Random walks on the BMW monoid: an algebraic approach

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Abstract

We consider Metropolis-based systematic scan algorithms for generating Birman–Murakami–Wenzl (BMW) monoid basis elements of the BMW algebra. As the BMW monoid consists of tangle diagrams, these scanning strategies can be rephrased as random walks on links and tangles. We also consider the Brauer algebra and use Metropolis-based scans to generate Brauer diagrams, giving rise to random walks on perfect matchings. Taking an algebraic perspective, we translate these walks into left multiplication operators in the BMW algebra and so give an algebraic interpretation of the Metropolis algorithm in this setting.

Keywords Metropolis algorithm · Systematic scans · Random walks · Representation theory · Semisimple algebras

1 Introduction

Studying the convergence of random walks on finite groups, and in particular, the problem of generating group elements according to a fixed probability distribution has a long history [6,9,11,22]. Of particular interest for the purposes of this paper is the important work of Diaconis and Ram [10], who compare systematic scanning techniques with random scanning techniques in the context of generating elements of a finite Coxeter group W using the Metropolis algorithm.

First introduced by Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller [19], the Metropolis algorithm gives a method for sampling from a probability distribution π by modifying an existing Markov chain to produce a new chain with stationary distribution π . This proves particularly useful for simulating configurations of particles with an associated energy (e.g., the influence that neighboring particles exert on each

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other). Later applications of the Metropolis algorithm include the simulation of Ising models, initially developed to model a ferromagnet, but (surprisingly) also of use in image analysis and Gibbs sampling [5,12]. See [16] for additional applications. The Metropolis algorithm has the advantage of being straightforward to construct and implement; however, in analyzing the rate of convergence to π rigorous bounds are often dependent on the specific situation. Further, these methods are most often examples of *random scan Markov chains* in that the process involved is that of selecting a site or set of sites to update at random. A more intuitively appealing and often more frequently used method in experimental work is that of a *systematic scan Markov chain*: a method to cycle through and update the sites in a deterministic order. While such scanning strategies may seem intuitive for use in sampling from π , they have proven difficult to analyze in many situations.

In [10] Diaconis and Ram use the Metropolis algorithm construction to produce Markov chains $K_1, K_2, \dots, K_{n-1}$ corresponding to multiplication by the generators $r_1, \dots, r_{n-1}$ of a Coxeter group W . For example, for each generator $r_i = (i \ i + 1)$ of the symmetric group S_n , let

$$Q_i(x, y) = \begin{cases} 1, & \text{if } y = r_i x, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for l_S the length function on words in S_n , let π be the probability distribution

$$\pi(x) = \frac{\theta^{-l_S(x)}}{\sum_{w \in S_n} \theta^{-l_S(w)}}.$$

The Metropolis algorithm construction, given explicitly in Sect. 2.2, modifies the chains Q_i to produce Markov chains $K_1, K_2, \dots, K_{n-1}$ that correspond to multiplication by the generators $r_1, \dots, r_{n-1}$. For K_i reversible, each with stationary distribution π , the following systematic scans produce reversible Markov chains with stationary distribution π :

$$\begin{aligned} & \frac{1}{n-1} \sum_{i=1}^{n-1} K_i && \text{(random scan),} \\ & K_1 K_2 \cdots K_{n-1} K_{n-1} \cdots K_2 K_1 && \text{(short systematic scan),} \\ & (K_1 \cdots K_{n-1} K_{n-1} \cdots K_1) \cdots (K_1 K_2 K_2 K_1) (K_1 K_1) && \text{(long systematic scan).} \end{aligned}$$

In the context of generation of dihedral group and hypercube elements, Diaconis and Ram [10] show that convergence of the short systematic scan for the distribution π above, with l_S replaced by the length function on the group, occurs in the same number of steps as that of a random scan, i.e., choosing a random sequence of indices $\{i_\ell\}_{\ell=1}^\infty$. However, results for different scanning techniques or probability distributions remain open.

The key insight that allows for analysis of the Metropolis scans is the translation of the Markov chains M_i into left multiplication operators in the corresponding

Iwahori–Hecke Algebra. As generators of the Hecke algebra can be represented by braid diagrams, a natural extension is to consider this problem in the setting of the BMW algebra, whose generators can be represented by *tangle diagrams* (see Sect. 4.3). In this paper, we show that the methodology of [10] generalizes to the setting of random walks on the Brauer and BMW monoids, showing that the Metropolis walks on tangles and matchings are algebrized by the BMW algebra.

The Metropolis algorithm in this context gives rise to systematic scanning strategies for generating basis elements via multiplication by generators. The corresponding random walk on the BMW and Brauer monoids is defined in Sect. 3.2. We show that the random walk is given by left multiplication in the BMW algebra, allowing for the translation to the algebraic setting that is key to the methodology of [10]. For B_n the set of Brauer diagrams on n dots, $\{T_{r_i}, T_{e_i} \mid 1 \leq i \leq n-1\}$ generators of the BMW algebra, $\mathcal{T}_n$ a basis for the BMW algebra (determined in Sect. 4.3), and $\tilde{K}_i$ the matrix of the random walk we show

Theorem 1 *Let $\mathcal{BMW}_n(m, l)$ be the BMW algebra with parameters $m = \theta^{-1} - 1$ and $l = 1$. Then,*

$$\sum_{y \in B_n} \tilde{K}_i(T_d, T_y) T_y = \begin{cases} T_{r_i} T_d, & \text{if } T_{r_i} T_d \in \mathcal{T}_n, \\ (\theta T_{r_i} + (1 - \theta) T_{e_i}) T_d, & \text{if } T_{r_i} T_d \notin \mathcal{T}_n. \end{cases}$$

After the algebraic translation of the problem, the main tools used in the analysis of [10] are a translation of the total variation norm into a trace norm on the Iwahori–Hecke algebra H . Rewriting this trace norm using characters of H allows for bounds to come from the dimensions and traces of representations of H , rather than computation of the eigenvalues. Theorem 1 provides the necessary algebrization of the problem; in future work, we plan to similarly use a trace norm on the BMW algebra to bound the total variation norm by dimensions and traces of representations of the BMW algebra.

We begin in Sects. 2 and 3 with the preliminaries needed from probability theory and representation theory. We also give a presentation of the Brauer algebra. In Sect. 4, we describe the random walk arising from the Metropolis algorithm, give a presentation of the BMW algebra, and prove Theorem 1. We then conclude in Sect. 5 with further directions and questions.

2 Preliminaries: Markov chains and the Metropolis algorithm

We introduce the necessary probabilistic background, following the notation and outline of [10]. Further background can be found in [1, 4, 15].

2.1 Markov chains

A Markov chain on a finite set X is a matrix $K = (K(x, y))_{x, y \in X}$ such that $K(x, y) \in [0, 1]$ and

$$\text{if } x \in X \text{ then } \sum_{y \in X} K(x, y) = 1.$$

As a process on the finite set X , the value $K(x, y)$ gives the probability of moving from x to y in one step and $K^m(x, y)$ gives the probability of moving from x to y in m steps.

A stationary distribution for K is a function $\pi : X \rightarrow [0, 1]$, $\sum_{x \in X} \pi(x) = 1$ such that

$$\text{if } x \in X \text{ then } \pi(x) = \sum_{y \in X} \pi(y) K(y, x). \quad (1)$$

A symmetric Markov chain is a Markov chain K such that

$$\text{if } x, y \in X \text{ then } K(x, y) = K(y, x),$$

and a reversible Markov chain is a Markov chain K such that

$$\text{if } x, y \in X \text{ then } \pi(x) K(x, y) = \pi(y) K(y, x).$$

An irreducible Markov chain is a Markov chain K such that if $x, y \in X$, then there exists an integer m such that $K^m(x, y) > 0$. The Markov chain K is aperiodic if

$$\gcd\{m(x) \mid x \in X\} = 1, \text{ where } m(x) = \min\{m \in \mathbb{Z}_{>0} \mid K^m(x, x) > 0\}$$

for $x \in X$.

By the Perron–Frobenius theorem, irreducible aperiodic chains have a unique stationary distribution.

Let K_x^m denotes the probability distribution $K^m(x, \cdot)$. The total variation distance from K_x^m to π is

$$|K_x^m - \pi|_{TV} := \max_{A \subseteq X} \left| \sum_{y \in A} K^m(x, y) - \pi(y) \right|.$$

For $L^2(\pi)$ the space of functions $f : X \rightarrow \mathbb{R}$, equipped with the inner product

$$\langle f, g \rangle_2 = \sum f(x)g(x)\pi(x), \quad (2)$$

the total variation distance is bounded by the $L^2(\pi)$ norm:

Lemma 1 (see, e.g., [10](Lemma 2.3)) *For $f \in L^2(\pi)$,*

$$|f|_{TV}^2 \leq \frac{1}{4} \|f/\pi\|_2^2.$$

2.2 The Metropolis algorithm

Let $P : X \times X \rightarrow [0, 1]$ be a symmetric Markov chain, and let $\pi : X \rightarrow [0, 1]$ be a probability distribution. Define a new Markov chain $K : X \times X \rightarrow [0, 1]$ by

$$K(x, y) = \begin{cases} P(x, y), & \text{if } x \neq y \text{ and } \pi(y) \geq \pi(x), \\ P(x, y) \frac{\pi(y)}{\pi(x)}, & \text{if } x \neq y \text{ and } \pi(y) < \pi(x), \\ P(x, x) + \sum_{\pi(z) < \pi(x)} P(x, z) \left(1 - \frac{\pi(z)}{\pi(x)}\right), & \text{if } x = y. \end{cases} \quad (3)$$

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller [19] shows that K is a reversible Markov chain with stationary distribution π . Hammersley and Handscomb give a nice description of the Metropolis algorithm in [14].

While $K(x, y)$ has stationary distribution π , neither irreducibility nor aperiodicity is guaranteed. In particular, the Markov chains we consider in Sect. 3.2 are aperiodic, but not irreducible. To understand these chains, we consider their *closed communication classes*.

Definition 1 Let K be a Markov chain with state space X . For $x, y \in X$, y is accessible from x , denoted $x \rightarrow y$, if $K(x, y)^n > 0$ for some $n \in \mathbb{Z}_{>0}$. x communicates with y , denoted $x \leftrightarrow y$, if $x \rightarrow y$ and $y \rightarrow x$. The equivalence classes under the relation $\leftrightarrow$ are the communication classes of K . A communication class C is closed if whenever $x \in C$ and $x \rightarrow y$ for some $y \in X$, then $y \in C$.

3 Preliminaries: the Brauer algebra

3.1 Fourier inversion and Plancherel

Let A be a semisimple algebra, $\{a_i\}_{i \in I}$ a basis for A and $f = \sum_{i \in I} f(a_i) a_i \in A$.

Definition 2 Let ρ be a matrix representation of A . The Fourier transform of f at ρ , denoted $\hat{f}(\rho)$, is the matrix sum

$$\hat{f}(\rho) = \sum_{i \in I} f(a_i) \rho(a_i).$$

For G a finite group and $\mathbb{C}[G]$ the associated group algebra with basis the elements of G , we recover the Fourier transform of G as studied in [7, 8, 17]. For more background on Fourier transforms in the semisimple algebra setting see [10, 18].

Random walks on groups are frequently studied using Fourier analysis. For example, for a finite group G and $Q : G \rightarrow \mathbb{C}$, $\sum_{g \in G} Q(g) = 1$ a probability distribution on G , let $\hat{Q}$ denotes the Fourier transform of Q . Let U denotes the uniform distribution on G , i.e., $U(g) = \frac{1}{|G|}$ for each $g \in G$. The following theorem, specialized to the

Fig. 1 A Brauer diagram d

finite group setting, can be found in [9]. The theorem in the semisimple algebra setting appears in [10].

Theorem 2 (Diaconis, [9]) *For G a finite group and Q a probability distribution on G ,*

$$|Q - U|_{TV}^2 \leq \frac{1}{4} \sum_{\rho} d_{\rho} \operatorname{Tr}(\hat{Q}(\rho) \hat{Q}(\rho)^*),$$

where $*$ denotes conjugate transpose, and the sum is over all nontrivial irreducible representations ρ of G .

A trace on a semisimple algebra A is a $\mathbb{C}$ -linear function $\tau : A \rightarrow \mathbb{C}$ such that if $a, b \in A$, then $\tau(ab) = \tau(ba)$. A trace function τ on A gives rise to a symmetric bilinear form $\langle \cdot, \cdot \rangle_{\tau} : A \times A \rightarrow \mathbb{C}$ by letting

$$\langle a, b \rangle_{\tau} = \tau(ab), \quad (4)$$

for $a, b \in A$.

Further recall that for any trace function τ and set R of inequivalent irreducible representations of A , there exist constants $t_{\rho} \in \mathbb{C}$ such that

$$\tau = \sum_{\rho \in R} t_{\rho} \chi_{\rho},$$

where for $a \in A$, $\chi_{\rho}(a) = \operatorname{Tr}(\rho(a))$.

3.2 The Brauer algebra

Let n be a positive integer and q an indeterminate. A Brauer diagram on n dots is a graph on $2n$ vertices $\{1, \dots, n, 1', \dots, n'\}$ with edges partitioning the vertices into pairs (Fig. 1).

Edges that pair two dots on the same row are arcs. The set B_n of Brauer diagrams on n dots has cardinality $(2n - 1)!! = (2n - 1)(2n - 3) \cdots 5 \cdot 3 \cdot 1$.

Let x and y be two Brauer diagrams on n dots. Place x above y and identify the vertices in the bottom row of x with those in the top row of y . Let c be the number of closed loops obtained in this concatenation. Then, the product xy is given by q^c times the resulting diagram without loops (Fig. 2).

The Brauer algebra $\mathcal{B}r_n(q)$ is the $\mathbb{C}[q]$ -span of the Brauer diagrams under the $\mathbb{C}$ -linear extension of diagram multiplication. The Brauer algebra is generated by

Fig. 2 $xy = q^1 z$ Fig. 3 $r_i, e_i, id \in Br_n$

$\{r_i, e_i \mid 1 \leq i \leq n-1\}$, for r_i and e_i as in Fig. 3. The identity element of $\mathcal{B}r_n(q)$ is given in Fig. 3.

Benkart, Shader, and Ram note in [2] that the following presentation of the Brauer algebra is “oft-quoted,” (see, for example, [21]), but we are not aware of a proof in the literature. In the interests of space, we shall continue the tradition and not include a proof here either.” We follow their example and note the following presentation of the Brauer algebra.

Theorem 3 *The Brauer algebra $\mathcal{B}r_n$ has presentation given by generating set $S = \{r_i, e_i \mid 1 \leq i \leq n-1\}$ and relations*

$$\begin{aligned}
 (B1) \quad r_i^2 &= 1, & (B2) \quad r_i r_j &= r_j r_i, \quad r_i e_j = e_j r_i, \quad e_i e_j = e_j e_i, \\
 & & & |i - j| > 1 \\
 (B3) \quad e_i^2 &= q e_i, & (B4) \quad e_i r_i &= r_i e_i = e_i, \\
 (B5) \quad r_i r_{i+1} r_i &= r_{i+1} r_i r_{i+1}, & (B6) \quad e_i e_{i+1} e_i &= e_i, \quad e_{i+1} e_i e_{i+1} = e_{i+1}, \\
 (B7) \quad r_i e_{i+1} e_i &= r_{i+1} e_i, & (B8) \quad e_{i+1} e_i r_{i+1} &= e_{i+1} r_i.
 \end{aligned}$$

4 Length in B_n and the walk

4.1 Reduced expressions and length

Definition 3 A word $W = s_{i_1} \dots s_{i_n}$ with $s_{i_j} \in S$ has an occurrence of $e_{i+1} r_i$ if a subword of W has form $w = e_{i+1} s_{j_1} \dots s_{j_k} r_i$, where $j_1, \dots, j_k \neq i, i+1$.

By Brauer relation (B2), the subword w can be rewritten $w = s_{i_1} \dots s_{i_l} e_{i+1} r_i s_{i_{l+1}} \dots s_{i_k}$ for some $1 \leq l \leq k$. We remove the occurrence of $e_{i+1} r_i$ using Brauer relation (B8); replace w with $w' = s_{i_1} \dots s_{i_l} e_{i+1} r_{i+1} s_{i_{l+1}} \dots s_{i_k}$.

Example 1 Let $W = e_6 e_3 r_4 r_5 e_1 r_2$, which has an occurrence of $e_3 r_2$: $w = e_3 r_4 r_5 e_1 r_2$. By Brauer relation (B2), $w = e_3 e_1 r_4 r_5 r_2 = e_1 e_3 r_2 r_4 r_5$. Replace w with $w' = e_1 e_3 e_2 r_3 r_4 r_5$. Note that $W = e_6 e_1 e_3 e_2 r_3 r_4 r_5$ has no occurrence of $e_3 r_2$.

Fig. 4 $r_2e_1r_2$

The e_{i+1} term in w' cannot form a new occurrence of $e_{i+1}r_i$ in W as it has e_i directly to its right. The e_i term, however, could form a new occurrence of $e_i r_{i-1}$ in W .

Example 2 Let $W = e_3r_4e_5r_2r_1$, which has an occurrence of e_3r_2 through subword $w = e_3r_4e_5r_2 = e_3r_2r_4e_5$. We remove this occurrence by replacing w with $w' = e_3e_2r_3r_4e_5$. Then $W = e_3e_2r_3r_4e_5r_1$, which has an occurrence of e_2r_1 .

As replacing an occurrence of $e_{i+1}r_i$ can at worst create a new occurrence of $e_i r_{i-1}$, the process of removing all possible occurrences of $e_{i+1}r_i$, $1 \leq i \leq n-2$ from W terminates after finitely many removals.

Example 3 Let $W = e_3r_4e_5r_2r_1$ as in Example 2. After removing the occurrence of e_3r_2 , $W = e_3e_2r_3r_4e_5r_1$, with subword $w = e_2r_3r_4e_5r_1 = e_2r_1r_3r_4e_5$. Replacing w with $w' = e_2e_1r_2r_3r_4e_5$ yields $W = e_3e_2e_1r_2r_3r_4e_5$ which has no occurrence of $e_{i+1}r_i$ for any i .

Definition 4 Let $d \in B_n$. A reduced expression for d is a word $d = s_{i_1} \cdots s_{i_n}$ with no occurrences of $e_{i+1}r_i$ for all $1 \leq i \leq n-2$.

Example 4 Let $d = r_2e_1r_2 \in B_3$ be the diagram of Fig. 4. This expression for d has no occurrence of e_2r_1 , so it is a reduced expression for d . Note that another expression for d is given by $d = r_1e_2r_1$; however, this is not a reduced expression for d as it has an occurrence of e_2r_1 . Indeed, removing the occurrence of e_2r_1 yields $d = r_1e_2e_1r_2 = r_2e_1r_2$, with the last equality due to Brauer relation (B7).

Define the length of a word $s_{i_1} \cdots s_{i_n}$ to be

$$l(s_{i_1} \cdots s_{i_n}) = \#\{r_i \text{ terms}\} + 2\#\{e_i \text{ terms}\}.$$

Define the length function on the set of Brauer diagrams $L : B_n \rightarrow \mathbb{Z}_{\geq 0}$ by

$$L(d) = \min\{l(W) \mid W \text{ a reduced expression for } d\}$$

Example 5 Let $d = r_2e_1r_2$ as in Fig. 4. Then $L(d) = 4$. From this example, it may seem that the length function gives the number of crossings of d plus the number of arcs of d ; however, this is not the case. For example, the element $d' = e_1e_2$ has two arcs and zero crossings, but $l(d') = 4$ (Fig. 5).

Fig. 5 $d' = e_1 e_2$ Fig. 6 $r_2 d$

4.2 The walk

The symmetric group S_n acts on the set B_n of Brauer diagrams by permuting the vertices $1, \dots, n$. Let r_i be the transposition in S_n that switches vertex i with vertex $i + 1$. Then r_i is as in Fig. 3, with the action $r_i d$ given by the multiplication $r_i d$ in the Brauer algebra. (see Fig. 6).

Let $\theta \in [0, 1]$ and define a distribution $\pi : B_n \rightarrow [0, 1]$ by

$$\pi(d) = \frac{1}{Z} \theta^{-L(d)}, \quad \text{where } Z = \sum_{z \in B_n} \theta^{-L(z)}.$$

Define $P_i : B_n \times B_n \rightarrow [0, 1]$ by

$$P_i(d, y) = \begin{cases} 1, & \text{if } y = r_i d, \\ 0, & \text{if } y \neq r_i d. \end{cases}$$

As $r_i^{-1} = r_i$, $P(d, y) = P(y, d)$ for all $d, y \in B_n$. The chain P_i corresponds to the simple random walk on the graph with vertices labeled by diagrams $d \in B_n$ and edge set $\{\{d, y\} \mid y = r_i d\}$.

Applying the Metropolis algorithm (3) to P_i and π yields $K_i : B_n \times B_n \rightarrow [0, 1]$ where

$$K_i(d, y) = \begin{cases} 1, & \text{if } y = r_i d \text{ and } L(d) \leq L(y), \\ \theta, & \text{if } y = r_i d \text{ and } L(d) > L(y), \\ 1 - \theta, & \text{if } y = d \text{ and } L(d) > L(r_i d), \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

In Appendix A, we give an example of the chain $K_1 : B_3 \times B_3 \rightarrow [0, 1]$. Interpreted as a random walk on B_n , the chain K_i describes the process:

From $d \in B_n$ act on the left by r_i . If the length increases, (†)
move to $r_i d$. If the length decreases, flip a θ -coin and if
heads move to $r_i d$. If tails, remain at d .

Fig. 7 $T_{r_i}, T_{e_i}, T_{r_i}^{-1}$

Fig. 8 Untangling relations

Note 1 Note that $S_n \subset B_n$ as S_n is the set of Brauer diagrams without arcs. For $d \in S_n$, $L(d) = \ell(d)$ for $\ell : S_n \rightarrow \mathbb{Z}$ the length function on S_n that gives the minimum number of transpositions r_i needed to express d . The submatrix of K_i corresponding to $\{d \mid d \in S_n\}$ is exactly the chain M_i of [10].

4.3 The BMW algebra

For m, ℓ, q parameters satisfying $q = (\ell - \ell^{-1})(m - m^{-1})^{-1} + 1$, the BMW algebra $\mathcal{BMW}_n(m, \ell)$ is given by generating set $\{T_{e_i}, T_{r_i}, T_{r_i}^{-1} \mid 1 \leq i \leq n-1\}$ and relations

$$\begin{aligned}
 (A1) \quad T_{r_i} &= T_{r_i}^{-1} + mT_{id} - mT_{e_i}, & (A2) \quad T_{r_i}^{\pm 1} T_{r_j}^{\pm 1} &= T_{r_j}^{\pm 1} T_{r_i}^{\pm 1}, \quad T_{r_i}^{\pm 1} T_{e_j} = T_{e_j} T_{r_i}^{\pm 1}, \\
 & & & T_{e_i} T_{e_j} = T_{e_j} T_{e_i}, \quad |i - j| > 1, \\
 (A3) \quad T_{e_i}^2 &= qT_{e_i}, & (A4) \quad T_{e_i} T_{r_i} &= T_{r_i} T_{e_i} = \ell^{-1} T_{e_i}, \\
 (A5) \quad T_{r_i} T_{r_{i+1}} T_{r_i} &= T_{r_{i+1}} T_{r_i} T_{r_{i+1}}, & (A6) \quad T_{e_i} T_{e_{i \pm 1}} T_{e_i} &= T_{e_i}, \\
 (A7) \quad T_{r_i} T_{r_{i \pm 1}} T_{e_i} &= T_{e_{i \pm 1}} T_{e_i} = T_{e_{i \pm 1}} T_{r_i} T_{r_{i \pm 1}}, & (A8) \quad T_{e_i} T_{r_{i \pm 1}} T_{e_i} &= \ell T_{e_i},
 \end{aligned}$$

(see, for example [13]). The standard pictorial representation of the generators is given in Fig. 7 and, pictorially, the relations are (Fig. 8).

Fig. 9 Lifts of $r_2e_1r_2$

Let $d \in B_n$. An element of $\mathcal{BMW}_n(m, l)$ is a lift of d if it is achieved by drawing d so that for each crossing of d , one edge is drawn to pass above the other. See Fig. 9 for two of the four possible lifts of $d = r_2e_1r_2$.

Given $d \in B_n$, choose a sequence of nodes of d consisting of one end point of each edge. An element of $\mathcal{BMW}_n(m, l)$ is totally descending (with this choice of nodes) if it is the result of drawing each edge of d in order, so that when meeting an edge that has already been drawn, the new edge crosses under the existing one.

Theorem 4 (Morton, Wasserman [20]) $\mathcal{BMW}_n(m, l)$ is spanned by its totally descending elements.

Note that this holds for any fixed choice of sequence of nodes of d , and thus, $\mathcal{BMW}_n(m, l)$ is spanned by any fixed set of lifts of B_n to $\mathcal{BMW}_n(m, l)$.

For a Brauer diagram $d \in B_n$, let $d = s_{i_1} \cdots s_{i_n}$ be a reduced expression for d of minimal length and let

$$T_d = T_{s_{i_1}} \cdots T_{s_{i_n}}.$$

This is well defined as the only length-preserving relations in the Brauer algebra are (B2) and (B5) and these relations hold in the BMW algebra as (A2) and (A5). Let $\mathcal{T}_n = \{T_d \mid d \in B_n\}$. By Theorem 4, $\mathcal{T}_n$ spans $\mathcal{BMW}_n(m, l)$.

Lemma 2 Let $BMW_n(m, l)$ be the BMW algebra with parameter $l = 1$. Let $T_d \in \mathcal{T}_n$. Then

$$L(r_i d) < L(d) \iff T_{r_i} T_d \notin \mathcal{T}_n$$

and in this case $T_{r_i}^{-1} T_d = T_{r_i d}$.

Proof Let $d = s_{i_1} \cdots s_{i_k} \in B_n$.

Case 1: $L(r_i d) > L(d)$

As multiplication by r_i cannot introduce an occurrence of $e_j r_{j-1}$, if $L(r_i d) > L(d)$, then $L(r_i d) = L(d) + 1$ and $r_i s_{i_1} \cdots s_{i_k}$ is a reduced expression for $r_i d$. Thus, $T_{r_i} T_d = T_{r_i} T_{s_{i_1}} \cdots T_{s_{i_k}} = T_{r_i d}$.

Case 2: $L(r_i d) = L(d)$

As Brauer relation (B4) is the only length-preserving multiplication by r_i , if $L(r_i d) = L(d)$, then $r_i d = s_{i_1} \cdots r_i e_i s_{i_{j+1}} \cdots s_{i_k}$ for some $1 \leq j \leq k$, with $|i_l - i| > 1$ for $l < j$ (using Brauer relation (B2)). By Brauer relation (B4), $r_i d = d$ and so a reduced expression for $r_i d$ is $s_{i_1} \cdots s_{i_k}$. By BMW relations (A2) and (A4),

$$T_{r_i} T_d = T_{r_i} T_{s_{i_1}} \cdots T_{s_{i_k}} = T_{s_{i_1}} \cdots T_{r_i} T_{e_i} T_{s_{i_{j+1}}} \cdots T_{s_{i_k}} = l^{-1} T_d.$$

As $l^{-1} = 1$ and $r_i d = d$, $T_{r_i} T_d = T_{r_i d}$.

Case 3: $L(r_id) < L(d)$

As Brauer relation (B1) and Brauer relation (B7) give the only possibilities for multiplication by r_i to reduce length, if $L(r_id) < L(d)$ then $r_id = s_{i_1} \cdots r_i r_i s_{i_{j+1}} \cdots s_{i_k}$ for some $1 \leq j \leq k$ with $|i_l - i| > 1$ for $l < j$ or $r_id = s_{i_1} \cdots r_i e_{i+1} e_i s_{i_{j+2}} \cdots s_{i_k}$ for some $1 \leq j \leq k-1$ with $|i_l - i| > 1$ for $l < j$.

Case 3a: $r_id = s_{i_1} \cdots r_i r_i s_{i_{j+1}} \cdots s_{i_k}$

Then by Brauer relation (B1), $r_id = s_{i_1} \cdots s_{i_{j-1}} s_{i_{j+1}} \cdots s_{i_k}$. In particular, $r_i s_{i_1} \cdots s_{i_k}$ is not a minimal length expression for r_id . As Brauer relation (B1) does not hold in the BMW algebra,

$$T_{r_i} T_d = T_{s_{i_1}} \cdots T_{r_i} T_{r_i} T_{s_{i_{j+1}}} \cdots T_{s_{i_k}} \neq T_{s_{i_1}} \cdots T_{s_{i_{j-1}}} T_{s_{i_{j+1}}} \cdots T_{s_{i_k}},$$

and so $T_{r_i} T_d \neq T_{r_i} d$. Note, however, that

$$T_{r_i}^{-1} T_d = T_{s_{i_1}} \cdots T_{r_i}^{-1} T_{r_i} T_{s_{i_{j+1}}} \cdots T_{s_{i_k}} = T_{r_i} d$$

Case 3b: $r_id = s_{i_1} \cdots r_i e_{i+1} e_i s_{i_{j+2}} \cdots s_{i_k}$

As in the previous case, $r_i s_{i_1} \cdots s_{i_k}$ is not a minimal length expression for r_id as $r_id = s_{i_1} \cdots r_i e_{i+1} e_i s_{i_{j+2}} \cdots s_{i_k} = s_{i_1} \cdots r_{i+1} e_i s_{i_{j+2}} \cdots s_{i_k}$ (using Brauer relation (B7)). Brauer relation (B7) does not hold in the BMW algebra, so $T_{r_i} T_d \neq T_{r_i} d$. Note, however, that by BMW relation (A7) $T_{r_i}^{-1} T_{e_{i+1}} T_{e_i} = T_{r_i}^{-1} T_{r_i} T_{r_{i+1}} T_{e_i} = T_{r_{i+1}} T_{e_i}$, and so $T_{r_i}^{-1} T_d = T_{r_i} d$.

□

For K_i the Markov chain defined by Equation (5), define $\tilde{K}_i : \mathcal{T}_n \times \mathcal{T}_n \rightarrow [0, 1]$ by

$$\tilde{K}_i(T_d, T_y) = K_i(d, y).$$

We now prove Theorem 1, giving an interpretation of the Markov chains K_i in terms of multiplication in the BMW algebra.

Theorem 5 1.1 Fix $\theta \in [0, 1]$ and $i \in \{1, \dots, n-1\}$. Let $\mathcal{BMW}_n(m, l)$ be the BMW algebra with parameters

$$m = \theta^{-1} - 1 \text{ and } l = 1.$$

Then

$$\sum_{y \in B_n} \tilde{K}_i(T_d, T_y) T_y = \begin{cases} T_{r_i} T_d, & \text{if } T_{r_i} T_d \in \mathcal{T}_n, \\ (\theta T_{r_i} + (1 - \theta) T_{e_i}) T_d, & \text{if } T_{r_i} T_d \notin \mathcal{T}_n. \end{cases}$$

Proof Suppose $T_{r_i} T_d \in \mathcal{T}_n$. Then by Lemma 2, $L(r_id) \geq L(d)$, meaning that $\tilde{K}_i(T_d, T_y) = 0$ for $y \neq r_id$ and $\tilde{K}_i(T_d, T_{r_id}) = 1$. Since $T_{r_i} T_d \in \mathcal{T}_n$, $T_{r_i} T_d = T_{d'} = T_{s_{i_1}} \cdots T_{s_{i_n}}$ for some reduced expression $s_{i_1} \cdots s_{i_n} = r_id$ of $d' = r_id \in B_n$ and so $T_{r_i} T_d = T_{r_i} d$

Now suppose $T_{r_i}T_d \notin \mathcal{T}_n$. Then by Lemma 2, $L(r_id) < L(d)$, meaning that

$$\sum_{y \in B_n} \tilde{K}_i(T_d, T_y)T_y = \theta T_{r_id} + (1 - \theta)T_d.$$

Also by Lemma 2, $T_{r_i}^{-1}T_d = T_{r_id}$ and so by BMW relation (A1),

$$\theta T_{r_i}T_d = \theta(T_{r_i}^{-1}T_d + mT_d - mT_{e_i}T_d) = \theta T_{r_id} + (1 - \theta)T_d - (1 - \theta)T_{e_i}T_d.$$

Adding $(1 - \theta)T_{e_i}T_d$ gives the desired result. $\square$

5 Further directions

In this paper, we extend the methodology of [10] to consider random walks on the BMW monoid arising from the Metropolis algorithm. We provide an algebraic translation of the problem, showing that Metropolis walks on the Brauer and BMW monoids are algebraized by the BMW algebra. An important next step would be to perform the mixing time analysis and use it to compare different scanning techniques.

We provide algebraic methods and tools for studying Metropolis-based random walks on the BMW monoid. As the BMW algebra is but one of a wealth of examples of diagram algebras, it would be interesting to explore Metropolis-based walks on other diagrams, as well as different random walks encoding statistics on these diagrams. For example, random walks on subalgebras of $\mathbb{C}[S_n]$ have been used to study permutation statistics [3]. Such walks could be extended to study statistics on perfect matchings and partitions. As the Brauer algebra (respectively, the partition algebra) is generated by diagrams encoding perfect matchings (respectively, set partitions), algebraic translation of such walks would allow for the use of the representation theory of the Brauer and partition algebras to study these statistics.

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A Example of the walk in $\mathcal{BMW}_3$

Example 6 In $\mathcal{BMW}_3$,

$$\mathcal{B}_3 = \mathbf{R} \cup \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}_3,$$

for

$$\begin{aligned} \mathbf{R} &= \{T_{id}, T_{r_1}, T_{r_2}, T_{r_1}T_{r_2}, T_{r_2}T_{r_1}, T_{r_1}T_{r_2}T_{r_1}\}, & \mathbf{E}_1 &= \{T_{e_1}, T_{r_2}T_{e_1}, T_{e_2}T_{e_1}\}, \\ \mathbf{E}_2 &= \{T_{e_2}, T_{r_1}T_{e_2}, T_{e_1}T_{e_2}\}, & \mathbf{E}_3 &= \{T_{e_1}T_{r_2}, T_{r_2}T_{e_1}T_{r_2}, T_{e_2}T_{e_1}T_{r_2}\}. \end{aligned}$$

The Markov chain K_1 has form

$$R \bigoplus E_1 \bigoplus E_2 \bigoplus E_3,$$

for

$$R = \begin{matrix} & \begin{matrix} id & r_1 & r_2 & r_1 r_2 & r_2 r_1 & r_1 r_2 r_1 \end{matrix} \\ \begin{matrix} id \\ r_1 \\ r_2 \\ r_1 r_2 \\ r_2 r_1 \\ r_1 r_2 r_1 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \theta & 1 - \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \theta & 1 - \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \theta & 1 - \theta \end{pmatrix} \end{matrix},$$

$$E_1 = \begin{matrix} & \begin{matrix} e_1 & r_2 e_1 & e_2 e_1 \end{matrix} \\ \begin{matrix} e_1 \\ r_2 e_1 \\ e_2 e_1 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \theta & 1 - \theta \end{pmatrix} \end{matrix}, \quad E_2 = \begin{matrix} & \begin{matrix} e_2 & r_1 e_2 & e_1 e_2 \end{matrix} \\ \begin{matrix} e_2 \\ r_1 e_2 \\ e_1 e_2 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ \theta & 1 - \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix},$$

$$E_3 = \begin{matrix} & \begin{matrix} e_1 r_2 & r_2 e_1 r_2 & e_2 e_1 r_2 \end{matrix} \\ \begin{matrix} e_1 r_2 \\ r_2 e_1 r_2 \\ e_2 e_1 r_2 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \theta & 1 - \theta \end{pmatrix} \end{matrix}.$$

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