Determinants of representations of Coxeter groups

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Abstract

In Ayer et al. (J Comb Theory Ser A 150:208–232, 2017), the authors characterize the partitions of n whose corresponding representations of S_n have nontrivial determinant. The present paper extends this work to all irreducible finite Coxeter groups W . Namely, given a nontrivial multiplicative character ω of W , we give a closed formula for the number of irreducible representations of W with determinant ω . For Coxeter groups of type B_n and D_n , this is accomplished by characterizing the bipartitions associated to such representations.

Keywords Coxeter groups · Hyperoctahedral · 2-Core towers · Specht modules · Core and quotient of partitions · Determinant of representations · Representation theory of symmetric group

1 Introduction

The tools of this paper, and [1], have their genesis in the paper [5] of Macdonald, who developed the arithmetic of partitions to give a closed formula for the number $A(n)$ of odd-dimensional Specht modules (irreducible representations) of S_n . Briefly, if n is expressed in binary as a sum of powers of 2, then $A(n)$ is the product of those powers of 2. This formula comes from characterizing the 2-core tower of ‘odd’ partitions; we review this notion in Sect. 5.1. In [1], the authors describe a simple way to read off the determinant of a representation of S_n from its 2-core tower and in particular give a closed formula for the number $B(n)$ of Specht modules whose determinant is the sign character. Since the number $p(n)$ of partitions of n itself does not have a nice closed formula, it is remarkable that such formulas for $A(n)$ and $B(n)$ exist.

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Most of this paper is devoted to the Coxeter group of type B_n , which we write as ‘ $\mathbb{B}_n$ ’ for simplicity. These are the ‘hyperoctahedral groups.’ Here, $\mathbb{B}_n$ is the wreath product of $\mathbb{Z}/2\mathbb{Z}$ by S_n . There are four multiplicative characters of $\mathbb{B}_n$, which we denote by $\omega = 1$, sgn^0 , ε , and sgn^1 , described as follows. The character sgn^0 is the composition of the projection to S_n with sgn , the character ε is $(-1)^m$, where m is the sum of the entries of the $(\mathbb{Z}/2\mathbb{Z})^n$ -factor, and finally $\text{sgn}^1 = \varepsilon \cdot \text{sgn}^0$.

Irreducible representations of $\mathbb{B}_n$, written $\rho_{\alpha\beta}$, are parametrized by pairs (α, β) of partitions with $|\alpha| + |\beta| = n$, called bipartitions of n . The representations are induced by certain ‘Young Subgroups’ $\mathbb{B}_a \times \mathbb{B}_b \leq \mathbb{B}_n$. For each ω , we characterize the 2-core towers of α, β so that $\det \rho_{\alpha\beta} = \omega$. For $\omega \neq 1$, this leads to closed formulas for

$$N_\omega(n) = \#\{(\alpha, \beta) : |\alpha| + |\beta| = n, \det \rho_{\alpha\beta} = \omega\}.$$

Moreover, we prove:

Theorem 1 *Let $n \geq 10$.*

- (1) *If n is odd, then $N_\varepsilon(n) = N_{\text{sgn}^1}(n) < N_{\text{sgn}^0}(n) < N_1(n)$.*
- (2) *If n is even, then $N_\varepsilon(n) = N_{\text{sgn}^0}(n) < N_{\text{sgn}^1}(n) < N_1(n)$.*

In Sect. 2, after establishing notations, we give formulas for the determinant of a representation of a Coxeter group in terms of the character. In Sect. 3, for completeness, we indicate the solution to the determinant problem for dihedral groups. Section 4 reviews the work of [5] and [1] for S_n and develops it further for application to the hyperoctahedral case.

Most of the work is in Sect. 5, where we first compute $\det \rho_{\alpha\beta}$ in terms of known quantities (‘ f_α ’ the degree of the Specht module ρ_α , and ‘ g_α ’ the multiplicity of -1 as an eigenvalue of $\rho_\alpha(12)$), and similarly for β). We use a well-known formula for the determinant of an induced representation. We then provide a table and a log plot illustrating the values of $N_\omega(n)$ for small n . This is followed by a conceptual proof of the identity $N_{\text{sgn}^1}(n) = N_\varepsilon(n)$ when $n \geq 3$ is odd. No such ‘pure thought’ proof is apparent for the other identity $N_{\text{sgn}^0}(n) = N_\varepsilon(n)$ when n is even. Next, we calculate each

$$N_\omega(a, b) = \{(\alpha, \beta) : |\alpha| = a, |\beta| = b, \det \rho_{\alpha\beta} = \omega\}.$$

In Sect. 5.6, we compute $N_\omega(n) = \sum_{a+b=n} N_\omega(a, b)$. Considerable work goes into simplifying these sums to arrive at closed formulas. These final formulas take the form

$$N_\omega(n) = A(2n) \times f_\omega(k),$$

where $f_\omega(k)$ is a function only depending on $k = \text{ord}_2(n)$, for n even, and similarly in terms of $\text{ord}_2(n-1)$ when n is odd.

Next we apply Clifford Theory to the Coxeter group of type D_n . This group is the kernel of ε , and we denote it by $\mathbb{D}_n$ (not to be confused with the dihedral groups, of type $I_2(n)$). This group has two multiplicative characters, the nontrivial one being the restriction of sgn^0 , which we denote as ‘ sgn .’ We give a closed formula for $N'_{\text{sgn}}(n)$, the number of irreducible representations of $\mathbb{D}_n$ with determinant sgn , in terms of our earlier formulas for $N_\omega(n)$.

We similarly treat the remaining (exceptional) types of finite irreducible Coxeter groups. Given the formulas of Sect. 2, this comes to a finite calculation with the character tables of [3].

2 Notation and preliminaries

2.1 Binary notation

The nonnegative integer n and its binary digits play a substantial role in this paper, so it is convenient to fix notation here. Put

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r}, \quad (1)$$

with ϵ either 0 or 1, and $1 \leq k_1 < k_2 < \cdots < k_r$. We will sometimes write $k = k_1$, and $n' = n - \epsilon$. Let $v(n)$ denote the number of 1's in the binary expansion of n ; in the above $v(n) = r + \epsilon$.

Write $\text{bin}(n)$ for the set of powers of 2 in the binary expansion of n ; in the above $\text{bin}(n) = \{k_1, \dots, k_r\}$, together with 0 if n is odd.

Given a natural number a , write $\text{ord}(a)$ for the highest power of 2 dividing a . In the above, $k = k_1 = \text{ord}(n')$. (It is traditional to write ' $\text{ord}_2(n)$,' but in this paper only '2' matters.) Note for later that if $0 \leq j \leq k$, then

$$v(n - 2^j) = v(n) + k - j - 1. \quad (2)$$

Definition 1 Let a, b be nonnegative integers. We say ' $\text{sum}(a, b)$ is neat,' provided that there is no carry when adding a to b in binary. We also write ' $a + b \doteq n$ ' to denote that $a + b = n$ and the sum is neat. Similarly we say ' $\text{sum}(a, b)$ is messy,' provided that there is a carry.

Thus $\text{sum}(a, b)$ is neat iff $\text{bin}(a)$ and $\text{bin}(b)$ are disjoint. It is a classical observation that the binomial coefficient $\binom{n}{a,b} = \frac{n!}{a!b!}$ is odd iff $\text{sum}(a, b)$ is neat; we will make extensive use of this without further comment. More generally, $\text{ord}\left(\binom{n}{a,b}\right)$ is equal to the number of carries when adding a to b in binary.

2.2 Partition notation

The notation ' $\lambda \vdash n$ ' means that λ is a partition of n . We also write $|\lambda| = n$ in this case. Write λ' for the conjugate partition; its Young diagram is the transpose of the Young diagram of λ . A 'hook' is a partition of the form $\lambda = (a, 1^b)$ for some nonnegative integers a, b .

A bipartition of a number n is a pair of partitions (α, β) with $|\alpha| + |\beta| = n$. The notation ' $(\alpha, \beta) \models n$ ' means that (α, β) is a bipartition of n .

Write $p(n)$ for the number of partitions of n , and $p_2(n)$ for the number of bipartitions of n . Write Λ for the set of partitions, Λ_n for the set of partitions of n , and $\text{Bip}(n)$ for the set of bipartitions of n .

2.3 Group theory notation

For a group G , write $D(G)$ for its derived (commutator) subgroup, and G_{ab} for its abelianization $G/D(G)$.

There are four infinite families of finite irreducible Coxeter groups, namely ones of type A_n , B_n , D_n , and $I_2(n)$ (the dihedral groups). The exceptional types are E_6 , E_7 , E_8 , F_4 , H_3 , H_4 . (See [6]).

All representations considered in this paper are finite-dimensional and complex. Write $\text{Irr}(G)$ for the set of isomorphism classes of irreducible representations of G . If π is a representation, we write χ_π for its character.

By ‘multiplicative character’ we mean a group homomorphism $W \rightarrow \mathbb{C}^\times$, which since W is generated by involutions, must take values in $\{\pm 1\}$. For a representation (π, V) of a group G , write $\det \pi$ for the composition of $\pi : G \rightarrow \text{GL}(V)$ with the determinant map. Then $\det \pi$ is a multiplicative character of G .

If (π_1, V_1) and (π_2, V_2) are representations of groups G_1 and G_2 , write ‘ $\pi_1 \boxtimes \pi_2$ ’ for the external tensor product representation of $G_1 \times G_2$ on $V_1 \otimes_{\mathbb{C}} V_2$. On the other hand, when $G_1 = G_2 = G$, we write ‘ $\pi_1 \otimes \pi_2$ ’ for the internal tensor product representation of G , with G acting diagonally on $V_1 \otimes_{\mathbb{C}} V_2$.

2.4 Solomon principle

Let π be a representation of a Coxeter group W . In this section, we show how to infer $\det \pi$ from its character. So let (W, S) be a Coxeter group (S is a certain set of generators of order 2; see [6]). There is a unique multiplicative character ε_W so that $\varepsilon_W(s) = -1$ for each $s \in S$, namely

$$\varepsilon_W(w) = (-1)^{\ell(w)},$$

where $\ell(w)$ is the length of w with respect to S .

For the Coxeter groups of type A_n , D_n , E_6 , E_7 , E_8 , H_3 , H_4 , and $I_2(p)$ with p odd, the trivial character and ε_W are the only multiplicative characters. This is equivalent to the abelianization W_{ab} having order 2.

Proposition 1 *Suppose $|W_{\text{ab}}| = 2$, and let $s \in S$. If π is a representation of W , then*

$$\det \pi = (\varepsilon_W)^b,$$

where

$$b = \frac{\dim \pi - \chi_\pi(s)}{2}.$$

Proof Let a be the multiplicity of 1 as an eigenvalue of $\pi(s)$, and b be the multiplicity of -1 . Then $\dim \pi = a + b$ and $\chi_\pi(s) = a - b$. $\square$

We attribute this approach to L. Solomon; see [10], Exercise 7.55.

The abelianization of the other Coxeter groups (B_n , F_4 , and $I_2(p)$ for p even) are Klein 4 groups. For these, fix two nonconjugate simple reflections $s_1, s_2 \in S$,

and multiplicative characters ω_1, ω_2 so that $\omega_1(s_1) = -1$, $\omega_1(s_2) = 1$, $\omega_2(s_1) = 1$, $\omega_2(s_2) = -1$. Then $\varepsilon_W = \omega_1 \cdot \omega_2$ and the multiplicative characters of W are $\{1, \omega_1, \omega_2, \omega_1 \cdot \omega_2\}$.

Proposition 2 Suppose $|W_{\text{ab}}| = 4$, and let $s_1, s_2, \omega_1, \omega_2$ be as above. If π is a representation of W , then

$$\det \pi = (\omega_1)^{x_1} (\omega_2)^{x_2},$$

where

$$x_1 = \frac{\dim \pi - \chi_\pi(s_1)}{2},$$

and

$$x_2 = \frac{\dim \pi - \chi_\pi(s_2)}{2}.$$

Proof Similar to the proof of Proposition 1. $\square$

3 Type $I_2(p)$: Dihedral groups

Let us sketch the case of dihedral groups. Let $p \geq 1$ be a positive integer, and let $W = D_p$ be the dihedral group of order $2p$. The irreducible representations of W are well known: they are either one-dimensional, or induced from the normal cyclic subgroup of order p , thus two-dimensional. It is elementary to check that the determinant of each two-dimensional irreducible representation is ε_W .

Case p odd: In this case $|W_{\text{ab}}| = 2$, so 1 and ε_W are the only multiplicative characters. From the above,

$$N_1(p) = \#\{\pi \in \text{Irr}(D_p) \mid \det \pi = 1\} = 1$$

and

$$N_{\varepsilon_W}(p) = \#\{\pi \in \text{Irr}(D_p) \mid \det \pi = \varepsilon_W\} = \frac{p+1}{2}.$$

Case p even Now $|W_{\text{ab}}| = 4$, so there are four multiplicative characters. We have $N_{\varepsilon_W}(p) = \frac{p}{2}$ and $N_\omega(p) = 1$ for $\omega \neq \varepsilon_W$.

4 Type A_n : symmetric groups

In this section, we review and develop material from [5,7], and [1] for S_n , which we will apply to the Type B_n Coxeter groups in the next section. The irreducible representations of S_n are indexed by partitions λ of n . Given $\lambda \vdash n$, we denote by $(\rho_\lambda, V_\lambda)$ the corresponding representation of S_n .

4.1 Cores, quotients, and towers

Let us quickly recall the theory of 2-cores and 2-quotients of partitions. Details may be found in [7].

Definition 2 We say that a partition λ is a 2-core partition, provided that none of its hook lengths are even. The set of 2-core partitions is denoted by $\mathcal{C}_2$.

There is assigned to any partition λ a partition $\text{core}_2 \lambda \in \mathcal{C}_2$ called the 2-core of λ , and a bipartition (α, β) , called the 2-quotient of λ . These have the property that

$$|\lambda| = |\text{core}_2 \lambda| + 2(|\alpha| + |\beta|). \quad (3)$$

Moreover, this assignment is a bijection between Λ and $\mathcal{C}_2 \times \Lambda \times \Lambda$. In what follows, we iterate the core-quotient procedure to obtain what is called the 2-core tower of λ . This is an infinite binary tree, with each node labeled with a 2-core partition. For uniformity, let us put $\lambda_0 = \alpha$ and $\lambda_1 = \beta$. The tower is organized into rows, one for each nonnegative integer. Its 0th row is the root of the tree, labeled with $\alpha_\emptyset := \text{core}_2 \lambda$. Its first row comprises two nodes

$$\alpha_0, \alpha_1,$$

where, if $\text{quo}_2 \lambda = (\lambda_0, \lambda_1)$, then $\alpha_i = \text{core}_2 \lambda_i$. Let $\text{quo}_2 \lambda_i = (\lambda_{i0}, \lambda_{i1})$, and define $\alpha_{ij} = \text{core}_2 \lambda_{ij}$. The second row is

$$\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}.$$

Recursively, having defined partitions λ_x for a binary sequence x , define the partitions λ_{x0} and λ_{x1} by

$$\text{quo}_2 \lambda_x = (\lambda_{x0}, \lambda_{x1}), \quad (4)$$

and let $\alpha_{x\epsilon} = \text{core}_2 \lambda_{x\epsilon}$ for $\epsilon = 0, 1$. The i th row of the 2-core tower of λ consists of the partitions α_x , where x runs over the set of all 2^i binary sequences of length i , listed from left to right in lexicographic order.

Thus the 2-core tower is:

This is called the 2-core tower of λ , which we denote by ‘ T_λ .’ Note that T_λ has nonempty partitions at only finitely many nodes. Moreover, λ can be reconstructed from T_λ .

If $w_i(\lambda)$ is the sum of the sizes of the partitions in the i th row of T_λ , then iterating (3) gives

$$|\lambda| = \sum_{i=0}^{\infty} w_i(\lambda) 2^i. \quad (5)$$

Example Let $\lambda = (12, 2, 1, 1)$. The 2-core of λ is empty and its 2-quotient is $((1, 1), (6))$. The 2-core of $(1, 1)$ is also empty, and its 2-quotient is $((1), \emptyset)$. The partition (6) has empty 2-core, and 2-quotient $(\emptyset, (3))$. Finally, (3) has 2-core (1) and 2-quotient $((1), \emptyset)$. From this we can write down T_λ :

The fourth row and below are empty.

We also define a map $\phi : \Lambda_a \times \Lambda_b \rightarrow \Lambda_{2(a+b)}$ so that $\phi(\alpha, \beta)$ has trivial 2-core, and its 2-quotient is (α, β) . Note $T_{\phi(\alpha, \beta)}$ is obtained simply by joining T_α and T_β side-by-side with root node labeled ‘ $\emptyset$.’ For example, $(12, 2, 1, 1) = \phi((1, 1), (6))$ and $(6) = \phi(\emptyset, (3))$. All partitions of an even number n with trivial 2-core are uniquely of the form $\phi(\alpha, \beta)$ for some $(\alpha, \beta) \models \frac{n}{2}$.

4.2 Review of Macdonald

Definition 3 Write f_λ for the dimension of V_λ . Say that a partition λ is odd, provided that f_λ is odd. Otherwise say that λ is even. Write Λ^{odd} for the set of odd partitions, and Λ_n^{odd} for the set of odd partitions of n . Write $A(n) = |\Lambda_n^{\text{odd}}|$.

The main result of [5] is:

Theorem 2 A partition is odd precisely when $w_i(\lambda) \leq 1$ for all $i \geq 0$.

In other words, it is odd when its 2-core tower has at most one cell in each row. For example, from the towers we see that the partition $(12, 2, 1, 1)$ is even, while the partitions $(1, 1)$ and (6) are odd.

By (5), if $\lambda \vdash n$ is an odd partition, then $\text{bin}(n)$ is equal to the set of indices of the nontrivial rows of T_λ .

Corollary 1 *For n with binary expansion*

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r},$$

put

$$\alpha(n) = k_1 + k_2 + \cdots + k_r.$$

Then $A(n) = 2^{\alpha(n)}$.

In other words, if we express n in binary as a sum of powers of 2, then $A(n)$ is the product of those powers of 2. In particular, if n is a power of 2, then $A(n) = n$. In this case, the odd partitions of n are precisely the hooks of length n .

Let us record some properties of $A(n)$ for later use.

Lemma 1 (1) *If $x + y \doteq n$, then $A(n) = A(x)A(y)$.*

(2) *If n is odd, then $A(n - 1) = A(n)$.*

(3) *More generally, if $0 \leq j \leq k = \text{ord}(n')$, then*

$$A(n - 2^j) = 2^{-k + \binom{k}{2} - \binom{j}{2}} A(n).$$

(4) *For all n we have*

$$A(2n) = 2^{v(n)} A(n).$$

Proof For the third part, we have

$$\begin{aligned} \alpha(n) &= \alpha(n - 2^k) + \alpha(2^k) \\ &= \alpha(n - 2^k) + k, \end{aligned}$$

and

$$\alpha(n - 2^j) = \alpha(n - 2^k) + \alpha(2^k - 2^j).$$

Now

$$\begin{aligned} \alpha(2^k - 2^j) &= j + (j + 1) + \cdots + (k - 1) \\ &= \binom{k}{2} - \binom{j}{2}. \end{aligned}$$

Thus

$$\alpha(n - 2^j) = \alpha(n) - k + \binom{k}{2} - \binom{j}{2}.$$

The result follows. The other parts are straightforward. Also see Proposition 5 for a bijective proof of the first part. $\square$

Here are simple bounds for $A(n)$:

Lemma 2 *Let n be a natural number. We have*

$$\frac{1}{2}n < A(n) \leq n^{\frac{1}{2}(\log_2 n + 1)}.$$

Proof We have

$$\begin{aligned} \log_2 n - 1 < [\log_2 n] &\leq \alpha(n) \leq 1 + 2 + \cdots + [\log_2 n] \\ &= \frac{[\log_2 n]([\log_2 n] + 1)}{2} \\ &\leq \frac{1}{2} \log_2 n (\log_2 n + 1). \end{aligned}$$

Since $A(n) = 2^{\alpha(n)}$, this gives the lemma. $\square$

4.3 Review of Ayyer–Prasad–Spallone

Put $s_1 = (12) \in S_n$ for $n \geq 2$.

Definition 4 Given $\lambda \vdash n$, put

$$g_\lambda = \frac{f_\lambda - \chi_\lambda(s_1)}{2}.$$

Say that a partition λ is chiral, provided that $\det \rho_\lambda = \text{sgn}$. Write $B(n)$ for the number of chiral partitions of n .

By Proposition 1, λ is chiral iff g_λ is odd. Recall the notation ‘ord,’ ‘bin,’ and $k = \text{ord}(n')$ from Sect. 2.1.

Theorem 3 [1, Theorem 6] *Let λ be a partition with 2-quotient (α, β) with $|\alpha| = a$ and $|\beta| = b$. Then λ is chiral if and only if one of the following holds:*

- (1) λ is odd, and
 - (a) if n is even, then $k - 1 \in \text{bin}(a)$.
 - (b) if n is odd, then $k - 1 \in \text{bin}(b)$.
- (2) $\text{core}_2 \lambda = \emptyset$ or (1), and
 - (a) α and β are odd,
 - (b) $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$.
- (3) $\text{core}_2 \lambda = (2, 1)$ and $\phi(\alpha, \beta)$ is odd.

Corollary 2 (1) $\#\{\lambda \vdash n \mid f_\lambda \equiv g_\lambda \equiv 1\} = \frac{1}{2}A(n)$.

(2) $\#\{\lambda \vdash n \mid f_\lambda \equiv 1, g_\lambda \equiv 0\} = \frac{1}{2}A(n)$.

(3) $\#\{\lambda \vdash n \mid f_\lambda \equiv 0, g_\lambda \equiv 1\} = B(n) - \frac{1}{2}A(n)$.

Let us give special attention to the chiral partitions of type (2) in Theorem 3. The corresponding towers have at most one cell in each row, except for the j th row which has precisely two cells, one in the left half, and another in the right half. Necessarily $j \geq 1$. Moreover, there are no cells in rows 1 through $j - 1$. Its root is labeled $\emptyset$ or (1). Such towers we call ‘ j -domino towers,’ or ‘domino towers’ if we forget the j . The above tower of $\lambda = (12, 2, 1, 1)$ is a 2-domino tower.

Write Λ^D for the set of domino towers, and Λ^j for the set of j -domino towers. Let $D(n) = |\Lambda_n^D|$ and $D_j(n) = |\Lambda_n^j|$. Note that $D(n) = 0$ if $n' \equiv 2 \pmod{4}$, and $D(n) = \sum_{j=1}^{k-1} D_j(n)$ otherwise.

It is easy to see that

$$\begin{aligned} D_j(n) &= 2^{2(j-1)} A\left(n - 2^{j+1}\right) \\ &= 2^{2(j-1)-k+\binom{k}{2}-\binom{j+1}{2}} A(n) \\ &= 2^{\binom{k}{2}-\binom{j}{2}-k+j-2} A(n), \end{aligned} \quad (6)$$

using Lemma 1.

Corollary 3 [1, Theorem 1] *For $n \geq 2$ and $k = \text{ord}(n')$, the number of chiral partitions is given by*

$$B(n) = A(n) \left(\frac{1}{2} + \epsilon \cdot 2^{\binom{k}{2}-k} + \sum_{j=1}^{k-1} 2^{\binom{k}{2}-\binom{j}{2}-k+j-2} \right).$$

The sum is understood to be 0 if $k = 1$.

Proof The number of partitions satisfying (1) in Theorem 3 is $\frac{1}{2}A(n)$, and the number satisfying (3) is

$$\epsilon A(n-3) = \epsilon A(n) 2^{\binom{k}{2}-k}.$$

So

$$\begin{aligned} B(n) &= \frac{1}{2}A(n) + D(n) + \epsilon A(n) 2^{\binom{k}{2}-k} \\ &= \frac{1}{2}A(n) + \left(\sum_{j=1}^{k-1} D_j(n) \right) + \epsilon A(n) 2^{\binom{k}{2}-k}, \end{aligned}$$

and the final formula follows from (6). $\square$

4.4 Case of trivial 2-core

Consider the case of partitions of the form $\phi(\alpha, \beta) \vdash 2n$. (See Sect. 4.1.) Note that $\phi(\alpha, \beta)$ is odd iff α and β are odd and $\text{sum}(a, b)$ is neat. Theorem 3 says:

Proposition 3 *The partition $\lambda = \phi(\alpha, \beta)$ is chiral iff both α and β are odd, and one of the following holds:*

- (1) $\text{bin}(a) \cap \text{bin}(b) = \emptyset$ and $\text{ord}(n) \in \text{bin}(a)$.

(2) $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$.

The disjunction of the two conditions of the proposition is equivalent to a certain binomial coefficient being odd.

Proposition 4 *Let $a + b = n$. The quantity $\binom{n-1}{a-1, b}$ is odd iff one of the following holds:*

- (1) $\text{bin}(a) \cap \text{bin}(b) = \emptyset$ and $\text{ord}(n) \in \text{bin}(a)$.
- (2) $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$.

The proof is left to the reader. □

Corollary 4 *Let $(\alpha, \beta) \models n$ with $|\alpha| = a$ and $|\beta| = b$. Then $\lambda = \phi(\alpha, \beta)$ is chiral iff the quantity*

$$f_\alpha f_\beta \binom{n-1}{a-1, b}$$

is odd.

4.5 Tower merging

Suppose λ and μ are partitions, with the property that for all $i \geq 0$, either the i th row of T_λ is empty, or the i th row of T_μ is empty. In this situation, one can define a partition ν of $|\lambda| + |\mu|$ as follows: For all $i \geq 0$, if the i th row of T_λ (resp., T_μ) is nonempty, let the i th row of T_ν be equal to the i th row of T_λ (resp. of T_μ). If the i th rows of both T_λ and T_μ are empty, then let the i th row of T_ν be empty. We will refer to this process as ‘merging’ the towers T_λ and T_μ .

For example, $\lambda = (3, 3, 2)$, whose 2-core tower is

can be merged with $\mu = (3, 1, 1, 1, 1)$, whose tower is

to yield $\nu = (11, 3, 2)$, with tower

Write ' $a + b \stackrel{D}{=} n$ ' when $a + b = n$, and $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$. Put $D_{<j}(n) = \sum_{i=1}^{j-1} D_i(n)$.

Proposition 5 *Let $a + b = n$ with $k = \text{ord}(n)$.*

- (1) *If $a + b \stackrel{D}{=} n$, then $A(n) = A(a)A(b)$.*
- (2) *If $a + b \stackrel{D}{=} n$ with $k = \text{ord}(b)$, then $A(a)D(b) = D(n)$.*
- (3) *If $a + b \stackrel{D}{=} n$ with $\text{ord}(a) = \text{ord}(b) = j$, then $A(a)D(b) = D_{<j}(n)$.*

Proof In each case, a bijective proof is given by the merging of towers: In the first case, merging gives a bijection from $\Lambda_a^{\text{odd}} \times \Lambda_b^{\text{odd}}$ to Λ_n^{odd} . In the second case, it gives a bijection from $\Lambda_a^{\text{odd}} \times \Lambda_b^D$ to Λ_n^D . In the last case, it gives a bijection from $\Lambda_a^{\text{odd}} \times \Lambda_b^D$ to $\bigcup_{i < j} \Lambda_n^i$. $\square$

5 Type B_n : hyperoctahedral groups

The Coxeter group of type B_n , which we denote by $\mathbb{B}_n$, is the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr S_n$. It is traditionally called the n th hyperoctahedral group, being the symmetries of

the standard hyperoctahedron (or standard hypercube) in $\mathbb{R}^n$. For $a + b = n$, there is a Young subgroup $\mathbb{B}_a \times \mathbb{B}_b \leq \mathbb{B}_n$ in the evident way.

The four multiplicative characters of $\mathbb{B}_n$ may be described as follows. Write $\varepsilon : (\mathbb{Z}/2\mathbb{Z})^n \rightarrow \{\pm 1\}$ for the character whose restriction to each factor $\mathbb{Z}/2\mathbb{Z}$ is nontrivial. As it is S_n -invariant, it extends to a multiplicative character $\varepsilon : \mathbb{B}_n \rightarrow \{\pm 1\}$. Write sgn^0 for the composition of the projection $\mathbb{B}_n \rightarrow S_n$ with the sign character of S_n . Finally, write $\text{sgn}^1 = \varepsilon \cdot \text{sgn}^0$. The multiplicative characters of $\mathbb{B}_n$ are then $1, \varepsilon, \text{sgn}^0$, and sgn^1 .

5.1 Irreducible representations of $\mathbb{B}_n$

Let $\lambda \vdash n$. We consider two extensions of the representation ρ_λ of S_n to $\mathbb{B}_n$, namely

$$\rho_\lambda^0(x; w) = \rho_\lambda(w) \text{ and } \rho_\lambda^1(x; w) = \varepsilon(x)\rho_\lambda(w),$$

for $x \in (\mathbb{Z}/2\mathbb{Z})^n$ and $w \in S_n$. For $(\alpha, \beta) \models n$, define

$$\rho_{\alpha\beta} = \text{Ind}_{\mathbb{B}_a \times \mathbb{B}_b}^{\mathbb{B}_n} \rho_\alpha^0 \boxtimes \rho_\beta^1.$$

Then

$$\text{Irr}(\mathbb{B}_n) = \{\rho_{\alpha\beta} \mid (\alpha, \beta) \models n\}.$$

Put

$$f_{\alpha\beta} = \dim \rho_{\alpha\beta} = \binom{n}{a, b} f_\alpha f_\beta.$$

For a multiplicative character ω of $\mathbb{B}_n$, we define

$$\text{Bip}_\omega(n) = \{(\alpha, \beta) \models n \mid \det \rho_{\alpha\beta} = \omega\},$$

and $N_\omega(n) = |\text{Bip}_\omega(n)|$. Similarly, for $a + b = n$ we put

$$\text{Bip}_\omega(a, b) = \{(\alpha, \beta) \models n \mid \alpha \vdash a, \beta \vdash b, \det \rho_{\alpha\beta} = \omega\},$$

and $N_\omega(a, b) = |\text{Bip}_\omega(a, b)|$.

5.2 The determinant of $\rho_{\alpha\beta}$

Proposition 29.2 in [2] gives the following formula for the determinant of an induced representation:

Proposition 6 *Let G be a finite group, $H \leq G$ a subgroup, and ρ a representation of H . If $\pi = \text{Ind}_H^G \rho$, then*

$$\det \pi = \det(\mathbb{C}[G/H])^{\dim \rho} \otimes (\det \rho \circ \text{ver}_{G/H}).$$

Here $\text{ver}_{G/H} : G_{\text{ab}} \rightarrow H_{\text{ab}}$ is the ‘Verlagerung’ map between the abelianizations of G and H , and is the main part of our calculation.

Recall the definition of $\text{ver}_{G/H}$: Let $t : G/H \rightarrow G$ be a section of the canonical projection. Given $g \in G$, for each $x \in G/H$ we have $gt(x) = t(y)h_{x,g}$ for some $y \in G/H$ and $h_{x,g} \in H$. Note that here $y = {}^g x$, with G/H considered as a G -set. Then

$$\text{ver}_{G/H}(g \bmod D(G)) = \prod_{x \in G/H}^m h_{x,g} \bmod D(H).$$

Let us first treat the case where $G = S_n$ and H is the Young subgroup $S_a \times S_b$, with $a + b = n$. Then $G_{\text{ab}} = S_n/A_n$, where A_n is the alternating group, and we may identify H_{ab} with $S_a/A_a \times S_b/A_b$.

Proposition 7 *Let $a + b = n$. Then the map*

$$\text{ver} = \text{ver}_{S_n/(S_a \times S_b)} : S_n/A_n \rightarrow S_a/A_a \times S_b/A_b$$

is given by

$$\text{ver}(\tau_n) = \left(\tau_a^{\binom{n-2}{a-2}}, \tau_b^{\binom{n-2}{b-2}} \right).$$

Here τ_n is any transposition in S_n , or trivial if $n < 2$.

Proof Let $J_n = \{1, 2, \dots, n\}$, and write ‘ $\wp_a(J_n)$ ’ for the set of subsets of J_n of cardinality a . For instance $J_a \in \wp_a(J_n)$. The map $S_n \rightarrow \wp_a(J_n)$ defined by $g \mapsto {}^g J_a$ descends to an isomorphism of $S_n/(S_a \times S_b)$ with $\wp_a(J_n)$, as S_n -sets. Pick any section $t : \wp_a(J_n) \rightarrow S_n$; thus $t^{(x)} J_a = x$ for all $x \in \wp_a(J_n)$.

For concreteness, let τ_n be the transposition $s_1 = (12)$. Since

$$h_{x,s_1} = t \left({}^{s_1} x \right)^{-1} s_1 t(x),$$

we have $h_{x,s_1} h_{s_1 x, s_1} = 1$. Thus

$$\begin{aligned} \text{ver}(\tau_n) &= \prod_{x \in \wp_a(J_n)} h_{x,s_1} \bmod A_n \\ &= \prod_{x \in \wp_a(J_n) \mid {}^{s_1} x = x} h_{x,s_1} \bmod A_n. \end{aligned}$$

Now ${}^{s_1} x = x$ iff $\{1, 2\}$ is a subset of x , or of the complement of x . In the first case, h_{x,s_1} is a transposition $\tau_a \in S_a$, and in the second case, h_{x,s_1} is a transposition $\tau_b \in S_b$. There are $\binom{n-2}{a-2}$ elements of $\wp_a(J_n)$ containing $\{1, 2\}$, and $\binom{n-2}{b-2}$ elements of $\wp_a(J_n)$ whose complement contains $\{1, 2\}$. The result follows. $\square$

Next we compute the Verlagerung for $G = \mathbb{B}_n$ and H the Young subgroup $\mathbb{B}_a \times \mathbb{B}_b$, with $a + b = n$. The derived quotient $(\mathbb{B}_n)_{\text{ab}} = \mathbb{B}_n/D(\mathbb{B}_n)$ is a Klein 4 group generated by the images of τ_n and e_n , where $e_n \in (\mathbb{Z}/2\mathbb{Z})^n$ is any vector with $\varepsilon(e_n) = -1$. We identify H_{ab} with $(\mathbb{B}_a)_{\text{ab}} \times (\mathbb{B}_b)_{\text{ab}}$.

Proposition 8 *The map*

$$\text{ver} = \text{ver}_{G/H} : (\mathbb{B}_n)_{\text{ab}} \rightarrow (\mathbb{B}_a)_{\text{ab}} \times (\mathbb{B}_b)_{\text{ab}}$$

is given by

$$\text{ver}(\tau_n) = \left(\tau_a^{\binom{n-2}{a-2}}, \tau_b^{\binom{n-2}{b-2}} \right)$$

and

$$\text{ver}(e_n) = \left(e_a^{\binom{n-1}{a-1}}, e_b^{\binom{n-1}{b-1}} \right).$$

Proof We may identify the quotient $\mathbb{B}_n/(\mathbb{B}_a \times \mathbb{B}_b)$ with $S_n/(S_a \times S_b)$, and in particular use a transversal $S_n/(S_a \times S_b) \rightarrow S_n$ to form a transversal $t : \mathbb{B}_n/(\mathbb{B}_a \times \mathbb{B}_b) \rightarrow \mathbb{B}_n$ whose image lies in S_n . The following diagram commutes:

$$\begin{array}{ccc} (S_n)_{\text{ab}} & \xrightarrow{\text{ver}} & (S_a)_{\text{ab}} \times (S_b)_{\text{ab}} \\ \downarrow & & \downarrow \\ (\mathbb{B}_n)_{\text{ab}} & \xrightarrow{\text{ver}} & (\mathbb{B}_a)_{\text{ab}} \times (\mathbb{B}_b)_{\text{ab}} \end{array}$$

Here the vertical maps are the obvious inclusions, and the top horizontal map was computed in the previous proposition. This gives the formula for $\text{ver}(\tau_n)$.

Now, with notation as in the previous proof, observe that

$$t(x)^{-1}x = J_a,$$

so

$$t(x)^{-1}(1) \in J_a \Leftrightarrow 1 \in x.$$

We have

$$h_{x,e_1} = t(x)^{-1}e_1t(x) = e_i,$$

with $i = t(x)^{-1}(1)$. Therefore, modulo $D(H)$, we may write

$$h_{x,e_1} = \begin{cases} e_a, & \text{if } 1 \in x \\ e_b, & \text{if } 1 \notin x. \end{cases}$$

There are $\binom{n-1}{a-1}$ subsets x of J_n with $1 \in x$, and $\binom{n-1}{b-1}$ with $1 \notin x$, thus the product of the h_{x,e_1} is $\left(e_a^{\binom{n-1}{a-1}}, e_b^{\binom{n-1}{b-1}} \right)$, giving the formula for $\text{ver}(e_n)$. $\square$

For i, j we have

$$\det \left(\rho_\alpha^0(\tau_a^i) \boxtimes \rho_\beta^1(\tau_b^j) \right) = (-1)^{g_\alpha f_\beta i + g_\beta f_\alpha j},$$

and so

$$\det \left(\rho_\alpha^0 \boxtimes \rho_\beta^1 \right) (\text{ver}(s_1)) = (-1)^{g_\alpha f_\beta \binom{n-2}{a-2} + g_\beta f_\alpha \binom{n-2}{b-2}}.$$

Similarly,

$$\det(\rho_\alpha^0(e_a^i) \boxtimes \rho_\beta^1(e_b^j)) = (-1)^{f_\alpha f_\beta j},$$

so

$$\det(\rho_\alpha^0 \boxtimes \rho_\beta^1)(\text{ver}(e_n)) = (-1)^{f_\alpha f_\beta \binom{n-1}{b-1}}.$$

The permutation module $\mathbb{C}[\wp_a(J_n)]$ coming from the action of $\mathbb{B}_n$ on $\wp_a(J_n)$ factors through the action of S_n . Thus e_n acts trivially, and s_1 acts by permuting these sets. The number of doubleton orbits of s_1 on this set is equal to the number of subsets of $\wp_a(J_n)$ which contain 1 but not 2, thus equals $\binom{n-2}{a-1}$. This gives

$$\det(\mathbb{C}[G/H])(e_n) = 1 \text{ and } \det(\mathbb{C}[G/H])(s_1) = (-1)^{\binom{n-2}{a-1}},$$

so that

$$\det(\mathbb{C}[G/H]) = (\text{sgn}^0)^{\binom{n-2}{a-1, b-1}}.$$

Let

$$x_{\alpha\beta} = f_\alpha f_\beta \binom{n-1}{a, b-1} \in \mathbb{Z}/2\mathbb{Z}$$

and

$$y_{\alpha\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} + f_\beta g_\alpha \binom{n-2}{a-2, b} + f_\alpha g_\beta \binom{n-2}{a, b-2} \in \mathbb{Z}/2\mathbb{Z}.$$

From the above, we deduce:

Theorem 4 For a bipartition (α, β) , we have

$$\det \rho_{\alpha\beta} = \varepsilon^{x_{\alpha\beta}} \cdot (\text{sgn}^0)^{y_{\alpha\beta}}.$$

Proof The preceding shows that

$$\det(\rho_{\alpha\beta}(e_n)) = (-1)^{x_{\alpha\beta}} \text{ and } \det(\rho_{\alpha\beta}(s_1)) = (-1)^{y_{\alpha\beta}},$$

and this gives the theorem. $\square$

An alternate proof of Theorem 4 can be given through Proposition 2, via the Frobenius character formula to compute $\chi_{\alpha\beta}(s_1)$ and $\chi_{\alpha\beta}(e_1)$. (Here $\chi_{\alpha\beta}$ denotes the character of $\rho_{\alpha\beta}$.) In fact,

$$\chi_{\alpha\beta}(s_1) = \binom{n-2}{a-2, b} f_\beta \chi_\alpha(s_1) + \binom{n-2}{a, b-2} f_\alpha \chi_\beta(s_1) \quad (7)$$

and

$$\chi_{\alpha\beta}(e_1) = f_\alpha f_\beta \left(2 \binom{n-1}{a-1, b} - \binom{n}{a, b} \right).$$

Details are left to the reader.

Computing the determinant of $\rho_{\alpha\beta}$ thus reduces to determining the parities of $x_{\alpha\beta}$ and $y_{\alpha\beta}$.

Theorem 5 *For $(\alpha, \beta) \models n$, the quantity $x_{\alpha\beta}$ is odd iff $\lambda = \phi(\beta, \alpha)$ is chiral. In other words, $g_{\phi(\beta, \alpha)} \equiv x_{\alpha\beta} \pmod{2}$.*

Proof This is immediate from Corollary 4. $\square$

Corollary 5 *For all n ,*

$$N_{\varepsilon}(n) + N_{\text{sgn}^1}(n) = B(2n).$$

An important case is when $\alpha = \beta$.

Proposition 9 *Let $a \geq 2$ and $\alpha \vdash a$. Then*

$$\det \rho_{\alpha\alpha} = \begin{cases} \varepsilon, & \text{if } a \text{ is a power of } 2 \text{ and } \alpha \text{ is a hook} \\ 1, & \text{otherwise.} \end{cases}$$

Proof In this case

$$x_{\alpha\alpha} \equiv f_{\alpha} \binom{2a-1}{a, a-1}, \text{ and } y_{\alpha\alpha} \equiv 0,$$

since $\binom{2a-2}{a-1, a-1}$ is always even. Now $\text{bin}(a) \cap \text{bin}(a-1) = \emptyset$ iff a is a power of 2, and in this case the odd partitions are precisely the hooks. This gives the proposition. $\square$

5.3 Values of $N_{\omega}(n)$ for small n

Using Theorem 4, one can compute values of $N_{\omega}(n)$ for small values of n . We offer below both a table of values, and a logplot, base 2, of each $N_{\omega}(n)$ for $2 \leq n \leq 65$. The horizontal axis is n . The orange line is $N_1(n)$, the green line is $N_{\text{sgn}^0}(n)$, the red line is $N_{\text{sgn}^1}(n)$, and the blue line is $N_{\varepsilon}(n)$.

n	$N_1(n)$	$N_{\text{sgn}^0}(n)$	$N_{\text{sgn}^1}(n)$	$N_\varepsilon(n)$
1	1	0	0	1
2	1	1	2	1
3	2	4	2	2
4	4	4	8	4
5	8	20	4	4
6	33	8	16	8
7	46	32	16	16
8	69	28	60	28
9	116	168	8	8
10	417	16	32	16
11	624	64	32	32
12	909	64	128	64
13	1322	320	64	64
14	2153	128	256	128
15	2932	512	256	256
16	4038	424	936	424

Note the compatibility with Theorem 1.

5.4 An involution of $\text{Bip}(n)$

Lemma 3 *Mod 2, we have $g_{\lambda'} = g_\lambda + f_\lambda$, and $f_{\phi(\alpha, \beta)} = f_{\alpha\beta}$.*

Proof For the first part, $\chi_{\lambda'}(s_1) = -\chi_\lambda(s_1)$, so $g_\lambda + g_{\lambda'} = f_\lambda$. The second part is immediate. $\square$

Given $(\alpha, \beta) \models n$, put $\sigma(\alpha, \beta) = (\alpha', \beta')$.

Proposition 10 *For $(\alpha, \beta) \models n$, we have*

- (1) $x_{\sigma(\alpha, \beta)} = x_{\alpha\beta}$,
- (2) $y_{\sigma(\alpha, \beta)} = y_{\alpha\beta} + f_{\alpha\beta}$.

Corollary 6 *Let $n \geq 3$ be odd. If $a + b = n$, then σ restricts to a bijection from $\text{Bip}_{\text{sgn}^1}(a, b)$ to $\text{Bip}_\varepsilon(b, a)$. Thus $N_{\text{sgn}^1}(b, a) = N_\varepsilon(a, b)$ and $N_{\text{sgn}^1}(n) = N_\varepsilon(n)$.*

Proof We must show that if $x_{\alpha\beta} = y_{\alpha\beta} = 1$, then $x_{\sigma(\alpha,\beta)} = 1$ and $y_{\sigma(\alpha,\beta)} = 0$. Now, if n odd and $\binom{n-1}{a,b-1}$ is odd, it is elementary to see that b is odd and $\binom{n}{a,b}$ is odd. By the previous proposition, we deduce that $y_{\sigma(\alpha,\beta)} = 0$. $\square$

Corollary 7 Let $n \geq 3$ be odd. Then $N_{\text{sgn}^1}(n) = N_\varepsilon(n) = \frac{1}{4}A(2n)$.

Proof By Corollaries 5 and 6, we have $N_{\text{sgn}^1}(n) = N_\varepsilon(n) = \frac{1}{2}B(2n)$. But by Corollary 3, $B(2n) = \frac{1}{2}A(2n)$. $\square$

5.5 Counting with fixed a, b .

Given $\omega \in \{1, \varepsilon, \text{sgn}^0, \text{sgn}^1\}$, and $a + b = n$, put

$$N_\omega(a, b) = \#\{\alpha \vdash a, \beta \vdash b \mid \det \rho_{\alpha\beta} = \omega\}.$$

We compute $N_\varepsilon(a, b)$ in Proposition 11, $N_{\text{sgn}^1}(a, b)$ in Proposition 12, and $N_{\text{sgn}^0}(a, b)$ in Proposition 13. Of course,

$$p(a)p(b) = N_1(a, b) + N_{\text{sgn}^0}(a, b) + N_{\text{sgn}^1}(a, b) + N_\varepsilon(a, b),$$

so one implicitly has an expression for $N_1(a, b)$ as well.

Recall that

$$x_{\alpha\beta} = f_\alpha f_\beta \binom{n-1}{a, b-1},$$

and

$$y_{\alpha\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} + f_\beta g_\alpha \binom{n-2}{a-2, b} + f_\alpha g_\beta \binom{n-2}{a, b-2}.$$

Lemma 4 Let $a + b = n$. The following conditions are equivalent:

- (1) $\binom{n-2}{a-1, b-1}$ is odd, $\binom{n-2}{a, b-2}$ is even, and $\binom{n-2}{a-2, b}$ is even.
- (2) n is even and $\binom{n-2}{a-1, b-1}$ is odd.

The proof is left to the reader. $\square$

Proposition 11 Let $a + b = n$. If n is even, then

$$N_\varepsilon(a, b) = \begin{cases} \frac{1}{2}A(a)A(b) & \text{if } \binom{n-1}{a, b-1} \text{ is odd and } \binom{n-2}{a-1, b-1} \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd, then

$$N_\varepsilon(a, b) = \begin{cases} \frac{1}{2}A(a)A(b) & \text{if } \binom{n-1}{a, b-1} \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof We must count $\alpha \vdash a, \beta \vdash b$ so that $x_{\alpha\beta} \equiv 1$ and $y_{\alpha\beta} \equiv 0$. For $x_{\alpha\beta}$ to be odd, we need f_α, f_β and $\binom{n-1}{a, b-1}$ to be odd.

In this case

$$y_{\alpha, \beta} = \binom{n-2}{a-1, b-1} + g_\alpha \binom{n-2}{a-2, b} + g_\beta \binom{n-2}{a, b-2}.$$

It is now clear that $\text{Bip}_\varepsilon(a, b) = \emptyset$ if either $\binom{n-1}{a, b-1}$ is even, or if n is even and $\binom{n-2}{a-1, b-1}$ is odd (using Lemma 4).

So henceforth assume that $\binom{n-1}{a, b-1}$ is odd, and that the conditions of Lemma 4 do not hold. Since

$$\binom{n-1}{a, b-1} = \binom{n-2}{a-1, b-1} + \binom{n-2}{a, b-2},$$

we consider two cases.

Case 1 $\binom{n-2}{a-1, b-1}$ is odd and $\binom{n-2}{a, b-2}$ is even. As we assume the conditions of Lemma 4 do not hold, further $\binom{n-2}{a-2, b}$ is odd. Thus

$$y_{\alpha\beta} \equiv 1 + g_\alpha \pmod{2}.$$

So for such a, b we have

$$\text{Bip}_\varepsilon(a, b) = \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is odd, } f_\beta \text{ is odd}\},$$

and by Corollary 2, we compute

$$N_\varepsilon(a, b) = \frac{1}{2} A(a)A(b).$$

Case 2 If $\binom{n-2}{a-1, b-1}$ is even and $\binom{n-2}{a, b-2}$ is odd, then

$$y_{\alpha\beta} \equiv g_\alpha \binom{n-2}{a-2, b} + g_\beta.$$

For $y_{\alpha\beta}$ to be even, we need $g_\alpha \binom{n-2}{a-2, b}$ and g_β to be both odd or both even. We again have two possibilities.

Case 2a If $\binom{n-2}{a-2, b}$ is even, then

$$\text{Bip}_\varepsilon(a, b) = \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } f_\beta \text{ is odd, } g_\beta \text{ is odd}\},$$

so $N_\varepsilon(a, b) = \frac{1}{2} A(a)A(b)$, as claimed.

Case 2b If $\binom{n-2}{a-2, b}$ is odd, then

$$\begin{aligned} \text{Bip}_\varepsilon(a, b) = & \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is even, } f_\beta \text{ is odd, } g_\beta \text{ is even}\} \\ & \cup \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is odd, } f_\beta \text{ is odd, } g_\beta \text{ is odd}\}. \end{aligned}$$

Thus

$$\begin{aligned} N_{\varepsilon}(a, b) &= A(a)A(b) - A(a)\frac{1}{2}A(b) - \frac{1}{2}A(a)A(b) + 2 \cdot \frac{1}{2}A(a)\frac{1}{2}A(b) \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

□

Proposition 12 For given a and b , we have

$$N_{\text{sgn}^1}(a, b) = \begin{cases} 0 & \text{if } \binom{n-1}{a, b-1} \text{ is even,} \\ A(a)A(b) & \text{if } n \text{ is even and } \binom{n-2}{a-1, b-1} \text{ is odd,} \\ \frac{1}{2}A(a)A(b) & \text{otherwise.} \end{cases} \quad (8)$$

Proof For $x_{\alpha\beta}$ to be odd, we need f_{α} , f_{β} and $\binom{n-1}{a, b-1}$ to be odd. Again there are two cases to consider.

Case 1 If $\binom{n-2}{a-1, b-1}$ is odd and $\binom{n-2}{a, b-2}$ is even, then

$$y_{\alpha\beta} \equiv 1 + g_{\alpha} \binom{n-2}{a-2, b} \pmod{2}.$$

For $y_{\alpha\beta}$ to be odd, we need $g_{\alpha} \binom{n-2}{a-2, b}$ to be even.

Case 1a If $\binom{n-2}{a-2, b}$ is even, then $N_{\text{sgn}^1}(a, b) = A(a)A(b)$.

Case 1b If $\binom{n-2}{a-2, b}$ is odd, then

$$\begin{aligned} N_{\text{sgn}^1}(a, b) &= \#\{f_{\alpha} \text{ is odd, } g_{\alpha} \text{ is even}\} \#\{f_{\beta} \text{ is odd}\} \\ &= [\#\{f_{\alpha} \text{ is odd}\} - \#\{f_{\alpha} \text{ is odd, } g_{\alpha} \text{ is odd}\}] \#\{f_{\beta} \text{ is odd}\} \\ &= \frac{1}{2}A(a)A(b). \end{aligned}$$

Case 2 If $\binom{n-2}{a-1, b-1}$ is even and $\binom{n-2}{a, b-2}$ is odd, then

$$y_{\alpha\beta} \equiv g_{\alpha} \binom{n-2}{a-2, b} + g_{\beta} \pmod{2}.$$

For $y_{\alpha\beta}$ to be odd, we have two cases:

Case 2a If $g_{\alpha} \binom{n-2}{a-2, b}$ is even and g_{β} is odd, then

$$y_{\alpha\beta} \equiv g_{\beta} \pmod{2}.$$

Thus, we have

$$\begin{aligned} N_{\text{sgn}^1}(a, b) &= \#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\} \#\{f_\alpha \text{ is odd}\} \\ &= \frac{1}{2} A(b) A(a). \end{aligned}$$

Case 2b If $g_\alpha \binom{n-2}{a-2, b}$ is odd and g_β is even, then

$$y_{\alpha\beta} \equiv g_\beta + g_\alpha \pmod{2}.$$

Thus, we have

$$\begin{aligned} N_{\text{sgn}^1}(a, b) &= \#\{f_\beta \text{ is odd}, g_\beta \text{ is odd}\} \#\{f_\alpha \text{ is odd}, g_\alpha \text{ is even}\} \\ &\quad + \#\{f_\beta \text{ is odd}, g_\beta \text{ is even}\} \#\{f_\alpha \text{ is odd}, g_\alpha \text{ is odd}\} \\ &= \frac{1}{2} A(b) A(a). \end{aligned}$$

□

Our formulas for $N_{\text{sgn}^0}(a, b)$, where $a + b = n$, depend on the parities of the binomial coefficients $\binom{n-2}{a-2, b}$, $\binom{n-2}{a-1, b-1}$, and $\binom{n-2}{a, b-2}$.

Proposition 13 *The following table computes $N_{\text{sgn}^0}(a, b)$.*

$\binom{n-2}{a-2, b}$	$\binom{n-2}{a-1, b-1}$	$\binom{n-2}{a, b-2}$	$N_{\text{sgn}^0}(a, b)$
0	0	0	0
0	1	0	0
1	0	0	$A(b)B(a)$
0	1	1	$A(a)B(b)$
1	1	1	$A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b)$
1	1	0	$A(b)B(a) - \frac{1}{2}A(a)A(b)$
0	0	1	$A(a)B(b) - \frac{1}{2}A(a)A(b)$
1	0	1	$A(a)B(b) + A(b)B(a) - A(a)A(b)$

In the table, we read the binomial coefficients mod 2

Proof Each of these eight cases we will refer to by the binary ‘code’ coming from the parity of the binomial coefficients. For instance, the first two cases correspond to the codes ‘000’ and ‘010.’ Of course, each code further determines the parity of the binomial coefficient $\binom{n-1}{a, b-1}$.

For the first two cases, suppose that $\binom{n-2}{a-2, b}$ and $\binom{n-2}{a, b-2}$ are even, but $x_{\alpha\beta} = 0$ and

$$y_{\alpha\beta} = f_\alpha f_\beta \binom{n-2}{a-1, b-1} = 1.$$

Then $f_\alpha f_\beta = 1$, so it must be that $\binom{n-1}{a, b-1} = 0$. But then necessarily $\binom{n-2}{a-1, b-1} = 0$, a contradiction. Thus $\text{Bip}_{\text{sgn}^0}(a, b) = \emptyset$ in the ‘000’ and ‘010’ cases.

For the ‘100’ case, we automatically have $x_{\alpha\beta} = 0$ and $y_{\alpha\beta} = f_\beta g_\alpha$. Thus $(\alpha, \beta) \in \text{Bip}_{\text{sgn}^0}(a, b)$ iff $f_\beta = g_\alpha = 1$. So $N_{\text{sgn}^0}(a, b) = A(b)B(a)$ here.

For the ‘011’ case, again $x_{\alpha\beta} = 0$ but

$$y_{\alpha\beta} = f_\alpha(f_\beta + g_\alpha),$$

which is odd iff f_α is odd and $f_\beta \neq g_\alpha$. By Corollary 2, this gives $N_{\text{sgn}^0}(a, b) = A(a)B(b)$.

Now consider the ‘111’ case, again with $x_{\alpha\beta} = 0$, but

$$y_{\alpha\beta} = f_\alpha f_\beta + f_\beta g_\alpha + f_\alpha g_\beta$$

must be odd. Note that f_α or g_α must be odd; we consider the three possibilities. For $f_\alpha = g_\alpha = 1$, then $y_{\alpha\beta} = g_\beta$; this gives $\frac{1}{2}A(a)B(b)$. For $f_\alpha = 0$, $g_\alpha = 1$, then $y_{\alpha\beta} = f_\beta$; this gives $(B(a) - \frac{1}{2}A(a))A(b)$. Next $f_\alpha = 1$, $g_\alpha = 0$ gives $y_{\alpha\beta} = f_\beta + g_\beta$; this gives $\frac{1}{2}A(a)B(b)$. The sum of these three quantities is equal to $A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b)$.

For the ‘110’ case, we must have $f_\alpha f_\beta = 0$ and so

$$y_{\alpha\beta} = f_\beta g_\alpha.$$

So $(\alpha, \beta) \in \text{Bip}_{\text{sgn}^0}$ iff $f_\alpha = 0$ and $f_\beta = g_\alpha = 1$. This gives

$$\left(B(a) - \frac{1}{2}A(a)\right)A(b)$$

possibilities.

For both ‘001,’ we must have $f_\beta = 0$, and $f_\alpha = g_\beta = 1$, giving

$$A(a)\left(B(b) - \frac{1}{2}A(b)\right).$$

For ‘101,’ to solve

$$y_{\alpha\beta} = f_\beta g_\alpha + f_\alpha g_\beta = 1$$

with $f_\alpha f_\beta = 0$, there are two possibilities. Either $f_\alpha = 0$, $f_\beta = g_\alpha = 1$, or $f_\beta = 0$, $f_\alpha = g_\beta = 1$. The first possibility gives $(B(a) - \frac{1}{2}A(a))A(b)$, and the second gives $(B(b) - \frac{1}{2}A(b))A(a)$. Adding gives the result. $\square$

5.6 Final count

We compute $N_\varepsilon(n)$ in Theorem 6, $N_{\text{sgn}^0}(n)$ in Theorem 7, and $N_{\text{sgn}^1}(n)$ in Theorem 8. The quantity $N_1(n)$ can be computed in principle from the formula

$$p_2(n) = N_1(n) + N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n) + N_\varepsilon(n),$$

where $p_2(n)$ denotes the number of bipartitions of n .

Theorem 6 For $n \geq 2$, and $k = \text{ord}(n)$, we have

$$N_\varepsilon(n) = \begin{cases} \frac{1}{4}A(2n) & \text{if } n \text{ is odd,} \\ \frac{1}{8}A(2n) \left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}}\right) & \text{if } n \text{ is even.} \end{cases}$$

Note that the sum is 0 when $k = 1$, and similarly for other sums in this paper.

Proof If n is odd, this follows from Corollary 7, so let n be even.

By Proposition 11, we have

$$N_\varepsilon(n) = \sum_{\substack{a+(b-1) \doteq n-1 \\ \text{sum}(a-1, b-1) \text{ messy}}} \frac{1}{2}A(a)A(b).$$

If a and b are odd, then $\text{sum}(a-1, b-1)$ is neat iff $\text{sum}(a, b-1)$ is neat, so $N_\varepsilon(a, b) = 0$. If a and b are even, then $\text{sum}(a-1, b-1)$ is automatically messy. Hence,

$$\begin{aligned} N_\varepsilon(n) &= \sum_{\substack{a+(b-1) \doteq n-1 \\ a, b \text{ even}}} \frac{1}{2}A(a)A(b) \\ &= \sum_{\substack{a+(b-1) \doteq n-1 \\ a, b \text{ even}}} \frac{1}{2}A(a)A(b-1)2^{\text{ord}(b) - \binom{\text{ord}(b)}{2}} \\ &= \frac{1}{2}A(n-1) \sum_{\substack{a+(b-1) \doteq n-1 \\ a, b \text{ even}}} 2^{\text{ord}(b) - \binom{\text{ord}(b)}{2}}. \end{aligned} \quad (9)$$

Now $\text{sum}(a, b-1)$ is neat iff $\text{ord}(a) \geq \text{ord}(b)$ and $\text{sum}(a, b-2^{\text{ord}(b)})$ is neat. There are two cases to consider. First if $\text{ord}(b) = k$, then we must also have $k < \text{ord}(a)$. The sum of these terms is

$$\frac{1}{2}A(n-1)2^{v(n-2^k)+k-\binom{k}{2}}.$$

The other case has $\text{ord}(a) = \text{ord}(b) < k$, and the sum over these a, b is

$$\frac{1}{2}A(n-1) \sum_{j=1}^{k-1} 2^{v(n-2^{j+1})+j-\binom{j}{2}}.$$

Further simplification comes by using (2):

$$N_{\varepsilon}(n) = \left(\frac{1}{2} A(n-1) 2^{v(n)+k-2} \sum_{j=1}^{k-1} 2^{-\binom{j}{2}} \right) + \frac{1}{2} A(n-1) 2^{v(n)+k-1-\binom{k}{2}}.$$

Now

$$A(n-1) = A(n) 2^{-k+\binom{k}{2}},$$

so we may rewrite this as

$$\begin{aligned} N_{\varepsilon}(n) &= \left(\frac{1}{2} A(n) 2^{v(n)-2} \sum_{j=1}^{k-1} 2^{\binom{k}{2}-\binom{j}{2}} \right) + \frac{1}{2} A(n) 2^{v(n)-1} \\ &= \left(\frac{1}{8} A(2n) \sum_{j=1}^{k-1} 2^{\binom{k}{2}-\binom{j}{2}} \right) + \frac{1}{4} A(2n). \end{aligned}$$

If $k = 1$, then the sum is empty, giving $N_{\varepsilon}(n) = \frac{1}{4} A(2n)$. $\square$

Theorem 7 *The following computes $N_{\text{sgn}^0}(n)$ in all cases. Let $k = \text{ord}(n')$.*

(1) *If $n \equiv 1 \pmod{4}$, then*

$$N_{\text{sgn}^0}(n) = \frac{1}{4} A(2n) \left(1 + 3 \cdot 2^{\binom{k}{2}-1} + 2^{\binom{k}{2}-k+1} + \sum_{j=2}^{k-1} \left(2^{\binom{k}{2}-\binom{j}{2}} + 2^{\binom{k}{2}-j} \right) \right).$$

(2) *If $n \equiv 3 \pmod{4}$, then $N_{\text{sgn}^0}(n) = \frac{1}{2} A(2n)$.*

(3) *If n is even, then*

$$N_{\text{sgn}^0}(n) = \frac{1}{8} A(2n) \left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2}-\binom{j}{2}} \right).$$

Proof **Case** $n \equiv 2 \pmod{4}$: The codes ‘011,’ ‘111,’ ‘101,’ and ‘110’ cannot occur in the $(n-2)$ nd row of Pascal’s triangle mod 2. Moreover, for a, b corresponding to the codes ‘000’ and ‘010’ we have $N_{\text{sgn}^0}(a, b) = 0$. So we are left with the codes ‘100’ and ‘001.’

Let us write

$$N_{\text{sgn}^0}(n)^{100} = \sum_{\substack{\text{sum}(a-2, b) \text{ neat} \\ \text{sum}(a-1, b-1) \text{ messy} \\ \text{sum}(a, b-2) \text{ messy}}} N_{\text{sgn}^0}(a, b),$$

and similarly for other codes. From Proposition 13, $N_{\text{sgn}^0}(a, b) = A(b)B(a)$. In this case, since $n-2 \equiv 0 \pmod{4}$, the first neatness condition on $a-2, b$ implies

the other two. Moreover, it implies that $a \equiv 2 \pmod{4}$, so that $B(a) = \frac{1}{2}A(a)$ and $A(a-2) = \frac{1}{2}A(a)$. It follows that

$$\begin{aligned} N_{\text{sgn}^0}(n)^{100} &= \sum_{(a-2)+b \doteq n-2} \frac{1}{2}A(a)A(b) \\ &= \sum_{(a-2)+b \doteq n-2} A(a-2)A(b) \\ &= 2^{v(n-2)}A(n-2) \\ &= \frac{1}{4}A(2n). \end{aligned}$$

Next, the condition that $\text{sum}(a, b-2)$ is neat implies that $b \equiv 2 \pmod{4}$, so $B(b) = \frac{1}{2}A(b)$. Thus

$$\begin{aligned} N_{\text{sgn}^0}(n)^{001} &= \sum_{a+(b-2) \doteq n-2} A(a)B(b) - \frac{1}{2}A(a)A(b) \\ &= 0. \end{aligned}$$

We conclude that $N_{\text{sgn}^0}(n) = \frac{1}{4}A(2n)$.

Case $n \equiv 3 \pmod{4}$: This time, the codes ‘111’ and ‘101’ do not occur in the $(n-2)$ nd row of Pascal’s triangle mod 2. It follows that

$$N_{\text{sgn}^0}(n) = N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{011} + N_{\text{sgn}^0}(n)^{110} + N_{\text{sgn}^0}(n)^{001}.$$

Now

$$N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{001} = \sum_{\substack{\text{sum}(a-2,b) \text{ neat} \\ \text{sum}(a-1,b-1) \text{ messy} \\ \text{sum}(a,b-2) \text{ messy}}} 2A(b)B(a) - \frac{1}{2}A(a)A(b), \quad (10)$$

and

$$N_{\text{sgn}^0}(n)^{011} + N_{\text{sgn}^0}(n)^{110} = \sum_{\substack{\text{sum}(a-2,b) \text{ messy} \\ \text{sum}(a-1,b-1) \text{ neat} \\ \text{sum}(a,b-2) \text{ neat}}} 2A(a)B(b) - \frac{1}{2}A(a)A(b). \quad (11)$$

Next, the conditions on a and b for the sum in (10) are equivalent to $(a-2)+b \doteq n-2$, $a \equiv 3 \pmod{4}$, and $b \equiv 0 \pmod{4}$. But then $B(a) = A(a)$, and $A(a) = 2A(a-2)$. So (10) becomes

$$3 \cdot 2^{v(n)-3}A(n).$$

Meanwhile, the conditions in (11) are equivalent to $(a-1)+(b-1) \doteq n-2$, $a \equiv 1 \pmod{4}$, and $b \equiv 2 \pmod{4}$. Now $B(b) = \frac{1}{2}A(b)$ and $A(b) = 2A(b-2)$. So

(11) becomes

$$2^{v(n)-3} A(n).$$

The result follows, since $A(n)2^{v(n)} = A(2n)$.

Case $n \equiv 1 \pmod{4}$: This time we have

$$N_{\text{sgn}^0}(n) = N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{011} + N_{\text{sgn}^0}(n)^{111} + N_{\text{sgn}^0}(n)^{110} + N_{\text{sgn}^0}(n)^{001}.$$

Again, Eqs. (10) and (11) hold, but we must add

$$N_{\text{sgn}^0}(n)^{111} = \sum_{\substack{\text{sum}(a-2,b) \text{ neat} \\ \text{sum}(a-1,b-1) \text{ neat} \\ \text{sum}(a,b-2) \text{ neat}}} A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b).$$

Now

$$N_{\text{sgn}^0}(n)^{111} = N_{\text{sgn}^0}(n)_0^{111} + N_{\text{sgn}^0}(n)_1^{111} + N_{\text{sgn}^0}(n)_2^{111} + N_{\text{sgn}^0}(n)_3^{111},$$

where $N_{\text{sgn}^0}(n)_i^{111}$ takes the sum over $a \equiv i \pmod{4}$. But since $N_{\text{sgn}^0}(a, b) = N_{\text{sgn}^0}(b, a)$ for such a, b , we also have $N_{\text{sgn}^0}(n)_3^{111} = N_{\text{sgn}^0}(n)_2^{111}$ and $N_{\text{sgn}^0}(n)_0^{111} = N_{\text{sgn}^0}(n)_1^{111}$.

We have

$$\begin{aligned} N_{\text{sgn}^0}(n)_2^{111} &= \sum_{\substack{\text{sum}(a-2,b) \text{ neat} \\ a \equiv 2 \pmod{4}}} A(a)A(b) \\ &= 2A(n-2)2^{v(n-5)} \\ &= 2^{\binom{k}{2}-3} A(2n), \end{aligned}$$

since $B(a) = \frac{1}{2}A(a)$, $B(b) = A(b)$, and $A(a) = 2A(a-2)$.

On the other hand, putting $k = \text{ord}(n')$,

$$N_{\text{sgn}^0}(n)_0^{111} = \sum_{j=2}^{k-1} \sum_{\substack{(a-2)+b \equiv n \\ \text{ord}(a)=j}} A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b).$$

Note that here, $\text{ord}(a) = \text{ord}(b-1) = j$. So $N_{\text{sgn}^0}(n)_0^{111}$ equals

$$\sum_{j=2}^{k-1} \sum_{\substack{(a-2)+b \equiv n \\ \text{ord}(a)=j}} A(a)A(b)2^{\binom{j}{2}-j} + A(a)D(b) + \frac{1}{2}A(a)A(b) + A(b)D(a).$$

Now $a + b \stackrel{D}{=} n$ with $\text{ord}(a) = \text{ord}(b) = j$, so by Proposition 5, $A(a)D(b) = A(b)D(a) = D_{<j}(n)$, so this gives

$$\sum_{j=2}^{k-1} \sum_{\substack{(a-2)+b \equiv n-2 \\ \text{ord}(a)=j}} A(a)A(b) \left(\frac{1}{2} + 2^{\binom{j}{2}-j} \right) + 2D_{<j}(n).$$

Since $A(a-2) = 2^{-j+\binom{j}{2}}A(a)$, we get

$$\sum_{j=2}^{k-1} \sum_{\substack{(a-2)+b \equiv n-2 \\ \text{ord}(a)=j}} A(n-2)2^{j-\binom{j}{2}} \left(\frac{1}{2} + 2^{\binom{j}{2}-j} \right) + 2D_{<j}(n),$$

which equals

$$\sum_{j=2}^{k-1} 2^{v(n)+k-j-3} \left[A(n-2)2^{j-\binom{j}{2}} \left(\frac{1}{2} + 2^{\binom{j}{2}-j} \right) + 2D_{<j}(n) \right].$$

and then

$$A(n)2^{-k+\binom{k}{2}} \sum_{j=2}^{k-1} 2^{v(n)+k-j-3} \left[2^{j-\binom{j}{2}} \left(\frac{1}{2} + 2^{\binom{j}{2}-j} \right) \right] + \sum_{j=2}^{k-1} 2^{v(n)+k-j-2} D_{<j}(n).$$

The first sum here is

$$A(2n)2^{\binom{k}{2}-3} \sum_{j=2}^{k-1} \left(2^{-\binom{j}{2}-1} + 2^{-j} \right),$$

which equals 0 if $k < 3$.

The second sum is

$$\begin{aligned} 2^{v(n)-2} \sum_{j=2}^{k-1} 2^{k-j} D_{<j}(n) &= 2^{v(n)-1} \sum_{i=1}^{k-1} (2^{k-i-1} - 1) D_i(n) \\ &= \frac{1}{16} A(2n) \left(\sum_{j=1}^{k-1} 2^{\binom{k}{2}-\binom{j}{2}} \right) - 2^{v(n)-1} D(n). \end{aligned}$$

Therefore, if $k \geq 3$ we have

$$N_{\text{sgn}^0}(n)_0^{111} = A(2n)2^{\binom{k}{2}} \sum_{j=2}^{k-1} \left(2^{-\binom{j}{2}-4} + 2^{-j-3} \right)$$

$$+ \frac{1}{16} A(2n) \left(\sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} \right) - 2^{v(n)-1} D(n);$$

note that it is 0 if $k = 2$.

From before,

$$N_{\text{sgn}^0}(n)^{111} = 2N_{\text{sgn}^0}(n)_0^{111} + 2N_{\text{sgn}^0}(n)_2^{111}.$$

If $k = 2$, then

$$N_{\text{sgn}^0}(n)^{111} = \frac{1}{2} A(2n).$$

If $k \geq 3$, then $N_{\text{sgn}^0}(n)^{111}$ equals

$$\begin{aligned} & 2^{\binom{k}{2}-2} A(2n) + A(2n) 2^{\binom{k}{2}} \sum_{j=2}^{k-1} \left(2^{-\binom{j}{2}-3} + 2^{-j-2} \right) \\ & + \frac{1}{8} A(2n) \left(\sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} \right) - 2^{v(n)} D(n). \end{aligned}$$

Meanwhile,

$$N_{\text{sgn}^0}(n)^{011+110} = \sum_{\substack{\text{sum}(a-2,b) \text{ messy} \\ \text{sum}(a-1,b-1) \text{ neat} \\ \text{sum}(a,b-2) \text{ neat}}} 2A(a)B(b) - \frac{1}{2} A(a)A(b).$$

The conditions on a, b imply that $\text{sum}(a, b)$ is neat, $a \equiv 1 \pmod{4}$, $b \equiv 0 \pmod{4}$, and $k \in \text{bin}(b)$. So

$$\begin{aligned} 2A(a)B(b) - \frac{1}{2} A(a)A(b) &= 2A(a)(D(b) + \frac{1}{2} A(b)) - \frac{1}{2} A(a)A(b) \\ &= 2D(n) + \frac{1}{2} A(n). \end{aligned}$$

The number of a, b satisfying this condition is $2^{v(n)-2}$, so we have

$$N_{\text{sgn}^0}(n)^{011+110} = 2^{v(n)-2} \left(2D(n) + \frac{1}{2} A(n) \right).$$

Next,

$$N_{\text{sgn}^0}(n)^{100+001} = \sum_{\substack{\text{sum}(a-2,b) \text{ neat} \\ \text{sum}(a-1,b-1) \text{ messy} \\ \text{sum}(a,b-2) \text{ messy}}} 2A(b)B(a) - \frac{1}{2} A(a)A(b).$$

The conditions on a, b again imply that $\text{sum}(a, b)$ is neat, $a \equiv 1 \pmod{4}$, $b \equiv 0 \pmod{4}$, and $k \in \text{bin}(a)$. So

$$2A(b)B(a) - \frac{1}{2}A(a)A(b) = \frac{1}{2}A(n) + 2^{\binom{k}{2}-k+1}A(n) + 2D(n).$$

The number of a, b satisfying this condition is again $2^{v(n)-2}$, so we have

$$N_{\text{sgn}^0}(n)^{100+001} = 2^{v(n)-2} \left(\frac{1}{2}A(n) + 2^{\binom{k}{2}-k+1}A(n) + 2D(n) \right).$$

Thus

$$N_{\text{sgn}^0}(n)^{011+110+100+001} = 2^{v(n)-2} \left(A(n) + 2^{\binom{k}{2}-k+1}A(n) + 4D(n) \right)$$

We obtain $N_{\text{sgn}^0}(n) = \frac{5}{4}A(2n)$, if $k = 2$. For $k \geq 3$ we finally obtain

$$N_{\text{sgn}^0}(n) = A(2n) \left(\frac{1}{4} + 3 \cdot 2^{\binom{k}{2}-3} + 2^{\binom{k}{2}-k-1} + \sum_{j=2}^{k-1} \left(2^{\binom{k}{2}-\binom{j}{2}-2} + 2^{\binom{k}{2}-j-2} \right) \right).$$

Case $n \equiv 0 \pmod{4}$: Now

$$N_{\text{sgn}^0}(n) = N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{001} + N_{\text{sgn}^0}(n)^{101}.$$

We have

$$\begin{aligned} N_{\text{sgn}^0}(n)^{100} &= \sum_{\substack{(a-2)+b \dot{=} n-2 \\ \text{sum}(a, b-2) \text{ messy}}} A(b)B(a), \\ N_{\text{sgn}^0}(n)^{001} &= \sum_{\substack{a+(b-2) \dot{=} n-2 \\ \text{sum}(a-2, b) \text{ messy}}} A(a)D(b). \end{aligned}$$

Since $B(n) = \frac{1}{2}A(n) + D(n)$ for n even, we may write

$$N_{\text{sgn}^0}(n)^{101} = \sum_{\substack{(a-2)+b \dot{=} n-2 \\ a+(b-2) \dot{=} n-2}} A(a)D(b) + A(b)D(a).$$

So

$$N_{\text{sgn}^0}(n)^{100} = \sum_{\substack{a+b \dot{=} n \\ k \in \text{bin}(a)}} A(b)B(a),$$

and

$$N_{\text{sgn}^0}(n)^{001} = \sum_{\substack{a+b \dot{=} n \\ k \in \text{bin}(b)}} A(a)D(b).$$

Thus

$$\begin{aligned} N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{001} &= \sum_{\substack{a+b \dot{=} n \\ k \in \text{bin}(b)}} A(a) \left(2D(b) + \frac{1}{2}A(b) \right) \\ &= \sum_{\substack{a+b \dot{=} n \\ k \in \text{bin}(b)}} 2D(n) + \frac{1}{2}A(n) \\ &= 2^{v(n)} \left(D(n) + \frac{1}{4}A(n) \right). \end{aligned} \quad (12)$$

We have used Proposition 5 for the equality $A(a)D(b) = D(n)$.

$$\begin{aligned} N_{\text{sgn}^0}(n)^{101} &= 2 \sum_{\substack{(a-2)+b \dot{=} n-2 \\ a+(b-2) \dot{=} n-2}} A(a)D(b) \\ &= 2 \sum_{j=2}^{k-1} \sum_{\substack{a+b \dot{=} n \\ \text{ord}(b)=j}} A(a)D(b) \\ &= 2 \sum_{j=2}^{k-1} 2^{v(n-2^{j+1})} D_{<j}(n) \\ &= 2 \sum_{j=2}^{k-1} 2^{v(n)+k-j-2} D_{<j}(n) \\ &= 2^{v(n)-1} \sum_{j=2}^{k-1} 2^{k-j} D_{<j}(n) \\ &= 2^{v(n)} \sum_{i=1}^{k-1} (1 + 2 + \cdots + 2^{k-i-2}) D_i(n). \end{aligned} \quad (13)$$

We have again used (2). Therefore

$$\begin{aligned}
 N_{\text{sgn}^0}(n) &= 2^{v(n)} \left(\sum_{j=1}^{k-1} (1 + 2 + \cdots + 2^{k-j-2}) D_j(n) + \left(\sum_{j=1}^{k-1} D_j(n) \right) + \frac{1}{4} A(n) \right) \\
 &= 2^{v(n)} \left(\frac{1}{4} A(n) + \sum_{j=1}^{k-1} 2^{k-j-1} D_j(n) \right) \\
 &= 2^{v(n)} A(n) \left(\frac{1}{4} + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2} - 3} \right) \\
 &= \frac{1}{8} A(2n) \left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} \right).
 \end{aligned}$$

□

Corollary 8 When n is even, we have $N_\varepsilon(n) = N_{\text{sgn}^0}(n)$.

□

Theorem 8 For $n \geq 2$, and $k = \text{ord}(n)$, we have

$$N_{\text{sgn}^1}(n) = \begin{cases} \frac{1}{4} A(2n) & \text{if } n \text{ is odd} \\ \frac{1}{8} A(2n) \left(2 + 2^k + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} (2^j - 1) \right) & \text{if } n \text{ is even.} \end{cases}$$

Proof For n odd, this is Corollary 7. Otherwise this follows from the identity $N_\varepsilon(n) + N_{\text{sgn}^1}(n) = B(2n)$, and the formulas for $N_\varepsilon(n)$ and $B(2n)$. □

5.7 Inequalities

In this section, we discuss how to establish, for $n \geq 10$:

- (1) $N_\varepsilon(n) = N_{\text{sgn}^1}(n) < N_{\text{sgn}^0}(n) < N_1(n)$, for n odd
- (2) $N_\varepsilon(n) = N_{\text{sgn}^0}(n) < N_{\text{sgn}^1}(n) < N_1(n)$, for n even,

from the formulas in the previous section. We have already established $N_\varepsilon(n) = N_{\text{sgn}^1}(n)$ when n is odd in Corollary 6, and $N_\varepsilon(n) = N_{\text{sgn}^0}(n)$ when n is even in Corollary 8.

If n is even, we have

$$\begin{aligned}
 N_\varepsilon(n) &= \frac{1}{8} A(2n) \left(2 + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} \right) \\
 &< \frac{1}{8} A(2n) \left(2 + 2^k + \sum_{j=1}^{k-1} 2^{\binom{k}{2} - \binom{j}{2}} (2^j - 1) \right) \\
 &= N_{\text{sgn}^1}(n).
 \end{aligned}$$

If n is odd, then $N_\varepsilon(n) = \frac{1}{4}A(2n)$, and it is obvious from the formulas that $N_\varepsilon(n) < N_{\text{sgn}^0}(n)$.

Comparing these formulas to

$$N_1(n) = p_2(n) - N_\varepsilon(n) - N_{\text{sgn}^0}(n) - N_{\text{sgn}^1}(n)$$

is considerably more difficult, since the partition function itself does not have an explicit formula. Let us sketch how this can be done. We will take n even for our sketch; the case n odd is similar. By the earlier inequalities, we have

$$\begin{aligned} \frac{N_1(n)}{N_{\text{sgn}^1}(n)} &> \frac{p_2(n)}{N_{\text{sgn}^1}(n)} - 3 \\ &> \frac{p_2(n)}{\frac{1}{8}A(2n)(2 + 2^{k+1})} - 3. \end{aligned}$$

So we must estimate $p_2(n)$ from below and $A(2n)$ from above; note that $2^k \leq n$. It is easy to see that $p_2(n) \geq 2p(n)$. Lemma 2 yields the (subexponential) upper bound

$$A(2n) \leq 2n^{\frac{1}{2}(\log_2 n + 3)}.$$

According to Hardy–Ramanujan [4], we have

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

as $n \rightarrow \infty$. From the above, it is straightforward to compute that

$$\lim_{n \rightarrow \infty} \frac{N_1(n)}{N_{\text{sgn}^0}(n)} = \infty.$$

But the inequality for all $n \geq 10$ takes more work. For this, one may start with the classical estimate

$$p(n) \geq \frac{n^{j-1}}{j!(j-1)!}$$

for all $1 \leq j \leq n$ (see [8]), and use the estimate

$$k! \leq \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} e^{\frac{1}{12k}}$$

of H. Robbins in [9]. With care one can prove from these estimates, and from computation for small n , that

$$p(n) \geq 5nA(n)$$

for $n \geq 64$. Our desired inequality ultimately follows from this.

This concludes our sketch of how to prove (for $n \geq 10$) that $N_{\text{sgn}^1}(n) < N_1(n)$ for n even, and similarly $N_{\text{sgn}^0}(n) < N_1(n)$ for n odd.

6 Type D_n : demihyperoctahedral groups

We denote by $\mathbb{D}_n$ the kernel of $\varepsilon : \mathbb{B}_n \rightarrow \{\pm 1\}$; it is the Weyl group of type D_n , called the demihyperoctahedral group. Since $\mathbb{D}_1$ is trivial, $\mathbb{D}_2$ is the Klein 4-group, and $\mathbb{D}_3$ is isomorphic to S_4 , we may take $n \geq 4$ when convenient.

The group $\mathbb{D}_n$, for $n \geq 2$, has two multiplicative characters: 1 and sgn , where sgn is the restriction of sgn^0 (or of sgn^1) to $\mathbb{D}_n$. To avoid confusion with earlier notation, let us write

$$N'_\omega(n) = \#\{\pi \in \text{Irr}(\mathbb{D}_n) \mid \det \pi = \omega\},$$

for $\omega = 1, \text{sgn}$.

6.1 Clifford theory

In this subsection, we assume Clifford Theory (especially the index 2 case) as well known and tacitly apply it to the subgroup $\mathbb{D}_n$ of $\mathbb{B}_n$.

Firstly, the restriction of $\rho_{\alpha, \beta}$ from $\mathbb{B}_n$ to $\mathbb{D}_n$ is irreducible iff $\alpha \neq \beta$. Let us call representations obtained this way ‘Type I’ representations. Moreover, $\text{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha_1, \beta_1} \cong \text{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha_2, \beta_2}$ iff (α_2, β_2) equals (α_1, β_1) or (β_1, α_1) .

Next, if n is even and $\alpha \vdash \frac{n}{2}$, then the restriction of $\rho_{\alpha, \alpha}$ to $\mathbb{D}_n$ is a direct sum of two nonisomorphic irreducible representations, say ρ_α^+ and ρ_α^- . Let us call these ‘Type II’ representations. The representation ρ_α^- is the twist of ρ_α^+ by $e_1 \in \mathbb{B}_n - \mathbb{D}_n$. We will write $\chi_\alpha^\pm$ for the character of $\rho_\alpha^\pm$. We have

$$\text{Irr}(\mathbb{D}_n) = \{\text{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha, \beta} \mid (\alpha, \beta) \models n, \alpha \neq \beta\} \coprod \left\{ \rho_\alpha^+, \rho_\alpha^- \mid \alpha \vdash \frac{n}{2} \right\},$$

with the second set nonempty only when n is even. The total number of representations of $\mathbb{D}_n$ is therefore equal to $\frac{1}{2}p_2(n)$ when n is odd, and equal to

$$\frac{1}{2}p_2(n) + \frac{3}{2}p\left(\frac{n}{2}\right)$$

when n is even. In particular, there are two multiplicative characters of $\mathbb{D}_n$: the trivial character, and the restriction of sgn^0 from $\mathbb{B}_n$, which we denote by ‘ sgn .’

The determinants of the Type I representations are given by

$$\det \text{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha \beta} = (\text{sgn})^{\gamma_{\alpha \beta}}.$$

So if n is odd, we have

$$N'_1(n) = \frac{1}{2}(N_1(n) + N_\varepsilon(n))$$

and

$$N'_{\text{sgn}}(n) = \frac{1}{2}(N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n)).$$

(The $N_\omega(n)$ were computed in the previous section).

6.2 Type II Determinants

Let $n = 2a$. For $\alpha \vdash a$, we have $\det \operatorname{Res}_{\mathbb{D}_n}^{\mathbb{B}_n} \rho_{\alpha, \alpha} = 1$ by Proposition 9. It follows that $\det \rho_\alpha^+ = \det \rho_\alpha^-$, so we need only compute $\det \rho_\alpha^+$.

Put

$$x_\alpha^+ = \frac{\dim \rho_\alpha^+ - \chi_\alpha^+(s_1)}{2} \in \mathbb{Z}/2\mathbb{Z}.$$

By Proposition 1,

$$\det \rho_\alpha^+ = (\operatorname{sgn})^{x_\alpha^+}.$$

If $n \geq 3$, it is not hard to see that $\operatorname{Int}(e_1)(s_1) = e_1 s_1 e_1^{-1}$ is conjugate in $\mathbb{D}_n$ to s_1 ; it follows that $\chi_\alpha^+(s_1) = \chi_\alpha^-(s_1)$.

Using Eq. (7) we compute

$$\begin{aligned} \chi_\alpha^+(s_1) &= \frac{1}{2} \chi_{\rho_{\alpha\alpha}}(s_1) \\ &= \binom{n-2}{a, a-2} f_\alpha \chi_\alpha(s_1) \\ &= \binom{n-2}{a, a-2} f_\alpha (f_\alpha - 2g_\alpha), \end{aligned}$$

and so

$$x_\alpha^+ = \frac{1}{4} \binom{n}{a, a} f_\alpha^2 - \frac{1}{2} \binom{n-2}{a-2, a} f_\alpha^2 + \binom{n-2}{a, a-2} f_\alpha g_\alpha.$$

If f_α is even, then clearly x_α^+ is even. If f_α is odd, then

$$\begin{aligned} x_\alpha^+ &= \frac{1}{4} \binom{n}{a, a} - \frac{1}{2} \binom{n-2}{a-2, a} + \binom{n-2}{a, a-2} g_\alpha \\ &= \frac{1}{2} \binom{n-2}{a-1, a-1} + \binom{n-2}{a, a-2} g_\alpha. \end{aligned}$$

Lemma 5 *Let $n = 2a \geq 6$ be an even integer.*

- (1) *If n is neither of the form 2^k nor $2^k + 2$, then $\binom{n-2}{a-1, a-1}$ is a multiple of 4, and $\binom{n-2}{a, a-2}$ is even.*
- (2) *If n is of the form 2^k , then $\binom{n-2}{a-1, a-1}$ is a multiple of 4, and $\binom{n-2}{a, a-2}$ is odd.*
- (3) *If n is of the form $2^k + 2$, then $\frac{1}{2} \binom{n-2}{a-1, a-1}$ is odd and $\binom{n-2}{a, a-2}$ is even.*

The proof is left to the reader. $\square$

Proposition 14 *Let $n \geq 6$ be an even integer.*

- (1) If n is not of the form 2^k or $2^k + 2$, then all Type II representations have determinant 1.
- (2) If n is of the form 2^k , then ρ_α^+ , ρ_α^- have determinant sgn iff f_α and g_α are odd.
- (3) If n is of the form $2^k + 2$, then ρ_α^+ , ρ_α^- have determinant sgn iff f_α is odd.

Theorem 9 Let $n \geq 4$. We have

$$N'_{\text{sgn}}(n) = \begin{cases} \frac{1}{2}(N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n)) + \frac{1}{2}n & \text{if } n = 2^k \text{ for some } k \geq 1, \\ \frac{1}{2}(N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n)) + n - 2 & \text{if } n = 2^k + 2 \text{ for some } k \geq 1, \\ \frac{1}{2}(N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n)) & \text{otherwise.} \end{cases}$$

This follows from the above and using $A(n/2) = \frac{1}{2}n$ when n is a power of 2, and $A(n/2) = \frac{1}{2}n - 1$ when $n = 2^k + 2$. (The case $n = 4$ is easily done separately, and agrees with the first two cases.)

Note that if n is even, then

$$N'_1(n) + N'_{\text{sgn}}(n) = \frac{1}{2}p_2(n) + \frac{3}{2}p(n),$$

so in principle we have a formula for $N'_1(n)$ as well.

7 Exceptional Coxeter groups

There are only finitely many irreducible finite Coxeter groups not of type A_n , B_n , D_n , or $I_2(n)$, namely the exceptional ones. For completeness, we give N_ω for each exceptional group. This is easily done by the propositions in Sect. 2.4, together with the character tables for these groups, which may be found in [3].

For the exceptional Coxeter groups W with $|W_{\text{ab}}| = 2$, we offer the following table. Below N_1 is the number of irreducible representations of W with trivial determinant, and N_{ε_W} is the number with ε_W as determinant.

Type	N_1	N_{ε_W}
H_3	6	4
H_4	18	16
E_6	13	12
E_7	44	16
E_8	64	48

Finally, for type F_4 , we have $N_1 = 9$, $N_{\varepsilon_W} = 8$, and $N_\omega = 4$ for $\omega \neq 1, \varepsilon_W$.

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