New families of optimal high-energy ternary sequences having good correlation properties

Jeffrey R. Hollon¹ · Muralidhar Rangaswamy² · Pawan Setlur³

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Abstract

We employ algebraic methods to provide some new constructions of what we call optimal high-energy ternary sequences (a sequence with entries in $\{0, 1, -1\}$ with a single zero, having optimal correlation properties). Our motivation for these constructions stems from their usefulness in several areas related to communication and radar systems.

Keywords Sequences · Difference sets · Correlation · Partial difference sets

1 Introduction

Sequences with ideal autocorrelation properties have many applications in spread spectrum communication systems such as code division multiple access (CDMA) systems, radar, signal processing, and source coding (audio, video or both etc.). Such sequences find many applications in signal processing and serve as aperture weighting functions in electromagnetic and acoustic imaging [7].

Sequences and their higher-dimensional counterparts (arrays) are critical in today's technological world where they are used in radar, error correction, digital communication, etc. Our work so far has resulted in the discovery of a few very rich classes of sequences all of whose out-of-phase autocorrelation values (side lobes) are very small. We call the constructed sequences as perfect sequences, and they serve as perfect algebraic/combinatorial objects in designing signals for communication and radar applications. Our work would be applicable in a much larger arena, opening doors for new application areas in the MIMO radar study as well.

✉ Jeffrey R. Hollon
jeffhollon@gmail.com

¹ Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA

² Air Force Research Lab, Wright Patterson Air Force Base, Building 620, 2241 Avionics Circle, Dayton, OH 45433-7132, USA

³ Radar Assessment and Development Lab, Riverside Research, Beavercreek, OH 45431, USA

In this paper, we employ algebraic methods to investigate perfect sequences and perfect arrays (binary and ternary) and construct what we call high-energy ternary sequences (a sequence with entries in $\{0, 1, -1\}$ with a single zero). Our motivation stems from their usefulness in several areas related to communication and radar systems: quantum computing, MC-CDMA systems, quasi-synchronous CDMA, multiple antenna wireless communication systems, FHSS which are widely used in military radios, CDMA and GSM networks, radars and sonars, and Bluetooth communications to name a few.

Our new sequences possess very desirable correlation properties:

1. The peak side lobes in the autocorrelation are small.
2. The sum of the squares of the side lobes in the autocorrelation is small.

Their ambiguity functions and Doppler tolerance [12] suggest that they are suitable for radar applications. While binary sequences/signals are typically sought after (having all its entries as 1 or -1), we only make a very modest assumption: allowing a SINGLE zero, thereby making them almost binary and achieving close to 100% energy. The gain is astounding: The sum of the squares of the side lobes in the autocorrelation is cut down by 50%. There are cases where we get other waveforms which are close to optimal but still look promising for radar use. Preliminary simulations validate our claim. Extension to two zeros is possible in a straightforward manner.

In Sect. 2, we provide basic definitions and algebraic preliminaries that pertain to group rings and combinatorial structures like Paley difference sets and their partial difference sets counterparts. Section 3 deals with our new constructions of what we call “optimal high-energy ternary sequences.” The adjective “optimal” pertains to attaining the theoretical minimum sum of squares of the side lobes in the autocorrelation; “high energy” qualifier refers to “almost” binary since only a single zero is present in our new sequences. We lay the foundation by starting from the more general class of abelian difference sets and partial differences and then obtaining as special cases that arise from the aforementioned classical Paley-type objects. Circulant conference matrices are known not to exist. Interestingly enough, in Sect. 3, we exhibit an almost-cyclically developed conference matrix using our OHETS sequences.

The results we have obtained for sequences generalize nicely to the more general domain of arrays. It is our hope that the rapid advancement of technology will soon allow for arrays to be used alongside sequences in various engineering applications. The advantage of arrays, over sequences, is that they embrace a much larger spectrum of parameters (e.g., size and energy of the array). In Sect. 4, we define some performance measures for sequences and compare these measures with ours vs previously known ones.

Section 5 improves upon and provides a modification of a patented algorithm of McLaughlin to generate “good” sets of high-energy sequences. We define a “good” set of sequences to have the properties: 1. Each sequence has a high Golay merit factor (GMF). 2. The aperiodic autocorrelation side lobes of each sequence are small. These properties make them suitable for MIMO radar applications. An application of our OHETS to frame synchronization is given in Sect. 6. In Sect. 7, we conclude with some remarks and open problems.

2 Preliminaries

A sequence $\mathbf{a} = (a_i)$, where $i = 0, 1, 2, \dots, N - 1$, is called periodic with period N provided that $a_i = a_{i+N}$ for all i . In this paper, we only consider real-valued sequences most of which will in fact be binary (respectively, ternary). That is all of their entries are either $\{+1, -1\}$ (respectively, $\{0, +1, -1\}$). The periodic cross-correlation function of the sequences $\mathbf{a}$ and $\mathbf{b}$ is defined by:

$$C(\tau) = \sum_{i=0}^{N-1} a_i b_{(i+\tau) \bmod N}^* \quad (1)$$

where b_i^* represents the complex conjugate of b_i . In this definition, if $\mathbf{a} = \mathbf{b}$, we call it the periodic autocorrelation function (ACF) of $\mathbf{a}$. Note that the sequence $C = C(t)$ is again periodic with period N , so that it suffices to consider the autocorrelation coefficients $C(t)$ for $t \in \{0, 1, \dots, N - 1\}$. The ACF measures how much the original sequence differs from its translates. The sequence $\mathbf{a}$ is said to have two-level ACF if its periodic ACF, C , satisfies:

$$C(\tau) = \begin{cases} E, & \tau = 0, \\ F, & \tau \neq 0, \end{cases} \quad (2)$$

where E and F are two constants, respectively, denoting the in-phase and out-phase ACF of the sequence $\mathbf{a}$. Let $\mathbf{a} = (a_i)$ be a binary sequence of period N . Define $D = \{0 \leq i \leq N - 1 : a_i = 1\}$ and $d_D(t) = |(t + D) \cap D|$ which is called the difference function of $D \subseteq \mathbb{Z}_N$. Then

$$C(t) = N - 4(k - d_D(t)) \quad (3)$$

where $k = |D|$. This would serve as a bridge between binary sequences and combinatorial designs. From Eq. (3), we readily get the following:

Proposition 1 $C(t) \equiv N \pmod{4}$

We call $C(N)$, $C(2N)$, $C(3N)$, $\dots$ the “main lobes” and the remaining $C(i)$ as “side lobes.” The sequence $\mathbf{a}$ is considered to have “ N elements,” one main lobe and $N - 1$ side lobes. It is well known that a sequence is said to have “good matched autocorrelation properties” if it satisfies:

1. The peak side lobes in the autocorrelation are small.
2. The sum of the squares of the side lobes in the autocorrelation is small.

Definition 1 Let G be the additively written cyclic group with v elements, (i.e., $G = \mathbb{Z}_v$) and D a subset of G with k elements. For any $\alpha \neq 0 \pmod{v}$, if the equation

$$d - d' \equiv \alpha \pmod{v}$$

has exactly λ solution pairs (d, d') with both d and d' in D , then the set D is called a (v, k, λ) difference set in G .

The set D defined in Definition 1 would work as the required cyclic difference set in the following result which is easy to prove.

Proposition 2 *A periodic binary sequence with period v , k entries $+1$ per period and 2-level ACF (with all nontrivial autocorrelation coefficients equal to r) is equivalent to a cyclic (v, k, λ) difference set where $r = v - 4(k - \lambda)$.*

The case where $r = 0$ corresponds to circulant Hadamard matrices of order v , where $v = 4u^2$, which are equivalent to cyclic $(4u^2, 2u^2 - u, u^2 - u)$ difference sets. The only known example of a circulant Hadamard matrix is when $v = 4$ and it is conjectured that there are no others which exist. Likewise, the cases $r = 1$ and $r = 2$ give rise to only a handful of cyclic (v, k, λ) difference sets. The case $r = -1$ would take us to a very fertile terrain where the examples are bountiful. In view of Proposition 2, these perfect binary sequences with $r = -1$ are equivalent to cyclic difference sets with parameters $(v, \frac{v-1}{2}, \frac{v-3}{4})$. These are commonly referred to as Paley–Hadamard difference sets. The famous m-sequences, also referred to as singer sequences, provide a large such class.

The variation we wish to employ in this research is to allow a single zero in the sequences, making them ternary and obtaining new families of sequences with the desirable properties mentioned above for “good matched autocorrelation properties.” The process of introducing a zero entry, where there was none before, we refer to as puncturing the sequence. A further property that these ternary sequences have is that of high energy, defined next.

Definition 2 The energy, E , of a sequence is calculated by $E = \sum_{i=0}^{N-1} |a_i|^2$.

Definition 3 The energy efficiency, E_{eff} , of a sequence is calculated by $E_{\text{eff}} = \frac{E}{N}$.

The sequences that are the main focus of this paper are ternary sequences with a single zero entry. Thus, the energy is maximized for ternary sequences (if the zero position were to be filled in, then we lose the ternary property). Further, it is desirable to have an energy efficiency which is close to 1. This is also closely met by the same reasoning: For our sequences, with a single zero entry, $E_{\text{eff}} = \frac{N-1}{N}$ which approaches 1 as the sequence’s period increases.

Perfect periodic autocorrelation sequences see applications in many areas, including spread spectrum communications [13], channel estimation and fast start-up equalization [10], pulse compression radars [5], sonar systems [16]. CDMA systems [6], system identification [15], and watermarking [14].

2.1 Group ring notation

We introduce the group ring notation which will be used in the proofs.

Definition 4 Let G be a finite group and R a ring where $G = \{g_0, g_1, g_2, \dots, g_{n-1}\}$. Then, the group ring of G over R is the set denoted by $R[G]$ defined as:

$$R[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in R \right\}.$$

$R[G]$ is also a free R -module of rank n and an algebra. So, $R[G]$ is also referred to as a group algebra. When working with the group ring notation, multiplication and addition are defined in a way similar to polynomials, and the group elements are written multiplicatively. We further define the *power* of a group ring element in the following way.

Definition 5 If $W = \sum_{g \in G} a_g g$ is an element of $R[G]$ and t some integer, then

$$\left(\sum_{g \in G} a_g g \right)^{(t)} = \sum_{g \in G} a_g g^t.$$

The following is well known and follows readily from the above definition.

Lemma 1 Let $D = \{d_0, d_1, \dots, d_{k-1}\}$. Then, D is a (v, k, λ) difference set if and only if

$$DD^{(-1)} = k - \lambda + \lambda G \text{ in } Z[G].$$

Remark Difference sets are studied in the more general group theoretic context. Since we are primarily dealing with “sequences,” we restrict our attention to “cyclic” groups Z_v . We use the term “array” for when the group in question is non-cyclic. For more on these and related studies that pertain to sequences and arrays and their interplay with group developed combinatorial designs refer to the survey article [1].

2.2 Abstraction of sequences to arrays

In most of this paper, we are concerned with sequences. However, it is important to also consider more general, higher-dimensional objects, called arrays for which the theorems can be generalized to. An r -dimensional matrix, $A = (a(j_1, j_2, \dots, j_r))$, with $0 \leq j_i \leq s_i - 1$ for $i = 1, 2, \dots, r$, is called an $s_1 \times s_2 \times \dots \times s_r$ array, where $s_1, s_2, \dots, s_r$ are all positive integers. $A(j_1, j_2, \dots, j_r)$ would denote the entry $a(j_1, j_2, \dots, j_r)$ of the matrix A at position $(j_1, j_2, \dots, j_r)$. We refer to this array as a sequence when $r = 1$.

These arrays have a similarly defined periodic autocorrelation function to sequences. The autocorrelation function γ at $\tau = (t_1, t_2, \dots, t_r)$ is given by

$$\gamma(\tau) = \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_r=0}^{s_r-1} a(j_1, j_2, \dots, j_r) \bar{a}(j_1 + t_1, j_2 + t_2, \dots, j_r + t_r)$$

where the i th subscript is taken modulo s_i and $\bar{a}$ denotes the complex conjugate of a . The array corresponds to the group ring element $R[G]$ where $G = Z_{s_1} \times Z_{s_2} \times \dots \times Z_{s_r} = \langle g_1 \rangle \times \langle g_2 \rangle \times \dots \times \langle g_r \rangle$ and g_i represents the generator of the cyclic group Z_{s_i} .

2.3 Difference sets of Paley type

Example 1 Paley difference set for prime power $q \equiv 3 \pmod{4}$.

Let q be a prime power and $q \equiv 3 \pmod{4}$. Let $GF(q)$ denote the finite field with q elements. Fix a primitive element α of $GF(q)$ and define:

$$S = \left\{ \alpha^{2i} \mid i = 0, 1, \dots, \frac{q-3}{2} \right\},$$

$$N = \left\{ \alpha^{2j+1} \mid j = 0, 1, \dots, \frac{q-3}{2} \right\}.$$

The sets S and N consist exactly of the square and non-square elements of $GF(q) \setminus \{0\}$. It is well known that S and N are themselves difference sets in the group $G = (GF(q), +)$ with parameters $(v, k, \lambda) = (q, \frac{q-1}{2}, \frac{q-3}{4})$. Hence, the following equations hold in the group ring $Z[G]$:

$$SS^{(-1)} = NN^{(-1)} = \left(\frac{q+1}{4} \right) + \left(\frac{q-3}{4} \right) G \quad (4)$$

$$N = S^{(-1)} \quad (5)$$

$$S = N^{(-1)} \quad (6)$$

$$1 + S + N = G. \quad (7)$$

The next example gives rise to the so-called partial difference set.

Definition 6 Let G be a multiplicatively written group of order v . A subset $D \subseteq G$ of size k is said to be a (v, k, λ, μ) partial difference set (PDS) in G if:

1. $1 \notin D$;
2. $D = D^{(-1)}$;
3. $DD^{(-1)} = D^2 = k + \lambda D + \mu(G - D - 1)$ in ZG .

Example 2 Paley partial difference set for prime power $q \equiv 1 \pmod{4}$

Let q be a prime power, $q \equiv 1 \pmod{4}$. Let $GF(q)$ denote the finite field with q elements. Fix a primitive element α of $GF(q)$ and define:

$$S = \left\{ \alpha^{2i} \mid i = 0, 1, \dots, \frac{q-3}{2} \right\},$$

$$N = \left\{ \alpha^{2j+1} \mid j = 0, 1, \dots, \frac{q-3}{2} \right\}.$$

Thus, S and N consist exactly of the square and non-square elements of $GF(q) \setminus \{0\}$. It is well known that S and N are themselves partial difference sets in the group $G = (GF(q), +)$ with parameters $(v, k, \lambda, \mu) = (q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$. Hence, the following equations hold in the group ring $Z[G]$:

$$SS^{(-1)} = S^2 = \left(\frac{q-1}{2}\right) + \left(\frac{q-5}{4}\right)S + \left(\frac{q-1}{4}\right)N \quad (8)$$

$$NN^{(-1)} = N^2 = \left(\frac{q-1}{2}\right) + \left(\frac{q-5}{4}\right)N + \left(\frac{q-1}{4}\right)S \quad (9)$$

$$S = S^{(-1)} \quad (10)$$

$$N = N^{(-1)} \quad (11)$$

$$1 + S + N = G. \quad (12)$$

More on the Paley sequences from their original source can be found in [11].

3 Main results

3.1 Optimal high-energy ternary sequences and arrays from difference sets and almost difference sets

The following proof examines the properties of difference and partial difference sets which are associated with an abelian group G and what occurs when a position, namely the identity, is punctured. We continue by examining each case separately. Note that in the following theorems, the term “sequence” refers to a cyclic group while “array” refers to an abelian group that is not cyclic. Two theorems are required for the completion of these high-energy sequences. The first is by Camion and Mann [4] and the second by Arasu et al. [3].

Theorem 1 (Camion and Mann, [4]) *Let D be an abelian (v, k, λ) difference set in G which satisfies $D + D^{(-1)} + 1 = G$ in $Z[G]$. Then, v is a prime power with $v \equiv 3 \pmod{4}$. If G is cyclic then v is prime and D consists of the nonzero squares or the non-squares in $GF(v)$.*

For partial difference sets, their parameters have been characterized in the following theorem using well-known equivalence of PDSs and strongly regular Cayley graphs.

Theorem 2 (Arasu et al. [3]) *Let Γ be a strongly regular Cayley graph based on an abelian group G with parameters (v, k, λ, μ) satisfying $\beta = \lambda - \mu = -1$. Then, up to complementation, Γ is either:*

1. *Of Paley type, that is it has the parameters of the type $\left(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4}\right)$ or;*
2. *it has parameters $(243, 22, 1, 2)$.*

3.1.1 Case: difference sets

Theorem 3 *Let D be a (v, k, λ) difference set in an abelian group G . Then, the sequence (array) defined by*

$$S = D - (G - D) + 1$$

is a high-energy sequence (array) with up to three periodic side lobes. In the special case where $D \cap D^{(-1)} = \emptyset$, the sequence (array) S will have only a two-valued periodic autocorrelation. Namely, the main lobe $v - 1$ with side lobes $v - 4(k - \lambda)$.

Proof Let D be a (v, k, λ) difference set in an abelian group G . Then

$$DD^{(-1)} = k - \lambda + \lambda G$$

and its complement, $G \setminus D$, is a $(v, v - k, v - k + 2\lambda)$ satisfying

$$(G - D)(G - D)^{(-1)} = k - \lambda + (v - 2k + \lambda)G.$$

Without loss of generality, we assume that $1 \notin D$. Consider the sequence (array) defined by $S = D - (G - D) + 1$ and examine its autocorrelation $SS^{(-1)}$.

$$\begin{aligned} \sum_{x \in G} C(x)x &= SS^{(-1)} = [(2D + 1) - G] \left[(2D^{(-1)} + 1) - G \right] \\ &= 4DD^{(-1)} + 2D + 2D^{(-1)} + 1 + [v - (4k + 2)]G \\ &= 2D + 2D^{(-1)} + 1 + 4[(k - \lambda) + \lambda G] + (v - 4k - 2)G \\ &= 2D + 2D^{(-1)} + 4(k - \lambda) + 1 + (v - 4k + 4\lambda - 2)G. \end{aligned}$$

This calculation shows that the side lobe values are dependent on the intersection of elements in D and $D^{(-1)}$. In particular, take the side lobe corresponding to the sequence element x and its side lobe value $C(x)$. By the previous calculation

$$C(x) = \begin{cases} v - 4(k - \lambda) - 2, & \text{if } x \notin D \cup D^{(-1)}, \\ v - 4(k - \lambda), & \text{if } x \in D \text{ or } x \in D^{(-1)}, \\ v - 4(k - \lambda) + 2, & \text{if } x \in D \cap D^{(-1)}. \end{cases}$$

In the special case where D satisfies $D + D^{(-1)} + 1 = G$, D is referred to as a skew symmetric difference set, then all side lobes collapse into a single value. In particular, all side lobe values will be $v - 4(k - \lambda) + 2$. Further, it is shown by use of Theorem 1 that this can only occur when the difference set has Paley parameters. $\square$

Example 3 As an example of Theorem 3, take the famous singer difference set (m-sequence) of length 15. This sequence is given by

$$D = [++++-++--+-+-----]$$

and its periodic correlation can be calculated to be

$$C(D) = [15-----].$$

Using Theorem 3,

$$S = [0 + + - + + - - + - + - - -]$$

whose periodic correlation is four-valued. This correlation is calculated to be

$$C(S) = [14 - - + - (-3) + - - + (-3) - + - -].$$

3.1.2 Case: partial difference sets

Theorem 4 *Let D be a (v, k, λ, μ) partial difference set in an abelian group G . Then, the sequence (array) defined by*

$$S = D - (G - D) + 1$$

is a high-energy sequence (array) with up to two periodic side lobes. In the special case where $\mu = \lambda + 1$, the sequence (array) S will have only a two-valued periodic autocorrelation function. Namely, the mainlobe $v - 1$ with side lobes given by $v + 4(\mu - k) - 2$ and $v + 4(\lambda - k) + 2$. In the special case where $\mu - \lambda = 1$, then a single-valued periodic side lobe will exist and the partial difference will either have Paley parameters or have the parameters $(243, 22, 1, 2)$.

Proof Let D be a (v, k, λ, μ) partial difference set in an abelian group G . Then

$$DD^{(-1)} = D^2 = k + \lambda D + \mu(G - D - 1).$$

Without loss of generality, we assume that $1 \notin D$ and examine the sequence (array) defined by

$$S = D - (G - D) + 1.$$

Sequence S is a high-energy sequence whose periodic autocorrelation, $SS^{(-1)}$, can be calculated by:

$$\begin{aligned} \sum_{x \in G} C(x)x &= SS^{(-1)} = [(2D + 1) - G] \left[(2D^{(-1)} + 1) - G \right] \\ &= 4DD^{(-1)} + 2D + 2D^{(-1)} + 1 + [v - (4k + 2)]G \\ &= 4D^2 + 2D + 2D + 1 + [v - (4k + 2)]G \\ &= 4D^2 + 4D + 1 + [v - (4k + 2)]G \\ &= 4[k - \mu + (\lambda - \mu)D + \mu G] + 4D + 1 + [v - 4k - 2]G \\ &= 4(k - \mu) + 1 + [4(\lambda - \mu) + 4]D + [v + 4\mu - 4k - 2]G. \end{aligned}$$

This calculation shows that the side lobe values will take on one of two values. One for entries in D and the other for when entries are not in D . In particular, take the side

lobe corresponding to the sequence element x and its side lobe value $C(x)$. By the previous calculation

$$C(x) = \begin{cases} v + 4(\mu - k) - 2, & \text{if } x \notin D, \\ v + 4(\lambda - k) + 2, & \text{if } x \in D. \end{cases}$$

In the special case where $\mu - \lambda = 1$, a single side lobe value will occur. Specifically, all side lobe values will have the single value given by $v + 4(\mu - k) - 2$. It has been shown by Theorem 2 that this condition only holds for when the partial difference set satisfies the Paley parameters or when D is the sporadic example with parameters $(243, 22, 1, 2)$. $\square$

Example 4 As an example of Theorem 4, take the famous Paley partial difference set of length 13. This binary sequence is given by

$$D = [- + - + + - - - - + + - +]$$

and its periodic correlation can be calculated to be

$$C(D) = [13 \ -3 \ + \ -3 \ -3 \ + \ + \ + \ + \ -3 \ -3 \ + \ -3].$$

Using Theorem 3,

$$S = [0 \ + \ - \ + \ + \ - \ - \ - \ - \ + \ + \ - \ +]$$

whose periodic correlation is two-valued. This correlation is calculated to be

$$C(S) = [12 \ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \ -].$$

3.2 Optimal high-energy ternary sequence construction (OHETS)

Take the constructions from Theorems 3 and 4. Using the cases where the sequences have the Paley parameters, we call the resulting sequence, S , which is created by puncturing, an optimal high-energy ternary sequence (OHETS). Before stating and proving the main theorem, we present a few necessary lemmas for its proof.

Lemma 2 For a prime power $q \equiv 3 \pmod{4}$, $SN^{(-1)} = \left(\frac{q+1}{4}\right)G - \left(\frac{q+1}{4}\right) - S$.

Proof By Eq. (6), $SN^{(-1)} = SS$. Further, $SS = S(G - N - 1)$ by Eq. (7) and $N = S^{(-1)}$ by Eq. (5). Now expand as follows:

$$\begin{aligned}
 SN^{(-1)} &= SS \\
 &= S(G - N - 1) \\
 &= SG - SN - S \\
 &= SG - SS^{(-1)} - S \\
 &= \left(\frac{q-1}{2}\right)G - \left(\frac{q+1}{4}\right) - \left(\frac{q-3}{4}\right)G - S \\
 &= \left(\frac{q+1}{4}\right)G - \left(\frac{q+1}{4}\right) - S.
 \end{aligned}$$

□

Lemma 3 For a prime power $q \equiv 3 \pmod{4}$, $S^{(-1)}N = \left(\frac{q+1}{4}\right)G - \left(\frac{q+1}{4}\right) - N$.

Proof Since $SN^{(-1)}$ and $S^{(-1)}N$ are inverses of one another, it is sufficient to compute the inverse of both sides of the result from Lemma 2. Note that the only change on the right side will be that S inverts to N . □

Lemma 4 For a prime power $q \equiv 1 \pmod{4}$, $SN = \left(\frac{q-1}{4}\right)G - \left(\frac{q-1}{4}\right)$.

Proof By using a very similar method as in Lemma 2, except using the equations presented in Example 2, we find the following:

$$\begin{aligned}
 SN &= S(G - S - 1) \\
 &= SG - SS - S \\
 &= SG - SS^{(-1)} - S \\
 &= \left(\frac{q-1}{2}\right)G - \left(\frac{q-1}{2}\right) - \left(\frac{q-5}{4}\right)S - \left(\frac{q-1}{4}\right)N - S \\
 &= \left(\frac{q-1}{2}\right)G - \left(\frac{q-1}{2}\right) - \left(\frac{q-1}{4}\right)(S + N) \\
 &= \left(\frac{q-1}{2}\right)G - \left(\frac{q-1}{2}\right) - \left(\frac{q-1}{4}\right)(G - 1) \\
 &= \left(\frac{q-1}{4}\right)G - \left(\frac{q-1}{4}\right).
 \end{aligned}$$

□

Theorem 5 Let q be an odd prime power. Let $G = (GF(q), +)$ where S and N denote the square and non-square elements of $GF(q) \setminus \{0\}$. Then, the ternary sequence (array) $\mathbf{a} = (a_i)$ indexed by $(GF(q), +)$ defined by:

$$a_i = \begin{cases} 0, & \text{if } i = 0, \\ +1, & \text{if } i \text{ is in } S, \\ -1, & \text{if } i \text{ is in } N, \end{cases}$$

satisfies that $\mathbf{aa}^{(-1)} = (q-1) - S - N$ in the group ring $Z[G]$. That is to say that all side lobes of $\mathbf{a}$ are -1 and the main lobe is $q-1$.

Proof We break this into two cases. Case 1 is for $q \equiv 3 \pmod{4}$ and Case 2 is for $q \equiv 1 \pmod{4}$.

Case 1 Assume that $q \equiv 3 \pmod{4}$. Then, the sequence $\mathbf{a} = S - N$ has an autocorrelation $\mathbf{aa}^{(-1)}$ computed by

$$\begin{aligned} \mathbf{aa}^{(-1)} &= (S - N)(S - N)^{(-1)} \\ &= SS^{(-1)} + NN^{(-1)} - SN^{(-1)} - NS^{(-1)}. \end{aligned}$$

Next, using Eq. (4) for the first two terms and Lemmas 2 and 3 for the last two terms we get

$$\begin{aligned} \mathbf{aa}^{(-1)} &= 2 \left[\left(\frac{q+1}{4} \right) + \left(\frac{q-3}{4} \right) G \right] - \left(\frac{q+1}{2} \right) G + \left(\frac{q+1}{2} \right) + (G-1) \\ &= \left(\frac{q+1}{2} \right) + \left(\frac{q-3}{2} \right) G - \left(\frac{q+1}{2} \right) G + \left(\frac{q+1}{2} \right) + (G-1) \\ &= \left(\frac{2q+2+2q+2-4}{4} \right) + \left(\frac{2q-6-2q-2+4}{4} \right) G \\ &= q - G. \end{aligned}$$

This finishes Case 1.

Case 2 Assume that $q \equiv 1 \pmod{4}$. Then, the sequence $\mathbf{a} = S - N$ and we examine its autocorrelation $\mathbf{aa}^{(-1)}$ directly. Note that

$$\begin{aligned} \mathbf{aa}^{(-1)} &= (S - N)(S - N)^{(-1)} \\ &= SS^{(-1)} + NN^{(-1)} - SN^{(-1)} - NS^{(-1)} \\ &= S^2 + N^2 - 2SN \end{aligned}$$

by Eqs. (10) and (11). Now expanding by using the equations with Example 2 and Lemma 4:

$$\begin{aligned} \mathbf{aa}^{(-1)} &= S^2 + N^2 - 2SN \\ &= (q-1) + \left(\frac{q-5}{4} \right) (S+N) + \left(\frac{q-1}{4} \right) (S+N) \\ &\quad - 2 \left[\left(\frac{q-1}{4} \right) G - \left(\frac{q-1}{4} \right) \right] \end{aligned}$$

Table 1 The first ten OHETS

Period	Sequence
3	[0 + −]
5	[0 + − − +]
7	[0 + + − + − −]
11	[0 + − + + + − − − + −]
13	[0 + − + + − − − + + − +]
17	[0 + + − + − − − + + − − + − + +]
19	[0 + − − + + + + − + − + − − − + + −]
23	[0 + + + + − + − + + − + − + − + − − −]
29	[0 + − − + + + + − + − − + − − + − − + − + + + − − +]
31	[0 + + − + + − + + + + − − − + − + − + + + − − − + − − + − −]

$$\begin{aligned}
&= (q-1) + \left(\frac{q-5}{4}\right)(G-1) + \left(\frac{q-1}{4}\right)(G-1) - \left(\frac{q-1}{2}\right)G \\
&\quad + \left(\frac{q-1}{2}\right) \\
&= \left(\frac{4q-4-q+5-q+1+2q-2}{4}\right) + \left(\frac{q-5+q-1-2q+2}{4}\right)G \\
&= q - G.
\end{aligned}$$

□

Remark The case for $q \equiv 1 \pmod{4}$ above provides optimal sequences. That is to say that these sequences provide the minimum possible sum of squares in the coefficients of their ACF. This is important as the Paley sequences for $q \equiv 3 \pmod{4}$ (See Example 1) give ACF coefficients of -1 and the case for $q \equiv 1 \pmod{4}$ (See Example 2) give 1 s and -3 s. Thus, our sequences provide a better (smaller) sum of squares for the $q \equiv 1 \pmod{4}$ case.

Proof For the OHETSs above, we assume prime power period q with a single punctured position at $i = 0$. Thus, the dot product of the original sequence $\mathbf{a}$ with any of its translates will result in exactly two entries being multiplied by the 0. This implies exactly two values of the dot product will cancel out, leaving $q - 2$ terms each of which is ± 1 . This means that an odd number of ± 1 s are being summed and the final value can not be even (i.e., can not be 0). Therefore, these new sequences provide the optimal value for the sum of squares since $(\pm 1)^2$ is the smallest odd integer. □

3.2.1 List of optimal high-energy ternary sequences and arrays

In Table 1, we show the OHETS sequences for the first lengths up to $N = 31$. Note that these ternary sequences are clearly high in energy.

These OHETS sequences have optimal periodic autocorrelation functions, as we proved; however, their aperiodic autocorrelation functions, as in Definition 7, are not

Table 2 Aperiodic autocorrelations of the first ten OHETS (using the shift with highest GMF)

Period	Aperiodic autocorrelation
3	[2 -1 0]
5	[4 0 0 -1 -1]
7	[6 0 -1 -1 0 0 -1]
11	[10 0 -1 -1 -2 -1 0 1 0 0 -1]
13	[12 -2 1 -2 0 -1 0 -1 0 -1 -2 1]
17	[16 0 -1 0 -1 1 2 -1 0 -1 0 -3 -2 0 -1 0 -1]
19	[18 0 1 0 0 -1 -2 -1 -2 -1 0 1 0 1 0 -1 -1 -2 -1]
23	[22 0 -3 0 1 -2 -3 -1 0 -1 -2 1 -2 1 0 -1 0 2 1 -2 -1 2 -1]
29	[28 0 -1 0 1 0 1 2 -2 -1 0 3 0 1 0 -1 -2 -1 -4 -1 0 1 -3 -2 -1 -2 -1 0 -1]
31	[30 0 1 2 1 0 -1 0 1 -4 -1 -1 -2 1 0 -1 0 -1 -2 1 0 0 3 -2 -1 0 -1 -2 -3 -2 -1]

Table 3 First two optimal high-energy arrays

Period	Array
3×3	$\begin{bmatrix} 0 & - & - \\ + & + & - \\ + & - & + \end{bmatrix}$
5×5	$\begin{bmatrix} 0 & - & - & - & - \\ + & + & - & + & - \\ + & + & + & - & - \\ + & - & - & + & + \\ + & - & + & - & + \end{bmatrix}$

optimal. We present them in Table 2. For the aperiodic autocorrelation function, we present the one associated with the shift, of the original sequence in Table 1, providing the largest Golay merit factor.

In Table 3, we show the first two OHETS-based arrays whose periodic autocorrelation functions are again optimal.

3.2.2 Generation of conference matrices with group developed core

A conference matrix, C , is a square ternary sequence with exactly one zero entry per row (column) satisfying $CC^T = kI$ for some constant k and identity matrix I . These objects are known not to exist in general if they are cyclically developed; however, we can generate an almost-cyclically developed conference matrix using our OHETS sequences.

Theorem 6 *A conference matrix, C , of size $q + 1$, where q is an odd prime, exists where its border consists of all $+1$ s and the core (non-border entries) is a cyclically developed matrix from an OHETS.*

Proof Let $\mathbf{a}$ be any OHETS of length q . Define A as the matrix consisting of the sequence $\mathbf{a}$ and all of its cyclic shifts. Thus, A is a cyclically developed square matrix

Fig. 1 Construction of conference matrix with group developed core

$$C = \begin{bmatrix} 0 & + & \cdots & + \\ + & & & \\ \vdots & & A & \\ + & & & \end{bmatrix}$$

of size $q \times q$. Define a new matrix C by taking A and bordering it on the left and the top by a column and row of $+1$ s and let $C_{1,1} = 0$. C can be visualized as in Fig. 1. The matrix C has the property that $CC^T = qI_{q+1}$ where I_n is the identity matrix of size n . This can be easily verified by looking at the dot product of any two rows of C . Let i and j represent the row and column indices of C , respectively.

Case 1 ($i = j$) It is clear when any row of C is dotted with itself that the answer is q . This is trivially true for the first row of C by construction. For all other rows, by definition A has $q - 1$ nonzero entries and one more is added to the left. Thus, any row dotted with itself is q .

Case 2 ($i \neq 1, j \neq 1$) For any two rows, both of which are not the first of C , there is exactly one extra copy of $+1$ to the dot product. Since all other terms of the product come from a OHETS and its shift, then their dot product is -1 by construction. Then, $+1 + -1 = 0$ is the dot product of any two such rows.

Case 3 ($i = 1$ and $j > 1$) If the first row is involved, then note the 0 at the beginning means the dot product will only consist of the sum of entries in the original OHETS. Thus, the product will simply be the sum of terms in the sequence $\mathbf{a}$. By construction, these OHETS are balanced when looking at the number of ± 1 s, and thus, the dot product is again 0.

This proves that $CC^T = qI_{q+1}$. Further, it is a conference matrix as each row contains exactly one 0 element. $\square$

Example 5 Using the data provided in Table 1, we can construct such a conference matrix of size 6. Let $\mathbf{a} = [0 + - - +]$. Then, we can “develop” A cyclically so that

$$A = \begin{bmatrix} 0 & + & - & - & + \\ + & 0 & + & - & - \\ - & + & 0 & + & - \\ - & - & + & 0 & + \\ + & - & - & + & 0 \end{bmatrix}$$

and then construct C by

$$C = \begin{bmatrix} 0 & + & + & + & + & + \\ + & 0 & + & - & - & + \\ + & + & 0 & + & - & - \\ + & - & + & 0 & + & - \\ + & - & - & + & 0 & + \\ + & + & - & - & + & 0 \end{bmatrix}.$$

It can be verified that $CC^T = 5I_6$.

3.3 High-energy binary peak shifting sequence construction (HEBPSS)

Using the properties of the perfect squares in the field $GF(q)$ as above, we prove the existence of another infinite family of sequences which reside in the group $H = Z_2 \times G$. Here, we have $q \equiv 1 \pmod{4}$ is a prime and $G = GF((q), +)$. This family has the property that their main lobe is of size q , with exactly two side lobes of size $\frac{3(q-1)}{4}$ and all remaining side lobes of $\frac{q-1}{2}$. Further, these sequences are flexible in that the position of the $\frac{3(q-1)}{4}$ side lobe pair can be moved around, in some sense. Thus, the term “peak shifting.” We first show the proof for this family of binary sequences. In the following theorem, we use that for a subset S of G , define $S(x^j)$ to be the resulting set in Z_{2q} after embedding $G = Z_q$ in Z_{2q} and applying the function $x \rightarrow x^j$ to the elements of S .

Theorem 7 *Let $q \equiv 1 \pmod{4}$ be a prime and let P be the set of all nonzero squares in $GF(q) - \{0\}$. Then, working in the group Z_{2q} , we define the binary sequence given by $\mathbf{b} = P(x^2)(1 + x^{2k+1}) + x^{2k+1}$, of period $2q$, where we fix a value $k \in \{0, 1, 2, \dots, \frac{q-1}{2}\}$. This sequence has a periodic autocorrelation function with the main lobe of q , exactly two side lobes of size $\frac{3(q-1)}{4}$ positioned at the entries x^{2k+1} and x^{-2k-1} , and all other side lobes have a value of $\frac{q-1}{2}$. In the special case where $k = \frac{q-1}{2}$, then all side lobes have a value of $\frac{q-1}{2}$ except the midpoint which will take on the value $q - 1$.*

Proof Let q be a prime where $q \equiv 1 \pmod{4}$ and assume that P is the set of all nonzero squares in Z_q . Let $\mathbf{b} = P(x^2)(1 + y) + y$, where $y = x^{2k+1}$ to make the following proof easier to read. We proceed by computing the autocorrelation function $\mathbf{bb}^{(-1)}$ directly and noting that $P(x^2) = P^{(-1)}(x^2)$ by Eq. (10).

$$\begin{aligned}\mathbf{bb}^{(-1)} &= [P(x^2)(1 + y) + y][P^{(-1)}(x^2)(1 + y^{-1}) + y^{-1}] \\ &= P(x^2)P(x^2)(2 + y + y^{-1}) + P(x^2)(2 + y + y^{-1}) + 1 \\ &= [P(x^2)P(x^2) + P(x^2)][2 + y + y^{-1}] + 1.\end{aligned}$$

Now, use Eq. (8) for the expansion of $P(x^2)P(x^2)$:

$$\begin{aligned}\mathbf{bb}^{(-1)} &= [P(x^2)P(x^2) + P(x^2)][2 + y + y^{-1}] + 1 \\ &= \left[\left[\frac{q-1}{2} + \frac{q-5}{4}P(x^2) + \frac{q-1}{4}N(x^2) \right] + P(x^2) \right] [2 + y + y^{-1}] + 1 \\ &= q + \frac{q-1}{2}y + P(x^2) \left(\frac{q-1}{2} + \frac{q-1}{4}y + \frac{q-1}{4}y^{-1} \right) \\ &\quad + N(x^2) \left(\frac{q-1}{2} + \frac{q-1}{4}y + \frac{q-1}{4}y^{-1} \right) + \frac{q-1}{2}y^{-1} \\ &= q + \frac{q-1}{2}y + \frac{q-1}{2}(P(x^2) + N(x^2)) + \frac{q-1}{4}(P(x^2) \\ &\quad + N(x^2))(y + y^{-1}) + \frac{q-1}{2}y^{-1}.\end{aligned}\tag{13}$$

At this point, note that the term $P(x^2) + N(x^2)$ contains every nonzero even power of x exactly once. Next we examine the fourth part of the expansion above, whose coefficient is $\frac{q-1}{4}$ separately. We will prove that $(P(x^2) + N(x^2))(y + y^{-1})$ contains y and y^{-1} exactly once each and all other odd powers of x exactly twice. First, by Eq. (12), we have that $G = P + N + 1$ in the group Z_q . Thus, in the group Z_{2q} , we have that $G(x^2) = P(x^2) + N(x^2) + 1$. This allows us to write:

$$\begin{aligned}(P(x^2) + N(x^2))(y + y^{-1}) &= (P(x^2) + G(x^2) - P(x^2) - 1)(y + y^{-1}) \\ &= (G(x^2) - 1)(y + y^{-1}) \\ &= G(x^2)y + G(x^2)y^{-1} - y - y^{-1}.\end{aligned}\quad (14)$$

Since $G(x^2)$ clearly contains all even powers of x , then $G(x^2)y$ and $G(x^2)y^{-1}$ must contain all odd powers of x each. Recall that $y = x^{2k+1}$ is itself an odd power of x . Thus,

$$G(x^2)y + G(x^2)y^{-1} - y - y^{-1} = y + y^{-1} + 2 \sum_{m \neq k} x^{2m+1}. \quad (15)$$

We can now substitute Eq. (15) into Eq. (13) yielding

$$\begin{aligned}\mathbf{bb}^{(-1)} &= q + \frac{q-1}{2}y + \frac{q-1}{2}(P(x^2) + N(x^2)) + \frac{q-1}{4}(P(x^2) \\ &\quad + N(x^2))(y + y^{-1}) + \frac{q-1}{2}y^{-1} \\ &= q + \frac{q-1}{2}y + \frac{q-1}{2}(P(x^2) + N(x^2)) \\ &\quad + \frac{q-1}{4} \left(y + y^{-1} + 2 \sum_{m \neq k} x^{2m+1} \right) + \frac{q-1}{2}y^{-1} \\ &= q + \frac{3(q-1)}{4}y + \frac{q-1}{2}(P(x^2) + N(x^2)) \\ &\quad + \frac{q-1}{2} \left(\sum_{m \neq k} x^{2m+1} \right) + \frac{3(q-1)}{4}y^{-1}.\end{aligned}$$

This shows the sequence $\mathbf{b}$ has the main lobe q , all side lobes are $\frac{q-1}{2}$ except exactly the side lobe pair y and y^{-1} whose values are $\frac{3(q-1)}{4}$. Finally, look at the special case where $k = \frac{q-1}{2}$. This implies that $y = y^{-1}$ as $y = x^{2k+1} = x^q = x^{-q} = y^{-1}$ where powers are computed modulo $2q$. This results in a change to Eq. (14) making

$$\begin{aligned}(P(x^2) + N(x^2))(y + y^{-1}) &= (P(x^2) + G(x^2) - P(x^2) - 1)(y + y^{-1}) \\ &= (G(x^2) - 1)(y + y) \\ &= 2y(G(x^2) - 1).\end{aligned}\quad (16)$$

When Eq. (16) is substituted into Eq. (13), we now get

$$\begin{aligned} \mathbf{bb}^{(-1)} &= q + \frac{q-1}{2}y + \frac{q-1}{2}(P(x^2) + N(x^2)) + \frac{q-1}{4}(2y(G(x^2) - 1)) \\ &\quad + \frac{q-1}{2}y \\ &= q + \frac{q-1}{2}y + \frac{q-1}{2}(P(x^2) + N(x^2)) + \frac{q-1}{2}(y(G(x^2) - 1)) \\ &\quad + \frac{q-1}{2}y \\ &= q + (q-1)y + \frac{q-1}{2} \left[(P(x^2) + N(x^2) + (y(G(x^2) - 1))) \right]. \end{aligned}$$

This calculation shows the main lobe of size q , all side lobes (excluding y) of size $\frac{q-1}{2}$, and a single peak at the midpoint y of size $q-1$. $\square$

The previous construction works for sequences, but generalizes nicely to the more general domain of arrays which we show next. It is our hope that with the rapid advancement of technology that engineering applications that employ sequences could easily be adapted using their higher-dimensional “array” counterparts instead; the advantage being “arrays” embrace a much larger spectrum, in comparison with their “sequence” compeers.

Theorem 8 *Let $H = Z_2 \times G$ be the group where $Z_2 = \langle t \rangle$ and G is the additive group of the finite field $GF(q)$ where $q \equiv 1 \pmod{4}$ is a prime power. Let g be any element of G and P represent the set of all nonzero squares in $GF(q)$. Then, the group ring object, in $Z[H]$ where we write the group H multiplicatively, $W = P(1+tg) + tg$ satisfies that its periodic autocorrelation has the main lobe of size q , side lobes of size $\frac{q-1}{2}$ and exactly two side lobes represented by the elements tg and tg^{-1} of size $\frac{3(q-1)}{4}$. If $g = g^{-1}$ then the two larger side lobes merge together with size $q-1$ at t because $g = g^{-1}$ implies $g = 1$ since $|G|$ is odd and hence $tg = t$.*

Proof Compute the autocorrelation function directly using virtually identical arguments as in Theorem 7. We note that $gG = G$ and $G + tG = H$.

$$\begin{aligned} WW^{(-1)} &= (P(1+tg) + tg) \left(P(1+tg^{-1}) + tg^{-1} \right) \\ &= \left[\left(\frac{q-1}{2} \right) + \left(\frac{q-1}{4} \right) (G-1) \right] [2 + tg + tg^{-1}] + 1 \\ &= \left(\frac{q+1}{2} \right) + \left(\frac{q-1}{2} \right) H + \left(\frac{q-1}{4} \right) (tg + tg^{-1}) \end{aligned}$$

which shows that the main lobe has a value of q , every side lobe that is not represented by tg or tg^{-1} has a value of $\frac{q-1}{2}$ and the two side lobes represented by tg and tg^{-1} have a value of $\frac{3(q-1)}{4}$ and in the case where $tg = tg^{-1} = t$ we get a single side lobe with value $q-1$. Thus, W is the array we set out to describe. $\square$

Table 4 List of first five HEBPSSs with $k = 0$

q (Period = $2q$)	Sequence
5	[0 + + + 0 0 0 + +]
13	[0 + + + 0 0 + + + + 0 0 0 0 0 0 0 + + + + 0 0 + +]
17	[0 + + + + + 0 0 + + 0 0 0 0 0 0 + + + + 0 0 0 0 0 0 + + 0 0 + + + +]
29	[0 + + + 0 0 0 0 + + + + + + + + 0 0 + + 0 0 0 0 0 0 + + 0 0 0 0 + + 0 0 0 0 0 0 + + 0 0 + + + + + + + + 0 0 0 0 + +]
37	[0 + + + 0 0 + + + + 0 0 0 0 + + 0 0 + + + + + + + + 0 0 0 0 0 0 + + 0 0 0 0 0 0 0 0 + + 0 0 0 0 0 0 + + + + + + + + 0 0 + + 0 0 0 0 + + + + 0 0 + +]

Table 5 List of all peak shifts of HEBPSSs with period $2q = 10$

k	Sequence	Periodic autocorrelation
0	[0 + + + 0 0 0 0 + +]	[5 3 2 2 2 2 2 2 2 3]
1	[0 + + + 0 + 0 0 + 0]	[5 2 2 3 2 2 2 3 2 2]
2	[0 0 + + 0 + 0 + + 0]	[5 2 2 2 2 4 2 2 2 2]

Table 6 First two high-energy binary peak shifting arrays

Period	Array
3×6	$\begin{bmatrix} 0 + + + + + \\ 0 0 0 0 + + \\ 0 0 + + 0 0 \end{bmatrix}$
5×10	$\begin{bmatrix} 0 + + + + + + + + + \\ 0 0 0 0 + + 0 0 + + \\ 0 0 0 0 0 0 + + + + \\ 0 0 + + + + 0 0 0 0 \\ 0 0 + + 0 0 + + 0 0 \end{bmatrix}$

3.3.1 List of high-energy binary peak shifting sequences and arrays

In Table 4, we show the first five HEBPS sequences explicitly as binary, $\{0, 1\}$, objects.

To demonstrate the “peak shifting” capability of this construction, in Table 5, we examine the three distinct shifts of the length 10 HEBPSS. Note the translation of the periodic autocorrelation side lobe of value 3 as it approaches the midpoint.

Lastly, we show in Table 6 the first two binary HEBPS arrays.

3.3.2 Conversion from binary $\{0, 1\}$ to $\{-1, 1\}$ sequences (arrays)

The sequences given in Theorem 7 are binary objects and have elements $\{0, 1\}$. It is sometimes useful to convert these $\{0, 1\}$ objects to $\{-1, 1\}$ objects. They are essentially the same sequences where all 0s have been replaced with -1 s. The following proposition gives a connection between the periodic side lobes of binary sequences which have been converted from a $\{0, 1\}$ to $\{-1, 1\}$ sequence.

Proposition 3 A binary $\{0, 1\}$ sequence $\mathbf{a}$ whose periodic autocorrelation function is given by $\mathbf{a}\mathbf{a}^{(-1)} = \sum \lambda_g g$ has a corresponding binary $\{-1, 1\}$ autocorrelation function $\tilde{\mathbf{a}}\tilde{\mathbf{a}}^{(-1)} = \sum \mu_g g$ where

$$\mu_g = |G| - 4(\chi_0(\mathbf{a}) - \lambda_g).$$

In this formula, $|G|$ represents the order of the group G and χ_0 is the principal character of the group G .

Proof Let $\mathbf{a} = \sum a_i g_i$ be a $(0, 1)$ sequence where g_i represents the elements of a group G . Define $P = \{g_i | a_i = 1\}$ and $N = \{g_i | a_i = 0\}$. Note that $\mathbf{a} = P$. Further, let the autocorrelation function of $\mathbf{a}$ be given by $\mathbf{a}\mathbf{a}^{(-1)} = \sum \lambda_g g$. Now, define a new sequence $\tilde{\mathbf{a}} = P - N$ and label its autocorrelation function $\tilde{\mathbf{a}}\tilde{\mathbf{a}}^{(-1)} = \sum \mu_g g$. Now we compute $\tilde{\mathbf{a}}\tilde{\mathbf{a}}^{(-1)}$:

$$\begin{aligned} \tilde{\mathbf{a}}\tilde{\mathbf{a}}^{(-1)} &= \sum \mu_g g \\ &= (2P - G)(2P - G)^{(-1)} \\ &= (2P - G)(2P^{(-1)} - G) \\ &= 4PP^{(-1)} - 2PG - 2P^{(-1)}G + G^2 \\ &= 4\mathbf{a}\mathbf{a}^{(-1)} - 2|P|G - 2|P|G + |G|G \\ &= 4\mathbf{a}\mathbf{a}^{(-1)} - 4|P|G + |G|G \\ &= |G|G - 4\left(|P|G - \mathbf{a}\mathbf{a}^{(-1)}\right) \\ &= |G|G - 4\left(\chi_0(\mathbf{a}) - \sum \lambda_g g\right). \end{aligned}$$

This shows that

$$\sum \mu_g g = |G|G - 4\left(\chi_0(\mathbf{a})G - \sum \lambda_g g\right).$$

We finish by fixing an element $g^* \in G$.

$$\mu_{g^*g^*} = |G|g^* - 4\left(\chi_0(\mathbf{a})g^* - \lambda_{g^*}g^*\right)$$

which shows that the correlations are related to one another by $\mu_{g^*} = |G| - 4(\chi_0(\mathbf{a}) - \lambda_{g^*})$ as claimed. $\square$

We finish this section by writing the sequences in Tables 4, 5, and 6 as binary $\{-1, 1\}$ objects.

Table 7 List of first five HEBPSSs with $k = 0$

q (Period = $2q$)	Sequence
5	$[- + + + - - - - + +]$
13	$[- + + + - - + + + - - - - - + + + - - + +]$
17	$[- + + + + - - + + - - - - - + + + - - - - + + + +]$
29	$[- + + + - - - + + + + + + - - + + - - - - + + -$ $- - - + + - - - - + + - + + + + + + - - - + +]$
37	$[- + + + - + + + - - - + + - + + + + + + - - - - + + - -$ $- - - - + + - - - - + + + + + + - + + - - - + + + - + +]$

Table 8 List of all peak shifts of HEBPSSs with period $2q = 10$

k	Sequence	Periodic autocorrelation
0	$[- + + + - - - + +]$	$[10 +2 -2 -2 -2 -2 -2 -2 +2]$
1	$[- + + + - + - - + -]$	$[10 -2 -2 +2 -2 -2 -2 +2 -2 -2]$
2	$[- - + + - + - + + -]$	$[10 -2 -2 -2 -2 +6 -2 -2 -2 -2]$

Table 9 First two high-energy binary peak shifting arrays

Period	Array
3×6	$\begin{bmatrix} - & + & + & + & + & + \\ - & - & - & - & + & + \\ - & - & + & + & - & - \end{bmatrix}$
5×10	$\begin{bmatrix} - & + & + & + & + & + & + & + & + & + \\ - & - & - & - & + & + & - & - & + & + \\ - & - & - & - & - & + & + & + & + & + \\ - & - & + & + & + & + & - & - & - & - \\ - & - & + & + & - & - & + & + & - & - \end{bmatrix}$

4 Sequences and their performance measures

In this section, we provide our new sequences given above, as well as other known sequences, and present them with computational properties. The properties we provide are: length, energy, energy efficiency, Golay merit factor, peak-to-max side lobe ratio, peak-to-average side lobe ratio, peak-to-neighbor side lobe ratio, spectral peak-to-average ratio, and Doppler tolerance. Many of these properties require the so-called aperiodic correlation function to be calculated for a given sequence. We define this first (Tables 7, 8 and 9).

Definition 7 The aperiodic cross-correlation function of two sequences $\mathbf{a}$ and $\mathbf{b}$, C_{aper} , is defined by

$$C_{\text{aper}}(l) = \sum_{i=0}^{n-1-l} a_i b_{i+l}^*$$

where a_i^* represents the complex conjugate of a_i and $0 \leq l < n$. If $\mathbf{a} = \mathbf{b}$, then this is referred to as the aperiodic autocorrelation function.

Definition 8 The Golay merit factor (GMF) of a sequence $\mathbf{a}$ is defined by

$$\text{GMF} = \frac{C_{\text{aper}}^2(0)}{2 \sum_{l=1}^{n-1} C_{\text{aper}}^2(l)}.$$

Definition 9 The peak-to-max side lobe ratio (PMSR) of a sequence is defined to be

$$\text{PMSR} = \frac{C_{\text{aper}}^2(0)}{\max\{C_{\text{aper}}^2(m)\}}$$

where $m \neq 0$.

Definition 10 The peak-to-average side lobe ratio (PASR) of a sequence is defined to be

$$\text{PASR} = \frac{C_{\text{aper}}^2(0)}{\text{mean}\{C_{\text{aper}}^2(m)\}}.$$

Definition 11 The peak-to-neighbor side lobe ratio (PNSR) of a sequence is defined to be

$$\text{PNSR} = \frac{C_{\text{aper}}(0)}{C_{\text{aper}}^2(1)}.$$

Definition 12 The spectral peak-to-average ratio (SPAR), as described in [9], is calculated in the following way: First we pad the sequence $\mathbf{a}$ with zeros on both sides. Call this new sequence $\mathbf{b} = [0 \ \mathbf{a} \ 0]$. The number of zeros, on both sides, is equal to the sequence length itself. Second, create the vector $F = |fft(\mathbf{b})|^2$ where fft is the fast Fourier transform function in MATLAB. Finally, the SPAR can be found by:

$$\text{SPAR} = \frac{\max\{F\}}{\text{mean}\{\hat{F}\}}$$

where $\hat{F} = F \setminus \max\{F\}$.

Definition 13 The correlation energy of a sequence is defined as

$$\text{CE} = \sum_{l=1}^{n-1} C^2(l).$$

Definition 14 The periodic Golay merit factor (pGMF) of a sequence $\mathbf{a}$ is defined by

$$\text{pGMF} = \frac{C^2(0)}{\sum_{l=1}^{n-1} C^2(l)}.$$

4.1 The performance measures of selected sequences

The data in Table 10 present all of the relevant properties mentioned above for selected sequences. Note that since the aperiodic correlation function of a function is dependent on the sequence's shift, of which there are n , these values have been calculated using the shift providing the largest GMF. From this table, we seek to examine sequences which achieve large E , E_{eff} , GMF, PMSR, PASR, PNSR, and pGMF. We seek all other values to be minimized.

From the sequences examined, we can see that our OHETS sequences provide a good balance between the sequence's energy and its Golay merit factor. In particular, when comparing the OHETS to the Paley it will frequently improve the GMF—never losing much if any. Another big advantage comes from our correlation energy and periodic GMF when examining the Paley sequence to the OHETS sequence of the same length. This is especially true for lengths which are congruent to 1 (*mod* 4) where we see large improvements in performance.

5 Sets of sequences with good properties

5.1 McLaughlin's algorithm for generating good sets of sequences

In this section, we take a look at an algorithm proposed by McLaughlin [9]. Given a set of sequences, it is good for applications if this set has an overall large average GMF, small average SPAR, and small average pairwise aperiodic cross-correlation. Ideally, each sequence should have perfect periodic autocorrelations meaning the periodic side lobes are all zero. McLaughlin's patent suggests to start with a set of m perfect sequences which are all inequivalent in the sense that no sequence in the set can be shifted to be another. For each of these m sequences, he looks for the w shifts which give the smallest values of $\frac{\text{SPAR}}{\text{GMF}}$. The number of shifts, w , which are chosen for each sequence can be made as large or small as wanted though choosing $w > 2$ greatly slows the computational process. Once the best w shifts for each sequence are found, he computes all possible w^m (aperiodic) cross-correlations and in the end chooses the set with the smallest average cross-correlation. It is important to note here that we were not able to match his values for SPAR exactly, and thus, we proceed by taking his starting set and using our SPAR program to generate our own final set of sequences. This algorithm can be summarized by the following steps:

1. Obtain a set of sequences which are pairwise inequivalent by shifting.
2. For each sequence, find the w shifts with smallest $\frac{\text{SPAR}}{\text{GMF}}$.
3. Generate all possible sets of sequences by using the shifts found in step 2.
4. For each set, compute all possible pairwise cross-correlations.

Table 10 Select sequences and their performance measures

Sequence	Length	E	E_{eff}	GMF	PMSR	PASR	PNSR	SPAR	CE	pGMF
HEBPSS	10	10	1	3.8462	25	69.2308	100	2.1454	36	12.7778
Barker	13	13	1	14.0833	169	338	∞	1.9710	12	14.0833
OHETS	13	12	0.9231	4	36	96	36	2.9226	12	12
Singer	15	15	1	4.1667	25	116.6667	∞	2.5079	14	16.0714
OHETS	17	16	0.9412	5.3333	28.4444	170.6667	∞	1.8302	16	16
HEBPSS	26	26	1	2.0485	8.3457	102.4242	8.3457	2.7324	292	2.3151
OHETS	29	28	0.9655	5.9394	49	332.601	∞	2.1505	28	28
Paley	29	29	1	5.128	52.5625	287.1707	210.25	1.8244	140	6.0071
OHETS	31	30	0.9677	6.338	56.25	380.2817	∞	2.0991	30	30
Singer	31	31	1	4.665	38.44	279.9029	240.25	2.2203	30	32.0333
HEBPSS	34	34	1	2	6.8402	132	6.8402	2.7637	516	2.2403
HEBPSS	58	58	1	1.6474	5.3824	187.8041	5.3824	3.2845	1572	2.1399
OHETS	61	60	0.9836	5.7325	100	687.8981	∞	2.2783	60	60
Paley	61	61	1	5.8506	103.3611	702.0755	930.25	2.1296	300	12.4033
Singer	63	63	1	4.6475	110.25	576.2951	992.25	2.9235	62	64.0161
HEBPSS	82	82	1	1.5219	4.9116	246.5568	4.9116	4.0397	3204	2.0986
OHETS	89	88	0.9888	6.1266	215.1111	1078.2784	∞	2.4777	88	88
Paley	89	89	1	5.7565	123.7656	1013.1512	1980.25	1.9846	440	18.0023
HEBPSS	122	122	1	1.4431	4.5811	349.2271	4.5811	3.9264	7204	2.0661
OHETS	127	126	0.9921	5.6579	158.76	1425.7847	∞	2.6818	126	126
Singer	127	127	1	4.2112	133.2975	1061.2292	∞	2.5391	126	128.0079

5. Compare the average cross-correlations and accept the set with smallest average cross-correlation as the final, improved, set.

We start by looking at one example set given by McLaughlin: Below is McLaughlin's starting set of 20 perfect periodic ternary sequences with the main lobe 25 (Tables 11, 12 and 13).

```

- - + + - + + + - + 0 + + + 0 - 0 - + 0 - - + - + + 0 0 + -
- 0 0 - - + - + + 0 + - 0 + 0 + + + 0 - - + + + - + + + - + -
- + - + - 0 + 0 0 + + - - 0 - + + + + + - + + + - 0 + + - 0 -
- + - + - + + - - + 0 + - - - 0 + 0 + + + + + 0 - + 0 0 + + -
- + + + - 0 + 0 0 + - + + 0 + - + - + + + + + - - 0 - - + 0 -
- + + + - + + 0 0 + - 0 + 0 + + - + + + 0 - - - - + - + 0 + -
0 - - - 0 + - + + - 0 + - + - + + + - - + + + 0 0 + 0 + + + -
0 - + + 0 - + + + - + + + + + - 0 - - + + 0 0 + 0 - + - + - -
0 + - - 0 - - + + + + + - + - + 0 + + - + 0 0 + 0 - + + + - -
0 + + - 0 + - + + + 0 - + + + + - + + - - - - 0 0 + 0 + - + -
+ - - + + + + + + - 0 + - + - 0 - 0 - + + - - 0 + + 0 0 + + -
+ - + 0 + 0 0 - - - - + + - + + + + - 0 + + + - + 0 - + + 0 -
+ - + + + - + + + - - 0 + + + 0 + 0 - + 0 + + - + - - 0 0 - -
+ 0 - + + + + - - 0 - + - - - + 0 + 0 + + 0 0 - + + - + + + -
+ 0 0 + + - + - - 0 + - 0 - 0 + + + 0 + - + + + + - + + - -
+ 0 + - + - - - - 0 + + + - + 0 + 0 - + 0 0 + + - + + + - -
+ + 0 0 + - 0 + + + + + 0 + 0 - - - + 0 + - - + + - + - + -
+ + 0 0 + + 0 - - + + - 0 - 0 - + - + 0 - + + + + + + - - + -
+ + + - + + - 0 0 + + 0 + 0 + - - - + - 0 - - + + + + - 0 + -
+ + + 0 + 0 0 + + + - - + + + - + - + 0 - + + - + 0 - - - 0 -

```

McLaughlin's ending set of 20 perfect ternary sequences where $w = 1$ after processing with our program.

```

- + 0 - - + - + + 0 0 + - - - + + - + + + + - + 0 + + + 0 - 0
+ - + + 0 + - 0 + 0 + + + 0 - - + + + - + + + - + - - 0 0 - -
+ + - - 0 - + + + + + - + + + - 0 + + - 0 - - + - + - 0 + 0 0
- - - 0 + 0 + + + + + 0 - + 0 0 + + - + - + - + + - - + 0 +
+ + + + - - 0 - - + 0 - - + + + - 0 + 0 0 + - + + 0 + - + +
+ 0 0 + - 0 + 0 + + - + + + 0 - - - - + - + 0 + - - + + + - +
- + + + - - + + + 0 0 + 0 + + + - 0 - - - 0 + - + + - 0 + - +
0 0 + 0 - + - + - - 0 - + + 0 - + + + - + + + + + - 0 - - + +
+ - + - + 0 + + - + 0 0 + 0 - + + + - - 0 + - - 0 - - + + + +
- 0 0 + 0 + - + - 0 + + - 0 + - + + + 0 - + + + + - + + - -
+ + - - 0 + + 0 0 + + - + - - + + + + + + - 0 + - + - 0 - 0 -
- - - + + - + + + + - 0 + + + - + 0 - + + 0 - + - + 0 + 0 0 -

```

Table 11 McLaughlin starting versus ending set (perfect ternary sequences, period 31, main lobe 25)

Measure	Starting set	Ending set
Average GMF	2.8169	5.3937
Average SPAR	3.0346	2.1086
Average aperiodic cross-correlation	2.4806	2.3919

- - 0 0 - - + - + + + - + + + - - 0 + + + 0 + 0 - + 0 + + - +
 0 - + + - + + + - + 0 - + + + + - - 0 - + - - - + 0 + 0 + + 0
 - 0 + + + 0 + - + + + + - + + - - + 0 0 + + - + - - 0 + - 0
 + - + + + - - + 0 + - + - - - 0 + + + - + + 0 + 0 - + 0 0 +
 + 0 + - - + + - + - + - - + + 0 0 + - 0 + + + + 0 + 0 - - -
 - 0 - 0 - + - + 0 - + + + + + - - + - + + 0 0 + + 0 - - + +
 + + 0 + 0 + - - - + - 0 - - + + + + - 0 + - + + + - + + - 0 0
 + - + 0 - + + - + 0 - - - 0 - + + + 0 + 0 0 + + + - - + + + -

Next, we wish to compare a set of our OHETSs to see if we can match, or beat, the McLaughlin ending set shown above. We create our starting set by selecting our OHETS of period 31 from the construction in Theorem 5. This sequence is given by

[0 + + - + + - + + + + - - - + - + - + + + - - - - + - - + - -].

We then add to this single sequence another which was found by computer search. The second sequence, given by

[0 + + + - + + - - - + + + + - + - - - + - + - - + + + + - + +]

has the property that all of its periodic side lobes are either 1 or -1 (the previous sequence has periodic side lobes which are all -1). We generate our starting set by computing all possible combinations of shifts, negations, and automorphisms of the two sequences and reducing the resulting list by removal of any sequences which are cyclic shifts of another in the list. The definitions for these operations are as follows:

Definition 15 The (cyclic) shift by n places of a sequence $\mathbf{a}$ of period N , written as the group ring object $A = \sum_{m=0}^{N-1} a_m x^m$ where $G = \langle x \rangle$, is

$$Ax^n = \sum_{m=0}^{N-1} a_m x^{m+n}$$

for some integer n .

Definition 16 The negation of a sequence $\mathbf{a}$ of period N , written as the group ring object $A = \sum_{m=0}^{N-1} a_m x^m$ where $G = \langle x \rangle$, is

$$-A = \sum_{m=0}^{N-1} -a_m x^m.$$

Definition 17 An automorphism of a sequence $\mathbf{a}$ of period N , written as the group ring object $A = \sum_{m=0}^{N-1} a_m x^m$ where $G = \langle x \rangle$, is

$$A(x^n) = \sum_{m=0}^{N-1} a_m x^{mn}$$

for some integer n where $\gcd(n, N) = 1$.

These are well-known operations and will preserve the properties of a sequence's periodic autocorrelation function. Using our program based on McLaughlin's algorithm described above, we arrived at our starting set of 20 sequences, from the original two, shown here:

```

0 + + - + + - + + + + - - - + - + - + + + - - - - + - - + - -
0 + + + - + + - - - + + + + - + - - - + - + - - + + + - + +
0 - + - + - + + - - + + + - - - - + - + + + + + + - + + - + +
0 + + + - + + - + + + - - + + + + - + + - - - - + + - + - +
0 - - + + - - + + + - + + + + + - + - - + + + + - - + - + -
0 - - + - - + - + + - + - + + + + + + - - - + + + - + + + - +
0 - + - + - - + + + + + - + - + + + + + - + + + - - + + - -
0 + + - + + - + + + + + + - + - - - + + + - - + + - + - + -
0 + - + - + + - - - - + + - + + + + - + - + + - + - - + + +
0 + - + + + - + + + - - - + + + + + + - + - + + - + - - + -
0 - - - + - - + + + - - - - + - + + + - + - + + - - - + - -
0 + - + - + - - + + - - - + + + + - + - - - - + - - + - -
0 - - + - - + - - - - + + - + - + + + - + - - - + + + - + -
0 + + - - + + - - - + - - - - + - + + - - - - + + - + - +
0 - - + - - - - + + - + - + + + - + - - - + + + - - + - +
0 - + - + - - + + + + + - + - - - - + + - - - + - - + - -
0 - + - - - + - - - + + + - - - - - + - + - - + - + + - + +

```

Table 12 OHETSs starting versus ending set (period 31)

Measure	Starting set	Ending set
Average GMF	2.9640	5.3189
Average SPAR	2.7618	1.9463
Average aperiodic cross-correlation	3.0871	3.0452

The result of our program for producing the ending set produces:

```

- - - + + + + - + - - - + - + - - + + + + - + + 0 + + + - + +
- - + - + + + + + - + + - + + 0 - + - + - + + - - + + + - -
+ + - + + + - - + + + + - + + - - - - + + - + - + 0 + + + -
+ + - + - - + + + + + - - + - + - 0 - - + + - - + + + - + + +
+ - + + - + - + + + + + - - - + + + - + + + - + 0 - - + - -
+ + + - + + + - - + + - - 0 - + - + - - + + + + + - + - + +
- - + + + - - + + - + - + - - 0 + + - + + - + + + + + - + - -
- + + + 0 + - + - + + - - - - - + + - + + + + - - + + + - + +
- - + - - 0 + - + + + + - + + + - - - + + + + + + - + - + + - +
+ + + - - - - + - + + + - + - + + - - - - + - - 0 - - - + - -
+ + - + - - - - - + - - + - - 0 + - + - + - - + + - - - + +
- - + - - - + + - - - - + - - + + + + + - - + - + - 0 - - - +
- - + - + + - - - - - + + - + - + 0 + + - - + + - - - + - - -
- - + - - - 0 - - + - - - - + + - + - + + + - + - - - - + + +
- + - - + - + - - - - - + + + - - - + - - - + - 0 + + - + +
- - - + - - - + + - - + + 0 + - + - + + - - - - - + + - + - -
+ + - - - + + - - + - + - + 0 - - + - - + - - - - - + - + +
+ - - - 0 - + - + - - + + + + - - + - - - - + + - - - + - -
+ + - + + 0 - + - - - + - - - + + + - - - - - + + - - - + - -
- - - - + - - + - - 0 + + - + + - + + + + - - - + - + - + + +

```

The relevant properties for these sets are as follows:

Thus, we are able to provide a set of 20 near-perfect (periodic side lobes are ± 1 instead of 0) sequences with higher energy, lower SPAR, and comparable average GMF and cross-correlations using our OHETSs when compared to McLaughlin's example.

In the same patent, McLaughlin also gives an example of 12 perfect ternary sequences with the main lobe of size 16. His starting set is given by:

```

- - 0 - + 0 0 - + + 0 0 0 0 + - 0 + 0 + 0 0 + 0 0 0 + 0 + + -
- - + - 0 + + - 0 0 0 + 0 + 0 - + 0 + 0 0 0 0 + + 0 0 + 0 0 -
0 0 - 0 0 - + 0 0 0 0 - 0 + + 0 - 0 + 0 0 0 + - + 0 + + + + -

```

Table 13 McLaughlin starting versus ending set (perfect ternary sequences, period 31, main lobe 16)

Measure	Starting set	Ending set
Average GMF	2.5514	5.5013
Average SPAR	2.9595	2.1521
Average aperiodic cross-correlation	1.4409	1.3710

00 − 0 + − 00 + + + − + 000 − + 0 + + + 0 − 0 + 0000 −
 0000 − 0 + 0 − − + 0 + + 000 − + − + + 00 + + 0 + 00 −
 0000 + 0 − 0 + + + 0 + − 000 + − + + + 00 − + 0 − 00 −
 00 + 00 + + 0000 + 0 + − 0 + 0 + 000 − + + 0 − + − − −
 00 + 0 + + 00 + + − + − 000 + + 0 + − − 0 + 0 − 0000 −
 + + 0 + − 00 + − − 0000 + + 0 − 0 − 00 + 000 + 0 + + −
 + + 0 + 000 + 00 − 0 − 0 + + 0000 − − + 00 − + 0 + + −
 + + 0 + 000 + 00 + 0 + 0 − + 0000 + + − 00 + − 0 − − −
 + + + + 0 + − + 000 + 0 − 0 + + 0 − 0000 + − 00 − 00 −

and his ending set, after running through our program, gives:

+ 0000 + − 0 + 0 + 00 + 000 + 0 + + − − − 0 − + 00 − +
 + + 00 + 00 − − − + − 0 + + − 000 + 0 + 0 − + 0 + 0000
 00 − 0 + + 0 − 0 + 000 + − + 0 + + + + − 00 − 00 − + 00
 − 0 + − 00 + + + − + 000 − + 0 + + + 0 − 0 + 0000 − 00
 − + 0 + + 000 − + − + + 00 + + 0 + 00 − 0000 − 0 + 0 −
 00 − 0000 + 0 − 0 + + + 0 + − 000 + − + + + 00 − + 0 −
 0 + 0 + − 0 + 0 + 000 − + + 0 − + − − − 00 + 00 + + 000
 − 0 + 0 − 0000 − 00 + 0 + + 00 + + − + − 000 + + 0 + −
 + 0 + + − + + 0 + − 00 + − − 0000 + + 0 − 0 − 00 + 000
 + 00 − 0 − 0 + + 0000 − − + 00 − + 0 + + − + + 0 + 000
 + − 00 + − 0 − − − + + 0 + 000 + 00 + 0 + 0 − + 0000 +
 + − 00 − 00 − + + + + 0 + − + 000 + 0 − 0 + + 0 − 0000

We attempt to better his set by taking two inequivalent perfect ternary sequences of period 31 with the main lobe 16. These two sequences are given by

$$[- - - 0 - + 00 - + + 0000 + - 0 + 0 + 00 + 000 + 0 + +]$$

and

$$[- 00 + 00 + + 0000 + 0 + - 0 + 0 + 000 - + + 0 - + - -].$$

Using our method of finding all possible shifts, negations, and automorphisms, followed by removing any sequences which are cyclic shifts of another, we can create a starting set of 22 such perfect ternary sequences given by:

```

0 + + - - - 0 - + 0 0 - + + 0 0 0 0 + - 0 + 0 + 0 0 + 0 0 0 +
0 + - + 0 + + + + - 0 0 - 0 0 - + 0 0 0 0 - 0 + + 0 - 0 + 0 0
0 0 - + - + + 0 0 + + 0 + 0 0 - 0 0 0 0 - 0 + 0 - - + 0 + + 0
0 0 + 0 - 0 + + 0 - 0 0 0 0 + - 0 0 - 0 0 - + + + + 0 + - + 0
0 0 - 0 0 + 0 + + 0 0 + + - + - 0 0 0 + + 0 + - - 0 + 0 - 0 0
0 - - + + + 0 + - 0 0 + + - - 0 0 0 0 - + 0 - 0 - 0 0 - 0 0 0 -
0 - + - 0 - - - - + 0 0 + 0 0 + - 0 0 0 0 + 0 - - 0 + 0 - 0 0
0 0 + - + - - 0 0 - - 0 - 0 0 + 0 0 0 0 + 0 - 0 + + - 0 - - 0
0 0 - 0 + 0 - - 0 + 0 0 0 0 - + 0 0 + 0 0 + - - - - 0 - + - 0
0 0 + 0 0 - 0 - - 0 0 - - + - + 0 0 0 - - 0 - + + 0 - 0 + 0 0
0 0 0 0 - - + 0 0 - + 0 + + + - - 0 - 0 0 0 - 0 0 - 0 - 0 + -
0 - + - - - 0 0 + 0 0 + + 0 0 0 0 + 0 + - 0 + 0 + 0 0 0 - + +
0 0 + - + + + 0 0 - + 0 - 0 0 - 0 0 0 0 + 0 - 0 + + + 0 + - 0
0 0 + 0 0 - 0 - 0 + + 0 0 0 0 - - + 0 0 - + 0 + + - + + 0 + 0
0 0 0 0 - 0 0 - 0 + - 0 0 + + + - + 0 0 0 - + 0 + + + 0 - 0 +
0 + - - - 0 0 0 0 + + 0 - 0 - 0 0 + 0 0 0 + 0 + + - + + 0 + - 0
0 + - + + + 0 0 - 0 0 - - 0 0 0 0 - 0 - + 0 - 0 - 0 0 0 + - -
0 0 - + - - - 0 0 + - 0 + 0 0 + 0 0 0 0 - 0 + 0 - - - 0 - + 0
0 0 - 0 0 + 0 + 0 - - 0 0 0 0 + + - 0 0 + - 0 - - + - - 0 - 0
0 0 0 0 + 0 0 + 0 - + 0 0 - - - + - 0 0 0 + - 0 - - - 0 + 0 -
0 - + + 0 0 0 0 - - 0 + 0 + 0 0 - 0 0 0 - 0 - - + - - 0 - + 0
0 - 0 0 0 0 - - 0 0 - 0 0 + + + - + 0 - - + 0 0 0 - 0 - 0 + -

```

After this starting set is processed by our program, we get the following set of 22 perfect ternary sequences:

```

+ 0 0 0 0 + - 0 + 0 + 0 0 + 0 0 0 + 0 + + - - - 0 - + 0 0 - +
0 0 - 0 + + 0 - 0 + 0 0 0 + - + 0 + + + + - 0 0 - 0 0 - + 0 0
- + 0 + + 0 0 0 - + - + + 0 0 + + 0 + 0 0 - 0 0 0 0 - 0 + 0 -
+ - 0 0 - 0 0 - + + + + 0 + - + 0 0 0 + 0 - 0 + + 0 - 0 0 0 0
- 0 + 0 - 0 0 0 0 - 0 0 + 0 + + 0 0 + + - + - 0 0 0 + + 0 + -
- 0 0 0 0 - + 0 - 0 - 0 0 - 0 0 0 - 0 - - + + + 0 + - 0 0 + -
0 0 + 0 - - 0 + 0 - 0 0 0 - + - 0 - - - - + 0 0 + 0 0 + - 0 0
+ - 0 - - 0 0 0 + - + - - 0 0 - - 0 - 0 0 + 0 0 0 0 + 0 - 0 +
- + 0 0 + 0 0 + - - - - 0 - + - 0 0 0 - 0 + 0 - - 0 + 0 0 0 0
+ 0 - 0 + 0 0 0 0 + 0 0 - 0 - - 0 0 - - + - + 0 0 0 - - 0 - +
- + 0 0 - + 0 + + + - - 0 - 0 0 0 - 0 0 - 0 - 0 + - 0 0 0 0 -

```

Table 14 Our starting versus ending set (perfect ternary sequences, period 31, main lobe 16)

Measure	Starting set	Ending set
Average GMF	3.6494	5.4460
Average SPAR	2.6113	2.1714
Average aperiodic cross-correlation	1.6100	1.3783

0 + 0 + - 0 + 0 + 0 0 0 - + + 0 - + - - - 0 0 + 0 0 + + 0 0 0
 0 0 - 0 0 0 0 + 0 - 0 + + + 0 + - 0 0 0 + - + + + 0 0 - + 0 -
 0 0 + 0 0 - 0 - 0 + + 0 0 0 0 - - + 0 0 - + 0 + + - + + 0 + 0
 - 0 + - 0 0 + + + - + 0 0 0 - + 0 + + + 0 - 0 + 0 0 0 0 - 0 0
 + 0 + + - + + 0 + - 0 0 + - - 0 0 0 0 + + 0 - 0 - 0 0 + 0 0 0
 0 - 0 - + 0 - 0 - 0 0 0 + - - 0 + - + + + 0 0 - 0 0 - - 0 0 0
 0 0 + 0 0 0 0 - 0 + 0 - - - 0 - + 0 0 0 - + - - - 0 0 + - 0 +
 0 0 - 0 0 + 0 + 0 - - 0 0 0 0 + + - 0 0 + - 0 - - + - - 0 - 0
 + 0 - + 0 0 - - - + - 0 0 0 + - 0 - - - 0 + 0 - 0 0 0 0 + 0 0
 - 0 - - + - - 0 - + 0 0 - + + 0 0 0 0 - - 0 + 0 + 0 0 - 0 0 0
 - - 0 0 - 0 0 + + + - + 0 - - + 0 0 0 - 0 - 0 + - 0 - 0 0 0 0

The relevant properties are given by Table 14.

In this case, we achieve very similar values to what McLaughlin was able to get. However, we are able to introduce 10 more sequences into the set by looking for negations, shifts, and automorphisms of two inequivalent sequences, thus providing a larger set to work with.

5.2 Modification to McLaughlin's algorithm

Here, we describe our attempt to use a modified version of McLaughlin's algorithm above to generate "good" sets of high-energy sequences. We define a "good" set of sequences to have the properties:

1. Each sequence has a high GMF (a threshold GMF value must be defined).
2. The aperiodic autocorrelation side lobes of each sequence are small (see Eq. 17).
3. The sequences are pairwise orthogonal (See Proposition 4).
4. All sequences in the set are high in energy (we allow a single 0 entry).

To obtain such a set of sequences, we start with a sequence (of OHETS type) from the construction in Theorem 5. This sequence is turned into a set of n sequences, where n is the length of the sequence, by cyclic shifts. This will generate n different sequences. From this set, we remove any sequences whose GMF is below a threshold value. Next, we further trim the list by choosing only sequences whose autocorrelations satisfy that their side lobes all lie at, or less than, -13 dB below the main lobe. That is, we wish to have aperiodic autocorrelation side lobes satisfying, at worse:

$$10 \log \left(\frac{C_{\text{aper}}(i)}{C_{\text{aper}}(0)} \right) \leq -13 \quad (17)$$

where $i = 1, 2, \dots, n - 1$. The sequences which meet this condition are kept, the others are removed. The resulting set is our “good” set. We aim to generate sets where this dB threshold is -15 dB, but will accept a value as large as -13 dB.

Proposition 4 *The nearly pairwise orthogonality of our “good” set is met when the set contains shifts of an OHETS sequence.*

Proof Orthogonality is measured by the usual dot product of two sequences. Thus, we take two sequences in our set, say S_1 and S_2 , and compute their dot product. This is equivalent to finding the main lobe of their cross-correlation. Since these two sequences are shifts of one another, then $S_2 = S_1 x^t$ for some integer t . Thus,

$$S_1 S_2 = S_1 S_1 x^t.$$

It follows that the dot product will simply be a shift, by t units, of any such sequence’s autocorrelation. That is to say that the dot product of two such sequences will be a side lobe of their periodic ACF. By Theorem 5, all periodic side lobes, of a given OHETS, are -1 . Hence, the dot product of two such sequences is -1 . $\square$

Remark 1. Strict orthogonality is when the dot product is 0, but for our sequences this dot product is guaranteed to always be -1 , regardless of length. Thus, as the sequence length increases, the set of sequences become closer to the ideal definition of pairwise orthogonality in an asymptotic sense.

2. The condition of Eq. (17) can be restated in terms of a fraction of the sequence length as it implies that $C_{\text{aper}}(i) = 10^{-1.3} C(0) = \frac{n-1}{10^{1.3}} \approx \frac{n-1}{20}$. Thus, we are looking for side lobes which are no larger than $\frac{1}{20}$ of the main lobe’s value.

5.2.1 Tables of some good high-energy sets

The following tables will provide examples of our so-called good sets from the previous section. We specifically have run exhaustive searches for where the GMF is no lower than 5 and the autocorrelation ratios are no higher than -13 dB. For lengths where multiple good sets are found, we show the one with the smallest autocorrelation ratios. Due to the length of the sequences, we compact them by writing them in hexadecimal format. To achieve this, we follow these steps:

1. Take the first sequence in the set and find the single 0 value.
2. Replace all -1 s in this sequence by 0.
3. Convert this binary $\{0, 1\}$ sequence to a hexadecimal form by using MATLAB’s `binaryVectorToHex()` function.
4. Find the zero locations of the remaining sequences in the set to determine how many positions they are right-cyclically shifted from the first.

Using this process allows us to present only the first sequence in the set. By reversing the process, one can generate the entire set (convert the starting sequence back to binary

Table 15 Starting sequences for good high-energy sets

Length	Starting sequence (hex)
181	1B1508FD1D5C5F88546F3EC99AD251E1021E292D664DF3
251	5916C4D79D95D22B230A6E4BB27D6038E2121F2950883EEF56B07B7B8E3F941
509	1BD68BFED8FC0B750F08E3C19ECD29CA72966F3078E21E15DA07E2F6FFA2D7 B9232657580A76EAB0C8088E5E693A2CA5345C967A7110130D576E501AEA64C49
521	07BAF6477E451B05A34EBEB531A6752F36D9E95CCB195AFAE58B41B144FDC4DEB BC8479221C58112985FD17030D3FC4D475AE2B23FCB0C0E8BFA194881A38449E21
1021	0B97207700E564168D4BC2A3035D79DF2E3176CA45D27B475716F25D129B7463 A7DCF5D6062A1E958B4135380770274EBB5BC99EB41DA775ED873FA2923394AB EA0246BF9E577C4C80C6659168F0F459C67EE020DA1743240930BA16C101DF98 E68BC3C5A26998C04C8FBA9E7F589015F54A7312517F386DEBB96E0B5E64F6B7
1091	02A9C54A5CF2D11F04D981AE7392C04D8F85F349F3D5C299F51CB49D94F4D5CD D5D22A262A686B236963A8335E2A1836982F0726FE5B18C53F326F83BA5862D6 AE355FF84A28304578D30AD7A0E8CA110F0B3D2013FE471328D2ED2B74CC2620 7A0998910403FDF76E66FA1FB9BCCD12B48B4EB371D8037FB432F0F77ACE8FA1 4AF34E15DF3EBADE0
2039	04A8FCE0F8A3909A12E4AE7ED7BE713E3D689E520FD4A8B3BB3B62C73DE7D1 D9AC4E3B9F7596D8AE37B49AF5CBCBBF01161628536909C5724B288311C6E53 23A0C218E5C91911F1756A07DAC374A1C1B8C10A40C56C5BD37B1D707C60756 FB027E732C9D006FB2CCE00B7A71D953A4528E80F04705CD6445F53778E5338 AA64B7FDB06E72B406BF52C102A232CFBCB444C7EAD2F1B48F483A3051128BB DB59E1B0F811558B7BDD2A57F6C2764F9971660D91BC9015AB44212E5577E0F 278652422EB775F3A3ED0ED270B4A81CDD2C20CB3BABF7CB5029FD2B189F24 012D9AAE3358E113505DD94C5F1DF0FE8EB5DA356471A12FF8CCB209FF46CB3 181BF2

form, replace all 0s with -1, puncture the sequence in its “zero location,” and then generate the other by cyclically shifting to the right.). Two tables are presented. The first gives the starting sequence for each length, and the second gives the remaining information (Tables 15 and 16).

All shifts for length 1021:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57
58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84
85 86 87 88 91 92 95 96 98 99 117 118 119 120 121 122 123 124 125 513 514 515
516 517 518 519 520 521 539 540 542 543 546 547 550 551 552 553 554 555 556
557 558 559 560 561 562 563 564 565 566 567 568 569 570 571 572 573 574 575
576 577 578 579 580 581 582 583 584 585 586 587 588 589 590 591 592 593 594
595 596 597 598 599 600 601 602 603 604 605 606 607 608 609 610 611 612 613
614 615 616 617 618 619 620 621 622 623 624 625 626 627 628 629 630 631 632
633 634 635 636 637 638

Table 16 Information for good high-energy sets

Length	Set size	Min GMF	Max corr ratio (dB)	Zero position	Shifts
181	4	5.2976	−13.0103	38	0,10,96,106
251	4	5.0233	−13.1876	52	0,26,122,148
509	8	6.0032	−14.5059	127	0,1,2,4,252,254,255,256
521	2	6.0068	−15.1188	133	0,256
1021	208	5.0204	−15.0345	192	See Below
1091	8	6.0017	−16.0606	268	0,1,2,5,551,554,555,556
2039	504	5.0004	−16.1023	380	See Below

All shifts for length 2039:

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 33
41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67
68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94
95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115
116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134
135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153
154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172
173 174 175 176 177 178 179 180 181 182 183 184 185 186 187 188 189 190 191
192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208 209 210
211 212 213 214 215 216 217 218 219 220 221 222 223 224 225 226 227 228 229
230 231 232 233 234 235 236 237 238 239 240 241 242 243 244 245 246 247 248
249 250 251 252 253 254 255 256 257 258 259 260 261 1019 1020 1021 1022 1023
1024 1025 1026 1027 1028 1029 1030 1031 1032 1033 1034 1035 1036 1037 1038
1039 1040 1041 1042 1043 1044 1045 1046 1047 1048 1049 1050 1051 1052 1053
1054 1055 1056 1057 1058 1059 1060 1061 1062 1063 1064 1065 1066 1067 1068
1069 1070 1071 1072 1073 1074 1075 1076 1077 1078 1079 1080 1081 1082 1083
1084 1085 1086 1087 1088 1089 1090 1091 1092 1093 1094 1095 1096 1097 1098
1099 1100 1101 1102 1103 1104 1105 1106 1107 1108 1109 1110 1111 1112 1113
1114 1115 1116 1117 1118 1119 1120 1121 1122 1123 1124 1125 1126 1127 1128
1129 1130 1131 1132 1133 1134 1135 1136 1137 1138 1139 1140 1141 1142 1143
1144 1145 1146 1147 1148 1149 1150 1151 1152 1153 1154 1155 1156 1157 1158
1159 1160 1161 1162 1163 1164 1165 1166 1167 1168 1169 1170 1171 1172 1173
1174 1175 1176 1177 1178 1179 1180 1181 1182 1183 1184 1185 1186 1187 1188
1189 1190 1191 1192 1193 1194 1195 1196 1197 1198 1199 1200 1201 1202 1203
1204 1205 1206 1207 1208 1209 1210 1211 1212 1213 1214 1215 1216 1217 1218
1219 1220 1221 1222 1223 1224 1225 1226 1227 1228 1229 1230 1231 1232 1233
1234 1235 1236 1237 1238 1239 1247 1251 1252 1253 1254 1255 1256 1257 1258
1259 1260 1261 1262 1263 1264 1265 1266 1267 1268 1269 1270 1271 1272 1273
1274 1275 1276 1277 1278 1279 1280

An exhaustive computer search was used to produce all possible sets of OHETS sequences up to length 2039. Such sets have been created computationally for all prime lengths up to 2048. The above sets were chosen to display because:

- Length 181 is the first set with a maximum correlation ratio of -13dB.
- Lengths 251, 509, 1021, and 2039 are near powers of 2.
- Length 521 is the first set with a maximum correlation ratio of -15dB.
- Length 1091 is the first set with maximum correlation ratio of -16dB.

Remark The above sets would be computationally challenging to get without the theoretical results from Theorem 5. In fact, the theorems presented in this paper were generated after doing exhaustive computational searches for preferable sequences and observing patterns found within them. These initial searches only included lengths up through 32 as the computational complexity grows rapidly. Such searches require the generation and examination of all 2^{n-1} possible sequences of each length n . Once generated, each sequence's autocorrelation function must be calculated, further slowing the process. The theorems above allow for these sets to be generated nearly instantaneously (relatively speaking).

6 Application of OHETSs to frame synchronization

Synchronization technology is an indispensable tool in wireless communication. Good metrics to evaluate the performance of synchronization include: the probability of missing synchronization and false synchronization, and frame synchronization time. In this section, we utilize our new sequences and compare their synchronization performance with known sequences including the well-known Barker codes. The simulation results show that the new sequences do out-perform their previously known counterparts.

Frame synchronization is a method used in digital communication systems to find valid data in a transmission that consists of data frames. To realize frame synchronization, the transmitter inserts a fixed data pattern at the start of each data frame to mark the start of valid data. The receiver searches for the fixed pattern in each data frame and achieves frame synchronization when the correlation between the input data and the fixed pattern is ideal. The probability of missing synchronization and false synchronization, and frame synchronization time are the main factors to influence the performance of the synchronization. See [17] and [8] for more information. For some recent results on comparative analysis of performance of different codes/sequences for frame synchronization based on probability of missing synchronization, see [2].

A MATLAB program was used to simulate the chance of missed synchronization from 0 to 12 dB in steps of 0.5 dB. For each step, a simulation of size 10,000 was used in a Monte Carlo approach. The sequences being compared are in the following table followed by the results.

As Figs. 2 and 3 show, the OHETSs offer performance advantages over both the Barker sequence and the perfect pair sequences of equal lengths (Table 17). This advantage is seen throughout the dB range where the OHETS consistently give a lower probability of missed synchronization. This suggests promising applications of these new OHETS in wireless communication systems.

Fig. 2 Frame synchronization of length 13 OHETS and Barker codes

Fig. 3 Frame synchronization of length 23 OHETS and perfect pair

Table 17 Sequences compared for frame synchronization

Name	Sequence
Barker length 13	[++++-+--+]
OHETS length 13	[0+-+---+--+]
OHETS length 23	[0++++-+--+--]
Perfect pair length 23	$A = [+---+--+--+--+--+--+]$ $B = [+0000+0+00++00++0+0+++]$

Table 18 Optimal sequences of prime lengths with two zeros

Length	Sequence
4	[+ - 0 0]
5	[+ - - 0 0]
6	[+ - 0 - - 0]
7	[+ - - - 0 - 0]
8	[+ - - - 0 + 0 -]
9	[+ - - - - 0 + 0 -]
11	[+ - - - - - + 0 - + 0]
13	[+ - - - - - + - + 0 - 0 +]
15	[+ - - - - - 0 0 - + - + - +]
16	[+ - - - - - 0 + - - 0 + - + - +]
25	[+ 0 + + 0 - - - - + + - + + - + + - + - + -]
27	[+ 0 + + 0 - - - - + - + - - + + - + + + + - + - - +]

7 Remarks and problems

- Remark 1.** We also have found by computer search some sporadic examples of lengths 9 and 15 all of whose side lobes are 1 or -1, making these optimal. The sequences are given by [+ - - - - + - 0 +] and [+ - - - - - + - + + - + 0], respectively.
- By allowing exactly two zeros, we obtain further examples of such ternary sequences of lengths 4, 5, 6, 7, 8, 9, 11, 13, 15, 16, 25, 27 by computer search. The following table contains such examples by length (Table 18):
 - We believe allowing for more zeros would give rise to more such sequences.
 - Circulant weighing matrices (also known as Ipotov's ternary sequences) are infinite families of such sequences whose lengths are $\frac{q^n-1}{q-1}$ where q is any prime power, $n > 1$ is any odd integer, and the number of zeros in the sequence is $\frac{q^{n-1}-1}{q-1}$.
 - Ipotov's sequences are "perfect," in the sense that all side lobes are zero.

Problems

- Generalize the sporadic examples found of lengths 9 and 15 (with a single zero) to infinite families.
- Obtain at least one infinite family of such sequences with exactly two zeros.

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