Primitive permutation groups satisfying the small orbit property and a problem of Bourgain and Kalai

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Abstract

We classify the primitive but not affine permutation subgroups of S_n with the following property: There exists some unbounded subset X of size at most $\text{clog} n$ whose orbit is polynomially bounded (in terms of n). This class of subgroups arises naturally in the study of threshold behavior of G -symmetric properties of Boolean functions. Our result answers a problem of Bourgain and Kalai. Contrary to a suggestion in their paper, we give an example for a primitive permutation group with this property, which contains A_m as a section for $m \rightarrow \infty$.

Keywords Permutation groups · Small orbit property · Primitive groups

1 Introduction

Let $I = \{n_t | t \in \mathbb{N}\}$ be a sequence of the natural numbers $\mathbb{N}$. For a sequence $G_t \leq S_{n_t}$ ($t \in \mathbb{N}$) of primitive permutation groups, we say that $\{G_t\}$ has the *small orbit property* if there exist some constants b and c and some subsets $X_t \subseteq \{1, \dots, n_t\}$ such that $\{|X_t|\}$ is not bounded, for all t $|X_t| \leq \text{clog} n_t$, and $|G_t(X_t)| \leq n_t^b$.

Consider the following example:

Let G be A_n or S_n acting on $\Omega = \{1, \dots, n\}$ ($n \geq 3$). Then, G acts transitively on the set of all subsets of Ω of size k for any k , and therefore, the size of the orbit of any subset X of size k is $\binom{n}{k}$. Clearly, the *small orbit property* is not satisfied.

A more interesting example is the case where S_n acts on $\Omega = \binom{\{1, 2, \dots, n\}}{2}$ (namely all 2-subsets of $\{1, \dots, n\}$), and in this case, the orbits correspond to isomorphism classes of graphs. Note that also here the *small orbit property* is not satisfied.

Let G_t ($t \in \mathbb{N}$) be a sequence of finite groups. Let $I = \{n_t | t \in \mathbb{N}\}$. Bourgain and Kalai posed the question of understanding primitive permutation groups with the

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small orbit property (see [1]). They proposed that the following are equivalent for an infinite set I of natural numbers and a sequence of groups G_t ($t \in I$):

1. For any embedding of G_t in S_{n_t} such that the action on $\{1, \dots, n_t\}$ is primitive, G_t satisfies the *small orbit property*.
2. There exists some M , such that G_t does not have A_m as a composition factor for m greater than M .

The motivation for studying permutation groups with the *small orbit property* comes from studies of threshold phenomena under symmetry and, in particular, a paper by Bourgain and Kalai [1].

Consider a monotone Boolean function $f(x_1, x_2, \dots, x_n)$. Assume that f is invariant under the action of some primitive group G . Let $\mu_p(f)$ be the probability that $f = 1$ when every variable x_i equals 1 with probability p and equals 0 with probability $1 - p$ (independently). Unless f is a constant function, $\mu_p(f)$ is a strictly monotone function of p . Define the critical probability $p_c = p_c(f)$ by $\mu_{p_c}(f) = 1/2$. Let $\epsilon > 0$ be a fixed tiny real number (say $\epsilon = 1/1000$) and suppose that $\mu_{p_1}(f)\epsilon$ and $\mu_{p_2}(f) = 1 - \epsilon$. The interval $[p_1, p_2]$ is called the threshold interval for the Boolean function f . Some results of Bourgain, Friedgut and Kalai show that if G satisfies the condition that the orbits of “large” subsets are “large,” then the threshold is sharp (see [2]), i.e., $p_2 - p_1 = o(1)$ (as a function of n). Friedgut and Kalai [2] described an upper bound for the threshold interval for Boolean functions which is invariant under a transitive group G . Friedgut and Kalai gave an upper bound for the length of the threshold interval when the function is invariant under a transitive permutation group and showed that when $\log p_c / \log n \rightarrow 0$, then the length of the threshold interval is actually $o(p_c)$.

Bourgain and Kalai developed a method to show stronger results for permutation groups when there are no large sets with small orbits. They observed that no further improvement over the bounds given in [2] is possible when the group satisfies the *small orbit property*. This leads naturally to the question considered here.

In this paper, we give a classification of all primitive but not affine groups which satisfy the *small orbit property*. This classification shows that the two properties studied by Bourgain and Kalai are incomparable: namely, the first property of Bourgain and Kalai does not imply the second, and the second property does not imply the first.

Theorem *Let $G_t \leq S_{n_t}$ be a sequence of primitive groups. Let $G = G_t$ for t large enough. Assuming G_t are not affine groups, the following two conditions hold: $\{G_t\}$ satisfies the *small orbit property* if and only if for all sufficient large t G_t is one of the following :*

1. $G_t \leq S_{k_t} \wr S_{m_t}$ (product type) where m_t is not bounded.
2. $G_t \leq H_t \wr S_{m_t}$ (product type) where m_t is bounded and H_t is a primitive permutation group satisfying the *small orbit property*.
3. G_t is of diagonal type.
4. G_t is of wreath product type.
5. G_t is an almost simple exceptional group of Lie type.
6. G_t is an almost simple classical group and any point stabilizer of G_t is an irreducible group.

7. G_t is an almost simple classical group and $\text{Soc}(G_t) = \text{PSL}(k_t, q_t), \text{PSp}(2k_t, q_t), \text{PU}(k_t, q_t^2), \text{PO}(2k_t + 1, q_t), \text{PO}^+(2k_t, q_t)$ or $\text{PO}^-(2k_t, q_t)$, where q_t is not bounded.

Remark We do not have a full characterization of affine groups that satisfying the *small orbit property*.

Remark In our proof, we show that if $\{G_t\}$ has the *small orbit property*, then there exist some subset $X_t \subseteq \{1, \dots, n_t\}$ such that $\{|X_t|\}$ is not bounded, $|X_t| \leq \log n_t$ for all t and $|G_t(X_t)| \leq n_t^5$.

In particular, the two properties of Bourgain and Kalai are incomparable. Indeed, the following are immediate consequences of the theorem.

Corollary 1 *The sequence $S_{k_t} \text{wr} S_{m_t}$, where m_t is not bounded, satisfies the first property of Bourgain and Kalai, but not the second.*

Corollary 2 *The sequence $\text{PSL}(k_t, q_t)$, where q_t is bounded satisfies the second property of Bourgain and Kalai, but not the first.*

2 Proof of the theorem

Recall that by the O’Nan Scott Theorem (see [4]) every finite primitive group G is one of the following:

- (1) G is an almost simple group.
- (2) $G \leq \text{AGL}(V)$ where V is a vector space of dimension k over a field F of size q .
- (3) G is of simple diagonal type.
- (4) G is of product type.
- (5) G is of twisted wreath product type.

Let $G_t \leq S_{n_t}$ be a sequence of primitive groups. Assuming G_t is not of type (2), then G_t must be of type (1), (3), (4) or (5). Whenever G_t satisfies the *small orbit property*, we construct a subset X_t whose size is not bounded but at most $\log n_t$, but whose orbit size is less than n_t^5 . Otherwise, we give a proof that G_t does not satisfy the *small orbit property*. Note that if $|G_t| < n_t^5$, then we may take any subset X_t such that $|X_t|$ is not bounded but of size at most $\log n_t$ and obtain $|G_t(X_t)| \leq |G_t| < n_t^5$. Hence, in what follows, we may assume that $|G_t| > n_t^5$.

Since by enlarging the group the size of the orbits cannot go down, then whenever G_t has the *small orbit property*, we may replace G_t by a maximal subgroup of S_{n_t} of the same type.

Case 1 Assume that G_t is an almost simple group. Then, by the classification theorem, either $\text{Soc}(G_t) = A_{k_t}$ for some k_t , or $\text{Soc}(G_t)$ is a simple group of Lie type, or $\text{Soc}(G_t)$ is a sporadic simple group. By taking t large enough, we may exclude the last case. So suppose now that $\text{Soc}(G_t)$ is a Lie type group. We shall use the following.

Lemma 1 *Let $G \leq S_n$ be an almost simple exceptional group of Lie type. Suppose G is primitive and n is large enough. Then, $|G| < n^5$.*

Proof Suppose first that G is simple. Denote by $m(G)$ the minimal index of a proper subgroup of G . Estimates for $m(G)$ in the various cases are given in Lemma 6.4 of [6]. Comparing them to the known orders of the groups G , one easily sees that $|G| < m(G)^{5-\epsilon}$ for some fixed $\epsilon > 0$. Since $m(G) \leq n$, this yields $|G| < n^{5-\epsilon}$ whenever G is an exceptional simple transitive subgroup of S_n .

Now, if G is not simple, let K be its simple socle. Then, $K \leq S_n$ is transitive, so we have $|K| \leq n^{5-\epsilon}$. It is easy to see that $|\text{Out}(K)| \leq \log |K|$ and so $|\text{Out}(K)| \leq 5 \log n < n^\epsilon$ for large enough n . It follows that $|G| \leq |K| |\text{Out}(K)| < n^{5-\epsilon} n^\epsilon = n^5$, as required. $\square$

Thus, any exceptional almost simple group of Lie type satisfies the *small orbit property*. Assume now that G_t is not exceptional. Hence, G_t is an almost simple classical group. Moreover, the values of $|G_t|$ and $m(G_t)$ (see [3] pages 170, 175) imply that k_t , the rank of G_t , is at least 3. Let $H_t \leq G_t$ be a point stabilizer. Assume first that H_t is irreducible. By a theorem of M. Liebeck and J. Saxl [5], we have $|G_t| \leq n_t^5$, and therefore, G_t satisfies the *small orbit property*. Assume now that H_t is a reducible group. Clearly, H_t is a maximal subgroup of G_t , since G_t is primitive. The maximal reducible subgroups of the classical groups are well known, and we shall use their description in constructing the required subset X_t .

Using the reducibility of the point stabilizer, it follows that there exist some d_t, q_t and $k_t \geq 3$ such that $\text{Soc}(G_t)$ is one of the following:

- (A) a) $\text{PSL}(k_t, q_t)$ acting on all d_t -dimensional subspaces of $V_t = (F_{q_t})^{k_t}$.
- b) $\text{PSL}(k_t, q_t)$ acting on pairs of subspaces of complementary dimensions.
- (B) a) $\text{PSp}(2k_t, q_t)$ acting on all d_t -dimensional non-degenerate subspaces of $V_t = (F_{q_t})^{2k_t}$.
- b) $\text{PSp}(2k_t, q_t)$ acting on all d_t -dimensional totally singular subspaces of $V_t = (F_{q_t})^{2k_t}$.
- c) $\text{PSp}(2k_t, q_t)$ acting on pairs of subspaces of complementary dimensions.
- (C) a) $\text{PU}(k_t, q_t^2)$ acting on all d_t -dimensional non-degenerate subspaces of $V_t = (F_{q_t^2})^{k_t}$.
- b) $\text{PU}(k_t, q_t^2)$ acting on all d_t -dimensional totally singular subspaces of $V_t = (F_{q_t^2})^{k_t}$.
- c) $\text{PU}(k_t, q_t^2)$ acting on pairs of subspaces of complementary dimension.

- (D) a)(1) $\text{PO}(2k_t + 1, q_t)$ acting on all d_t -dimensional non-degenerate subspaces of $V_t = (F_{q_t})^{2k_t+1}$.
 a)(2) $\text{PO}(2k_t + 1, q_t)$ acting on all d_t -dimensional totally singular subspaces of $V_t = (F_{q_t})^{2k_t+1}$.
 b)(1) $\text{PO}^+(2k_t, q_t)$ acting on all d_t -dimensional non-degenerate subspaces of $V_t = (F_{q_t})^{2k_t}$.
 b)(2) $\text{PO}^+(2k_t, q_t)$ acting on all d_t -dimensional totally singular subspaces of $V_t = (F_{q_t})^{2k_t}$.
 c)(1) $\text{PO}^-(2k_t, q_t)$ acting on all d_t -dimensional non-degenerate subspaces of $V_t = (F_{q_t})^{2k_t}$.
 c)(2) $\text{PO}^-(2k_t, q_t)$ acting on all d_t -dimensional totally singular subspaces of $V_t = (F_{q_t})^{2k_t}$.
 c)(3) $\text{PO}(2k_t + 1, q_t)$ where q_t is even and G_t acts on all one-dimensional non-singular subspaces.
 d) $\text{PO}(2k_t, q_t)$ or $\text{PO}(2k_t + 1, q_t)$ acting on pairs of subspaces of complementary dimensions.

Note that any sequence G_t of almost simple classical groups with the above action has the *small orbit property* if and only if its socle has this property. This follows from the inequality:

$$|G_t(X_t)| \leq |\text{Soc}(G_t)(X_t)| |\text{Out}(G_t)| \leq |\text{Soc}(G_t)| \cdot n_t.$$

Therefore, we may restrict to the socle and assume G_t is simple. So let G_t be a sequence of simple classical groups with one of the actions above. We will show that if q_t is not bounded, then G_t satisfies the *small orbit property*, and if q_t is bounded, then G_t does not satisfy the *small orbit property*. Assume first that q_t is not bounded. Suppose first that $G_t = \text{PGL}_{k_t}(q_t)$. Let $\{v_{t,1}, \dots, v_{t,k_t}\}$ be a basis for V_t and let $H_t \leq G_t$ be a point stabilizer. Then, H_t is a maximal subgroup of G_t . Assume first that G_t acts on the set of all d_t -dimensional subspaces of $F_{q_t}^{k_t}$ for some d_t between 1 and $k_t/2$. Here $n_t = \frac{(q_t^{k_t}-1)(q_t^{k_t-1}-1)\dots(q_t^{d_t+1}-1)}{(q_t^{d_t}-1)(q_t^{d_t-1}-1)\dots(q_t-1)} \sim q_t^{d_t(k_t-d_t)}$ and $\log n_t \sim d_t(k_t - d_t) \log q_t$. We now construct the subset X_t . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t} \in F_{q_t}$, let

$$U_{t,i} = \text{Sp}(v_{t,1}, \dots, v_{t,d_t-1}, v_{t,d_t} + a_{t,i}v_{t,d_t+1}).$$

Let

$$X_t = \{U_{t,1}, \dots, U_{t,\log q_t}\}.$$

Then, $\log q_t - 1 \leq |X_t| \leq \log q_t \leq \log n_t$, and since q_t is not bounded, $|X_t|$ is not bounded as well. It is easy to see that any $A_t \in \text{GL}_{k_t}(q_t)$ which stabilizes the subspace $\text{Sp}(v_{t,1}, \dots, v_{t,d_t-1})$ and the vectors v_{t,d_t}, v_{t,d_t+1} also stabilizes X_t , and therefore,

$$|G_t X_t| \geq (q_t^{d_t-1} - 1)(q_t^{d_t-1} - q_t) \cdots (q_t^{d_t-1} - q_t^{d_t-2}) \\ (q_t^{k_t-(d_t+1)} - 1) \cdots (q_t^{k_t-(d_t+1)} - q_t^{k_t-d_t}).$$

It follows that

$$|G_t(X_t)| = \frac{(q_t^{k_t} - 1)(q_t^{k_t} - q_t) \cdots (q_t^{k_t} - q_t^{k_t-1})}{|G_t X_t|} \leq n_t^2.$$

Thus, G_t satisfies the *small orbit property*.

Assume now that G_t acts on the set of all pairs of subspaces (U_t, W_t) such that $U_t \oplus W_t = V_t$ and $\dim U_t = d_t$, where $d_t \leq k_t/2$. Then, $n_t = q_t^{2d_t(k_t-d_t)}$ and $\log n_t = 2d_t(k_t - d_t) \log q_t$. We now construct the subset X_t . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t} \in F_{q_t}$, let

$$U_{t,i} = \text{Sp}(v_{t,1}, \dots, v_{t,d_t-1}, v_{t,d_t} + a_{t,i}v_{t,d_t+1})$$

and let

$$W_t = \text{Sp}(v_{t,d_t}, v_{t,d_t+2}, \dots, v_{t,k_t}).$$

Let

$$X_t = \{U_{t,1} \oplus W_t, \dots, U_{t,\log q_t} \oplus W_t\}.$$

It is easy to see that any $A_t \in \text{GL}_{k_t}(q_t)$ which stabilizes the subspace $\text{Sp}(v_{t,1}, \dots, v_{t,d_t-1})$, the vectors v_{t,d_t}, v_{t,d_t+1} and the subspace $\text{Sp}(v_{t,d_t+2}, \dots, v_{t,k_t})$ also stabilizes X_t , and therefore,

$$|(G_t)_{X_t}| \geq (q_t^{d_t-1} - 1)(q_t^{d_t-1} - q_t) \cdots (q_t^{d_t-1} - q_t^{d_t-2}) \\ (q_t^{k_t-(d_t+1)} - 1) \cdots (q_t^{k_t-(d_t+1)} - q_t^{k_t-d_t}).$$

It follows that

$$|G_t(X_t)| = \frac{(q_t^{k_t} - 1)(q_t^{k_t} - q_t) \cdots (q_t^{k_t} - q_t^{k_t-1})}{|G_t X_t|} \leq n_t.$$

Thus, G_t satisfies the *small orbit property*.

Assume now that G_t acts on the set of all pairs of subspaces (U_t, W_t) such that $U_t \oplus W_t = V_t$, $\dim U_t = d_t$, where $d_t \leq k_t/2$. Then, $n_t = q_t^{d_t(2k_t-3d_t)}$ and $\log n_t = d_t(2k_t - 3d_t) \log q_t$. We now construct the subset X_t . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t}$, let

$$U_{t,i} = \text{Sp}(v_{t,1}, \dots, v_{t,d_t-1}, v_{t,d_t} + a_{t,i}v_{t,d_t+1})$$

and let

$$W_t = \text{Sp}(v_{t,d_t}, v_{t,d_t+2}, \dots, v_{t,k_t}).$$

Let

$$X_t = \{(U_{t,1}, W_t), \dots, (U_{t,\log q_t}, W_t)\}.$$

It is easy to see that any $A_t \in \text{GL}_{k_t}(q_t)$ which stabilizes the subspace $\text{Sp}(v_{t,1}, \dots, v_{t,d_t-1})$, the vectors v_{t,d_t}, v_{t,d_t+1} and the subspace $\text{Sp}(v_{t,d_t+2}, \dots, v_{t,k_t})$ also stabilizes X_t , and therefore,

$$|(G_t)_{X_t}| \geq (q_t^{d_t-1} - 1)(q_t^{d_t-1} - q_t) \cdots (q_t^{d_t-1} - q_t^{d_t-2}) \\ (q_t^{k_t-(d_t+1)} - 1) \cdots (q_t^{k_t-(d_t+1)} - q_t^{k_t-d_t}).$$

It follows that

$$|G_t(X_t)| = \frac{(q_t^{k_t} - 1)(q_t^{k_t} - q_t) \cdots (q_t^{k_t} - q_t^{k_t-1})}{|(G_t)_{X_t}|} \leq n_t$$

Thus, G_t satisfies the *small orbit property*.

Assume now that G_t acts on the set of all pairs of subspaces (U_t, W_t) such that $U_t \subseteq W_t$, $\dim U_t = d_t$, and $\dim W_t = k_t - d_t$, where $d_t \leq k_t/2$. Then, $n_t = q_t^{d_t(2k_t-3d_t)}$ and $\log n_t = d_t(2k_t - 3d_t)\log q_t$. We now construct the subset X_t . Let $\{v_1, \dots, v_k\}$ be a basis for V . We now construct the subset X . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t}$, let

$$U_{t,i} = \text{Sp}(v_{t,1}, \dots, v_{t,d_t-1}, v_{t,d_t} + a_{t,i}v_{t,d_t+1})$$

and let

$$W_t = \text{Sp}(v_{t,1}, \dots, v_{t,d_t-1}, v_{t,d_t}, v_{t,d_t+1}, v_{t,d_t+2}, \dots, v_{t,k_t-d_t}).$$

Let

$$X_t = \{(U_{t,1}, W_t), \dots, (U_{t,\log q_t}, W_t)\}.$$

It is easy to see that for any $A_t \in \text{GL}_{k_t}(q_t)$ which stabilizes the subspace $\text{Sp}(v_{t,1}, \dots, v_{t,d_t-1})$, the vectors v_{t,d_t}, v_{t,d_t+1} and the subspace $\text{Sp}(v_{t,d_t+2}, \dots, v_{t,k_t})$ also stabilizes X_t , and therefore,

$$|(G_t)_{X_t}| \geq (q_t^{d_t-1} - 1)(q_t^{d_t-1} - q_t) \cdots (q_t^{d_t-1} - q_t^{d_t-2}) \\ (q_t^{k_t-(d_t+1)} - 1) \cdots (q_t^{k_t-(d_t+1)} - q_t^{k_t-d_t}).$$

It follows that

$$|G_t(X_t)| = \frac{(q_t^{k_t} - 1)(q_t^{k_t} - q_t) \cdots (q_t^{k_t} - q_t^{k_t-1})}{|(G_t)_{X_t}|} \leq n_t^2.$$

Thus, G_t satisfies the *small orbit property*.

Assume now that G_t is any group not of types $D(c)(3)$ or A . Assume first that G_t acts on the set of all d_t -dimensional totally singular subspaces for some d_t between 1 and $k_t/2 - 1$. Let $\{e_{t,1}, f_{t,1}, \dots, e_{t,k_t/2}, f_{t,k_t/2}\}$ be a standard basis for V_t , i.e., $(e_{t,i}, e_{t,j} = (f_{t,i}, f_{t,j} = 1$, and $(e_{t,i} = f_{t,j} = \delta_{i,j}$ (it might be the case that instead of $e_{t,k_t/2}, f_{t,k_t/2}$ we will have basis elements x_t, y_t , with different relations, or that the dimension is odd, and we will have an additional basis element x_t , (see chapter 2 of [3]), but for our purpose it will not make a difference). For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t}$, let

$$U_{t,i} = \text{Sp}(e_{t,1}, \dots, e_{t,d_t-1}, e_{t,d_t} + a_{t,i}e_{t,d_t+1}).$$

Let

$$X_t = \{U_{t,1}, \dots, U_{t,\log q_t}\}.$$

It is easy to see that any $A_t \in G_t$ which stabilizes the subspace $\text{Sp}(e_{t,d_t}, e_{t,d_t+1})$ also stabilizes X_t . Using the results of chapter 2 of [3], we get that $|G_t(X_t)| \leq n_t^2$. Assume now that $d_t = k_t/2$. For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t} \in F_{q_t}$, let

$$U_{t,i} = \text{Sp}(e_{t,1}, \dots, e_{t,k_t/2-2}, e_{k_t/2-1} + a_i f_{k_t/2}, e_{k_t/2} - a_i f_{k_t/2-1})$$

(in the symplectic case, we take $e_{t,k_t/2} + a_{t,i} f_{t,k_t/2-1}$ instead of $e_{t,k_t/2} - a_{t,i} f_{t,k_t/2-1}$). Let

$$X_t = \{U_{t,1}, \dots, U_{t,\log q_t}\}.$$

As before, we get that $|G_t(X_t)| \leq n_t^2$. Assume now that G_t acts on the set of all d_t -dimensional non-degenerate subspaces of V_t . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t} \in F_{q_t}$, let

$$U_{t,i} = \{e_{t,1}, f_{t,1}, \dots, e_{t,d_t/2-1}, f_{t,d_t/2-1}, e_{t,d_t/2}, f_{t,d_t/2} + a_{t,i}e_{t,d_t/2+1}\}.$$

Let

$$X_t = \{U_{t,1}, \dots, U_{t,\log q_t}\}.$$

Thus, we get that $|G_t(X_t)| \leq n_t^2$. Assume now that G_t is of type $D(c)(3)$ above. Let $\{e_{t,1}, f_{t,1}, \dots, e_{t,k_t/2}, f_{t,k_t/2}\}$ be a symplectic basis for V_t . For $1 \leq i \leq \log q_t$ and $a_{t,1}, \dots, a_{t,\log q_t}$, let

$$U_{t,i} = \text{Sp}(e_{t,1} + a_{t,i}f_{t,1}).$$

Let

$$X_t = \{U_{t,1}, \dots, U_{t,\log q_t}\}.$$

It follows that $|G_t(X_t)| \leq n_t^2$. Hence, whenever q_t is not bounded, G_t satisfies the *small orbit property*. Assume now that q_t is bounded. Let G_t be any group of types $A - D$ above. We will show that under the action of G_t on the set of all one-dimensional totally singular subspaces of V_t , the size of the orbit of any unbounded subset of size at most $c \log n_t$ is not polynomial. Let X_t be any unbounded set of vectors of size at most $c \log n_t$. Let $r_t = \dim \text{Span}(X_t)$. Clearly, r_t is not bounded (since q_t is bounded). Let $\{x_{t,1}, \dots, x_{t,r_t}\} \subseteq X_t$ be a base for $\text{Span}(X_t)$, and $\{v_{t,r_t+1}, \dots, v_{t,k_t}\}$ its complement to a base for V_t . Let $g_t \in (G_t)_{X_t}$. Then, for all $1 \leq i \leq r_t$ $g_t(x_{t,i}) \in X_t$ and $g_t(v_{t,r_t+i}) \notin \text{Span}(X_t)$. Therefore,

$$\begin{aligned} |G_t X_t| &\leq |X_t|(|X_t| - 1) \cdots (|X_t| - (r_t - 1))(q_t^{k_t} - q_t^{r_t}) \cdots (q_t^{k_t} - q_t^{k_t-1}) \\ &\leq |X_t|^{r_t} (q_t^{k_t} - q_t^{r_t}) \cdots (q_t^{k_t} - q_t^{k_t-1}). \end{aligned}$$

It follows that

$$\begin{aligned} |G_t(X_t)| &= |G_t : (G_t)_{X_t}| \geq \frac{(q_t^{k_t} - 1) \cdots (q_t^{k_t} - q_t^{k_t-1})}{|X_t|^{r_t} (q_t^{k_t} - q_t^{r_t}) \cdots (q_t^{k_t} - q_t^{k_t-1})} \\ &= \frac{(q_t^{k_t} - 1) \cdots (q_t^{k_t} - q_t^{r_t-1})}{|X_t|^{r_t}} \geq \frac{(q_t^{k_t} - q_t^{r_t-1})^{r_t}}{(ck_t \log q_t)^{r_t}} \geq \frac{(q_t^{k_t-1})^{r_t}}{(ck_t \log q_t)^{r_t}} \\ &= \left(\frac{(q_t^{k_t-1})}{ck_t \log q_t} \right)^{r_t} > \left(\frac{(q_t^{k_t-2})}{ck_t} \right)^{r_t}. \end{aligned}$$

Since r_t is not bounded, it now follows that $|G_t(X_t)|$ is not polynomial in $n_t = q_t^{k_t}$, as desired.

Case 3 Assume that G_t is of diagonal type. We will show that G_t satisfies the *small orbit property*. We may assume that there exists some integer k_t such that $G_t = H_t^{k_t} \cdot (\text{Out}(H_t) \times S_{k_t})$, where H_t is a non-abelian simple group, k_t is a positive integer and G_t acts on the set of all cosets of the diagonal group $D_t = \{(h_t, \dots, h_t) : h_t \in H_t\}$. Denote this set by Ω . Here $n_t = |H_t^{k_t-1}|$ and $\log n_t = (k_t - 1) \log |H_t|$. Since $|G_t| = |H_t^{k_t}| |H_t| k_t! \leq n_t^2 k_t!$, then we may assume that k_t is not bounded. Let $L_t < H_t$ be a nontrivial subgroup of H_t and let $1 \neq l_t \in L_t$. Let

$$X_t = \{(l_t, 1, \dots, 1)D_t, (1, l_t, 1, \dots, 1)D_t, \dots, (1, \dots, 1, l_t)D_t\}.$$

Then, $|X_t| = k_t \leq \log n_t$. Since $|S_{k_t}(X_t)| = 1$ we have

$$|G_t(X_t)| = |H_t^{k_t} \cdot (\text{Out}(H_t) \times S_{k_t})(X_t)| \leq n_t^2 |S_{k_t}(X_t)| = n_t^2.$$

Case 4 Assume that G_t is of product type. Let H_t be a primitive permutation group of type (1) or (3) acting on a set $\Gamma_t = \{1, \dots, k_t\}$. For $m_t > 1$, let $W_t = H_t \text{wr} S_{m_t}$

and take W_t to act on $\Omega_t = \Gamma_t^{m_t}$ in its natural product action. Then, $G_t \leq H_t \wr S_{m_t}$, and $n_t = k_t^{m_t}$. Assume first that m_t is not bounded. We will show that G_t satisfies the *small orbit property*. Let $x_t, y_t \in \Gamma_t$ and let

$$X_t = \{(x_t, y_t, \dots, y_t), (y_t, x_t, y_t, \dots, y_t), \dots, (y_t, \dots, y_t, x_t)\}.$$

Then, $|X_t| = m_t \leq \log n_t$. It is easy to see that $|S_{k_t}^{m_t}(X_t)| \leq (k_t^{m_t})^2 = n_t^2$. Since $|S_{m_t}(X_t)| = 1$, we have $|G_t(X_t)| \leq |S_{k_t}^{m_t} \cdot S_{m_t}(X_t)| \leq n_t^2 |S_{m_t}(X_t)| = n_t^2$. Assume now that m_t is bounded. We will show that G_t has the *small orbit property* if and only if H_t has the *small orbit property*: Assume first that H_t satisfies the *small orbit property*. As we have shown (see cases 1 and 3), there exists some subset Y_t which is not bounded but of size at most $\log k_t$ such that $|H_t(Y_t)| \leq k_t^5$. Denote $Y_t = \{y_t^1, \dots, y_t^{|Y_t|}\}$. Let

$$X_t = \{(y_t^1, \dots, y_t^1), (y_t^1, \dots, y_t^1), \dots, (y_t^{|Y_t|}, \dots, y_t^{|Y_t|})\}.$$

Then, X_t is not bounded but of size at most $\log n_t$. Since $|H_t(Y_t)| \leq k_t^5$, then $|H_t^{m_t}(X_t)| \leq (k_t^5)^{m_t} = (k_t^{m_t})^5 = n_t^5$. Clearly, $|S_{m_t}(X_t)| = 1$, and therefore, $|H_t^{m_t} \cdot S_{m_t}(X_t)| \leq n_t^5$.

Assume now that H_t does not satisfy the *small orbit property*. By our classification (see cases 1 and 3), H_t must be an almost simple group. A previous argument (see case 1) implies that $\text{Soc}(H_t)$ does not satisfy the *small orbit property* as well. We will show now that G_t does not satisfy the *small orbit property* as well: Let X_t be any unbounded subset of Ω_t of size at most $c \log n_t$. Denote by X_t^i the projection on the i 'th axis. Since m_t is bounded, then there exists some $1 \leq i \leq m_t$ such that X_t^i is an unbounded set of size at most $c m_t \log k_t$. Hence, under our assumption $|(\text{Soc}(H_t))(X_t^i)|$ is not polynomial in k_t , and since m_t is bounded, then $|(\text{Soc}(H_t))(X_t^i)|$ is not polynomial in n_t . It now follows that $|(\text{Soc}(H_t))^{m_t}(X_t)|$ is not polynomial in n_t . Since $(\text{Soc}(H_t))^{m_t} \leq G_t$, then we get now that $|G_t(X_t)|$ is not polynomial in n_t , as desired.

Case 5 Any group G_t of type 5 satisfies the *small orbit property*, since any twisted wreath product group is contained in a group of product type of the form $H_t \wr S_{m_t}$ where H_t is of diagonal type (see [4]).

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