **green** (respectively, **red**) if $\varepsilon_\sigma(j) = 1$ (respectively, $\varepsilon_\sigma(j) = -1$).

Note that, by Theorem 2.4, the column index of a *C-matrix* C_σ is either green or red.

Definition 2.7 Let B be a skew-symmetrizable matrix, and $\mathbf{k} = (k_1, \dots, k_s)$ be a sequence of column indices of B. Denote by C_{σ_j} the C-matrix of B given by $\sigma_j := \mu_{k_j} \dots \mu_{k_2}\mu_{k_1}$.

(i) $\mathbf{k} = (k_1, \dots, k_s)$ is called a **green-to-red sequence** of B if each column index of the C-matrix C_{σ_s} is red, i.e., $C_{\sigma_s} \in M_{n\times n}(\mathbb{Z}_{\le 0})$.

(ii) $\mathbf{k} = (k_1, \dots, k_s)$ is called a **green sequence** of B if k_i is green in the C-matrix $C_{\sigma_{i-1}}$ for $i = 2, 3, \dots, s$.

(iii) $\mathbf{k} = (k_1, \dots, k_s)$ is called **maximal green sequence** of B if it is both a green sequence and a green-to-red sequence of B.

Example 2.8 Let $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, and $\mathbf{k} = (2, 3, 1, 2)$.

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mu_3} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\mu_1} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ \hline -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\mu_2} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ \hline -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hence, $\mathbf{k} = (2, 3, 1, 2)$ is a maximal green sequence of B.

3 Uniform column sign-coherence of B_2

In this section, we give a method to produce uniformly column sign-coherent matrices from a known one (Theorem 3.2). Then it is shown that all nonnegative matrices and rank ≤ 1 column sign-coherent matrices are uniformly column sign-coherent (Corollary 3.3 and Corollary 3.4).

Lemma 3.1 *Let $P = (p_{ij}) \in M_{p\times m}(\mathbb{Z}_{\ge 0})$, $p, m > 0$, and B_1 be an $n \times n$ skew-symmetrizable matrix. If $B_2 \in M_{m\times n}(\mathbb{Z})$ is column sign-coherent, then for $1 \le k \le n$,*

$$\mu_k\left(\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right) = \mu_k\begin{pmatrix} B_1 \\ PB_2 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}\mu_k\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Proof Denote by $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (b_{ij})$, $\mu_k \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = (b'_{ij})$, $\begin{pmatrix} B_1 \\ PB_2 \end{pmatrix} = (a_{ij})$, $\mu_k \begin{pmatrix} B_1 \\ PB_2 \end{pmatrix} = (a'_{ij})$. Clearly, the principal parts of $\mu_k \begin{pmatrix} B_1 \\ PB_2 \end{pmatrix}$ and $\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix} \mu_k \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ are equal. It suffices to show the lower parts of $\mu_k \begin{pmatrix} B_1 \\ PB_2 \end{pmatrix}$ and $\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix} \mu_k \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ are equal. We know that for $i > n$, $a_{ij} = \sum_{l=1}^{m} p_{il} b_{n+l,j}$. By Eq. (1), for $i > n$,

$$a'_{ij} = a_{ij} + \operatorname{sgn}(a_{ik}) \max(a_{ik}b_{kj}, 0) = \sum_{l=1}^{m} p_{il} b_{n+l,j} + \operatorname{sgn}\left(\sum_{l=1}^{m} p_{il} b_{n+l,k}\right) \max\left(\sum_{l=1}^{m} p_{il} b_{n+l,k} b_{kj}, 0\right).$$

Because B_2 is column sign-coherent and $P \in M_{p\times m}(\mathbb{Z}_{\geq 0})$, we know that $(p_{il_1} b_{n+l_1,k})(p_{il_2} b_{n+l_2,k}) \geq 0, 1 \leq l_1, l_2 \leq m$. Thus, if $p_{il_1} b_{n+l_1,k} \neq 0$, then $\operatorname{sgn}(p_{il_1} b_{n+l_1,k}) = \operatorname{sgn}(\sum_{l=1}^{m} p_{il} b_{n+l,k})$. So

$$\begin{aligned} a'_{ij} &= \sum_{l=1}^{m} p_{il} b_{n+l,j} + \operatorname{sgn}\left(\sum_{l=1}^{m} p_{il} b_{n+l,k}\right) \max\left(\sum_{l=1}^{m} p_{il} b_{n+l,k} b_{kj}, 0\right) \\ &= \sum_{l=1}^{m} p_{il} b_{n+l,j} + \sum_{l=1}^{m} \operatorname{sgn}(p_{il} b_{n+l,k}) \max(p_{il} b_{n+l,k} b_{kj}, 0) \\ &= \sum_{l=1}^{m} p_{il} (b_{n+l,j} + \operatorname{sgn}(b_{n+l,k}) \max(b_{n+l,k} b_{kj}, 0)), \\ &= \sum_{l=1}^{m} p_{il} b'_{n+l,j}. \end{aligned}$$

Then the result follows. □

Theorem 3.2 *Let $P \in M_{p\times m}(\mathbb{Z}_{\geq 0})$ for $p, m > 0$, and B_1 be an $n \times n$ skew-symmetrizable matrix. If $B_2 \in M_{m\times n}(\mathbb{Z})$ is uniformly column sign-coherent with respect to B_1, then so is PB_2.*

Proof For any sequence of mutations $\mu_{k_s} \ldots \mu_{k_2}\mu_{k_1}$, the lower part of $\mu_{k_s} \ldots \mu_{k_2}\mu_{k_1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is column sign-coherent, by the uniform column sign-coherence of B_2 with respect to B_1. Clearly, the lower part of $\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix} \mu_{k_s} \ldots \mu_{k_2}\mu_{k_1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ is also column sign-coherent. By Lemma 3.1, we have

$$\mu_{k_s} \ldots \mu_{k_2}\mu_{k_1} \left(\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) = \begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix} \mu_{k_s} \ldots \mu_{k_2}\mu_{k_1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

So the lower part of $\mu_{k_s} \dots \mu_{k_2}\mu_{k_1}\left(\begin{pmatrix} I_n & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right)$ is also column sign-coherent. Thus, PB_2 is uniformly column sign-coherent with respect to B_1. □

Corollary 3.3 *Let B_1 be an $n \times n$ skew-symmetrizable matrix. Then any matrix $P \in M_{m\times n}(\mathbb{Z}_{\geq 0})$ is uniformly column sign-coherent with respect to B_1.*

Proof By Remark 2.5, I_n is uniformly column sign-coherent with respect to B_1. Then the result follows from Theorem 3.2 since $P = PI_n$. □

Corollary 3.4 *Let B_1 be an $n \times n$ skew-symmetrizable matrix, and B_2 be an $m \times n$ column sign-coherent integer matrix. If* $\operatorname{rank}(B_2) \leq 1$*, then B_2 is uniformly column sign-coherent with respect to B_1.*

Proof Because $\operatorname{rank}(B_2) \leq 1$, B_2 has the form of

$$B_2 = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \alpha,$$

where α is a row vector and $c_1, c_2, \dots, c_m \in \mathbb{Q}$. Because B_2 is column sign-coherent, we can assume that $c_1, c_2, \dots, c_m \geq 0$. Clearly, α is uniformly column sign-coherent with respect to B_1. Then by Theorem 3.2, B_2 is uniformly column sign-coherent with respect to B_1. □

By the two corollaries, we can construct many matrices which are uniformly column sign-coherent with respect to a given skew-symmetrizable matrix. Now we give an example showing that there does exist a matrix which is column sign-coherent but not uniformly column sign-coherent.

Example 3.5

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ \hdashline 2 & -1 \\ 1 & -2 \end{pmatrix} \xrightarrow{\mu_1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ \hdashline -2 & 1 \\ -1 & -1 \end{pmatrix}.$$

It can be seen that $\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$ is column sign-coherent but it is not uniformly column sign-coherent with respect to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is natural to consider the following problem.

Problem 3.6 *For a given skew-symmetrizable matrix B_1, which matrices are uniformly column sign-coherent with respect to B_1?*

In the following proposition, we give a characterization for those matrices which are uniformly column sign-coherent with respect to B_1.

Proposition 3.7 *Let* $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ *be a skew-symmetrizable matrix with* $B_1 \in M_n(\mathbb{Z})$ *and* $B_4 \in M_m(\mathbb{Z})$*,* $m > 0$*. Then* B_2 *is uniformly column sign-coherent with respect to* B_1 *if and only if* B_4 *is invariant under any sequence of mutations* $\mu_{k_s} \ldots \mu_{k_2}\mu_{k_1}$ *with* $1 \leq k_i \leq n, i = 1, 2, \ldots, s$.

Proof Let $B = (b_{ij})$, and $\mu_k(B) = (b'_{ij})$, $1 \leq k \leq n$. We know for any i, j,

$$b'_{ij} = b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik}b_{kj}, 0).$$

Then $b'_{ij} = b_{ij}$ if and only if $b_{ik}b_{kj} \leq 0$, and then if and only if $b_{ik}b_{jk} \geq 0$ because either $b_{kj}b_{jk} < 0$ or $b_{kj} = b_{jk} = 0$ holds.

So B_4 is invariant under the mutation $\mu_k(B) = (b'_{ij})$, $1 \leq k \leq n$ if and only if $b'_{ij} = b_{ij}$ for $n+1 \leq i, j \leq n+m$, and then if and only if $b_{ik}b_{jk} \geq 0$ for $n+1 \leq i, j \leq n+m$, $1 \leq k \leq n$, which means that B_2 is column sign-coherent. The result follows. □

4 The existence of maximal green sequences

Based on the discussion about uniform column sign-coherence, in this section, we reduce the existence of maximal green sequences for skew-symmetrizable matrices to the existence of maximal green sequences for irreducible skew-symmetrizable matrices.

4.1 Irreducible skew-symmetrizable matrices

In this subsection, we give the definition of irreducible skew-symmetrizable matrices and their characterization.

Let $B = (b_{ij})_{n\times n}$ be a matrix, and n_1, n_2 be two positive integers. For $1 \leq i_1 < \cdots < i_{n_2} \leq n$ and $1 \leq j_1 < \cdots < j_{n_1} \leq n$, denote by $B^{i_1,\ldots,i_{n_2}}_{j_1,\ldots,j_{n_1}}$ the submatrix of B with entries b_{ij}, where $i = i_1, \ldots, i_{n_2}$ and $j = j_1, \ldots, j_{n_1}$. If $n_2 < n$ or $n_1 < n$, the corresponding submatrix $B^{i_1,\ldots,i_{n_2}}_{j_1,\ldots,j_{n_1}}$ is a proper submatrix of B. If $n_2 = n_1$ and $\{i_1, \ldots, i_{n_2}\} = \{j_1, \ldots, j_{n_1}\}$, the corresponding submatrix is a principal submatrix of B. Clearly, any principal submatrix of a skew-symmetrizable matrix is still skew-symmetrizable.

Definition 4.1 A skew-symmetrizable matrix $B = (b_{ij})_{n\times n}$ is called **reducible**, if B has a proper submatrix $B^{i_1,\ldots,i_{n_2}}_{j_1,\ldots,j_{n_1}}$ satisfying

(i) $B^{i_1,\ldots,i_{n_2}}_{j_1,\ldots,j_{n_1}}$ is a nonnegative matrix, i.e., $B^{i_1,\ldots,i_{n_2}}_{j_1,\ldots,j_{n_1}} \in M_{n_2\times n_1}(\mathbb{Z}_{\geq 0})$.

(ii) $\{i_1, \ldots, i_{n_2}\} \cup \{j_1, \ldots, j_{n_1}\} = \{1, 2, \ldots, n\}$ and $\{i_1, \ldots, i_{n_2}\} \cap \{j_1, \ldots, j_{n_1}\} = \phi$.

Otherwise, B is said to be **irreducible** if such proper submatrix does not exist.

Clearly, B is reducible if and only if up to renumbering the row–column indices of B, B can be written as a block matrix as follows

$$B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$$

with $B_1 \in M_{n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2}(\mathbb{Z})$ such that the proper submatrix B_2 of B is a nonnegative matrix, i.e., $B_2 \in M_{n_2 \times n_1}(\mathbb{Z}_{\geq 0})$.

In the skew-symmetric case the definition of irreducibility for quiver version has been given in [8].

For a skew-symmetrizable matrix B, we can encode the sign pattern of entries of B by the quiver $\Gamma(B)$ with the vertices $1, 2, \ldots, n$ and the arrows $i \to j$ for $b_{ij} > 0$. We call $\Gamma(B)$ the **underlying quiver of** B. If $\Gamma(B)$ is an acyclic quiver, then B is said to be **acyclic**. If $\Gamma(B)$ is a connected quiver, then B is said to be **connected**. Clearly, if B is an irreducible skew-symmetrizable matrix, then it must be connected.

For a quiver Q, if there exists a path from a vertex a to a vertex b, then a is said to be a **predecessor** of b, and b is said to be a **successor** of a. For a vertex a in Q, denote by $M(a)$, $N(a)$ the set of predecessors of a and the set of successors of a, respectively. By viewing a vertex a as a trivial path from a to a, we know that $a \in M(a) \cap N(a)$.

Proposition 4.2 *Let $B = (b_{ij})_{n \times n}$ be a connected skew-symmetrizable matrix. Then B is irreducible if and only if each arrow of the quiver $\Gamma(B)$ is in some oriented cycle.*

Proof Suppose that B is reducible, then B can be written as a block matrix

$$B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$$

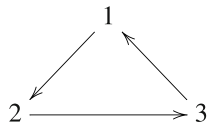
with $B_1 \in M_{n_1}(\mathbb{Z})$ and $B_4 \in M_{n_2}(\mathbb{Z})$ such that the proper submatrix $B_2 \in M_{n_2 \times n_1}(\mathbb{Z}_{\geq 0})$, up to renumbering the row–column indices of B. Since B is connected, B_2 cannot be a zero matrix. So there exist $i > n_1$, $j \leq n_1$ such that $b_{ij} \neq 0$. In fact $b_{ij} > 0$, since $B_2 \in M_{n_2 \times n_1}(\mathbb{Z}_{\geq 0})$. We know that the arrow $i \to j$ is not in any oriented cycle of $\Gamma(B)$, because $B_2 \in M_{n_2 \times n_1}(\mathbb{Z}_{\geq 0})$.

Suppose that there exists an arrow $i \to j$ is not in any oriented cycle of $\Gamma(B)$. We know that i cannot be a successor of j, i.e., $i \notin N(j)$. Let n_1 be the number of elements of $N(j)$. Clearly, $1 \leq n_1 \leq n-1$. We can renumber the row–column indices of B such that the elements of $N(j)$ are indexed by $1, 2, \ldots, n_1$. B can be written as a block matrix

$$B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}.$$

We claim that $B_2 \in M_{(n-n_1) \times n_1}(\mathbb{Z}_{\geq 0})$. Otherwise, there exists $k_1 > n_1$ and $k_2 \leq n_1$, i.e., $k_1 \notin N(j)$, $k_2 \in N(j)$ such that $b_{k_1 k_2} < 0$. Thus, k_1 is a successor of k_2, so is a successor of j, by $k_2 \in N(j)$. This contradicts $k_1 \notin N(j)$. So $B_2 \in M_{(n-n_1) \times n_1}(\mathbb{Z}_{\geq 0})$ and B is reducible. The proof is finished. □

Example 4.3 Let $B = \begin{pmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$. It is a skew-symmetrizable matrix with skew-symmetrizer $S = \text{diag}\{2, 1, 1\}$. The underlying quiver $\Gamma(B)$ is as follows.



Since any arrow of $\Gamma(B)$ is in an oriented cycle, B is irreducible.

4.2 Reduction of the existence of maximal green sequences

In this subsection, we reduce the existence of maximal green sequences for skew-symmetrizable matrices to the existence of maximal green sequences for irreducible skew-symmetrizable matrices.

Lemma 4.4 *Let B be a skew-symmetrizable matrix and $\sigma_{s+1} := (k_1, \ldots, k_{s+1})$ be a sequence of column indices of B. Denote by $\tilde{B}_{\sigma_i} = \begin{pmatrix} B_{\sigma_i} \\ C_{\sigma_i} \end{pmatrix} := \mu_{k_i} \ldots \mu_{k_2} \mu_{k_1} \begin{pmatrix} B \\ I_n \end{pmatrix}$, $i = 1, \ldots, s+1$. If k_{s+1} is a green column index of C_{σ_s}, then any green column index j of C_{σ_s}, with $j \neq k_{s+1}$, must be green in $C_{\sigma_{s+1}}$.*

Proof The proof is the same as that of Lemma 2.16 of [2]. For the convenience of readers, we give the proof here.

Because j and k_{s+1} are green column indices of C_{σ_s}, we know that $(C_{\sigma_s})_{ij} \geq 0$ and $(C_{\sigma_s})_{ik_{s+1}} \geq 0$. By the definition of mutation, we have

$$\begin{aligned} (C_{\sigma_{s+1}})_{ij} &= (C_{\sigma_s})_{ij} + \text{sgn}((C_{\sigma_s})_{ik_{s+1}}) \max((C_{\sigma_s})_{ik_{s+1}} (C_{\sigma_s})_{k_{s+1}j}, 0) \\ &\geq (C_{\sigma_s})_{ij} \geq 0. \end{aligned}$$

So, j is green in $C_{\sigma_{s+1}}$. □

Theorem 4.5 (Direct sum formula) *Let $B = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix} = (b_{ij})$ be a skew-symmetrizable matrix with $B_1 \in M_n(\mathbb{Z})$ and $B_4 \in M_m(\mathbb{Z})$, and $\tilde{\mathbf{k}}$ be a sequence $\tilde{\mathbf{k}} = (k_1, \ldots, k_s, k_{s+1}, \ldots, k_{s+p})$, with $1 \leq k_i \leq n$, and $n+1 \leq k_j \leq m+n$ for $i = 1, \ldots, s$, and $j = s+1, \ldots, s+p$. If B_2 is a matrix in $M_{m \times n}(\mathbb{Z}_{\geq 0})$, then $\tilde{\mathbf{k}}$ is a maximal green sequence of B if and only if $\mathbf{k} := (k_1, \ldots, k_s)$ (respectively, $\mathbf{j} := (k_{s+1}, \ldots, k_{s+p})$) is a maximal green sequence of B_1 (respectively, B_4).*

Proof Let $\tilde{B}=\begin{pmatrix} B_1 & B_3\\ B_2 & B_4\\ I_n & 0\\ 0 & I_m\end{pmatrix}$ and $B_{\sigma_i}=\mu_{k_i}\ldots\mu_{k_2}\mu_{k_1}(\tilde{B}), i=1,\ldots,s,s+1,\ldots,s+p$. By $B_2\in M_{m\times n}(\mathbb{Z}_{\geq 0})$ and Corollary 3.3, we know that $\begin{pmatrix} B_2\\ I_n\\ 0\end{pmatrix}$ is uniformly column sign-coherent with respect to B_1. By the same argument in Proposition 3.7, we know that the submatrix $\begin{pmatrix} B_4\\ 0\\ I_m\end{pmatrix}$ of $\tilde{B}$ is invariant under the sequence of mutations $\mu_{k_s}\ldots\mu_{k_2}\mu_{k_1}$, $1\leq k_i\leq n$ for $i=1,2,\ldots,s$. So for $i\leq s$ the matrix B_{σ_i} has the form of

$$B_{\sigma_i}=\begin{pmatrix} B_{1;\sigma_i} & B_{3;\sigma_i}\\ B_{2;\sigma_i} & B_4\\ C_{1;\sigma_i} & 0\\ 0 & I_m\end{pmatrix}. \tag{2}$$

“ $\Longleftarrow$”: Because $\mathbf{k}=(k_1,k_2,\ldots,k_s)$ is a maximal green sequence of B_1, we know that $C_{1;\sigma_s}\in M_{n\times n}(\mathbb{Z}_{\leq 0})$. Thus, by the uniform column sign-coherence of $\begin{pmatrix} B_2\\ I_n\\ 0\end{pmatrix}$ with respect to B_1, we know that $\begin{pmatrix} B_{2;\sigma_s}\\ C_{1;\sigma_s}\\ 0\end{pmatrix}\in M_{(2m+n)\times n}(\mathbb{Z}_{\leq 0})$. By $B_{2;\sigma_s}\in M_{m\times n}(\mathbb{Z}_{\leq 0})$ and that the principal part of B_{σ_s} is skew-symmetrizable, we can know $B_{3;\sigma_s}\in M_{n\times m}(\mathbb{Z}_{\geq 0})$. Then by Corollary 3.3, we know that $\begin{pmatrix} B_{3;\sigma_s}\\ 0\\ I_m\end{pmatrix}\in M_{(2n+m)\times m}(\mathbb{Z}_{\geq 0})$ is uniformly column sign-coherent with respect to B_4. By the same argument in Proposition 3.7 again, we know that the submatrix $\begin{pmatrix} B_{1;\sigma_s}\\ C_{1;\sigma_s}\\ 0\end{pmatrix}$ of B_{σ_s} is invariant under the sequences of mutations $\mu_{k_{s+p}}\ldots\mu_{k_{s+2}}\mu_{k_{s+1}}(B_{\sigma_s})$, $n+1\leq k_i\leq n+m$ for $i=s+1,\ldots,s+p$. So for $i\geq s+1$, the matrix B_{σ_i} has the form of

$$B_{\sigma_i}=\begin{pmatrix} B_{1;\sigma_s} & B_{3;\sigma_i}\\ B_{2;\sigma_i} & B_{4;\sigma_i}\\ C_{1;\sigma_s} & 0\\ 0 & C_{4;\sigma_i}\end{pmatrix}.$$

Because $\mathbf{j}=(j_1,j_2,\ldots,j_p)$ is a maximal green sequence of B_4, we know that $C_{4;\sigma_{s+p}}\in M_{m\times m}(\mathbb{Z}_{\leq 0})$. Thus, the lower part of $B_{\sigma_{s+p}}$ is $\begin{pmatrix} C_{1;\sigma_s} & 0\\ 0 & C_{4;\sigma_{s+p}}\end{pmatrix}\in$

$M_{(m+n)\times(m+n)}(\mathbb{Z}_{\leq 0})$. It can be seen that $\tilde{\mathbf{k}} = (\mathbf{k}, \mathbf{j})$ is a green sequence of B, so it is maximal.

" $\Longrightarrow$" By (2), $B_{\sigma_s} = \begin{pmatrix} B_{1;\sigma_s} & B_{3;\sigma_s} \\ B_{2;\sigma_s} & B_4 \\ C_{1;\sigma_s} & 0 \\ 0 & I_m \end{pmatrix}$. Clearly, $\mathbf{k} = (k_1, \ldots, k_s)$ is a green sequence of B_1 and $\mathbf{j} = (k_{s+1}, \ldots, k_{s+p})$ is a maximal green sequence of B_4.

We claim that each $l \in \{1, 2, \ldots, n\}$ is red in $C_{1;\sigma_s}$, i.e., $C_{1;\sigma_s} \in M_{n\times n}(\mathbb{Z}_{\leq 0})$, and thus, $\mathbf{k} = (k_1, \ldots, k_s)$ is a maximal green sequence of B_1. Otherwise, there will exist a $l_0 \in \{1, 2, \ldots, n\}$ which is green in $C_{1;\sigma_s}$. Thus, l_0 is green in $\begin{pmatrix} C_{1;\sigma_s} & 0 \\ 0 & I_m \end{pmatrix}$ the lower part of B_{σ_s}. By Lemma 4.4 and $l_0 \leq n < k_{s+i}, i = 1, 2, \ldots, p$, we know that l_0 will remain green in $\begin{pmatrix} C_{1;\sigma_{s+p}} & C_{3;\sigma_{s+p}} \\ C_{2;\sigma_{s+p}} & C_{4;\sigma_{s+p}} \end{pmatrix}$ the lower part of $B_{\sigma_{s+p}}$. It is impossible since $(k_1, \ldots, k_s, k_{s+1}, \ldots, k_{s+p})$ is a maximal green sequence of B. □

When B is skew-symmetric and B_2 is a matrix over $\{0, 1\}$, the above theorem has been actually given in [8, Theorem 3.12]. The authors of [8] believe that the result also holds for $B_2 \in M_{m\times n}(\mathbb{Z}_{\geq 0})$, but they did not have a proof. In fact, we have given the proof for this in the skew-symmetrizable case.

Remark 4.6 Note that the " $\Longleftarrow$" part of the proof of the above theorem also holds if we replace maximal green sequences with green-to-red sequences, and the proof is identical. We are thankful to Fan Qin for pointing out this.

Example 4.7 Let $B = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$. Here $B_1 = 0 = B_4$, $B_2 = 3 \geq 0$. The column index set of B_1 is $\{1\}$, and the column index set of B_4 is $\{2\}$. It is known that (1) is a maximal green sequence of B_1 and (2) is a maximal green sequence of B_4. Then by Theorem 4.5, (1, 2) is a maximal green sequence of B. Indeed,

$$\begin{pmatrix} 0 & -2 \\ 3 & 0 \\ \hdashline 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\mu_1} \begin{pmatrix} 0 & 2 \\ -3 & 0 \\ \hdashline -1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\mu_2} \begin{pmatrix} 0 & -2 \\ 3 & 0 \\ \hdashline -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example 4.8 Let $B = \left(\begin{array}{ccc:cc} 0 & 1 & -1 & -2 & -2 \\ -1 & 0 & 1 & 0 & -4 \\ 1 & -1 & 0 & -3 & 0 \\ \hdashline 2 & 0 & 3 & 0 & -2 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix}$ where B_1 is of order 3×3 and B_4 is of order 2×2. Clearly, B is skew-symmetrizable with skew-symmetrizer $S = \text{diag}\{1, 1, 1, 1, 2\}$ and $B_2 \in M_{2\times 3}(\mathbb{Z}_{\geq 0})$. The column index set of B_1 is $\{1, 2, 3\}$, and the column index set of B_4 is $\{4, 5\}$. By Example 2.8 (respectively, Example 4.7), (2, 3, 1, 2) (respectively, (4, 5)) is a maximal green sequence of B_1 (respectively, B_4). Then by Theorem 4.5, (2, 3, 1, 2, 4, 5) is a maximal green sequence of B. Indeed,

$$\tilde{B} := \left(\begin{array}{ccc|cc} 0 & 1 & -1 & -2 & -2 \\ -1 & 0 & 1 & 0 & -4 \\ 1 & -1 & 0 & -3 & 0 \\ \hline 2 & 0 & 3 & 0 & -2 \\ 1 & 2 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_2} \left(\begin{array}{ccc|cc} 0 & -1 & 0 & -2 & -2 \\ 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & -3 & -4 \\ \hline 2 & 0 & 3 & 0 & -2 \\ 1 & -2 & 2 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_3} \left(\begin{array}{ccc|cc} 0 & -1 & 0 & -2 & -2 \\ 1 & 0 & 1 & -3 & 0 \\ 0 & -1 & 0 & 3 & 4 \\ \hline 2 & 3 & -3 & 0 & -2 \\ 1 & 0 & -2 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_1}$$

$$\left(\begin{array}{ccc|cc} 0 & 1 & 0 & 2 & 2 \\ -1 & 0 & 1 & -3 & 0 \\ 0 & -1 & 0 & 3 & 4 \\ \hline -2 & 3 & -3 & 0 & -2 \\ -1 & 0 & -2 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_2} \left(\begin{array}{ccc|cc} 0 & -1 & 1 & 2 & 2 \\ 1 & 0 & -1 & 3 & 0 \\ -1 & 1 & 0 & 0 & 4 \\ \hline -2 & -3 & 0 & 0 & -2 \\ -1 & 0 & -2 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_4} \left(\begin{array}{ccc|cc} 0 & -1 & 1 & -2 & 2 \\ 1 & 0 & -1 & -3 & 0 \\ -1 & 1 & 0 & 0 & 4 \\ \hline 2 & 3 & 0 & 0 & 2 \\ -1 & 0 & -2 & -1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\mu_5}$$

$$\left(\begin{array}{ccc|cc} 0 & -1 & 1 & -2 & -2 \\ 1 & 0 & -1 & -3 & 0 \\ -1 & 1 & 0 & 0 & -4 \\ \hline 2 & 3 & 0 & 0 & -2 \\ 1 & 0 & 2 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right).$$

Denote by $\tilde{B}' = \mu_2\mu_1\mu_3\mu_2(\tilde{B})$. It is can be seen that the submatrix $\tilde{B}^{4,5,6,7,8,9,10}_{4,5}$ of $\tilde{B}$ is invariant along the mutation sequence $(2, 3, 1, 2)$ and the submatrix ${\tilde{B}'}^{1,2,3,6,7,8,9,10}_{1,2,3}$ of $\tilde{B}'$ is invariant along the mutation sequence $(4, 5)$.

The following lemma is the skew-symmetrizable version of [12, Theorem 9, Theorem 17] about induced subquivers. Although corresponding result in [12] was verified for the situation of quivers, or say, in skew-symmetric case, the method of its proof can be naturally extended to the skew-symmetrizable case.

Lemma 4.9 *Let B be a skew-symmetrizable matrix. If B admits a maximal green sequence (respectively, green-to-red sequence), then any principal submatrix of B also has a maximal green sequence (respectively, green-to-red sequence).*

Theorem 4.10 *Let B be a skew-symmetrizable matrix. Then B has a maximal green sequence (respectively, green-to-red sequence) if and only if any irreducible principal submatrix of B has a maximal green sequence (respectively, green- to-red sequence).*

Proof It follows from Lemma 4.9, Theorem 4.5 and Remark 4.6. □

Remark 4.11 By the above theorem, we can give our explanation of the existence of maximal green sequences for acyclic skew-symmetrizable matrices. Because any irreducible principal submatrix of an acyclic skew-symmetrizable matrix B is only a

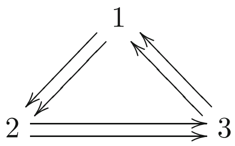


Fig. 1 Markov quiver

1×1 zero matrix, and it always has a maximal green sequence, we then know that by Theorem 4.10 any acyclic skew-symmetrizable matrix admits a maximal green sequence.

By Theorem 4.10, we reduce the existence of maximal green sequences (respectively, green-to-red sequences) for skew-symmetrizable matrices to the existence of maximal green sequences (respectively, green-to-red sequences) for irreducible skew-symmetrizable matrices B, i.e., those B whose all arrows of $\Gamma(B)$ are in oriented cycles, by Proposition 4.2. So it is natural to ask that

Problem 4.12 *Which irreducible skew-symmetrizable matrices admit maximal green sequences (respectively, green-to-red sequences)?*

Note that the existence of green-to-red sequences is mutation invariant [12] and acyclic skew-symmetrizable matrices always have a green-to-red sequences (Remark 4.11). So the irreducible skew-symmetrizable matrices which are mutation equivalent to acyclic matrices always admit a green-to-red sequences.

It is known that the existence of maximal green sequences for quivers of finite type, or quivers of finite mutation type is mutation invariant (see [2,8,13]). So the existence of maximal green sequences for irreducible subquivers of a quiver of finite type, or quivers of finite mutation type has a clear answer from these references.

In [2,12], the authors have shown that the Markov quiver (Fig. 1) has no maximal green sequence. This is an example of irreducible quiver with no maximal green sequence. More generally, the authors in [2, Proposition 8.1] proved that if a quiver Q has a non-degenerate potential such that the corresponding quiver is Jacobi-infinite, then Q has no maximal green sequences.

4.3 An application

There are two ways to understand Theorem 4.5, i.e., the direct sum formula. On the one hand, by direct sum formula, we can reduce the existence of maximal green sequences for skew-symmetrizable matrices to the existence of maximal green sequences for irreducible cases (Theorem 4.10). On the other hand, by direct sum formula, we can use the known irreducible matrices which have maximal green sequences to construct more matrices which have maximal green sequences. The existence of maximal green sequence for many of these matrices is not known previously. This is the value of our direct sum formulas. In this subsection, we will talk about the second understanding in detail.

Let $\{B_\lambda : \lambda \in \Lambda\}$ be a set of irreducible skew-symmetrizable matrices, denote by $\langle B_\lambda : \lambda \in \Lambda\rangle$ the set of skew-symmetrizable matrices whose irreducible principal submatrices are all contained in the set $\{B_\lambda : \lambda \in \Lambda\}$. The following corollary follows directly from Theorem 4.10.

Corollary 4.13 *If each B_λ in $\{B_\lambda : \lambda \in \Lambda\}$ has a maximal green sequence, so does any skew-symmetrizable matrix in $\langle B_\lambda : \lambda \in \Lambda\rangle$.*

The existence of maximal green sequences for quivers (or say skew-symmetric matrices) of finite type or from surfaces has a clear answer (see [2,13]). Now we use Corollary 4.13 to give many skew-symmetric matrices which are not of finite type or from surfaces but each of them admits a maximal green sequence.

Let $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, which is irreducible. By Example 2.8, $\mathbf{k} = (2, 3, 1, 2)$ is a maximal green sequence of B. It is known that any skew-symmetric matrix from a surface has entries $\pm 2, \pm 1, 0$ (see [5]), and any skew-symmetric matrix of finite type has entries $\pm 1, 0$ (see [6]). Thus, $\langle B\rangle$ contains many matrices which are not of finite type or from surface. For example, $\begin{pmatrix} B & mI_3 \\ -mI_3 & B \end{pmatrix} \in \langle B\rangle$ is not of finite type or from surface for $m \geq 3$. By Corollary 4.13, we can get each of such matrices has a maximal green sequence. The existence of maximal green sequences for such matrices is not clear previously.

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