New Cameron–Liebler line classes with parameter $\frac{q^2+1}{2}$

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Abstract New families of Cameron–Liebler line classes of $\text{PG}(3, q)$, $q \geq 7$ odd, with parameter $(q^2 + 1)/2$ are constructed.

Keywords Cameron–Liebler line class · Tight set · Klein quadric

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1 Introduction

In this paper, we deal with Cameron–Liebler line classes of $\text{PG}(3, q)$. The notion of Cameron–Liebler line class was introduced in the seminal paper [5] with the purpose of classifying those collineation groups of $\text{PG}(3, q)$ having the same number of orbits on points and lines. On the other hand, a classification of Cameron–Liebler line classes would yield a classification of symmetric tactical decompositions of points and lines of $\text{PG}(n, q)$ and that of certain families of weighted point sets of $\text{PG}(3, q)$ [4, 5, 14]. Cameron–Liebler line classes are also related to other combinatorial structures such as two-intersection sets, strongly regular graphs and projective two-weight codes [1, 15].

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Definition 1.1 [5, 14] In $\text{PG}(3, q)$, a *Cameron–Liebler line class* $\mathcal{L}$ with parameter x is a set of lines such that every spread of $\text{PG}(3, q)$ contains exactly x lines of $\mathcal{L}$.

The size of a Cameron–Liebler line class $\mathcal{L}$ equals $x(q^2 + q + 1)$, and the number of lines of $\mathcal{L}$ intersecting a given line ℓ only depends on whether ℓ belongs to $\mathcal{L}$ or does not.

Under the Klein correspondence between the lines of $\text{PG}(3, q)$ and points of a Klein quadric $\mathcal{Q}^+(5, q)$, a Cameron–Liebler line class with parameter x is equivalent to a so-called *x -tight set* of $\mathcal{Q}^+(5, q)$.

Definition 1.2 A set $\mathcal{T}$ of points of $\mathcal{Q}^+(5, q)$ is said to be *x -tight* if

$$|P^\perp \cap \mathcal{T}| = \begin{cases} x(q+1) + q^2 & \text{if } P \in \mathcal{T} \\ x(q+1) & \text{if } P \notin \mathcal{T} \end{cases},$$

where $\perp$ denotes the polarity of $\text{PG}(5, q)$ associated with $\mathcal{Q}^+(5, q)$.

For more results on tight sets of polar spaces, see [1].

There exist trivial examples of Cameron–Liebler line classes $\mathcal{L}$ with parameters $x = 1, 2$ and $x = q^2, q^2 - 1$. A Cameron–Liebler line class with parameter $x = 1$ is either the set of lines through a point or the set of lines in a plane. A Cameron–Liebler line class with parameter $x = 2$ is the union of the two previous examples, if the point is not in the plane [5, 14]. In general, the complement of a Cameron–Liebler line class with parameter x is a Cameron–Liebler line class with parameter $q^2 + 1 - x$ and the union of two disjoint Cameron–Liebler line classes with parameters x and y , respectively, is a Cameron–Liebler line class with parameter $x + y$.

It was conjectured that no other examples of Cameron–Liebler line classes exist [5], but Bruen and Drudge [3] were the first to find a counterexample, namely an infinite family of Cameron–Liebler line classes with parameter $x = (q^2 + 1)/2$, q odd. Bruen–Drudge’s example admits the group $G = \text{P}\Omega^-(4, q)$, stabilizing an elliptic quadric $\mathcal{Q}^-(3, q)$ of $\text{PG}(3, q)$, as an automorphism group.

In [11], Govaerts and Penttila found a Cameron–Liebler line class with parameter $x = 7$ in $\text{PG}(3, 4)$.

Infinite families of Cameron–Liebler line classes with parameter $(q^2 - 1)/2$ were found for $q \equiv 5$ or $9 \pmod{12}$ in [7, 8]. By construction, for a line class $\mathcal{X}$ of such a family there is a fixed plane Π and a fixed point $z \notin \Pi$ such that $\mathcal{X}$ never contains the lines $\mathcal{Y}$ of the plane Π and the lines $\mathcal{Z}$ through the point z . Therefore, $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Z}$ are both examples of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$.

For nonexistence results of Cameron–Liebler line classes, see [10, 13] and references therein.

Recently, by perturbing the Bruen–Drudge’s example, a new infinite family of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, $q \geq 5$ odd, has been constructed independently in [6] and [9]. Such a family admits the stabilizer of a point of $\mathcal{Q}^-(3, q)$ in the group G , say K .

In this paper, we will introduce a new derivation technique for Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, see Theorem 3.9. Applying such a derivation

to the Bruen–Drudge’s example, we construct a new family of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, $q \geq 7$ odd, not equivalent to the examples known so far and admitting a subgroup of K of order $q^2(q + 1)$.

Throughout the paper q is a power of an odd prime.

2 The Bruen–Drudge’s example

In this section, we give a brief description of Bruen–Drudge’s example [3, 6] and of the example constructed in [6, 9]. Let $X_1, \dots, X_4$ be homogeneous projective coordinates in $\text{PG}(3, q)$. Let ω be a non-square element of $\text{GF}(q)$. Let $\mathcal{E}$ be the elliptic quadric of $\text{PG}(3, q)$ with equation $X_1^2 - \omega X_2^2 + X_3 X_4 = 0$ and quadratic form Q . Each point $P \in \mathcal{E}$ lies on q^2 secant lines to $\mathcal{E}$ and so lies on $q + 1$ tangent lines. Let $\mathcal{L}_P$ be a set of $(q + 1)/2$ tangent lines to $\mathcal{E}$ through P , and let E be the set of external lines to $\mathcal{E}$; then the set

$$\bigcup_{P \in \mathcal{E}} \mathcal{L}_P \cup E$$

has size $(q^2 + 1)(q^2 + q + 1)/2$ which is the number of lines of $\text{PG}(3, q)$ in a Cameron–Liebler line class with parameter $(q^2 + 1)/2$. By suitably selecting the sets $\mathcal{L}_P$, it is in fact a Cameron–Liebler line class.

Let $G = \text{P}\Omega^-(4, q)$ be the commutator subgroup of the full stabilizer of $\mathcal{E}$ in $\text{PGL}(4, q)$.

Lemma 2.1 *The group G has three orbits on points of $\text{PG}(3, q)$, i.e., the points of $\mathcal{E}$ and other two orbits O_s and O_n , both of size $q^2(q^2 + 1)/2$.*

The two orbits O_s , O_n correspond to points of $\text{PG}(3, q)$ such that the evaluation of the quadratic form Q is a nonzero square or a non-square in $\text{GF}(q)$, respectively. We say that a point X of $\text{PG}(3, q)$ is a *square point* or a *non-square point* with respect to a given quadratic form F according as the evaluation $F(X)$ is a nonzero square or a non-square of $\text{GF}(q)$.

Lemma 2.2 *In its action on lines of $\text{PG}(3, q)$, the group G has four orbits: two orbits, say $\mathcal{L}_0$ and $\mathcal{L}_1$, both of size $(q + 1)(q^2 + 1)/2$, consisting of lines tangent to $\mathcal{E}$ and two orbits, say $\mathcal{L}_2$ and $\mathcal{L}_3$, both of size $q^2(q^2 + 1)/2$ consisting of lines secant or external to $\mathcal{E}$, respectively.*

The block-tactical decomposition matrix for this orbit decomposition is

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ q & 0 & \frac{q-1}{2} & \frac{q+1}{2} \\ 0 & q & \frac{q-1}{2} & \frac{q+1}{2} \end{bmatrix},$$

and hence the point-tactical decomposition matrix is

$$\begin{bmatrix} \frac{q+1}{2} & \frac{q+1}{2} & q^2 & 0 \\ q+1 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \\ 0 & q+1 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \end{bmatrix}.$$

Simple group-theoretic arguments show that a line of $\mathcal{L}_0$ ($\mathcal{L}_1$) contains q points of O_s (O_n), that a line secant to $\mathcal{E}$ always contains $(q-1)/2$ points of O_s and $(q-1)/2$ points of O_n and that a line external to $\mathcal{E}$ contains $(q+1)/2$ points of O_s and $(q+1)/2$ points of O_n . From the orbit-decompositions above, it is an easy matter to prove the following results:

Lemma 2.3 *Let ℓ be a line that is either secant or external to $\mathcal{E}$. Then, the number of lines of $\mathcal{L}_0$ (or of $\mathcal{L}_1$) meeting ℓ equals $(q+1)^2/2$.*

Lemma 2.4 *Let ℓ be a line of $\mathcal{L}_0$ (resp. $\mathcal{L}_1$). Then,*

- *the number of lines of $\mathcal{L}_0$ (resp. $\mathcal{L}_1$) meeting ℓ in a point equals $q^2 + (q-1)/2$;*
- *the number of lines of $\mathcal{L}_1$ (resp. $\mathcal{L}_0$) meeting ℓ in a point equals $(q+1)/2$;*

Let $\mathcal{L}' = \mathcal{L}_0 \cup \mathcal{L}_3$. Using Lemmas 2.3 and 2.4, it can be seen that $\mathcal{L}'$ is the Bruen–Drudge’s Cameron–Liebler line class constructed in [3].

In particular, $\mathcal{L}'$ has the following three characters with respect to line-sets in planes of $\text{PG}(3, q)$:

$$q^2 + \frac{q+1}{2}, \frac{q(q-1)}{2}, \frac{q(q+1)}{2} + 1,$$

and

$$\frac{q+1}{2}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2} + q + 1,$$

with respect to line-stars of $\text{PG}(3, q)$.

Consider a point of $\mathcal{E}$; since G is transitive on $\mathcal{E}$, we can choose it as the point $U_3 = (0, 0, 1, 0)$. Let π be the plane with equation $X_4 = 0$. Then, π is tangent to $\mathcal{E}$ at the point U_3 . The plane π contains a set $\mathcal{L}'_3$ consisting of q^2 lines of $\mathcal{L}_3$. Let $\mathcal{L}'_2$ be the set of q^2 lines of $\mathcal{L}_2$ through U_3 . In [6], we showed that $\mathcal{L}'' = (\mathcal{L}' \setminus \mathcal{L}'_3) \cup \mathcal{L}'_2$ is a Cameron–Liebler line class with parameter $(q^2 + 1)/2$, admitting the stabilizer of the point U_3 in G , K , as an automorphism group.

In particular, $\mathcal{L}''$ has the following five characters with respect to line-sets in planes of $\text{PG}(3, q)$:

$$\frac{q+1}{2}, \frac{q(q-1)}{2} - 1, \frac{q(q+1)}{2}, \frac{q(q+1)}{2} + q + 1, q^2 + \frac{q-1}{2},$$

and

$$\frac{q+3}{2}, \frac{q(q-1)}{2}, \frac{q(q+1)}{2} + 1, \frac{q(q+1)}{2} + q + 2, q^2 + \frac{q+1}{2},$$

with respect to line-stars of $\text{PG}(3, q)$. It turns out that, if $q > 3$, these characters are distinct from those of a Bruen–Drudge Cameron–Liebler line class.

3 The new family

In this section, we introduce a new derivation technique which will allow us to construct a new infinite family of Cameron–Liebler line classes with parameter $(q^2 + 1)/2$, $q \geq 7$.

With the notation introduced in the previous section, let $\mathcal{E}_\lambda$ be the elliptic quadric with equation $X_1^2 - \omega X_2^2 + \lambda X_4^2 + X_3 X_4 = 0$, $\lambda \in \text{GF}(q)$. Then, the non-degenerate quadrics $\mathcal{E} = \mathcal{E}_0, \mathcal{E}_\lambda, \lambda \in \text{GF}(q) \setminus \{0\}$, together with the plane π satisfying the equation $X_4 = 0$, form a pencil $\mathcal{P}$ of $\text{PG}(3, q)$. The base locus of $\mathcal{P}$ is the point $U_3 = (0, 0, 1, 0)$. Let $\perp$ (resp. $\perp_\lambda$) be the orthogonal polarity associated with $\mathcal{E}$ (resp. $\mathcal{E}_\lambda$). Note that $U_3^{\perp_\lambda} = \pi$, $\forall \lambda \in \text{GF}(q)$.

With a slight abuse of notation, we identify a plane with the set of lines contained in it, and we will denote them in the same manner.

Lemma 3.1 *If $P \in O_s$, then*

$$|P^\perp \cap \mathcal{L}_0| = \begin{cases} q+1 & \text{if } q \equiv -1 \pmod{4} \\ 0 & \text{if } q \equiv 1 \pmod{4} \end{cases},$$

$$|P^\perp \cap \mathcal{L}_1| = \begin{cases} 0 & \text{if } q \equiv -1 \pmod{4} \\ q+1 & \text{if } q \equiv 1 \pmod{4} \end{cases}.$$

If $P \in O_n$, then

$$|P^\perp \cap \mathcal{L}_0| = \begin{cases} 0 & \text{if } q \equiv -1 \pmod{4} \\ q+1 & \text{if } q \equiv 1 \pmod{4} \end{cases},$$

$$|P^\perp \cap \mathcal{L}_1| = \begin{cases} q+1 & \text{if } q \equiv -1 \pmod{4} \\ 0 & \text{if } q \equiv 1 \pmod{4} \end{cases}.$$

Proof It is enough to show that if $t \in \mathcal{L}_0$, then $t^\perp$ belongs either to $\mathcal{L}_0$ or to $\mathcal{L}_1$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$. Since G is transitive on elements of $\mathcal{L}_0$, let t be the line joining U_3 with $U_1 = (1, 0, 0, 0)$. Then, $t^\perp$ is the line joining U_3 with $U_2 = (0, 1, 0, 0)$, which belongs to $\mathcal{L}_0$ if and only if -1 is not a square in $\text{GF}(q)$, i.e., $q \equiv -1 \pmod{4}$. $\square$

Lemma 3.2 *Every line of $\text{PG}(3, q)$ not contained in π and not containing U_3 is tangent to exactly one elliptic quadric of $\mathcal{P}$.*

Proof Let P be a point of the elliptic quadric $\mathcal{E}_\lambda$ of $\mathcal{P}$, different from U_3 . Let T_P be the pencil of lines through P in $P^{\perp_\lambda}$. In order to prove the result, it is enough to show that a line ℓ of T_P is either secant or external to a non-degenerate quadric of $\mathcal{P}$ distinct from $\mathcal{E}_\lambda$. The plane $P^{\perp_\lambda}$ meets π in a line r and it meets $\mathcal{E}_{\lambda'}$, $\lambda' \neq \lambda$, in a

non-degenerate conic $\mathcal{C}_{\lambda'}$, $\lambda' \in \text{GF}(q) \setminus \{\lambda\}$. Then, $P, r, \mathcal{C}_{\lambda'}, \lambda' \in \text{GF}(q) \setminus \{\lambda\}$, form a pencil of quadrics of π . From [12, Table 7.7], r is the polar line of P with respect to $\mathcal{C}_{\lambda'}$. Hence, P is an interior point with respect to $\mathcal{C}_{\lambda'}$ and the result follows. $\square$

Remark 3.3 Note that $\mathcal{E}_\lambda \subseteq \mathcal{O}_s$ if and only if $-\lambda$ is a nonzero square in $\text{GF}(q)$.

Remark 3.4 Let Q_λ be the quadratic form associated with $\mathcal{E}_\lambda$ (then $Q = Q_0$). For a point of π , the evaluation of Q_λ is the same for all $\lambda \in \text{GF}(q)$.

Let $\pi_0 = \mathcal{O}_s \cap \pi$ and $\pi_1 = \mathcal{O}_n \cap \pi$. Then, $|\pi_0| = |\pi_1| = q(q+1)/2$. We need the following result.

Lemma 3.5 *Let R be a point of $\mathcal{E}_\lambda \setminus \{U_3\}$, $\lambda \neq 0$, let ℓ be a line meeting $\mathcal{E}_\lambda$ exactly in R and let $P = \pi \cap \ell$.*

If $P \in \pi_0$, then

$$|\ell \cap \mathcal{E}| = \begin{cases} 0 & \text{if } \lambda \text{ is a non-square in } \text{GF}(q) \\ 2 & \text{if } \lambda \text{ is a square in } \text{GF}(q) \end{cases}.$$

If $P \in \pi_1$, then

$$|\ell \cap \mathcal{E}| = \begin{cases} 2 & \text{if } \lambda \text{ is a non-square in } \text{GF}(q) \\ 0 & \text{if } \lambda \text{ is a square in } \text{GF}(q) \end{cases}.$$

Proof Since there exists a subgroup of K of order q^2 which permutes in a single orbit the q^2 points of an elliptic quadric of $\mathcal{P}$, w.l.o.g., we can choose the point R as the point $(0, 0, -\lambda, 1) \in \mathcal{E}_\lambda$. Then, ℓ is contained in $R^\perp$. Assume that $P \in \pi_0$. Straightforward calculations show that P is a point having coordinates $(x, y, 0, 0)$, where $x^2 - \omega y^2$ is a nonzero square in $\text{GF}(q)$ and the line ℓ , apart from P , contains the points $(\mu x, \mu y, -\lambda, 1)$, $\mu \in \text{GF}(q)$. Note that $(\mu x, \mu y, -\lambda, 1) \in \mathcal{E}$ if and only if $\lambda = \mu^2(x^2 - \omega y^2)$, that is if and only if λ is a square in $\text{GF}(q)$. Analogously, if $P \in \pi_1$. $\square$

Let $\lambda_1 \neq 0$ be a fixed square in $\text{GF}(q)$ and let λ_2 be a fixed non-square in $\text{GF}(q)$. Consider the following sets of lines:

$$\begin{aligned} t_0 &= \{r \in \mathcal{L}_0 : |r \cap \pi_0| = q\}, \quad t_1 = \{r \in \mathcal{L}_1 : |r \cap \pi_1| = q\}, \\ \mathcal{T}_{10} &= \{r \in \mathcal{L}_2 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_0| = 1\}, \\ \mathcal{T}_{11} &= \{r \in \mathcal{L}_3 : |r \cap \mathcal{E}_{\lambda_1}| = 1, |r \cap \pi_1| = 1\}, \\ \mathcal{T}_{20} &= \{r \in \mathcal{L}_3 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_0| = 1\}, \\ \mathcal{T}_{21} &= \{r \in \mathcal{L}_2 : |r \cap \mathcal{E}_{\lambda_2}| = 1, |r \cap \pi_1| = 1\}. \end{aligned}$$

Then, $|t_0| = |t_1| = (q+1)/2$ and $|\mathcal{T}_{10}| = |\mathcal{T}_{11}| = |\mathcal{T}_{20}| = |\mathcal{T}_{21}| = q^2(q+1)/2$.

Let $\mathcal{A} = \mathcal{T}_{11} \cup \mathcal{T}_{20}$ and $\mathcal{B} = \mathcal{T}_{10} \cup \mathcal{T}_{21}$. From Lemma 3.5, we have that $\mathcal{A}$ is a set consisting of $q^2(q+1)$ external lines to $\mathcal{E}$ and $\mathcal{B}$ is a set consisting of $q^2(q+1)$ secant lines to $\mathcal{E}$.

For a line ℓ of $\text{PG}(3, q)$, we define the following line-sets:

$$\mathcal{A}_\ell = \{r \in \mathcal{A} : |r \cap \ell| = 1\}, \mathcal{B}_\ell = \{r \in \mathcal{B} : |r \cap \ell| = 1\}.$$

Remark 3.6 Taking into account Remark 3.4, by construction, we have the following:

- the lines in $\mathcal{T}_{11} \cup t_1$ are all the $(q+1)(q^2+1)/2$ tangent lines to $\mathcal{E}_{\lambda_1}$ having q non-square points with respect to $\mathcal{Q}_{\lambda_1}$;
- the lines in $\mathcal{T}_{10} \cup t_0$ are all the $(q+1)(q^2+1)/2$ tangent lines to $\mathcal{E}_{\lambda_1}$ having q square points with respect to $\mathcal{Q}_{\lambda_1}$;
- the lines in $\mathcal{T}_{20} \cup t_0$ are all the $(q+1)(q^2+1)/2$ tangent lines to $\mathcal{E}_{\lambda_2}$ having q square points with respect to $\mathcal{Q}_{\lambda_2}$;
- the lines in $\mathcal{T}_{21} \cup t_1$ are all the $(q+1)(q^2+1)/2$ tangent lines to $\mathcal{E}_{\lambda_2}$ having q non-square points with respect to $\mathcal{Q}_{\lambda_2}$.

Lemma 3.7 Let ℓ be a line of $\text{PG}(3, q)$ such that $\ell \notin \mathcal{A} \cup \mathcal{B}$, then $|\mathcal{A}_\ell| = |\mathcal{B}_\ell|$.

Proof If ℓ contains U_3 , it is tangent to both $\mathcal{E}_{\lambda_1}$ and $\mathcal{E}_{\lambda_2}$ when $\ell \subset \pi$ and ℓ is not tangent to neither $\mathcal{E}_{\lambda_1}$ or $\mathcal{E}_{\lambda_2}$ when $\ell \not\subset \pi$. If ℓ does not contain U_3 , it is not tangent to $\mathcal{E}_{\lambda_1}$ or $\mathcal{E}_{\lambda_2}$, since ℓ is not contained in $\mathcal{A} \cup \mathcal{B}$. Observe that if ℓ is not tangent neither to $\mathcal{E}_{\lambda_1}$ nor to $\mathcal{E}_{\lambda_2}$, then, from Remark 3.6 and Lemma 2.3, each of the following line-sets: $\mathcal{T}_{11} \cup t_1$, $\mathcal{T}_{20} \cup t_0$, $\mathcal{T}_{10} \cup t_0$, $\mathcal{T}_{21} \cup t_1$, contains $(q+1)^2/2$ lines meeting ℓ in a point. We consider several cases.

Case 1): $|\ell \cap \mathcal{E}_{\lambda_1}| = |\ell \cap \mathcal{E}_{\lambda_2}| = 0$

- Assume first that $|\ell \cap \pi_0| = 1$. In this case, there is exactly one line of t_0 meeting ℓ in a point and there are no lines of t_1 meeting ℓ in a point. It follows that $|\mathcal{A}_\ell| = (q+1)^2/2 + (q+1)^2/2 - 1$. Analogously, there are $(q+1)^2/2$ lines in $\mathcal{T}_{10} \cup t_0$ meeting ℓ and $(q+1)^2/2$ lines in $\mathcal{T}_{21} \cup t_1$ meeting ℓ . Hence, $|\mathcal{B}_\ell| = (q+1)^2/2 + (q+1)^2/2 - 1$.
- If $|\ell \cap \pi_1| = 1$, then there is exactly one line of t_1 meeting ℓ in a point and there are no lines of t_0 meeting ℓ in a point. Hence, we get again $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q(q+2)$.
- If $|\ell \cap \pi_0| = |\ell \cap \pi_1| = (q+1)/2$, then there are $(q+1)/2$ lines of t_1 meeting ℓ in a point and $(q+1)/2$ lines of t_0 meeting ℓ in a point. It follows that $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = (q+1)^2/2 + (q+1)/2 - (q+1)^2/2 - (q+1)/2 = q(q+1)$.

Case 2): $|\ell \cap \mathcal{E}_{\lambda_1}| = 2$, $|\ell \cap \mathcal{E}_{\lambda_2}| = 0$ or $|\ell \cap \mathcal{E}_{\lambda_1}| = 0$, $|\ell \cap \mathcal{E}_{\lambda_2}| = 2$

Repeating the same argument as in Case 1), **a)**, or Case 1), **b)**, according as $|\ell \cap \pi_0| = 1$ or $|\ell \cap \pi_1| = 1$, we obtain $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q(q+2)$.

Case 3): $|\ell \cap \mathcal{E}_{\lambda_1}| = 2$, $|\ell \cap \mathcal{E}_{\lambda_2}| = 2$

Repeating the same argument as in Case 1), **a)**, or Case 1), **b)**, according as $|\ell \cap \pi_0| = 1$ or $|\ell \cap \pi_1| = 1$, we obtain $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q(q+2)$. If $U_3 \in \ell$, then there are $(q+1)/2$ lines of t_1 meeting ℓ in a point and $(q+1)/2$ lines of t_0 meeting ℓ in a point. It follows that $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q(q+1)$.

Case 4): $|\ell \cap \mathcal{E}_{\lambda_1}| = |\ell \cap \mathcal{E}_{\lambda_2}| = 1$

There are two possibilities: either $\ell \in t_0$ or $\ell \in t_1$.

- $\ell \in t_0$. In this case, from Remark 3.6 and Lemma 2.4, each of the following line-sets: $\mathcal{T}_{10} \cup t_0$, $\mathcal{T}_{20} \cup t_0$, contains $q^2 + (q-1)/2$ lines meeting ℓ in a point.

Analogously, each of the following line-sets: $\mathcal{T}_{11} \cup t_1$, $\mathcal{T}_{21} \cup t_1$, has $(q+1)/2$ elements meeting ℓ in a point. On the other hand, t_0 contains $(q-1)/2$ lines intersecting ℓ in a point and t_1 contains $(q+1)/2$ lines intersecting ℓ in a point. Hence, $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q^2$.

b) $\ell \in t_1$. As in the previous case, we get again $|\mathcal{A}_\ell| = |\mathcal{B}_\ell| = q^2$.

The proof is now complete. $\square$

Lemma 3.8 *Let ℓ be a line of $\text{PG}(3, q)$.*

- If $\ell \in \mathcal{A}$, then $|\mathcal{A}_\ell| = \frac{3q^2+3q-2}{2}$ and $|\mathcal{B}_\ell| = \frac{q^2+3q}{2}$.
- If $\ell \in \mathcal{B}$, then $|\mathcal{A}_\ell| = \frac{q^2+3q}{2}$ and $|\mathcal{B}_\ell| = \frac{3q^2+3q-2}{2}$.

Proof Assume first that $\ell \in \mathcal{A}$ and in particular that $\ell \in \mathcal{T}_{11}$. From Remark 3.6 and Lemma 2.4, there are $q^2 + (q-1)/2$ lines of $\mathcal{T}_{11} \cup t_1$ meeting ℓ in a point, whereas, there are $(q+1)/2$ lines of $\mathcal{T}_{10}$ meeting ℓ in a point. Also, from Remark 3.6 and Lemma 2.3, each of the following line-sets: $\mathcal{T}_{20} \cup t_0$, $\mathcal{T}_{21} \cup t_1$ contains $(q+1)^2/2$ lines meeting ℓ in a point. Since there is exactly one line of t_1 meeting ℓ in a point and there are no lines of t_0 meeting ℓ in a point, it follows that $|\mathcal{A}_\ell| = q^2 + (q-1)/2 - 1 + (q+1)^2/2 = (3q^2 + 3q - 2)/2$ and $|\mathcal{B}_\ell| = (q+1)^2/2 + (q+1)/2 - 1 = (q^2 + 3q)/2$. Similarly, if $\ell \in \mathcal{T}_{20}$.

If $\ell \in \mathcal{B}$, by repeating the previous arguments, we obtain the desired result.

We are ready to prove the following result.

Theorem 3.9 *Let $\mathcal{L}$ be a Cameron–Liebler line class with parameter $(q^2+1)/2$ such that $\mathcal{A} \subset \mathcal{L}$ and $|\mathcal{B} \cap \mathcal{L}| = 0$. Then, the set $\tilde{\mathcal{L}} = (\mathcal{L} \setminus \mathcal{A}) \cup \mathcal{B}$ is a Cameron–Liebler line class with parameter $(q^2+1)/2$.*

Proof Since $\mathcal{L}$ is a Cameron–Liebler line class with parameter $(q^2+1)/2$, we have that $|\{r \in \mathcal{L} : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q+1)(q^2+1)/2$ if $\ell \in \mathcal{L}$, or $(q+1)(q^2+1)/2$ if $\ell \notin \mathcal{L}$.

Let ℓ be a line of $\text{PG}(3, q)$.

- If $\ell \in \mathcal{L} \setminus (\mathcal{A} \cup \mathcal{B})$, then $\ell \in \tilde{\mathcal{L}}$. From Lemma 3.7, it follows that $|\{r \in \tilde{\mathcal{L}} : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q+1)(q^2+1)/2$.
- If $\ell \notin \mathcal{L} \cup \mathcal{A} \cup \mathcal{B}$, then $\ell \notin \tilde{\mathcal{L}}$. From Lemma 3.7, it follows that $|\{r \in \tilde{\mathcal{L}} : |r \cap \ell| \geq 1\}|$ equals $(q+1)(q^2+1)/2$.
- If $\ell \in \mathcal{A}$, then $\ell \in \mathcal{L} \setminus \tilde{\mathcal{L}}$. From Lemma 3.8, we have that $|\{r \in \tilde{\mathcal{L}} : |r \cap \ell| \geq 1\}|$ equals $q^2 + (q+1)(q^2+1)/2 - (3q^2 + 3q - 2)/2 + (q^2 + 3q)/2 - 1 = (q+1)(q^2+1)/2$.
- If $\ell \in \mathcal{B}$, then $\ell \in \tilde{\mathcal{L}} \setminus \mathcal{L}$. From Lemma 3.8, we have that $|\{r \in \tilde{\mathcal{L}} : |r \cap \ell| \geq 1\}|$ equals $(q+1)(q^2+1)/2 + (3q^2 + 3q - 2)/2 - (q^2 + 3q)/2 + 1 = q^2 + (q+1)(q^2+1)/2$.

The proof is now complete. $\square$

We consider $\mathcal{L}$ being $\mathcal{L}'$ and we denote by $\mathcal{L}''' = (\mathcal{L}' \setminus \mathcal{A}) \cup \mathcal{B}$. Then, from Theorem 3.9, $\mathcal{L}'''$ is a Cameron–Liebler line class with parameter $(q^2+1)/2$.

In what follows, we show that $\mathcal{L}'''$ is left invariant by a group of order $q^2(q+1)$. We shall find it helpful to associate to a projectivity of $\text{PGL}(4, q)$ a matrix of $\text{GL}(4, q)$. We shall consider the points as column vectors, with matrices acting on the left.

Proposition 3.10 *The Cameron–Liebler line class $\mathcal{L}'''$ admits a subgroup of K of order $q^2(q+1)$.*

Proof Let Ψ be the subgroup of $\text{PGL}(4, q)$ whose elements are associated to the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 2x & -2\omega y & 1 & \omega y^2 - x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $x, y \in \text{GF}(q)$. Then, Ψ is an elementary abelian group of order q^2 . Easy computations show that if $g \in \Psi$ and $P \in \mathcal{E}_\lambda$, then $P^g \in \mathcal{E}_\lambda$ and Ψ acts transitively on $\mathcal{E}_\lambda \setminus \{U_3\}$, $\lambda \in \text{GF}(q)$; furthermore, the evaluation of $Q_{\lambda'}$, $\lambda' \neq \lambda$, is the same for both P and P^g .

Let Φ be the subgroup of $\text{PGL}(4, q)$ whose elements are associated with the following matrices:

$$\begin{pmatrix} z & wt & 0 & 0 \\ t & z & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix},$$

where $z, t, u \in \text{GF}(q)$ are such that $z^2 - wt^2 = u^2$. Note that the previous equation holds true if and only if the line of $\text{PG}(2, q)$ joining the points $(0, 0, 1)$ and $(z, t, 0)$ is secant to the conic $\mathcal{D} : X_1^2 - wX_2^2 - X_3^2 = 0$. Since $(0, 0, 1)$ is an interior point with respect to $\mathcal{D}$, we have that up to a scalar factor there are exactly $q+1$ triples (z, t, u) such that $z^2 - wt^2 = u^2$. Hence, $|\Phi| = q+1$. Easy computations show that if $g \in \Phi$ and $P \in \mathcal{E}_\lambda$, then $P^g \in \mathcal{E}_\lambda$; furthermore, the evaluation of $Q_{\lambda'}$, $\lambda' \neq \lambda$, for the point P is a square if and only if it is a square for P^g .

Since both Ψ and Φ stabilize U_3 , we have that the direct product $\Gamma = \Psi \times \Phi$ is a group of order $q^2(q+1)$ fixing the point U_3 and the plane π . Hence, Γ is a subgroup of K .

The group Γ has the orbits π_0, π_1 and $\{U_3\}$ on π . It follows that the stabilizer in Γ of a point of $\pi \setminus \{U_3\}$ has order two. Let t be a line of $\text{PG}(3, q)$ not contained in π and not containing U_3 . From Lemma 3.2, the line t is tangent to exactly one elliptic quadric $\mathcal{E}_\lambda$ at a point R . The stabilizer of t in Γ has to fix R and $t \cap \pi$. Hence, it has order at most two. On the other hand, $(RU_3)^\perp$ does not depend on λ and Γ contains the involutory biaxial homology of G fixing pointwise the lines RU_3 and $(RU_3)^\perp$. Hence, $|t^\Gamma|$ equals $q^2(q+1)/2$ and Γ fixes $\mathcal{L}'''$. $\square$

Let us denote by O_s^i and O_n^i , the sets of size $q^2(q^2+1)/2$ corresponding to points of $\text{PG}(3, q)$ such that the evaluation of the quadratic form Q_{λ_i} , $i = 1, 2$, is a nonzero square or a non-square in $\text{GF}(q)$, respectively.

Proposition 3.11 *The characters of $\mathcal{L}'''$, with respect to line-sets in planes of $\text{PG}(3, q)$ form a subset of:*

$$\left\{ q^2 + \frac{q+1}{2}, q^2 - \frac{3(q+1)}{2}, \frac{q(q-1)}{2} + 3(q+1), \frac{q(q-1)}{2} + 2(q+1), \right. \\ \left. \frac{q(q-1)}{2} + q + 1, \frac{q(q-1)}{2}, \frac{q(q-1)}{2} - (q+1), \frac{q(q+1)}{2} - 2(q+1) \right\}.$$

Proof The plane π contains $q^2 + (q+1)/2$ lines of $\mathcal{L}'''$.

Let σ be a plane distinct from π .

If $\sigma \cap \pi \in \mathcal{L}_0$, then σ contains q lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$.

If $\sigma \cap \pi \in \mathcal{L}_1$, then σ contains q lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$.

If σ is a plane distinct from π and not containing U_3 , then $\sigma = P^{\perp_{\lambda_i}}$, for some $P \in \mathcal{E}_{\lambda_i} \setminus \{U_3\}$. In particular, we may assume that $P = (0, 0, -\lambda_i, 1)$.

Let $\sigma = P^{\perp}$, with $P \in \mathcal{E} \setminus \{U_3\}$.

Consider the tactical configuration whose points are the q^2 planes tangent to $\mathcal{E}$ at some point of $\mathcal{E} \setminus \{U_3\}$ and whose blocks are the $q^2(q+1)/2$ lines contained in $\mathcal{T}_{11}$ (in $\mathcal{T}_{20}$). It turns out that σ contains $q+1$ lines of $\mathcal{T}_{11}$ ($\mathcal{T}_{20}$). Since σ contains $q^2 + (q+1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q^2 - 3(q+1)/2$ lines of $\mathcal{L}'''$.

Let $\sigma = P^{\perp_{\lambda_1}}$, $P \in \mathcal{E}_{\lambda_1} \setminus \{U_3\}$.

Assume that $q \equiv 1 \pmod{4}$. Taking into account Lemma 3.1, we have that σ contains no line of $\mathcal{L}_0$.

If $\lambda_1 - \lambda_2$ is a non-square, then $(P^{\perp_{\lambda_1}})^{\perp_{\lambda_2}} \in O_n^2$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{21}$ and $q+1$ lines of $\mathcal{T}_{20}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$.

If $\lambda_1 - \lambda_2$ is a square, then $(P^{\perp_{\lambda_1}})^{\perp_{\lambda_2}} \in O_s^2$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{20}$ and $q+1$ lines of $\mathcal{T}_{21}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$.

Assume that $q \equiv -1 \pmod{4}$. Taking into account Lemma 3.1, we have that σ contains $q+1$ lines of $\mathcal{L}_0$.

If $\lambda_1 - \lambda_2$ is a non-square, then $(P^{\perp_{\lambda_1}})^{\perp_{\lambda_2}} \in O_n^2$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{21}$ and $q+1$ lines of $\mathcal{T}_{20}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$.

If $\lambda_1 - \lambda_2$ is a square, then $(P^{\perp_{\lambda_1}})^{\perp_{\lambda_2}} \in O_s^2$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{20}$ and $q+1$ lines of $\mathcal{T}_{21}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$.

Let $\sigma = P^{\perp_{\lambda_2}}$, $P \in \mathcal{E}_{\lambda_2} \setminus \{U_3\}$.

Assume that $q \equiv 1 \pmod{4}$. Taking into account Lemma 3.1, we have that σ contains $q+1$ lines of $\mathcal{L}_0$.

If $\lambda_2 - \lambda_1$ is a non-square, then $(P^{\perp_{\lambda_2}})^{\perp_{\lambda_1}} \in O_n^1$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{11}$ and $q+1$ lines of $\mathcal{T}_{10}$. Since σ contains

$q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + 2(q+1)$ lines of $\mathcal{L}'''$.

If $\lambda_2 - \lambda_1$ is a square, then $(P^{\perp_{\lambda_2}})^{\perp_{\lambda_1}} \in O_s^1$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{10}$ and $q+1$ lines of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$.

Assume that $q \equiv -1 \pmod{4}$. Taking into account Lemma 3.1, we have that σ contains no line of $\mathcal{L}_0$.

If $\lambda_2 - \lambda_1$ is a non-square, then $(P^{\perp_{\lambda_2}})^{\perp_{\lambda_1}} \in O_n^1$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{10}$ and $q+1$ lines of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 - (q+1)$ lines of $\mathcal{L}'''$.

If $\lambda_2 - \lambda_1$ is a square, then $(P^{\perp_{\lambda_2}})^{\perp_{\lambda_1}} \in O_s^1$. The plane σ contains $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{11}$ and $q+1$ lines of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$.

Let $\sigma = P^{\perp_{\lambda_3}}$, $P \in \mathcal{E}_{\lambda_3} \setminus \{U_3\}$, $\lambda_3 \in \text{GF}(q) \setminus \{0, \lambda_1, \lambda_2\}$.

Taking into account Lemma 3.1, we have that $(P^{\perp_{\lambda_3}})^{\perp} \in O_s$ if and only if λ_3 is a square in $\text{GF}(q)$.

Assume that λ_3 is a square in $\text{GF}(q)$. The following possibilities arise:

- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are squares in $\text{GF}(q)$ and then $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_s^1$ and $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_s^2$. If $q \equiv -1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$. If $q \equiv 1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$;
- $\lambda_3 - \lambda_1$ is a non-square and $\lambda_3 - \lambda_2$ is a square in $\text{GF}(q)$, then $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_n^1$ and $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_s^2$. If $q \equiv -1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{11}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 - (q+1)$ lines of $\mathcal{L}'''$. If $q \equiv 1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + 2(q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_3 - \lambda_1$ is a square and $\lambda_3 - \lambda_2$ is a non-square in $\text{GF}(q)$, then $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_s^1$ and $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_n^2$. If $q \equiv -1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + 3(q+1)$ lines of $\mathcal{L}'''$. If $q \equiv 1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{11}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 - 2(q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are non-squares in $\text{GF}(q)$ and then $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_1}} \in O_n^1$ and $(P^{\perp_{\lambda_3}})^{\perp_{\lambda_2}} \in O_n^2$. If $q \equiv -1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$. Since σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$. If $q \equiv 1 \pmod{4}$, σ contains $q+1$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{11}$. Since σ contains $q(q-1)/2$ lines of $\mathcal{L}'$, we have that σ contains $q(q-1)/2$ lines of $\mathcal{L}'''$;

Assume that λ_3 is a non-square in $\text{GF}(q)$. Arguing as above, we have the following possibilities:

- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are squares in $\text{GF}(q)$. In this case, σ contains $q(q-1)/2$ or $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1$ is a non-square and $\lambda_3 - \lambda_2$ is a square in $\text{GF}(q)$. In this case, σ contains $q(q-1)/2 - 2(q+1)$ or $q(q-1)/2 + 3(q+1)$ lines of $\mathcal{L}'''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1$ is a square and $\lambda_3 - \lambda_2$ is a non-square in $\text{GF}(q)$. In this case, σ contains $q(q-1)/2 + 2(q+1)$ or $q(q-1)/2 - (q+1)$ lines of $\mathcal{L}'''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
- $\lambda_3 - \lambda_1, \lambda_3 - \lambda_2$ are non-squares in $\text{GF}(q)$. In this case, σ contains $q(q-1)/2$ or $q(q-1)/2 + q + 1$ lines of $\mathcal{L}'''$, according as $q \equiv -1 \pmod{4}$ or $q \equiv 1 \pmod{4}$.

The proof is now complete. $\square$

Proposition 3.12 *The characters of $\mathcal{L}'''$, with respect to line-stars of $\text{PG}(3, q)$ form a subset of:*

$$\left\{ \frac{q+1}{2}, \frac{5(q+1)}{2}, \frac{q(q+1)}{2} - 2(q+1), \frac{q(q+1)}{2} - (q+1), \right. \\ \left. \frac{q(q+1)}{2}, \frac{q(q+1)}{2} + q + 1, \frac{q(q+1)}{2} + 2(q+1), \frac{q(q+1)}{2} + 3(q+1) \right\}.$$

Proof Through the point U_3 , there pass $(q+1)/2$ lines of $\mathcal{L}'''$.

Let $P \in \pi_0$. Through P , there pass q lines of $\mathcal{T}_{10}$ and of $\mathcal{T}_{20}$ and no line of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$. Since P is on $q(q+1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + q + 1$ lines of $\mathcal{L}'''$.

Let $P \in \pi_1$. Through P , there pass q lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$ and no line of $\mathcal{T}_{10}$ and of $\mathcal{T}_{20}$. Since P is on $q(q+1)/2$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2$ lines of $\mathcal{L}'''$.

Let $P \in \mathcal{E} \setminus \{U_3\}$. Consider the tactical configuration whose points are the q^2 points of $\mathcal{E} \setminus \{U_3\}$ and whose blocks are the $q^2(q+1)/2$ lines contained in $\mathcal{T}_{10}$ (in $\mathcal{T}_{21}$). It turns out that through P there pass $q+1$ lines of $\mathcal{T}_{10}$ ($\mathcal{T}_{21}$). Since P is on $q+1$ lines of $\mathcal{L}'$, we have that P is on $5(q+1)/2$ lines of $\mathcal{L}'''$.

Let $P \in \mathcal{E}_{\lambda_1} \setminus \{U_3\}$.

Assume that $q \equiv 1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_1} \subseteq O_s$.

If $\lambda_2 - \lambda_1$ is a square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through P , there pass $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{21}$ and $q+1$ lines of $\mathcal{T}_{20}$. Since P is on $q(q+1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2$ lines of $\mathcal{L}'''$.

If $\lambda_2 - \lambda_1$ is a non-square, then $\mathcal{E}_{\lambda_1} \subseteq O_n^2$. Through P , there pass $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{20}$ and $q+1$ lines of $\mathcal{T}_{21}$. Since P is on $q(q+1)/2 + q + 1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + 2(q+1)$ lines of $\mathcal{L}'''$.

Assume that $q \equiv -1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_1} \subseteq O_n$.

If $\lambda_2 - \lambda_1$ is a square, then $\mathcal{E}_{\lambda_1} \subseteq O_s^2$. Through P there pass $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{21}$ and $q+1$ lines of $\mathcal{T}_{20}$. Since P is on $q(q+1)/2$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 - (q+1)$ lines of $\mathcal{L}'''$.

If $\lambda_2 - \lambda_1$ is a non-square, then $\mathcal{E}_{\lambda_1} \subseteq O_n^2$. Through P there pass $(q+1)/2$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{10}$, no line of $\mathcal{T}_{20}$ and $q+1$ lines of $\mathcal{T}_{21}$. Since P is on $q(q+1)/2$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'''$.

Let $P \in \mathcal{E}_{\lambda_2} \setminus \{U_3\}$.

Assume that $q \equiv 1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_2} \subseteq O_n$.

If $\lambda_1 - \lambda_2$ is a square, then $\mathcal{E}_{\lambda_2} \subseteq O_s^1$. Through P , there pass $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{11}$ and $q+1$ lines of $\mathcal{T}_{10}$. Since P is on $q(q+1)/2$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'''$.

If $\lambda_1 - \lambda_2$ is a non-square, then $\mathcal{E}_{\lambda_2} \subseteq O_n^1$. Through P , there pass $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{10}$ and $q+1$ lines of $\mathcal{T}_{11}$. Since P is on $q(q+1)/2$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 - (q+1)$ lines of $\mathcal{L}'''$.

Assume that $q \equiv -1 \pmod{4}$. Taking into account Remark 3.3, we have that $\mathcal{E}_{\lambda_2} \subseteq O_s$.

If $\lambda_1 - \lambda_2$ is a square, then $\mathcal{E}_{\lambda_2} \subseteq O_s^1$. Through P , there pass $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{11}$ and $q+1$ lines of $\mathcal{T}_{10}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + 2(q+1)$ lines of $\mathcal{L}'''$.

If $\lambda_1 - \lambda_2$ is a non-square, then $\mathcal{E}_{\lambda_2} \subseteq O_n^1$. Through P there pass $(q+1)/2$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{21}$, no line of $\mathcal{T}_{10}$ and $q+1$ lines of $\mathcal{T}_{11}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2$ lines of $\mathcal{L}'''$.

Let $P \in \mathcal{E}_{\lambda_3} \setminus \{U_3\}$, $\lambda_3 \in \text{GF}(q) \setminus \{0, \lambda_1, \lambda_2\}$.

Taking into account Remark 3.3, we have that $P \in O_s$ if and only if $-\lambda_3$ is a square in $\text{GF}(q)$.

Assume that $-\lambda_3$ is a square in $\text{GF}(q)$. The following possibilities arise:

- $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are squares in $\text{GF}(q)$ and then $P \in O_s^1 \cap O_s^2$. Through P , there pass $q+1$ lines of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{11}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'''$;
- $\lambda_1 - \lambda_3$ is a square and $\lambda_2 - \lambda_3$ is a non-square in $\text{GF}(q)$ and then $P \in O_s^1 \cap O_n^2$. Through P , there pass $q+1$ lines of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{11}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + 3(q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_1 - \lambda_3$ is a non-square and $\lambda_2 - \lambda_3$ is a square in $\text{GF}(q)$ and then $P \in O_n^1 \cap O_s^2$. Through P , there pass $q+1$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{20}$ and no line of $\mathcal{T}_{21}$ and of $\mathcal{T}_{10}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 - (q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are non-squares in $\text{GF}(q)$ and then $P \in O_n^1 \cap O_n^2$. Through P , there pass $q+1$ lines of $\mathcal{T}_{11}$ and of $\mathcal{T}_{21}$ and no line of $\mathcal{T}_{20}$ and of $\mathcal{T}_{10}$. Since P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'$, we have that P is on $q(q+1)/2 + q+1$ lines of $\mathcal{L}'''$.

Assume that $-\lambda_3$ is a non-square in $\text{GF}(q)$. Arguing as above, we have the following possibilities:

- $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are squares in $\text{GF}(q)$ and then $P \in O_s^1 \cap O_s^2$. In this case, P is on $q(q+1)/2$ lines of $\mathcal{L}'''$;

- $\lambda_1 - \lambda_3$ is a square and $\lambda_2 - \lambda_3$ is a non-square in $\text{GF}(q)$ and then $P \in O_s^1 \cap O_n^2$. In this case, P is on $q(q+1)/2 + 2(q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_1 - \lambda_3$ is a non-square and $\lambda_2 - \lambda_3$ is a square in $\text{GF}(q)$ and then $P \in O_n^1 \cap O_s^2$. In this case, P is on $q(q+1)/2 - 2(q+1)$ lines of $\mathcal{L}'''$;
- $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$ are non-squares in $\text{GF}(q)$ and then $P \in O_n^1 \cap O_n^2$. In this case, P is on $q(q+1)/2$ lines of $\mathcal{L}'''$.

The proof is now complete. $\square$

Theorem 3.13 *If $q \geq 7$ odd, the Cameron–Liebler line class $\mathcal{L}'''$ is not equivalent to one of the previously known examples.*

Proof From the proof of Proposition 3.12, if $P \in \mathcal{E} \setminus \{U_3\}$, through P there pass $5(q+1)/2$ lines of $\mathcal{L}'''$. Since, for $q \geq 7$, the value $5(q+1)/2$ does not appear among the characters of $\mathcal{L}'$ and $\mathcal{L}''$, we may conclude that $\mathcal{L}'''$ is distinct from $\mathcal{L}'$ and $\mathcal{L}''$. On the other hand, both examples $\mathcal{X} \cup \mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Z}$ admit $q^2 + q + 1$ as a character, but from Propositions 3.11 and 3.12, such a value does not appear as a character of $\mathcal{L}'''$. $\square$

Remark 3.14 Let $\square_q$ denote the set of nonzero square elements of $\text{GF}(q)$. From the proof of Proposition 3.11 and of Proposition 3.12, if there exist $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \text{GF}(q) \setminus \{0, \lambda_1, \lambda_2\}$ such that $a_1, a_2, a_3, a_4 \in \square_q, b_1, b_2, b_3, b_4 \notin \square_q$ and $a_1 - \lambda_1 \in \square_q, a_1 - \lambda_2 \in \square_q, a_2 - \lambda_1 \in \square_q, a_2 - \lambda_2 \notin \square_q, a_3 - \lambda_1 \notin \square_q, a_3 - \lambda_2 \in \square_q, a_4 - \lambda_1 \notin \square_q, a_4 - \lambda_2 \notin \square_q, b_1 - \lambda_1 \in \square_q, b_1 - \lambda_2 \in \square_q, b_2 - \lambda_1 \in \square_q, b_2 - \lambda_2 \notin \square_q, b_3 - \lambda_1 \notin \square_q, b_3 - \lambda_2 \in \square_q, b_4 - \lambda_1 \notin \square_q, b_4 - \lambda_2 \notin \square_q$, then $\mathcal{L}'''$ has exactly eight characters with respect to line-sets in planes of $\text{PG}(3, q)$ (line-stars of $\text{PG}(3, q)$).

Remark 3.15 Note that both, $\mathcal{L}'$ and $\mathcal{L}''$, are Cameron–Liebler line classes satisfying the requirements of Theorem 3.9 and that, starting from $\mathcal{L}'$ or $\mathcal{L}''$, the replacement technique described in Theorem 3.9 can be iterated $(q-1)/2$ times.

Remark 3.16 Computations performed with Magma [2] suggest that starting from $\mathcal{L}'$ and applying Theorem 3.9 (multiple derivation is allowed), apart from Bruen–Drudge’s example and the example described in [6, 9], we get what follows. The notation α^i in the character strings below stands for: there are i planes containing α lines of the Cameron–Liebler line class or there are i line-stars containing α lines of the Cameron–Liebler line class.

$q = 7$: A new example arises having the characters:

- (i) $13^{49}, 21^{126}, 29^{77}, 37^{98}, 45^{49}, 53$ with respect to line-sets in planes of $\text{PG}(3, 7)$ and $4, 12^{49}, 20^{98}, 28^{77}, 36^{126}, 44^{49}$ with respect to line-stars of $\text{PG}(3, 7)$.

$q = 9$: Three new examples arise having the characters:

- (i) $16^{81}, 36^{207}, 46^{288}, 56^{81}, 66^{162}, 86$ with respect to line-sets in planes of $\text{PG}(3, 9)$ and $5, 25^{162}, 35^{81}, 45^{288}, 55^{207}, 75^{81}$ with respect to line-stars of $\text{PG}(3, 9)$;
 (ii) $26^{162}, 36^{207}, 46^{126}, 56^{162}, 66^{162}, 86$ with respect to line-sets in planes of $\text{PG}(3, 9)$ and $5, 25^{162}, 35^{162}, 45^{126}, 55^{207}, 65^{162}$ with respect to line-stars of $\text{PG}(3, 9)$;

- (iii) $26^{162}, 36^{126}, 46^{288}, 56^{162}, 76^{81}, 86$ with respect to line-sets in planes of $\text{PG}(3, 9)$ and $5, 15^{81}, 35^{162}, 45^{288}, 55^{126}, 65^{162}$ with respect to line-stars of $\text{PG}(3, 9)$.

$q = 11$: Five new examples arise having the characters:

- (i) $31^{121}, 43^{121}, 55^{308}, 67^{429}, 79^{242}, 91^{121}, 103^{121}, 127$ with respect to line-sets in planes of $\text{PG}(3, 11)$ and $6, 30^{121}, 42^{121}, 54^{242}, 66^{429}, 78^{308}, 90^{121}, 102^{121}$ with respect to line-stars of $\text{PG}(3, 11)$, see Remark 3.14;
- (ii) $43^{242}, 55^{550}, 67^{187}, 79^{121}, 91^{242}, 103^{121}, 127$ with respect to line-sets in planes of $\text{PG}(3, 11)$ and $6, 30^{121}, 42^{242}, 54^{121}, 66^{187}, 78^{550}, 90^{242}$ with respect to line-stars of $\text{PG}(3, 11)$;
- (iii) $43^{242}, 55^{429}, 67^{308}, 79^{363}, 115^{121}, 127$ with respect to line-sets in planes of $\text{PG}(3, 11)$ and $6, 18^{121}, 54^{363}, 66^{308}, 78^{429}, 90^{242}$ with respect to line-stars of $\text{PG}(3, 11)$;
- (iv) $31^{121}, 43^{242}, 55^{187}, 67^{187}, 79^{484}, 91^{242}, 127$ with respect to line-sets in planes of $\text{PG}(3, 11)$ and $6, 42^{242}, 54^{484}, 66^{187}, 78^{187}, 90^{242}, 102^{121}$ with respect to line-stars of $\text{PG}(3, 11)$;
- (v) $19^{121}, 55^{429}, 67^{308}, 79^{363}, 91^{242}, 127$ with respect to line-sets in planes of $\text{PG}(3, 11)$ and $6, 42^{242}, 54^{363}, 66^{308}, 78^{429}, 114^{121}$ with respect to line-stars of $\text{PG}(3, 11)$;

Interestingly, if we start from $\mathcal{L}''$, we get the complements of the abovementioned examples. In general, it seems a difficult task to determine how many inequivalent examples of Cameron–Liebler line classes arise from Theorem 3.9 and we leave it as an open problem.

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