A family of edge-transitive Cayley graphs

Jiangmin Pan¹ · Zhaofei Peng¹

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Abstract Edge-transitive graphs of order a prime or a product of two distinct primes with any positive integer valency, and of square-free order with valency at most 7 have been classified by a series of papers. In this paper, a complete classification is given of edge-transitive Cayley graphs of square-free order with valency less than the smallest prime divisor of the order. This leads to new constructions of infinite families of both arc-regular Cayley graphs and edge-regular Cayley graphs (so half-transitive). Also, as by-products, it is proved that, for any given positive integers $k, s \geq 1$ and $m, n \geq 2$, there are infinitely many arc-regular normal circulants of valency $2k$ and order a product of s primes, and there are infinitely many edge-regular normal metacirculants of valency $2m$ and order a product of n primes; such arc-regular and edge-regular examples are also specifically constructed.

Keywords Edge-transitive graph · Half-transitive graph · Arc-regular graph · Edge-regular graph · Normal Cayley graph

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✉ Jiangmin Pan
jmpan@ynu.edu.cn

¹ School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, People's Republic of China

1 Introduction

For a graph Γ , if its full automorphism group $\text{Aut}\Gamma$ is transitive on its vertex set, edge set or arc set, then Γ is called *vertex-transitive*, *edge-transitive* or *arc-transitive*, respectively. An arc-transitive graph is edge-transitive. In particular, if $\text{Aut}\Gamma$ is regular on the arc set or the edge set of Γ , then Γ is called *arc-regular* or *edge-regular*, respectively; if $\text{Aut}\Gamma$ is transitive on both vertex set and edge set of Γ , but intransitive on the arc set of Γ , then Γ is called *half-transitive*.

Edge-transitive graphs of square-free order (namely, not divisible by any prime square) provide a rich source of many interesting families of graphs, and have a rich research history. In 1971, Chao [6] classified arc-transitive graphs of prime order p , then Cheng and Oxley [7] classified edge-transitive graphs of order $2p$, and Wang and Xu [29] classified arc-transitive graphs of order $3p$. These results were generalized to the case of order a product of two distinct primes by Praeger et al. [26, 27]. Moreover, edge-transitive graphs of square-free order and valency at most 7 have been classified by [9, 17, 19, 20]. Quite recently, Li et al. [18] characterized the ‘basic’ edge-transitive graphs (namely, each nontrivial normal subgroup of the edge-transitive automorphism group has at most two orbits on the vertex set) of square-free order, with certain cases needing further research. It seems difficult to approach a general classification of edge-transitive graphs of square-free order. Actually, the classification of edge-transitive graphs of order $6p$ with p a prime has not been reached yet.

A graph Γ is called a *Cayley graph* of a group R if there is a subset $S \subseteq R \setminus \{1\}$, with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, such that its vertex set equals R and two vertices g and h are adjacent in Γ if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\text{Cay}(R, S)$. For convenience, a Cayley graph of a cyclic group is also called a *circulant*, and a Cayley graph of a metacyclic group is also called a *metacirculant*. It is well known that a graph Γ is isomorphic to a Cayley graph of a group R if and only if $\text{Aut}\Gamma$ contains a subgroup which is isomorphic to R and acts regularly on the vertex set of Γ , see [3, Proposition 16.3]. Let $\Gamma = \text{Cay}(R, S)$. Then, the right regular presentation

$$\hat{R} = \{\hat{g} \mid \hat{g} : x \mapsto xg, \text{ for all } g, x \in R\}$$

is a subgroup of $\text{Aut}\Gamma$, refer to [13]. If $\hat{R}$ is normal in an automorphism group $G \leq \text{Aut}\Gamma$, then Γ is called a *G -normal Cayley graph* of R . In particular, Γ is called a *normal Cayley graph* of R if $\hat{R}$ is normal in $\text{Aut}\Gamma$.

The purpose of this paper is to give a complete classification of edge-transitive Cayley graphs of square-free order with valency less than the smallest prime divisor of the order of the graph, which in particular leads to new constructions of infinite family of both arc-regular and edge-regular (so half-transitive) Cayley graphs. Also, by using the outcomes, the paper will prove two theorems regarding arc-regular and edge-regular normal Cayley graphs with any prescribed even valency.

To state our results, we need to introduce some terminologies and notations (which are standard) used in this paper. For example, for a positive integer n , we use $\mathbb{Z}_n$, D_n with n even, A_n and S_n to denote the cyclic group and the dihedral group of order n , the alternating group and the symmetric group of degree n , respectively, and use $\mathbb{Z}_n^*$ to denote the multiplicative group of the residue class ring modular n (then $\mathbb{Z}_n^* \cong \text{Aut}(\mathbb{Z}_n)$

is abelian). Given two groups N and H , denote by $N \times H$ the direct product of N and H , by $N.H$ an extension of N by H , and if such an extension is split, then we write $N : H$ instead of $N.H$. Also, denote by K_n the complete graph of order n , by $V\Gamma$ the vertex set of a graph Γ , by $\Gamma(\alpha)$ the set of vertices adjacent to the vertex α in Γ , and if Γ is regular, denote by $\text{val}(\Gamma)$ its valency.

The following is a complete classification of the edge-transitive Cayley graphs of our case. For convenience, certain graphs appearing in the theorem are introduced in Sect. 2.

Theorem 1.1 *Let $\Gamma = \text{Cay}(R, S)$ be a connected G -edge-transitive Cayley graph of square-free order n with $\hat{R} \leq G \leq \text{Aut}\Gamma$, and suppose the valency of Γ is less than the smallest prime divisor of n . Then one of the following statements holds.*

1. $R = \langle a \rangle \cong \mathbb{Z}_n$ is cyclic, and

$$\Gamma = \text{Cay}\left(R, a^H\right), \quad \text{with } \tau \in H \leq \text{Aut}(\langle a \rangle),$$

where τ is an automorphism of R mapping a to a^{-1} . Further, the following statements hold.

- (i) If n is a prime and $H \cong \mathbb{Z}_{n-1}$, then $\Gamma = K_n$, $G = \mathbb{Z}_n : \mathbb{Z}_{\frac{n}{2}}$ or $\mathbb{Z}_n : \mathbb{Z}_{n-1}$, and $\text{Aut}\Gamma = S_n$.
- (ii) If n is a prime and $H < \mathbb{Z}_{n-1}$ or n is not a prime, then Γ is an arc-regular normal Cayley graph of R , $\text{val}(\Gamma) = |H|$, and

$$\text{Aut}\Gamma = \hat{R} : H = \begin{cases} G, & \text{if } \tau \in G; \\ G : \langle \tau \rangle \cong G : \mathbb{Z}_2, & \text{if } \tau \notin G. \end{cases}$$

2. $R = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_h \times (\mathbb{Z}_t : \mathbb{Z}_l)$ such that the center $Z(\langle b \rangle : \langle c \rangle) = 1$,

$$\Gamma = \text{Cay}(R, \{ac^z, (ac^z)^{-1}\}^K) \quad \text{with } (z, l) = 1,$$

and $\text{val}(\Gamma) = 2|K|$, where $K \leq \mathbb{Z}_h^* \times \mathbb{Z}_t^*$ is a subgroup of $\text{Aut}(R)$. Further, $\text{Aut}\Gamma = G = \hat{R} : K$ and Γ is an edge-regular (so half-transitive) normal Cayley graph of R with the only exception: $(h, t, l) = (1, 11, 5)$ and $\text{val}(\Gamma) = 4$.

3. $R \cong \mathbb{Z}_{11} : \mathbb{Z}_5$, $G \cong \mathbb{Z}_{11} : \mathbb{Z}_{10}$, $\text{Aut}\Gamma \cong \text{PGL}(2, 11)$, and Γ is arc-transitive of valency 4.
4. G is almost simple, and $R, G, \text{val}(\Gamma)$ and $\text{Aut}\Gamma$ are listed in Table 1, where $G(q^d)$ is a regular subgroup of $\Gamma\text{L}(1, q^d)$, P_1 is a parabolic subgroup of $\text{PSL}(d, q)$, $q = p^e$, p is a prime and $\alpha \in V\Gamma$.
5. Either
 - (i) $G \cong (\text{D}_{2m} \times \text{PSL}(2, 23)).o$ with $o \leq \mathbb{Z}_2$, and $\Gamma \cong C_m \times \Sigma$ is of valency 8, where $m \mid n$ with the smallest prime divisor at least 13, C_m is a cycle of length m , and Σ satisfies Row 10 of Table 1 with valency 4; or
 - (ii) $G \cong (M \times \text{PSL}(2, 59)).o$ with $o \leq \mathbb{Z}_2$, and $\Gamma \cong \Delta \times \Sigma$, where (Δ, M) (as (Γ, G) there) satisfies part (1) or (2) with $\text{val}(\Delta) = 2$ or 4, and Σ satisfies Row 11 of Table 1 such that $\text{val}(\Delta) \cdot \text{val}(\Sigma)$ is less than the smallest prime divisor of $|V\Delta|$ and 29.

Table 1 G -edge-transitive graphs Γ with G almost simple

Row	G	R	G_α	Γ	$\text{Aut } \Gamma$	Conditions
1	A_n	$\mathbb{Z}_n$	A_{n-1}	K_n	S_n	n odd prime
2	S_n	$\mathbb{Z}_n$	S_{n-1}	K_n	S_n	
3	$\text{PSL}(d, q).o$	$G(q^d)$	$P_1.o$	$K_{\frac{q^d-1}{q-1}}$	$S_{\frac{q^d-1}{q-1}}$	$\frac{q^d-1}{q-1}$ odd prime, $o \leq \mathbb{Z}_{(d, q-1)}.\mathbb{Z}_e$
4	$\text{PSL}(2, 11)$	$\mathbb{Z}_{11}$	A_5	K_{11}	S_{11}	
5	M_{11}	$\mathbb{Z}_{11}$	M_{10}	K_{11}	S_{11}	
6	M_{23}	$\mathbb{Z}_{23}$	M_{22}	K_{23}	S_{23}	
7	$\text{PSL}(2, p)$	$\mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}$	D_{p+1}	$\text{val}(\Gamma) = \frac{p+1}{4}$	$\text{PGL}(2, p)$	$\frac{p-1}{2}$ odd prime
8	$\text{PSL}(2, 11)$	$\mathbb{Z}_{11} : \mathbb{Z}_5$	A_4	$\text{val}(\Gamma) = 4$	$\text{PGL}(2, 11)$	
9	$\text{PGL}(2, 11)$	$\mathbb{Z}_{11} : \mathbb{Z}_5$	S_4	$\text{val}(\Gamma) = 4$	$\text{PGL}(2, 11)$	
10	$\text{PSL}(2, 23)$	$\mathbb{Z}_{23} : \mathbb{Z}_{11}$	S_4	$\text{val}(\Gamma) = 4, 6, 8$	$\text{PSL}(2, 23)$	
11	$\text{PSL}(2, 59)$	$\mathbb{Z}_{59} : \mathbb{Z}_{29}$	A_5	$\text{val}(\Gamma) = 6, 10, 12, 20$	$\text{PSL}(2, 59)$	

Remark For an automorphism $\sigma \in \text{Aut}(R)$, the Cayley graphs $\text{Cay}(R, S)$ and $\text{Cay}(R, S^\sigma)$ are isomorphic. A Cayley graph $\Gamma = \text{Cay}(R, S)$ is called a *CI-graph* (‘CI’ stands for *Cayley Isomorphism*) if, whenever $\text{Cay}(R, T) \cong \text{Cay}(R, S)$, we have $T = S^\sigma$ for some automorphism $\sigma \in \text{Aut}(R)$. By [15, Theorem 4.1], if R is of odd order and $(|R|, |(\text{Aut } \Gamma)_\alpha|) = 1$, where α is a vertex of Γ , then Γ is a CI-graph. Therefore, graphs in Theorem 1.1 are CI-graphs, and consequently, one may enumerate the non-isomorphic arc-regular and edge-regular Cayley graphs of order n appearing in parts (1) and (2) of Theorem 1.1.

A graph Γ is called *locally primitive* if the vertex stabilizer $(\text{Aut } \Gamma)_\alpha$ is primitive on $\Gamma(\alpha)$ for each vertex α of Γ . Locally primitive graphs are edge-transitive. For a positive integer s , an s -arc of a graph Γ is a sequence $v_0, v_1, \dots, v_s$ of $s+1$ vertices such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. If $\text{Aut } \Gamma$ is transitive on the set of s -arcs of Γ , but not transitive on the set of $(s+1)$ -arcs, then Γ is simply called s -transitive.

Theorem 1.1 has the following corollary which shows that the connected locally primitive Cayley graphs of odd square-free order with ‘small’ valency are rare.

Corollary 1.2 *Let Γ be a connected locally primitive Cayley graph of square-free order n . Suppose further the valency of Γ is less than the smallest prime divisor of n . Then Γ is either 1-transitive or 2-transitive, and further, the triple $(\Gamma, n, \text{Aut } \Gamma)$ is listed in Table 2 for $\alpha \in V\Gamma$.*

By definition, half-transitive graphs, arc-regular graphs, edge-regular graphs and normal Cayley graphs are closely related to their full automorphism groups. Since determining the full automorphism groups (generally very difficult) of given graphs is one of the fundamental topics in the field of algebraic graph theory, constructing each of the above classes of graphs is an interesting topic in the literature. For arc-regular graphs, connected ones with valency 2 are just cycles, quite a lot is now known are with small valency (especially valency 3 and 4), see [2, 11, 12, 21, 23] for examples. For any even valency, there is an infinite family of arc-regular graphs of order a prime with cyclic stabilizer by Chao [6] and independently by Alspach [1].

Part (1)(ii) of Theorem 1.1 provides a new infinite family of arc-regular normal circulants of square-free order with abelian vertex stabilizer. We may actually further prove the following theorem.

Table 2 Locally primitive Cayley graphs Γ

Γ	n	$\text{Aut } \Gamma$	$(\text{Aut } \Gamma)_\alpha$	Transitivity	Conditions
K_n	n	S_n	S_{n-1}	2-transitive	n odd prime
$\text{val}(\Gamma) = 4$	55	$\text{PGL}(2, 11)$	S_4	2-transitive	
$\text{val}(\Gamma) = 4$	253	$\text{PSL}(2, 23)$	S_4	2-transitive	
$\text{val}(\Gamma) = 6, 10$	3481	$\text{PSL}(2, 59)$	A_5	2-transitive	
$\text{val}(\Gamma) = \frac{p+1}{4}$	$\frac{p(p-1)}{2}$	$\text{PGL}(2, p)$	D_{p+1}	1-transitive	$\frac{p-1}{2}, \frac{p+1}{4}$ primes

Theorem 1.3 *For any given positive integers k and s , there are infinitely many arc-regular normal circulants of valency $2k$ and order a product of s primes.*

We remark that the proof of Theorem 1.3 includes a specific construction of such arc-regular circulants, see Construction 7.2 below.

The known constructions of edge-regular Cayley graphs are very few. Clearly, edge-regular Cayley graphs are half-transitive, but the inverse is generally not true. The study of half-transitive graphs was initiated by a question of Tutte [30] regarding the existence of half-transitive graphs with even valency, after he proved that a vertex-transitive and edge-transitive graph with odd valency is arc-transitive. Since connected graphs with valency 2 are cycles, which are arc-transitive, the valency of half-transitive graphs must be even and at least 4. Quite a lot known examples of half-transitive graphs are in the 4-valent case, see [4, 8, 22, 23] for references.

Motivated by Theorem 1.3, it is natural to ask: Are there infinitely many edge-regular Cayley graphs with any even valency at least 4? The following theorem answers it affirmatively and the corresponding examples are constructed (see Construction 7.3 below). Note that edge-transitive Cayley graphs of abelian groups are arc-transitive and so never be edge-regular.

Theorem 1.4 *For any given positive integers $k, t \geq 2$, there are infinitely many edge-regular normal metacirculants (so half-transitive) of valency $2k$ and order a product of t primes.*

After this introductory section, some preliminary results are given in Sect. 2, and two technical lemmas regarding the normal subgroups of the edge-transitive automorphism groups are proved in Sect. 3. Then, by investigating the soluble group case and the almost simple group case in Sects. 4 and 5, we finish the proofs of Theorem 1.1 and Corollary 1.2 in Sect. 6. Finally, in Sect. 7, we present proofs of Theorems 1.3 and 1.4.

2 Preliminaries

2.1 Some background results

For a group G , the *Fitting subgroup* of G is the largest nilpotent normal subgroup of G . Clearly, the Fitting subgroup is a characteristic subgroup. The following is a basic property of the Fitting subgroups of soluble groups.

Lemma 2.1 [28, P. 30, Corollary] *Let G be a soluble group and F the Fitting subgroup of G . Then $F \neq 1$ and the centralizer $C_G(F) \leq F$.*

Given a group G and its subgroup H , let $[G : H] = \{Hg \mid g \in G\}$, the set of right cosets of H in G . The *coset action* of G on $[G : H]$ is defined by:

$$(Hg)^x = Hgx, \text{ for all } g, x \in G.$$

This coset action is faithful if and only if $\bigcap_{g \in G} H^g = 1$ (refer to [10, Example 1.3.4]). In this case, H is called *core-free* in G .

Suppose σ and τ are two actions of a group G on sets Ω and Δ , respectively. These actions are *permutation equivalent* if there is a bijection $\lambda : \Omega \rightarrow \Delta$ such that

$$\lambda(\alpha^{\sigma(g)}) = (\lambda(\alpha))^{\tau(g)}, \text{ for all } \alpha \in \Omega \text{ and } g \in G.$$

It is well known that each transitive action of a group G is permutation equivalent to the coset action of G on $[G : G_\alpha]$. The following observation will be used in the later discussion.

Lemma 2.2 *Let $G = H_1 \times \cdots \times H_s$ be a group acting transitively on a set Ω , with $s \geq 2$, such that $G_\alpha = (H_1)_\alpha \times \cdots \times (H_s)_\alpha$ for $\alpha \in \Omega$. Set $\Omega_i = [H_i : (H_i)_\alpha]$ for $i = 1, 2, \dots, s$. Then, the action of G on Ω is permutation equivalent to the following ‘product action’ of G on the cartesian product $\Omega_1 \times \cdots \times \Omega_s$:*

$$((H_1)_\alpha g_1, \dots, (H_s)_\alpha g_s)^{x_1 x_2 \cdots x_s} = ((H_1)_\alpha g_1 x_1, \dots, (H_s)_\alpha g_s x_s),$$

where $g_i, x_i \in H_i$ for $i = 1, 2, \dots, s$.

Actually, we may identify the action of G on Ω with the coset action of G on $[G : G_\alpha]$. By assumption, any elements $g, x \in G$ can be uniquely expressed as $g = g_1 g_2 \cdots g_s$ and $x = x_1 x_2 \cdots x_s$ with $g_i, x_i \in H_i$. Then, it is easy to show that

$$\lambda : G_\alpha g_1 g_2 \cdots g_s \rightarrow ((H_1)_\alpha g_1, (H_2)_\alpha g_2, \dots, (H_s)_\alpha g_s)$$

is a bijection of $[G : G_\alpha]$ to $\Omega_1 \times \cdots \times \Omega_s$ ensuring the equivalence.

Taking normal quotient graphs is a typical method for studying vertex-transitive graphs. Suppose Γ is a G -vertex-transitive graph, where $G \leq \text{Aut} \Gamma$ has an intransitive normal subgroup N . Denote by

$$V\Gamma_N := \{\alpha^N \mid \alpha \in V\Gamma\},$$

the set of N -orbits in $V\Gamma$. The *normal quotient graph* Γ_N of Γ induced by N is defined with vertex set $V\Gamma_N$ and two vertices $B, C \in V\Gamma_N$ are adjacent if and only if some vertex in B is adjacent in Γ to some vertex in C . If $\text{val}(\Gamma) = \text{val}(\Gamma_N)$, then Γ is called a *normal cover* (or *regular cover*) of Γ_N .

The truth of the following lemma is easily justified.

Lemma 2.3 *Let Γ be a connected G -vertex-transitive and G -edge-transitive graph, where $G \leq \text{Aut} \Gamma$ has an intransitive normal subgroup N . Then, the following statements hold:*

1. Γ_N is a connected G/K -edge-transitive graph, where K is the kernel of G acting on $V\Gamma_N$.
2. $K \leq \prod_{B \in V\Gamma_N} K^B$.
3. If Γ is a normal cover of Γ_N , then N is semi-regular on $V\Gamma$.

2.2 Normal Cayley graphs

For a Cayley graph $\Gamma = \text{Cay}(R, S)$, then $\hat{R}$ and

$$\text{Aut}(R, S) = \{\sigma \in \text{Aut}(R) \mid S^\sigma = S\}$$

are subgroups of $\text{Aut}\Gamma$ (note that $\text{Aut}(R, S)$ is also a subgroup of the automorphism group $\text{Aut}(R)$). Further, the following nice property holds.

Lemma 2.4 [13, Lemma 2.1] *Let $\Gamma = \text{Cay}(R, S)$ be a Cayley graph. Then, the normalizer $N_{\text{Aut}\Gamma}(\hat{R}) = \hat{R} : \text{Aut}(R, S)$.*

Now, suppose $\Gamma = \text{Cay}(R, S)$ is a G -normal edge-transitive Cayley graph, where $\hat{R} \triangleleft G \leq \text{Aut}\Gamma$. By Lemma 2.4, $G \leq \hat{R} : \text{Aut}(R, S)$, and so $G_1 \leq \text{Aut}(R, S)$ has at most two orbits on the neighbor set $\Gamma(\mathbf{1}) = S$, where $\mathbf{1}$ denotes the vertex of Γ corresponding to the identity element of R . Observing that, for $s \in S$, if there is $g \in G_1$ such that $s^g = s^{-1}$, then $(\mathbf{1}, s)^{g\hat{s}} = (\mathbf{1}, s^{-1})^{\hat{s}} = (s, \mathbf{1})$, it follows that Γ is G -arc-transitive. This suggests the following construction of normal edge-transitive Cayley graphs.

Construction 2.5 *Let R be a group. Take $H \leq \text{Aut}(R)$ and $s \in R \setminus \{1\}$. Let*

$$S = \left\{ s^H, \left(s^{-1} \right)^H \right\}, \quad \text{and } \Gamma = \text{Cay}(R, S).$$

Then Γ is a G -normal edge-transitive Cayley graph, where $G = \hat{R} : H$. Further, Γ is G -half-transitive if $s^{-1} \notin s^H$, and Γ is G -arc-transitive if $s^{-1} \in s^H$. In particular, Γ is connected if and only if $\langle s^H \rangle = R$.

Conversely, each normal edge-transitive Cayley graph can be constructed as above.

Construction 2.5 provides a method to construct ‘many’ normal edge-transitive Cayley graphs. However, it is generally difficult to know whether the graphs are really normal Cayley graphs or not, due to the difficulty for finding their full automorphism groups.

2.3 Product graphs

Let $\Delta_1, \Delta_2, \dots, \Delta_s$ be graphs with $s \geq 2$. Define the *product graph* $\Delta_1 \times \Delta_2 \times \dots \times \Delta_s$ with vertex set $V\Delta_1 \times V\Delta_2 \times \dots \times V\Delta_s$, and two vertices $u = (u_1, u_2, \dots, u_s)$ and $v = (v_1, v_2, \dots, v_s)$ are adjacent if and only if u_i is adjacent to v_i in Δ_i for each $i = 1, 2, \dots, s$.

By definition, it is easy to verify the following lemma, see [14] for more properties of edge-transitive product graphs.

Lemma 2.6 *The following statements hold:*

1. $\text{val}(\Delta_1 \times \Delta_2 \times \dots \times \Delta_s) = \text{val}(\Delta_1) \cdot \text{val}(\Delta_2) \cdot \dots \cdot \text{val}(\Delta_s)$.

2. $\text{Aut}(\Delta_1) \times \text{Aut}(\Delta_2) \times \cdots \times \text{Aut}(\Delta_s) \leq \text{Aut}(\Delta_1 \times \Delta_2 \times \cdots \times \Delta_s)$.
3. If each Δ_i is vertex-transitive and edge-transitive, and at most one of $\Delta_1, \Delta_2, \dots, \Delta_s$ is half-transitive, then $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_s$ is edge-transitive.

Edge-transitive graphs which are vertex-transitive can be constructed as fused-orbital graphs, stated as following. For a transitive permutation group G on a set Ω , an *orbital* of G is a digraph with vertex set Ω and arc set $(\alpha, \beta)^G$ where $\alpha, \beta \in \Omega$. If $(\alpha, \beta)^G = (\beta, \alpha)^G$, this orbital is called *self-paired*. A *fused-orbital* of G is a graph with vertex set Ω and arc set $(\alpha, \beta)^G \cup (\beta, \alpha)^G$. If $(\alpha, \beta)^G = (\beta, \alpha)^G$, this fused-orbital is a self-paired orbital and is G -arc-transitive; if $(\alpha, \beta)^G \neq (\beta, \alpha)^G$, then the fused-orbital is a G -half-transitive graph. Conversely, it is easy to know that each vertex-transitive and edge-transitive graph can be reconstructed in this way.

The following result is due to Higman, see [10, Theorem 3.2A].

Lemma 2.7 *Let G be a transitive permutation group. Then, G is primitive if and only if each nontrivial orbital is connected.*

Lemma 2.8 *Let Γ be a G -vertex-transitive and G -edge-transitive graph, where $G \leq \text{Aut}\Gamma$. Suppose further $G = H_1 \times H_2 \times \cdots \times H_l$ and $G_\alpha = (H_1)_\alpha \times (H_2)_\alpha \times \cdots \times (H_l)_\alpha$, where $l \geq 2$ and $\alpha \in V\Gamma$. Then*

$$\Gamma \cong \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_l,$$

where Γ_i is a fused-orbital graph of H_i (so H_i -edge-transitive) with vertex stabilizer $(H_i)_\alpha$ for $i = 1, 2, \dots, l$.

Proof Let $\Omega_i = [H_i : (H_i)_\alpha]$ for $i = 1, 2, \dots, l$. Since $G \leq \text{Aut}\Gamma$, G_α is core-free in G , it follows that $(H_i)_\alpha$ is core-free in H_i , so H_i can be viewed as a transitive permutation group on Ω_i by coset action. By Lemma 2.2, we may identify $V\Gamma$ with $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_l$, and identify the action of G on $V\Gamma$ with the product action of G on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_l$.

Now since Γ is a G -vertex-transitive and G -edge-transitive graph, Γ is a fused-orbital graph of G on $V\Gamma$, with arc set $(\alpha, \beta)^G \cup (\beta, \alpha)^G$ say, where $\beta \in \Gamma(\alpha)$. By the discussion above, we may label $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l)$ with $\alpha_i, \beta_i \in \Omega_i$. Then $G_\alpha = (H_1)_{\alpha_1} \times (H_2)_{\alpha_2} \times \cdots \times (H_s)_{\alpha_l}$.

First suppose G is arc-transitive on Γ . Then G_α is transitive on $\Gamma(\alpha)$, so

$$\Gamma(\alpha_1, \alpha_2, \dots, \alpha_l) = (\beta_1, \beta_2, \dots, \beta_l)^{G_\alpha} = \left(\beta_1^{(H_1)_{\alpha_1}}, \beta_2^{(H_2)_{\alpha_2}}, \dots, \beta_l^{(H_l)_{\alpha_l}} \right).$$

It follows $\Gamma \cong \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_l$, where Γ_i is a self-paired orbital graph of H_i on Ω_i with arc set $(\alpha_i, \beta_i)^{H_i}$. If G is half-transitive on Γ , then G_α has two orbits on $\Gamma(\alpha)$, that is $\Gamma(\alpha) = \beta^{G_\alpha} \cup \gamma^{G_\alpha}$ for some $\beta, \gamma \in \Gamma(\alpha)$. Then one may similarly prove $\Gamma \cong \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_l$, with each Γ_i a fused-orbital graph of H_i on Ω_i . $\square$

3 Normal subgroups

From now on, we begin the process for proving Theorem 1.1. For convenience, in Sects. 3–6 of this paper, we always make the following hypothesis (the notations used in the hypothesis are also fixed).

Hypothesis 3.1 Let $\Gamma = \text{Cay}(R, S)$ be a connected G -edge-transitive Cayley graph of square-free order n , where $\hat{R} \leq G \leq \text{Aut}\Gamma$ and $n = p_1 p_2 \dots p_s$, with primes $p_1 < p_2 < \dots < p_s$. Suppose further $k := \text{val}(\Gamma) < p_1$. Set $\pi = \{p_1, p_2, \dots, p_s\}$.

Then R is of square-free order n and so is a metacyclic group, and n is odd, hence k is even since an edge-transitive graph of odd order must have even valency. For $\alpha \in V\Gamma$, denote by $G_\alpha^{\Gamma(\alpha)}$ the induced permutation group of the action of G_α on $\Gamma(\alpha)$. Then $G_\alpha^{\Gamma(\alpha)} \leq S_k$. Since Γ is connected and G -vertex-transitive, $|G_\alpha|$ and $|G_\alpha^{\Gamma(\alpha)}|$ have the same prime divisors (see [6]), so each prime divisor of $|G_\alpha|$ is less than p_1 .

As usual, for a group H and a prime $p \mid |H|$, we use H_p to denote a Sylow p -subgroup of H . Moreover, if $|H| = r_1^{t_1} r_2^{t_2} \dots r_l^{t_l}$ is a prime factorization, let

$$|H|_\pi = \prod_{r_i \in \pi} r_i^{t_i}, \quad |H|_{\pi'} = \prod_{r_i \notin \pi} r_i^{t_i}.$$

Because $|G : G_\alpha| = n$, we obtain $|G|_\pi = n = p_1 p_2 \dots p_s$ and $|G_\alpha| = |G|_{\pi'}$.

The next lemma will be repeatedly used in the subsequent discussion.

Lemma 3.1 *The normal subgroups of G have the following properties, where $\alpha \in V\Gamma$.*

1. Each minimal normal subgroup of G is simple.
2. If $1 \neq N \triangleleft G$, then $|N_\alpha| = |N|_{\pi'}$, $|N \cap \hat{R}| = |\alpha^N| = |N : N_\alpha| = |N|_\pi \neq 1$, and $N \cap \hat{R}$ is regular on α^N . In particular, if N is transitive on $V\Gamma$, then $\hat{R} \leq N$.
3. Suppose $M, N \triangleleft G$ such that $M \cap N = 1$. Then $(|M|, |N|, n) = 1$ and $(|M : M_\alpha|, |N : N_\alpha|) = 1$.
4. Suppose $X = N_1 \times N_2 \times \dots \times N_l$, where $N_i \triangleleft G$ for $i = 1, 2, \dots, l$. Then $X_\alpha = (N_1)_\alpha \times (N_2)_\alpha \times \dots \times (N_l)_\alpha$.

Proof 1. Let M be a minimal normal subgroup of G . Then $M \cong T^d$ with T a simple group and $d \geq 1$. Since $|\alpha^M| = |M : M_\alpha| \neq 1$ and $|\alpha^M|$ divides $|V\Gamma|$ with $|V\Gamma| = n$, we have $|M|_\pi \neq 1$. Then, as $|M|_\pi = (|T|_\pi)^d$ divides $|G|_\pi = p_1 p_2 \dots p_s$, we obtain $d = 1$ and M is simple.

2. Since $N \neq 1$ is normal in G , $|N : N_\alpha| = |\alpha^N| \neq 1$ divides n , and since $N_\alpha \leq G_\alpha$ is a π' -subgroup, we have $|\alpha^N| = |N|_\pi$ and $|N_\alpha| = |N|_{\pi'}$. Let $Y = N\hat{R}$. Then $|Y| = \frac{|N||\hat{R}|}{|N \cap \hat{R}|}$ divides $|G|$, so $|N|_\pi = |N \cap \hat{R}|_\pi = |N \cap \hat{R}|$ and $N \cap \hat{R}$ is regular on α^N . Further, if N is transitive on $V\Gamma$, then $|N \cap \hat{R}| = |N|_\pi = n = |\hat{R}|$, so $\hat{R} \leq N$.

3. Now $MN = M \times N \triangleleft G$. Then, since $|M|_\pi |N|_\pi$ divides $|G|_\pi = p_1 p_2 \dots p_s$, $(|M|, |N|, n) = (|M|_\pi, |N|_\pi) = 1$, and by (2), $(|M : M_\alpha|, |N : N_\alpha|) = 1$.

4. Clearly $X_\alpha \geq (N_1)_\alpha \times (N_2)_\alpha \times \cdots \times (N_l)_\alpha$. On the other hand, part (2) implies

$$|X_\alpha| = |X|_{\pi'} = |N_1|_{\pi'} |N_2|_{\pi'} \cdots |N_l|_{\pi'} = |(N_1)_\alpha| |(N_2)_\alpha| \cdots |(N_l)_\alpha|,$$

hence $X_\alpha = (N_1)_\alpha \times (N_2)_\alpha \times \cdots \times (N_l)_\alpha$. $\square$

Let $T = \text{PSL}(d, p^e)$ be a nonabelian simple group with $d \geq 2$ and p a prime. It is known that T has a natural 2-transitive permutation representation on the set of projective points (1 dimensional subspaces) of d -dimensional linear space over p^e -element field, with degree $\frac{p^{de}-1}{p^e-1}$ and vertex stabilizer

$$P_1 := \begin{cases} \mathbb{Z}_p^e : \mathbb{Z}_{\frac{p^e-1}{(2, p^e-1)}}, & \text{if } d = 2; \\ \mathbb{Z}_p^{e(d-1)} \cdot \mathbb{Z}_{\frac{p^e-1}{(d, p^e-1)}} \cdot \text{PSL}(d-1, p^e) \cdot \mathbb{Z}_{(d-1, p^e-1)}, & \text{if } d \geq 3. \end{cases}$$

Also, the outer automorphism group

$$\text{Out}(T) = \begin{cases} \mathbb{Z}_{(2, p^e-1)} \times \mathbb{Z}_e, & \text{if } d = 2; \\ \mathbb{Z}_{(d, p^e-1)} \cdot \mathbb{Z}_e \cdot \mathbb{Z}_2, & \text{if } d \geq 3. \end{cases}$$

In particular, $|\text{Out}(T)| \leq 2e(d, p^e - 1)$.

Lemma 3.2 *Let T be a nonabelian simple normal subgroup of G , and suppose that T is intransitive on $V\Gamma$ and $p_1 \leq |\text{Out}(T)|$. Then $(T, T_\alpha) \neq (\text{PSL}(d, p^e), P_1)$ for $\alpha \in V\Gamma$.*

Proof Since Γ is connected and T is intransitive on $V\Gamma$, there exist two T -orbits B_1 and B_2 such that the induced subgraph $[B_1, B_2]$ is not an empty graph (recall that the graph $[B_1, B_2]$ is defined with vertex set $B_1 \cup B_2$ and edge set consisting of edges of Γ between B_1 and B_2). Since T is transitive on B_1 and B_2 , $[B_1, B_2]$ is a regular graph.

If $\text{val}([B_1, B_2]) = 1$, then $\text{val}(\Gamma) = \text{val}(\Gamma_T)$, Γ is a normal cover of Γ_T . By Lemma 2.3(3), T is semi-regular on $V\Gamma$ and so $|T| \mid n$, which is a contradiction. Thus, $\text{val}([B_1, B_2]) \geq 2$. Let $\alpha \in B_1$.

Suppose, on the contrary, $(T, T_\alpha) = (\text{PSL}(d, p^e), P_1)$. Then T is 2-transitive on B_i with degree $|B_i| = |T : T_\alpha| = \frac{p^{de}-1}{p^e-1}$ for $i = 1, 2$. Noting that $\text{PSL}(d, p^e)$ has unique 2-transitive permutation representation of degree $\frac{p^{de}-1}{p^e-1}$ if $d = 2$, and has exactly two 2-transitive permutation representations of degree $\frac{p^{de}-1}{p^e-1}$ if $d \geq 3$, namely, one is on the set of projective points and the another is on the set of hyperplanes ($(d-1)$ dimensional subspaces), see [5, Theorem 5.3 (S)].

Assume first T^{B_1} and T^{B_2} are permutation equivalent. Then T_α fixes a vertex $\beta \in B_2$ and so $T_\alpha = T_\beta$ as they have the same order, hence $T_\alpha = T_\beta$ is transitive on $B_2 \setminus \{\beta\}$ due to the 2-transitivity of T on B_2 . If $\beta \in \Gamma(\alpha)$, then $\text{val}([B_1, B_2]) = 1$, a contradiction. Hence there is a vertex $\gamma \in B_2 \setminus \{\beta\}$ which is adjacent to α . As T_α is transitive on $B_2 \setminus \{\beta\}$, it follows $[B_1, B_2] \cong K_{|B_1|, |B_1|} - |B_1|K_2$, and so

$$\text{val}(\Gamma) \geq \text{val}([B_1, B_2]) = |B_1| - 1 = \frac{p^{ed} - 1}{p^e - 1} - 1 = \frac{p^e (p^{e(d-1)} - 1)}{p^e - 1}.$$

By assumption, we obtain

$$\frac{p^e (p^{e(d-1)} - 1)}{p^e - 1} \leq \text{val}(\Gamma) < p_1 \leq |\text{Out}(T)| \leq 2e(d, p^e - 1),$$

which is not possible.

Assume now T^{B_1} and T^{B_2} are permutation inequivalent. Then, B_1 and B_2 can be identified with the set of projective points and the set of hyperplanes, respectively, and the graph $[B_1, B_2]$ is isomorphic to the point-hyperplane incidence graph or point-hyperplane non-incidence graph, see [24, p. 193] or [25]. Thus, if there is a hyperplane $\gamma \in \Gamma(\alpha)$ such that α is contained in γ , then $\text{val}([B_1, B_2]) = \frac{p^{e(d-1)} - 1}{p^e - 1}$, the number of 1-subspaces of a $(d-1)$ -subspace; if each hyperplane $\gamma \in \Gamma(\alpha)$ does not contain α , then $\text{val}([B_1, B_2]) = |B_1| - \frac{p^{e(d-1)} - 1}{p^e - 1} = p^{e(d-1)}$. Hence, $\text{val}(\Gamma) \geq \text{val}([B_1, B_2]) \geq \min\{\frac{p^{e(d-1)} - 1}{p^e - 1}, p^{e(d-1)}\}$, the minimal one of $\frac{p^{e(d-1)} - 1}{p^e - 1}$ and $p^{e(d-1)}$. By assumption, we have

$$\min\left\{\frac{p^{e(d-1)} - 1}{p^e - 1}, p^{e(d-1)}\right\} \leq \text{val}(\Gamma) < p_1 \leq |\text{Out}(T)| \leq 2e(d, p^e - 1),$$

which is a contradiction. $\square$

4 Soluble case

Let Γ, R, G satisfy Hypothesis 3.1. In this section, we investigate the case where G is soluble.

Lemma 4.1 *Suppose G is soluble. Then $\hat{R}$ is normal in G .*

Proof Let F be the Fitting subgroup of G . If a prime $p \mid |F|$, then $1 \neq F_p \triangleleft G$, by Lemma 3.1(2), $F_p \cap \hat{R} \neq 1$ and so p divides $|\hat{R}| = n$. Then since F is nilpotent, we conclude

$$F = \mathbf{O}_{p_1}(G) \times \mathbf{O}_{p_2}(G) \times \cdots \times \mathbf{O}_{p_s}(G),$$

where $\mathbf{O}_{p_i}(G)$ denotes the largest normal p_i -subgroup of G .

For each $i \in \{1, 2, \dots, s\}$, recall that $|G_{p_i}| = p_i$, we have $\mathbf{O}_{p_i}(G) \leq \mathbb{Z}_{p_i}$, so F is a cyclic group. Then by Lemma 2.1, $C_G(F) = F$, and $G/F = G/C_G(F) \leq \text{Aut}(F)$ is abelian. Further, if some $\mathbf{O}_{p_i}(G) \neq 1$, by Lemma 3.1(2), $\mathbf{O}_{p_i}(G) \cap \hat{R} \neq 1$, so $\mathbf{O}_{p_i}(G) \subseteq \hat{R}$. It follows $F \subseteq \hat{R}$. Now, as G/F is abelian and $\hat{R}/F \leq G/F$, we obtain $\hat{R}/F \triangleleft G/F$, and so $\hat{R} \triangleleft G$. $\square$

Let **1** denote the vertex of Γ corresponding to the identity element of R . By Lemmas 4.1, $\hat{R} \triangleleft G$, then Lemma 2.4 implies $G_1 \leq \text{Aut}(R, S) \leq \text{Aut}(R)$. We therefore need information of the automorphism group $\text{Aut}(R)$.

If $R \cong \mathbb{Z}_n$ is cyclic, then $\text{Aut}(R) \cong \mathbb{Z}_n^*$ is abelian. Suppose R is not cyclic. Since R is of square-free order, we may write $R = \langle a \rangle \times (\langle b \rangle : \langle c \rangle)$ with $Z(\langle b \rangle : \langle c \rangle) = 1$, where $Z(\langle b \rangle : \langle c \rangle)$ denotes the center of the group $\langle b \rangle : \langle c \rangle$.

Lemma 4.2 *Let $R = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_h \times (\mathbb{Z}_t : \mathbb{Z}_l)$ with $Z(\langle b \rangle : \langle c \rangle) = 1$. Then $\text{Aut}(R) \cong \mathbb{Z}_h^* \times (\mathbb{Z}_t : \mathbb{Z}_l^*)$, and further, each automorphism $\sigma \in \text{Aut}(R)$ has the form:*

$$\sigma : a \rightarrow a^x, b \rightarrow b^y, c \rightarrow b^z c, \text{ where } (x, h) = (y, t) = 1, 0 \leq z \leq t - 1.$$

In particular, an element of R with order l is not conjugate to its inverse under $\text{Aut}(R)$, and if $b^i c^j$ with i, j integers is of order l , then $b^i c^j$ is conjugate to c^j under $\text{Aut}(R)$.

Proof Since R is of square-free order, the subgroups $\langle a \rangle$, $\langle b \rangle$ and $\langle b, c \rangle$ are characteristic subgroups of R , so we may suppose $\sigma : a \rightarrow a^x, b \rightarrow b^y, c \rightarrow b^z c^m$, where $(x, h) = (y, t) = 1$. Since c normalizes $\langle b \rangle$, $b^c = b^r$, with $r^l \equiv 1 \pmod{t}$. Then $(b^y)^{c^m} = b^{yr^m}$, and $(b^y)^{c^m} = \sigma(b)^{\sigma(c)} = \sigma(b^c) = \sigma(b^r) = b^{yr}$, so $yr^m \equiv yr \pmod{t}$, and consequently $r^{m-1} \equiv 1 \pmod{t}$. It follows $r^{(l, m-1)} \equiv 1 \pmod{t}$. If $(l, m-1)$ is a proper divisor of l , then $c^{(l, m-1)} \neq 1$ is in the center of the group $\langle b \rangle : \langle c \rangle$, a contradiction. Thus, $l \mid (m-1)$, and $\sigma(c) = b^z c$.

Now, let

$$\begin{aligned} M &= \langle \sigma : a \rightarrow a^x, b \rightarrow b, c \rightarrow c \mid (x, h) = 1 \rangle, \\ N &= \langle \sigma : a \rightarrow a, b \rightarrow b, c \rightarrow b^z c \mid 0 \leq z \leq t - 1 \rangle, \\ K &= \langle \sigma : a \rightarrow a, b \rightarrow b^y, c \rightarrow c \mid (y, t) = 1 \rangle. \end{aligned}$$

Then it is easy to show that $\text{Aut}(R) = M \times (N : K) \cong \mathbb{Z}_h^* \times (\mathbb{Z}_t : \mathbb{Z}_t^*)$.

Suppose $x \in R$ is of order l . Then $x = b^i c^j$ for some integers i and j . Since l is odd and $(b^i c^j)^{-1} = b^{i'} c^{l-j}$ for some integer i' , the above discussion shows x is not conjugate to x^{-1} under $\text{Aut}(R)$. Since $\langle b^i c^j \rangle$ and $\langle c \rangle$ are Hall subgroups of order l of R , they are conjugate in R , it follows that $b^i c^j$ is conjugate to c^j under $\text{Aut}(R)$. $\square$

With Lemmas 4.1 and 4.2, we may here determine the graphs Γ for the case G is soluble. We remark, the full automorphism group $\text{Aut}\Gamma$ in the case will be determined in Lemma 6.4 since more results of general cases will be needed.

Lemma 4.3 *Assume G is soluble. Then Γ is a G -normal Cayley graph of R , and further, one of the following holds:*

1. $R = \langle a \rangle$ is cyclic, $S = a^H$ and Γ is $(\hat{R} : H)$ -arc-regular, where $H = \langle G_1, \tau \rangle \leq \text{Aut}(R)$ with τ mapping a to a^{-1} .
2. $R = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_h \times (\mathbb{Z}_t : \mathbb{Z}_l)$ with $Z(\langle b \rangle : \langle c \rangle) = 1$, $S = \{ac^z, (ac^z)^{-1}\}^K$ with $(z, l) = 1$ and $K = G_1 \leq \mathbb{Z}_h^* \times \mathbb{Z}_t^*$ a subgroup of $\text{Aut}(R)$, $\text{val}(\Gamma) = 2|G_1|$, and Γ is G -half-transitive and G -edge-regular.

Proof By Lemma 4.1, Γ is a G -normal edge-transitive Cayley graph of R , so $G_1 \leq \text{Aut}(R, S)$ has at most two orbits on $\Gamma(1) = S$, hence $S = \{s, s^{-1}\}^{G_1}$ for some $s \in S$ (see Construction 2.5). In particular, elements in S have the same order. As Γ is connected, $\langle S \rangle = R$, G_1 acts faithfully on S since $G_1 \leq \text{Aut}(R)$.

1. Suppose $R = \langle a \rangle \cong \mathbb{Z}_n$ is cyclic. Then, s is of order n and conjugate to a under $\text{Aut}(R)$, as $\text{Cay}(R, S) \cong \text{Cay}(R, S^\phi)$ for each $\phi \in \text{Aut}(R)$, we may suppose, upon the isomorphism, $s = a$. Since R is cyclic, R has an automorphism τ mapping a to a^{-1} , and since $S = S^{-1}$, $\tau \in \text{Aut}(R, S) \leq \text{Aut}\Gamma$, hence $S = a^H$, where $H = \langle G_1, \tau \rangle$. Further, since $H \leq \text{Aut}(R) \cong \mathbb{Z}_n^*$ is abelian and acts faithfully and transitively on S , H is regular on S , so $\text{val}(\Gamma) = |S| = |H|$ and Γ is $(\hat{R} : H)$ -arc-regular.
2. Suppose R is not cyclic. Then, we may write $R = \langle a \rangle \times (\langle b \rangle : \langle c \rangle) \cong \mathbb{Z}_h \times (\mathbb{Z}_t : \mathbb{Z}_l)$ with $Z(\langle b \rangle : \langle c \rangle) = 1$. Write $s = a^x b^y c^z$. For each $\sigma \in \text{Aut}(G)$, by Lemma 4.2, $\sigma(s) \in \langle a^x, b, c^z \rangle$ and so $S \subseteq \langle a^x, b, c^z \rangle$, by the connectivity of Γ , $G = \langle a^x, b, c^z \rangle$. It follows $(x, h) = (z, l) = 1$ and $b^y c^z$ is of order l . By Lemma 4.2, a is conjugate to a^x under $\text{Aut}(R)$, and $b^y c^z$ is conjugate to c^z under $\text{Aut}(R)$, we have $a^x b^y c^z$ is conjugate to $a c^z$ under $\text{Aut}(R)$. Hence, upon the isomorphism, we may assume $s = a c^z$, and so $S = \{a c^z, (a c^z)^{-1}\}^K$, where $K = G_1 \leq \text{Aut}(R)$. Again by Lemma 4.2, there is no $\sigma \in \text{Aut}(G)$ which maps s to s^{-1} , Γ is G -half-transitive. Recall $K \leq \text{Aut}(R) \cong \mathbb{Z}_h^* \times (\mathbb{Z}_t : \mathbb{Z}_l^*)$ and $(|K|, n) = 1$, we have $K \leq \mathbb{Z}_h^* \times \mathbb{Z}_l^*$ is abelian. Now, since $\langle s^K \rangle = \langle (s^{-1})^K \rangle = R$, K acts faithfully on s^K and $(s^{-1})^K$, so is regular. It follows that $\text{val}(\Gamma) = 2|K|$, and Γ is G -edge-regular. $\square$

5 Almost simple case

A transitive permutation group is called *quasiprimitive* if its minimal normal subgroups are transitive. Quasiprimitive permutation groups which are almost simple and contain a transitive metacyclic subgroup of odd order are known, see [16]. Then, we easily deduce the following result.

Lemma 5.1 *Let X be an almost simple quasiprimitive permutation group on a set Ω and contains a regular metacyclic subgroup K of odd order. Then either $(X, X_\alpha) = (A_m, A_{m-1})$ or (S_m, S_{m-1}) with m odd, or the triple (X, K, X_α) lies in Table 3, where p is a prime, $G(q^d)$ is a regular subgroup of $\Gamma\text{L}(1, q^d) = \mathbb{Z}_{q^d-1} : \mathbb{Z}_d$, and $\alpha \in \Omega$.*

Suppose Γ , R and G satisfy Hypothesis 3.1. In this section, we will work out the graphs Γ for the case where G is almost simple. Recall the *socle* of a group G , denoted by $\text{soc}(G)$, is the product of all minimal normal subgroups of G .

First, the following lemma determines the candidates of the group G .

Lemma 5.2 *Suppose G is almost simple. Then, G is quasiprimitive on $V\Gamma$, and the triple $(G, \hat{R}, G_\alpha)$ (as (X, K, X_α) there) satisfies Lemma 5.1, where $\alpha \in V\Gamma$.*

Proof Let $T = \text{soc}(G)$. Then, $G = T.o$ with $o \leq \text{Out}(T)$.

Assume, on the contrary, T is intransitive on $V\Gamma$. Let $B = \alpha^T$ be an orbit of T on $V\Gamma$. Then, G_B^B is almost simple with socle $T^B \cong T$, and by Lemma 3.1(2), $\hat{R} \cap T$ is regular on B . That is, G_B^B is an almost simple quasiprimitive permutation group on B containing a regular metacyclic subgroup $\hat{R} \cap T$. By Lemma 5.1, either

- (i) $(G_B^B, (G_B^B)_\alpha) = (A_m, A_{m-1})$ or (S_m, S_{m-1}) with m odd; or

Table 3 All most simple quasiprimitive permutation groups containing a regular metacyclic subgroup of odd order

Row	X	K	X_α	Conditions
1	$A_{p.o}$	$p:\frac{p-1}{2}$	$S_{p-2} \times o$	$o \leq 2, p \equiv 3(\bmod 4)$
2	$\text{PSL}(d, q).o$	$G(q^d)$	$P_1.o$	$q = p^e, o \leq (d, q-1).e$
3	$\text{PSL}(2, p).o$	$p:\frac{p-1}{2}$	$D_{(p+1)o}$	$o \leq 2, p \equiv 3(\bmod 4)$
4	$\text{PSL}(2, 11).o$	11:5	$A_4.o$	$o \leq 2$
5	$\text{PSL}(2, 11)$	11	A_5	
6	$\text{PSL}(2, 23)$	23:11	S_4	
7	$\text{PSL}(2, 29)$	29:7	A_5	
8	$\text{PSL}(2, 59)$	59:29	A_5	
9	$\text{PSL}(5, 2)$	31:5	$2^6 : (S_3 \times \text{PSL}(3, 2))$	
10	M_{11}	11	M_{10}	
11	M_{11}	11:5	$M_{9,2}$	
12	M_{23}	23	M_{22}	
13	M_{23}	23:11	$M_{21,2,2^4}.A_7$	

(ii) $(G_B^B, R \cap T, (G_B^B)_\alpha)$ lies in Table 3.

Since T is intransitive on $V\Gamma$, there is a prime $p_i \in \pi$ such that $p_i \nmid |T : T_\alpha|$. Then, since $T_\alpha \leq G_\alpha$ is a p' -subgroup, $p_i \nmid |T|$. It follows $p_i \mid |\text{Out}(T)|$ since $G \leq T.\text{Out}(T)$. Clearly, $T \neq A_m$ with m odd. Further, checking the candidates in Table 3, the only possibility is $(T, T_\alpha) = (\text{PSL}(d, q), P_1)$. By Lemma 3.2, it is a contradiction.

Thus, T is transitive on $V\Gamma$ and G is quasiprimitive on $V\Gamma$, hence the triple $(G, \hat{R}, G_\alpha)$ satisfies Lemma 5.1. $\square$

The next lemma further determines the graphs Γ in the almost simple case.

Lemma 5.3 Assume G is almost simple with socle T . Then Γ is T -edge-transitive and lies in Table 1.

Proof By Lemma 5.2, (G, R, G_α) satisfies Lemma 5.1. We analyze all the candidates there.

Suppose first G is 2-transitive on $V\Gamma$. Then $\Gamma \cong K_n$ is a complete graph, and by assumption, $n-1 = \text{val}(\Gamma)$ is less than the smallest prime divisor of n , n is a prime, which give rise to examples as in Rows 1–6 of Table 1.

Suppose now $(T, T_\alpha) = (A_p, S_{p-2})$, as in Row 1 of Table 3. Then $|V\Gamma| = |T : T_\alpha| = \frac{p(p-1)}{2}$, and the action of G on $V\Gamma$ is permutation equivalent to the action of G acting on $\Omega^{[2]}$ (the set of 2-subsets of Ω), induced by the natural action of G on $\Omega := \{1, 2, \dots, p\}$ of degree p . Identifying $V\Gamma$ with $\Omega^{[2]}$, and let $\alpha = \{1, 2\}$. Then, for $\beta = \{i, j\} \in \Omega^{[2]}$, if both i and j are not equal to 1 or 2, then $T_{\alpha\beta} \cong S_{p-4}.\mathbb{Z}_2$ and $|T_\alpha : T_{\alpha\beta}| = \frac{(p-2)(p-3)}{2}$. On the other hand, if one of i and j equals 1 or 2, then

$T_{\alpha\beta} \cong A_{p-3}$ and $|T_\alpha : T_{\alpha\beta}| = 2(p-2)$. Hence, $\text{val}(\Gamma) \geq \min\{2(p-2), \frac{(p-2)(p-3)}{2}\}$ is larger than the smallest prime divisor of $|V\Gamma| = \frac{p(p-1)}{2}$. This is a contradiction.

Suppose $(G, G_\alpha) = (\text{PSL}(2, p).o, D_{(p+1)o})$ with $o \leq \mathbb{Z}_2$, as in Row 3 of Table 3. Write $G_\alpha = \langle a \rangle : \langle b \rangle$ and let $\beta \in \Gamma(\alpha)$. If a^i with order $o(a^i) \geq 3$ lies in $G_{\alpha\beta}$, then $\langle a^i \rangle$ is normal in both G_α and G_β , note that $G_\alpha \cong G_\beta$ are maximal subgroup of G , it follows $G_\beta = N_G(\langle a^i \rangle) = G_\alpha$. In particular, G_α fix β . Since G acts edge-transitively on Γ , we obtain $\text{val}(\Gamma) = 2$, so Γ is a cycle and $\text{Aut}\Gamma$ is a dihedral group, a contradiction. Thus, $G_{\alpha\beta} \leq D_4$. If $G_{\alpha\beta} = 1$, then $\text{val}(\Gamma) \geq |G_\alpha : G_{\alpha\beta}| \geq p+1$ is bigger than the smallest prime divisor of $|V\Gamma| = \frac{p(p-1)}{2}$, which is not the case. Hence, $G_{\alpha\beta} \cong \mathbb{Z}_2$ or D_4 . Assume $G = \text{PSL}(2, p)$. Then, $G_\alpha \cong D_{p+1}$, and $\text{val}(\Gamma) \in \{\frac{p+1}{4}, \frac{p+1}{2}, p+1\}$. By the restriction of $\text{val}(\Gamma)$, the only possibility is $\text{val}(\Gamma) = \frac{p+1}{4}$ with $\frac{p-1}{2}$ a prime. If $G = \text{PGL}(2, p)$, then $G_\alpha \cong D_{2(p+1)}$, and $\text{val}(\Gamma) \geq \frac{p+1}{2}$ is bigger than the smallest prime divisor of $\frac{p(p-1)}{2}$, no graph Γ exists in the case.

For the remaining candidates, Γ is always of order a product of two distinct primes (note that for the case n is a prime, G is always 2-transitive which has been discussed above). Edge-transitive graphs of order a product of two distinct primes were classified in [26, 27]. Checking the list there, Γ lies in Rows 7–10 of Table 1.

Finally, for Rows 1–6 of Table 1, Γ is always a complete graph and T is 2-transitive on $V\Gamma$, so Γ is T -arc-transitive. For the other candidates in Table 1, one easily see that T_α has at most two orbits on $\Gamma(\alpha)$, Γ is T -edge-transitive. $\square$

6 Proofs of Theorem 1.1 and Corollary 1.2

With outcomes in Sects. 3, 4 and 5, we will complete the proofs of Theorem 1.1 and Corollary 1.2 in this section.

Assume Γ, R, G satisfy Hypothesis 3.1. Let M be the *soluble radical* of G , namely, the largest soluble normal subgroup of G . Then, M is a characteristic subgroup of G . Recall that $|R| = n = p_1 p_2 \cdots p_s$ with $p_1 < p_2 < \cdots < p_s$ primes and $\pi = \{p_1, p_2, \dots, p_s\}$.

We first consider the trivial soluble radical case.

Lemma 6.1 *Assume $M = 1$ and G is not almost simple. Then no graph Γ exists.*

Proof Let $N = \text{soc}(G)$. Since $M = 1$, by Lemma 3.1(1), each minimal normal subgroup of G is nonabelian simple, then Lemma 3.1(4) implies

$$N := T_1 \times T_2 \times \cdots \times T_l, \quad N_\alpha = (T_1)_\alpha \times (T_2)_\alpha \times \cdots \times (T_l)_\alpha,$$

where $l \geq 2$ and each T_i is nonabelian simple and normal in G . It then follows from [10, Theorem 4.3A] that $T_1, T_2, \dots, T_l$ are the only minimal normal subgroups of G . Further, since T_i is not isomorphic to T_j for $i \neq j$ by Lemma 3.1(3), we further conclude $T_1, T_2, \dots, T_l$ are characteristic subgroups of G . Noting that the centralizer $C_G(N) = 1$, we obtain $G \leq \text{Aut}(N) = (T_1.\text{Out}(T_1)) \times \cdots \times (T_l.\text{Out}(T_l))$.

Assume N is intransitive on $V\Gamma$. Then $n \nmid |N|$, as $n \mid |G|$, there exists some $p_i \in \pi$ such that $p_i \mid |\text{Out}(T_i)|$. Let $B = \alpha^{T_i}$, an orbit of T_i on $V\Gamma$. By Lemma 3.1(2), $T_i \cap \hat{R}$

is regular on B , so $T_i^B \cong T_i$ is quasiprimitive containing a regular metacyclic subgroup $T_i \cap \hat{R}$, and hence satisfies Lemma 5.1. Then since $p_i \mid |\text{Out}(T_i)|$, the only possibility is $(T_i, (T_i)_\alpha) = (\text{PSL}(d, q), P_1)$. However, by Lemma 3.2, it is a contradiction.

Therefore, N is transitive on $V\Gamma$, and $N \supseteq \hat{R}$ by Lemma 3.1(2). Now $\Sigma := \text{Cay}(R, \{s, s^{-1}\}^{N_1})$ with $s \in S$ is a subgraph of Γ and is N -edge-transitive. By Lemma 2.8, $\Sigma \cong \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_l$, where Σ_i is a fused-orbital of T_i on $\Omega_i := [T_i : (T_i)_\alpha]$ by coset action and is T_i -edge-transitive for $i = 1, 2, \dots, l$. Recall $(T_i, (T_i)_\alpha)$ satisfies Lemma 5.1. If $(T_i, (T_i)_\alpha) = (\text{PSL}(2, 11), A_4)$ for some i , then $|\alpha^{T_i}| = |T_i : (T_i)_\alpha| = 55$ divides n , so $p_1 \leq 5$ and $\text{val}(\Gamma) = 2$ or 4 . The former case is not possible as the full automorphism group of a cycle is soluble. If $\text{val}(\Gamma) = 4$, by the classification of tetravalent edge-transitive Cayley graphs of square-free order given in [17], each such graph does not admit an edge-transitive graph which has more than one nonabelian normal subgroups, a contradiction. Hence $(T_i, (T_i)_\alpha) \neq (\text{PSL}(2, 11), A_4)$ for each i . Now, checking the candidates in Lemma 5.1, $(T_i)_\alpha$ is always maximal in T_i , so T_i is primitive on Ω_i , and hence Σ_i is connected by Lemma 2.7. Further, as $T_i \cap \hat{R}$ is regular on Ω_i by Lemma 3.1(2), one easily sees that Σ_i satisfies Lemma 5.3, and consequently, lies in Table 1. Since $\text{val}(\Sigma) = \text{val}(\Sigma_1) \cdot \text{val}(\Sigma_2) \cdots \text{val}(\Sigma_l)$ and $l \geq 2$, by checking the valency, we always have $\text{val}(\Gamma) \geq \text{val}(\Sigma) > p_1$, which is a contradiction. $\square$

To treat the nontrivial soluble radical case, we need the following lemma.

Lemma 6.2 *Assume G is insoluble. Then $G = (M \times T).o$, where T is a nonabelian simple group and $o \leq \text{Out}(T)$.*

Proof We prove Lemma 6.2 by using induction on the order of Γ . If $M = 1$, Lemma 6.2 is true by Lemma 6.1.

Assume $M \neq 1$. Let F be the Fitting subgroup of M . Then F is clearly also the Fitting subgroup of G . Since $M \neq 1$, by Lemma 2.1, $F \neq 1$ and $C_M(F) \leq F$. As F is nilpotent, arguing similarly as in the proof of Lemma 4.1, we obtain $F \leq \mathbb{Z}_n$ is a subgroup of R , and so $C_M(F) = F$. If $F \cong \mathbb{Z}_n$, then F is regular on $V\Gamma$, Γ is a G -normal Cayley graph of F , by Lemma 2.4, $G \leq N_{\text{Aut}\Gamma}(\hat{F}) \leq \hat{F} : \text{Aut}(F) \cong \mathbb{Z}_n : \mathbb{Z}_n^*$ is soluble, a contradiction. Thus $F < \mathbb{Z}_n$ is intransitive on $V\Gamma$. Let K be the kernel of G acting on $V\Gamma_F$, the set of F -orbits on $V\Gamma$. Then $F \triangleleft K$, and for $B \in V\Gamma_F$, F is regular on B , by [10, Corollary 4.2B], $K^B \leq N_{\text{Sym}(B)}(F) = F : \text{Aut}(F)$ is soluble. It then follows from Lemma 2.3(2) that $K \leq \prod_{B \in V\Gamma_F} K^B$ is soluble. In particular, $K \leq M$.

Now Γ_F is a connected G/K -edge-transitive graph. Note that G/K has a transitive subgroup $\hat{R}K/K$ on $V\Gamma_F$. Since $|\hat{R}K/K| = |\hat{R}/(\hat{R} \cap K)|$ is a divisor of n and $\text{val}(\Gamma_F) \leq \text{val}(\Gamma) < p_1$, we have $(|\hat{R}K/K|, n) = 1$ for $\delta \in V\Gamma_F$, so $(\hat{R}K/K)_\delta = 1$ and $\hat{R}K/K$ is regular on $V\Gamma_F$, that is, Γ_F is a Cayley graph of $\hat{R}K/K$. Therefore, the triple $(\Gamma_F, \hat{R}K/K, G/K)$ satisfies Hypothesis 3.1. Since the soluble radical of G/K is M/K , by induction, we have $G/K = (M/K \times T).o$, and in turn, $G \cong M.T.o$, where T is nonabelian simple and $o \leq \text{Out}(T)$. In particular, T is the unique insoluble composition factor of G .

Let $C = C_G(F)$. Noting that F is a normal Hall subgroup of G (and so of C), by Schur-Zassenhaus theorem, F has a complementary subgroup in C , say H , hence $C = F \times H$.

Since $G/C \leq \text{Aut}(F)$ is abelian, C is insoluble, so is H . Let M_1 be the soluble radical of H , and let F_1 be the Fitting subgroup of M_1 . Since F_1 is characteristic in M_1 , M_1 is characteristic in H and H is normal in G , we have F_1 is normal in G . If $M_1 \neq 1$, by Lemma 2.1, $F_1 \neq 1$. It follows $\langle F, F_1 \rangle = F \times F_1 > F$ is a nilpotent normal subgroup of G , a contradiction. Hence $M_1 = 1$. Now, as G has unique insoluble composition factor T , it easily follows $\text{soc}(H) = T$. Thus, $\langle M, T \rangle = M \times T \triangleleft G$ and $G = (M \times T).o$. $\square$

With Lemma 6.2, we here determine the corresponding graphs Γ when $M \neq 1$.

Lemma 6.3 *Assume G is insoluble and $M \neq 1$. Then Γ satisfies part (5) of Theorem 1.1.*

Proof By Lemmas 6.2 and 3.1, $G = (M \times T).o$ and M is intransitive on $V\Gamma$. Let K be the kernel of G acting on $V\Gamma_M$. Then $M \triangleleft K \triangleleft G$, $\alpha^M = \alpha^K$ and $|M|_\pi = |K|_\pi$ by Lemma 3.1(2). If $M < K$, as $K/M \triangleleft G/M \cong T.o$, we have $|M|_\pi |T|_\pi$ divides $|K|_\pi$, implying $|T|_\pi = 1$, which is a contradiction again by Lemma 3.1(2). Hence $K = M$. Moreover, arguing similarly as in the proof of Lemma 6.2, we have $\hat{R}M/M$ is regular on $V\Gamma_M$, it follows that Γ_M is a connected G/M -edge-transitive Cayley graph of $\hat{R}M/M$, and satisfies Hypothesis 3.1.

Since G/M is almost simple with socle isomorphic to T (for convenience, we also use T to denote the socle of G/M), by Lemma 5.3, Γ_M is T -edge-transitive, and in turn, Γ is X -edge-transitive with $X = M \times T$. By Lemma 3.1(4), $X_\alpha = M_\alpha \times T_\alpha$, and then by Lemma 2.8, $\Gamma \cong \Delta \times \Sigma$, where Δ is a connected M -edge-transitive graph and Σ is a connected T -edge-transitive graph. Further, by Lemma 3.1(3), $\hat{R} \cap M$ is regular on $V\Delta$ and $\hat{R} \cap T$ is regular on $V\Sigma$, hence Δ satisfies Lemma 4.3, and Σ lies in Table 1 by Lemma 5.3. Now, since $\text{val}(\Gamma) = \text{val}(\Delta) \cdot \text{val}(\Sigma) < p_1$, checking the valency, there are only two possibilities:

- (a) $\text{val}(\Delta) = 2$, $T = \text{PSL}(2, 23)$ and $\text{val}(\Sigma) = 4$.
- (b) $\text{val}(\Delta) \leq 4$, $T = \text{PSL}(2, 59)$ and $\text{val}(\Sigma) \leq 12$.

For the former case, Δ is a cycle of length m say and $M \cong D_{2m}$, Γ satisfies part (5)(i) of Theorem 1.1. For case (b), $\text{val}(\Gamma) = \text{val}(\Delta) \cdot \text{val}(\Sigma)$ is less than the smallest prime divisor of $|V\Delta|$ and 29, Γ satisfies part (5)(ii) of Theorem 1.1. $\square$

We may now determine the full automorphism group $\text{Aut}\Gamma$ while G is soluble.

Lemma 6.4 *Let $A = \text{Aut}\Gamma$ and suppose G is soluble. Then one of the following statements holds.*

1. $A = G$;
2. $\Gamma \cong K_n$ with $n \geq 5$ a prime, $G \cong \mathbb{Z}_n : \mathbb{Z}_{\frac{n-1}{2}}$ or $\mathbb{Z}_n : \mathbb{Z}_{n-1}$ and $A = S_n$;
3. $R = \langle a \rangle \cong \mathbb{Z}_n$ with n not a prime, $\tau \notin G_1$ and $A = G : \langle \tau \rangle \cong G : \mathbb{Z}_2$, where $\tau \in \text{Aut}(R)$ maps a to a^{-1} ;
4. $R \cong \mathbb{Z}_{11} : \mathbb{Z}_5$, $G \cong \mathbb{Z}_{11} : \mathbb{Z}_{10}$, $A \cong \text{PGL}(2, 11)$ and Γ is of valency 4.

Proof Assume $A \neq G$. Then $R < G < A$, and Γ is A -edge-transitive.

Suppose first A is soluble. By Lemma 4.3, Γ is a normal Cayley graph, and satisfies part (1) or (2) of Lemma 4.3. For part (1), R is cyclic, and as $\tau \in A_1$, both $\hat{R} : \langle G_1, \tau \rangle$

and $A = \hat{R} : A_1$ are regular on the arc set of Γ , $\hat{R} : \langle G_1, \tau \rangle = \hat{R} : A_1$. Since $A_1 \leq \text{Aut}(R)$ is abelian, it follows $\tau \notin G_1$ and $A = G : \langle \tau \rangle \cong G : \mathbb{Z}_2$ since $A \neq G$. For part (2), both G and A are regular on the edge set of Γ , so $A = G$, which is a contradiction.

Suppose now A is insoluble. By Lemmas 5.3, 6.1 and 6.3, either Γ lies in Table 1 or satisfies parts (5) of Theorem 1.1. Then since A has a soluble edge-transitive subgroup G of Γ , either Γ is a complete graph K_n with n a prime, or $A = \text{PGL}(2, 11)$, $G \cong \mathbb{Z}_{11} : \mathbb{Z}_{10}$, $R \cong \mathbb{Z}_{11} : \mathbb{Z}_5$ and Γ is of valency 4. For the former case, as G is edge-transitive on Γ , G is 2-homogeneous of prime degree n , hence $G \cong \mathbb{Z}_n : \mathbb{Z}_{\frac{n}{2}}$ or $\mathbb{Z}_n : \mathbb{Z}_{n-1}$ since G is soluble. $\square$

Now, we are ready to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 Let $A = \text{Aut}\Gamma$. Then $R < G \leq A$ and Γ is A -edge-transitive.

Assume first G is soluble. If A is insoluble, then $G < A$, by Lemma 6.4, part (1)(i) or part (3) of Theorem 1.1 holds. If A is soluble, by Lemmas 4.3 and 6.4, Γ is a normal Cayley graph of R , and either $A = G : \mathbb{Z}_2$ and R is cyclic or $A = G$. It follows that Γ satisfies part (1)(ii) or part (2) of Theorem 1.1.

Assume now G is insoluble. Then A is insoluble. Let M denote the soluble radical of A . If $M = 1$, by Lemma 6.1, A is almost simple, then by Lemma 5.3, Γ lies in Table 1, part (4) of Theorem 1.1 holds. If $M \neq 1$, by Lemma 6.3, $\Gamma \cong \Delta \times \Sigma$ satisfies part (5) of Theorem 1.1. This completes the proof of Theorem 1.1. $\square$

Proof of Corollary 1.2 Since Γ is a locally primitive Cayley graph, Γ is arc-transitive, so Γ satisfies parts (1) or (3)–(5) of Theorem 1.1. Let $A = \text{Aut}\Gamma$. Then A_α with $\alpha \in V\Gamma$ is primitive on $\Gamma(\alpha)$.

If Γ is arc-regular, as in part (1)(ii) of Theorem 1.1, then since $\text{val}(\Gamma)$ is even and A_α is abelian, A_α is not primitive on $\Gamma(\alpha)$, which is not the case. If Γ lies in part (1)(i) or part (3) and (4), by the primitivity of A_α acting on $\Gamma(\alpha)$, Γ is listed in Table 2. Finally, one easily shows the graphs in part (5) of Theorem 1.1 are not locally primitive. $\square$

7 Proofs of Theorems 1.3 and 1.4

The following number-theoretic result is known and easy to prove.

Lemma 7.1 *For each positive integer $k \geq 2$, there are infinitely many primes p such that $k \mid (p - 1)$.*

We finally prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3 We first give a construction.

Construction 7.2 *Choose distinct primes $p_1 < p_2 < \dots < p_s$ such that $p_1 > 2k + 1$ and $2k \mid (p_i - 1)$ for $i = 1, 2, \dots, s$. Set $n = p_1 p_2 \dots p_s$ and let $R = \langle a \rangle$ be a cyclic group of order n . Then $\text{Aut}(R) \cong \mathbb{Z}_{p_1-1} \times \mathbb{Z}_{p_2-1} \times \dots \times \mathbb{Z}_{p_s-1}$ has a subgroup H of order $2k$ which contains the automorphism τ of R mapping a to a^{-1} .*

Let $S = a^H$ and set $\Gamma = \text{Cay}(R, S)$.

Since $\langle S \rangle = R$, Γ is connected. Also, Γ has a soluble arc-regular automorphism group $\hat{R} : H$, with valency $\text{val}(\Gamma) \leq |H| = 2k \leq p_1 - 2 \leq n - 2$, hence Γ is not a complete graph and satisfies Theorem 1.1. By checking the list there, Γ is an arc-regular normal Cayley graph of valency $2k$. Further, by Lemma 7.1, the primes $p_1, p_2, \dots, p_s$ above have infinitely many choices, Theorem 1.3 follows. $\square$

Proof of Theorem 1.4 We give the following construction.

Construction 7.3 Let $p_1 < p_2 < \dots < p_t$ be primes such that $5 \neq p_1 > 2k$ and $p_1 k \mid (p_i - 1)$ for $i = 2, \dots, t$. Write $m = p_2 p_3 \dots p_s$. Then there is a center free metacyclic group $R := \langle a \rangle : \langle b \rangle \cong \mathbb{Z}_m : \mathbb{Z}_{p_1}$, such that $\text{Aut}(R) \cong \mathbb{Z}_m : \mathbb{Z}_m^*$ (see Lemma 4.2) contains a subgroup K of order k such that $\langle b^K \rangle = R$.

Let $S = \{b, b^{-1}\}^K$ and $\Gamma = \text{Cay}(R, S)$.

Now, Γ is connected since $\langle b^K \rangle = R$. Since $\text{val}(\Gamma) = |S| \leq 2|K| < p_1$ and Γ has an edge-transitive automorphism group $\hat{R} : K$, Γ satisfies Theorem 1.1. Since $\hat{R} : H$ is soluble and $p_1 \neq 5$, by Theorem 1.1, Γ is an edge-regular normal metacirculant with valency $2k$. Because the primes $p_1, p_2, \dots, p_t$ above have infinitely many choices by Lemma 7.1, Theorem 1.4 follows. $\square$

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