Edge-transitive uniface embeddings of bipartite multi-graphs

Wenwen Fan¹ · Cai Heng Li² · Naer Wang³

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Abstract It is shown that a bipartite multi-graph Γ has an orientably edge-transitive embedding with a single face if and only if $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ such that $\gcd(m, n) = 1$ and mn is even whenever λ is even. A consequence of this shows that each orientable surface carries a simple edge-transitive map with a single face.

Keywords Edge-transitive · Uniface embedding · Complete bipartite multi-graphs

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✉ Wenwen Fan
fww0871@163.com

Cai Heng Li
lich@sustc.edu.cn

Naer Wang
wangnaer@zjou.edu.cn

¹ School of Mathematics, Yunnan Normal University, Kunming 650500, Yunnan, People's Republic of China

² Department of Mathematics, Southern University of Science and Technology, ShenZhen, People's Republic of China

³ School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316000, Zhejiang, People's Republic of China

1 Introduction

Let $\Gamma = (V, E)$ be a connected (multi)graph with vertex set V and edge set E . A 2-cell embedding of Γ in an orientable closed surface $\mathbb{S}$ decomposes $\mathbb{S}$ into open discs, called *faces*. The set of faces is denoted by F . A 2-cell embedding is sometimes simply called an *embedding*, or a *map*, and denoted by an incident triple $\mathcal{M} = (V, E, F)$. The graph Γ is called the *underlying graph*, and the surface $\mathbb{S}$ is called the *supporting surface* of the map. A map is said to be a *simple map* if the underlying graph is a simple graph, while a map called a *bipartite map* if it has a bipartite underlying graph. If $\mathcal{M}$ has only one face, then $\mathcal{M}$ is called a *uniface map* or a *uniface embedding*.

An automorphism of a map $\mathcal{M}$ is a permutation of $V \cup E \cup F$ which preserves V , E , and F (setwise) and the incidence relation between them. For a bipartite map $\mathcal{M} = (V, E, F)$, the vertex set V has a bipartition $V = U \cup W$ such that all edges of $\mathcal{M}$ lie between U and W . Let $\text{Aut}^\oplus \mathcal{M}$ be the subgroup of the automorphisms of $\mathcal{M}$ which preserves the orientation of the supporting surface and the bipartition of V . Then $\text{Aut}^\oplus \mathcal{M}$ is semiregular on the edge set E . If $\text{Aut}^\oplus \mathcal{M}$ is transitive on E , then $\mathcal{M}$ is called *orientably edge-transitive map* (or embedding). In this case, $\text{Aut}^\oplus \mathcal{M}$ is regular on E , and $\mathcal{M}$ is called an *orientably edge-regular map* (or embedding).

Classifying edge-transitive embeddings of complete bipartite (multi)graphs is an open problem proposed by Jones in 2007, see [8]. Some special families have been classified, refer to [2–5, 7]. In terms of [9], the maps we consider in this paper are of type 2ex. We denote by $\mathbf{K}_{m,n}^{(\lambda)}$ a *complete bipartite (multi)graph* with edge-multiplicity λ . The following theorem characterises orientably edge-transitive uniface embeddings of bipartite graphs.

Theorem 1.1 *A bipartite graph Γ has an orientably edge-transitive uniface embedding if and only if $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ such that $\gcd(m, n) = 1$, and mn is even whenever λ is even.*

Generally, a complete bipartite graph $\mathbf{K}_{m,n}^{(\lambda)}$ where $\gcd(m, n) = 1$ and mn is even whenever λ is even may have many non-isomorphic orientably edge-regular embeddings, refer to [4, 7], but only one of them is a uniface map. For convenience, we denote this unique uniface embedding of $\mathbf{K}_{m,n}^{(\lambda)}$ by

$$\mathcal{K}_{m,n}^{(\lambda)}, \text{ or } \mathcal{K}_{m,n} \text{ for } \lambda = 1.$$

Obviously, $\mathcal{K}_{m,n}^{(\lambda)} \cong \mathcal{K}_{n,m}^{(\lambda)}$ has face length $mn\lambda$. The genus of the supporting surface of a map is said to be the genus of the map.

Theorem 1.2 *The uniface map $\mathcal{K}_{m,n}^{(\lambda)}$ has genus $\frac{1}{2}(mn\lambda - (m + n) + 1)$; in particular, $\mathcal{K}_{m,n}$ has genus $\frac{1}{2}(m - 1)(n - 1)$.*

Any non-negative integer g can be written as $g = \frac{1}{2}(2 - 1)((2g + 1) - 1)$, and hence Theorem 1.2 tells us that the map $\mathcal{K}_{2,2g+1}$ lies on the surface $\mathbb{S}_g$. This provides us with a simple method for constructing orientable surfaces, associated with a cyclic group action, see Construction 3.5.

Corollary 1.3 *For any integer $g \geq 0$, the surface $\mathbb{S}_g$ carries an orientably edge-regular simple uniface map $\mathcal{K}_{m,n}$ such that $g = \frac{1}{2}(m-1)(n-1)$ with $\gcd(m, n) = 1$.*

It would be worth to point out that the conclusion of Corollary 1.3 is contrary to the conclusion for (arc)regular maps which states that there are infinite many surfaces that do not carry reflexible (arc) regular simple maps and chiral (arc) regular simple maps, refer to [1].

Given a positive integer g , it would be interesting to determine the number of the uniface maps $\mathcal{K}_{m,n}^{(\lambda)}$ on $\mathbb{S}_g$, which is equal to the number of the triples of positive integers (m, n, λ) such that

$$m \leq n, \gcd(m, n) = 1, \text{ and } mn \text{ is even whenever } \lambda \text{ is even.} \quad (1)$$

Problem A Given a positive integer g , determine the number of triples (m, n, λ) satisfying condition (1) such that $g = \frac{1}{2}(mn\lambda - (m+n) + 1)$. In particular, determine the number of pairs (m, n) such that $g = \frac{1}{2}(m-1)(n-1)$ with $m \leq n$ and $\gcd(m, n) = 1$.

For prime genus, we have the following conclusion.

Example 1.4 Let p be a prime. If $p = 3$ or $p \equiv 1 \pmod{3}$, then $\mathcal{K}_{2,2p+1}$ and $\mathcal{K}_{3,p+1}$ are the only orientably edge-regular simple uniface map on $\mathbb{S}_p$; if $p \equiv 2 \pmod{3}$, then $\mathcal{K}_{2,2p+1}$ is the only orientably edge-regular simple uniface map on $\mathbb{S}_p$.

Harer and Zagier [6] computed the number of bipartite uniface maps (dessins) based on a 2ℓ -gon. It is a natural problem to determine the number of orientably edge-regular uniface maps based on a 2ℓ -gon, leading to the next problem.

Problem B For a positive integer ℓ , find the number of factorisations $\ell = mn\lambda$ such that m, n, λ satisfy condition (1).

Example 1.5 Let $\ell = p^d$ with p prime. Then there are exactly $d + 1$ orientably edge-regular uniface maps that can be formed from a $2p^d$ -gon, which are $\mathcal{K}_{1,p^e}^{(p^{d-e})}$ with $0 \leq e \leq d$ and have genus $\frac{1}{2}(p^d - p^e)$.

2 Edge-transitive uniface bipartite maps

Let $\Gamma = (V, E)$ be a connected bipartite graph with bipartition $V = U \cup W$. Let $\mathcal{M} = (V, E, F)$ be an orientably edge-transitive embedding of Γ , and $G = \text{Aut}^\oplus \mathcal{M}$.

Lemma 2.1 *Assume that $\mathcal{M}$ has a unique face and $|E| = \ell$. Then G is a cyclic group of order ℓ and regular on the edges.*

Proof The group G is transitive on the edge set E by definition. Since $\mathcal{M}$ has a unique face f say, we have $G = G_f$. Since $\mathcal{M}$ is orientably edge-transitive, G_f is cyclic. As G preserves the bipartition $V = U \cup W$ and the orientation of the supporting surface, G is faithful on the edge set. Thus $G = G_f$ is regular on E , and $|G| = |E| = \ell$. $\square$

The next lemma shows that Γ is a complete bipartite multi-graph if it admits a cyclic edge-transitive group of automorphisms.

Lemma 2.2 Assume that a connected bipartite graph Γ with bipartition U, W has an edge-regular cyclic group G of automorphisms. Then $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ such that $\gcd(m, n) = 1$.

Proof Let $\alpha \in U$ and $\beta \in W$ be two adjacent vertices. Then the stabilisers $G_\alpha = \langle a \rangle$, and $G_\beta = \langle b \rangle$ for some elements $a, b \in G$ as G preserves the orientation of the supporting surface. Since Γ is a connected G -edge-transitive bipartite graph, $G = \langle G_\alpha, G_\beta \rangle = \langle a, b \rangle$. We may identify $U = [G : G_\alpha]$ and $W = [G : G_\beta]$ such that $G_\alpha x$ is adjacent to $G_\beta y$ if and only if $yx^{-1} \in G_\beta G_\alpha$, where $x, y \in G$.

As G is cyclic, we have $G = G_\alpha G_\beta$, and hence G_α is transitive on the set $W = [G : G_\beta]$. It follows that α is adjacent to all vertices in W . Noticing that G preserves U and W , and is transitive on U , we conclude that each vertex of U is adjacent to all vertices of W . So Γ is a complete bipartite multi-graph.

Let $\langle c \rangle = G_\alpha \cap G_\beta = \mathbb{Z}_\lambda$ for some integer λ . Then $G_\alpha = \langle a \rangle \cong \mathbb{Z}_{m\lambda}$ and $G_\beta = \langle b \rangle \cong \mathbb{Z}_{n\lambda}$ for some integers m, n , and the order $|G| = mn\lambda$. Since G is regular on the edges of the graph, $G_\alpha \cap G_\beta$ is regular on the set of edges between α and β . Therefore, the multi-graph Γ is isomorphic to $\mathbf{K}_{m,n}^{(\lambda)}$. Moreover, since G is cyclic, the factor group

$$G/\langle c \rangle = (\langle a \rangle \langle b \rangle) / \langle c \rangle = \langle \bar{a} \rangle \times \langle \bar{b} \rangle \cong \mathbb{Z}_m \times \mathbb{Z}_n$$

is cyclic, and hence $\gcd(m, n) = 1$. $\square$

Let e be an edge of Γ . If the end vertices of e are α and β , we sometimes write

$$e = [\alpha, e, \beta] = [\beta, e, \alpha].$$

For a vertex α of Γ , let $E(\alpha)$ denote the set of edges which are incident with α .

Lemma 2.3 Assume that $\mathcal{M}$ has a unique face. Then, for an edge $e = [\alpha, e, \beta]$, there exist $a, b \in G$ such that $G_\alpha = \langle a \rangle$, $G_\beta = \langle b \rangle$, and $G = \langle ba \rangle$.

Proof Since G is regular on the edges, we identify the edges of $\mathcal{M}$ with the elements of G such that G acts on the edge set by right multiplication. Let e be the edge represented by the identity $1 \in G$ with the end vertices being $\alpha \in U$ and $\beta \in W$, and we write

$$e = [\alpha, 1, \beta] = [\beta, 1, \alpha].$$

Let f, f' be the two faces which are incident with the edge e .

Let $(e_1, \alpha, e, \beta, e_2)$ be the 3-arc on the boundary of f , and $(e'_1, \alpha, e, \beta, e'_2)$ be the 3-arc on the boundary of f' . Since G_α is regular on $E(\alpha)$ and preserves the supporting surface, there exists an element $a \in G$ such that $G_\alpha = \langle a \rangle$, and a sends e to e_1 , and sends e'_1 to e . Similarly, there exists an element $b \in G$ such that $G_\beta = \langle b \rangle$, and b sends e to e'_2 , and sends e_2 to e . Thus, $e_1 = a$, $e'_1 = a^{-1}$, $e_2 = b^{-1}$ and $e'_2 = b$. Noticing that the rotation a sends the face f' to f , the image of the 2-arc $(1, b)$ under a is $(1, b)^a = (a, ba)$. On the other hand, the element ba sends the 2-arc $(b^{-1}, 1)$ to $(b^{-1}, 1)^{ba} = (a, ba)$. The two 2-arcs $(b^{-1}, 1)$ and (a, ba) lie on the boundary of

the face f , and each of them uniquely determines the face f . We conclude that the element ba fixes the face f . So does the subgroup $\langle ba \rangle$, and it follows that all the images of $(b^{-1}, 1)$ under $\langle ba \rangle$ form the boundary cycle of f :

$$C := (b^{-1}, 1, a, ba, \dots, b^{-1}(ba)^i, (ba)^i, a(ba)^i, \dots, (ba)^{\ell-1})$$

where $\ell = |ba|$. Since $\mathcal{M}$ has only one face, each edge of Γ appears on C exactly two times. It follows that

$$\begin{aligned} 1 &= a(ba)^i, \\ b^{-1} &= (ba)^j, \end{aligned}$$

for some integers i, j . Thus $a = (ba)^{\ell-i}$ and $b = (ba)^{\ell-j}$. Therefore, $G = \langle a, b \rangle = \langle ba \rangle$, completing the proof. $\square$

For an integer n and a prime p , let n_p be the p -part of n , that is, n_p is a power of p and divides n , and n/n_p is coprime to p . The factor n/n_p is denoted by $n_{p'}$.

For a group G and a prime divisor p of the order $|G|$, let G_p be a Sylow p -subgroup of G , and $G_{p'}$ a Hall p' -subgroup of G . For an element $g \in G$, let g_p be the p -part of g and $g_{p'}$ the p' -part, namely, $g = g_p g_{p'} = g_{p'} g_p$ such that $o(g_p)$ is a power of p , and $o(g_{p'})$ is coprime to p .

Lemma 2.4 *If $\mathbf{K}_{m,n}^{(\lambda)}$ has an orientably edge-regular uniface embedding, then $\gcd(m, n) = 1$, and mn is even whenever λ is even.*

Proof Let $\mathcal{M}$ be an orientably edge-regular uniface embedding of $\mathbf{K}_{m,n}^{(\lambda)}$. By Lemmas 2.1 and 2.2, we have $G = \langle a \rangle \langle b \rangle$, where $\langle a \rangle \cong \mathbb{Z}_{m\lambda}$, $\langle b \rangle \cong \mathbb{Z}_{n\lambda}$, $\langle a \rangle \cap \langle b \rangle \cong \mathbb{Z}_\lambda$. Let f be the unique face of $\mathcal{M}$. Then $G = G_f = \langle ba \rangle$ by Lemma 2.3.

Suppose that λ is even and both m, n are odd. Let a_2, b_2 be the 2-part of a, b , respectively. Since $o(a) = m\lambda$, $o(b) = n\lambda$ and $G = \langle ba \rangle$, both a_2 and b_2 have order λ_2 , the 2-part of λ , and hence they are generators of the unique Sylow 2-subgroup G_2 . Thus $a_2 = (ba)^i$ and $b_2 = (ba)^j$ for some odd integers i, j . Hence $i + j$ is even and $b_2 a_2 = (ba)^{i+j}$ is not a generator of G_2 , which contradicts $G = \langle ba \rangle$. Therefore, if λ is even, then one of m, n is even, and equivalently, mn is even. $\square$

For example, $\mathbf{K}_{3,5}^{(2)}$ does not have an edge-regular uniface embedding.

3 Proof of Theorems

The proof of Theorem 1.1 consists of a series of lemmas.

Lemma 3.1 *Let G be a cyclic group of order ℓ , and let $\ell = mn\lambda$ be such that $\gcd(m, n) = 1$, and mn is even whenever λ is even. Then there exist $a, b \in G$ such that $o(a) = m\lambda$, $o(b) = n\lambda$, and $G = \langle a \rangle \langle b \rangle = \langle ba \rangle$.*

Proof Let $p_1, p_2, \dots, p_t$ be the prime divisors of the integer ℓ . Since $\gcd(m, n) = 1$, for each p_i , the p_i -parts of m, n are such that either

- (i) $m_{p_i} = 1$, and $n_{p_i} = \ell_{p_i}/\lambda_{p_i}$, or
 (ii) $n_{p_i} = 1$, and $m_{p_i} = \ell_{p_i}/\lambda_{p_i}$.

Let g_i be an element of G of order ℓ_{p_i} , where $1 \leq i \leq t$. Then g_i is a generator of the unique Sylow p_i -subgroup of G . Let

$$\begin{cases} a = g_1^{n_{p_1}} g_2^{n_{p_2}} \cdots g_t^{n_{p_t}}, \\ b = g_1^{m_{p_1}} g_2^{m_{p_2}} \cdots g_t^{m_{p_t}}. \end{cases}$$

Then $\{g_i^{m_{p_i}}, g_i^{n_{p_i}}\} = \{g_i, g_i^{\ell_{p_i}/\lambda_{p_i}}\}$, and thus

$$g_i^{m_{p_i}} g_i^{n_{p_i}} = g_i^{m_{p_i} + n_{p_i}} = g_i^{1 + \ell_{p_i}/\lambda_{p_i}}, \text{ and } \langle g_i^{m_{p_i}} \rangle \cap \langle g_i^{n_{p_i}} \rangle = \langle g_i^{\ell_{p_i}/\lambda_{p_i}} \rangle.$$

Since mn is even whenever λ is even, if $\lambda_2 \neq 1$, then $m_2 n_2 \neq 1$. Thus ℓ_2/λ_2 is even, and $1 + \ell_2/\lambda_2$ is odd. It implies that, for every p_i , we have that $1 + \ell_{p_i}/\lambda_{p_i}$ is coprime to p_i . Thus $g_i^{m_{p_i}} g_i^{n_{p_i}} = g_i^{1 + \ell_{p_i}/\lambda_{p_i}}$ is a generator of the Sylow p_i -subgroup $\langle g_i \rangle$, and hence

$$ba = \left(g_1^{m_{p_1}} g_1^{n_{p_1}}\right) \left(g_2^{m_{p_2}} g_2^{n_{p_2}}\right) \cdots \left(g_t^{m_{p_t}} g_t^{n_{p_t}}\right)$$

is a generator of G . Furthermore, the intersection

$$\langle c \rangle = \langle a \rangle \cap \langle b \rangle = \langle g_1^{\ell_{p_1}/\lambda_{p_1}} g_2^{\ell_{p_2}/\lambda_{p_2}} \cdots g_t^{\ell_{p_t}/\lambda_{p_t}} \rangle \cong \mathbb{Z}_\lambda.$$

Since $m_{p_i} n_{p_i} = \ell_{p_i}/\lambda_{p_i}$ and $o(g_i^{mn_{p_i}}) = o(g_i^{m_{p_i} n_{p_i}}) = o(g_i^{\ell_{p_i}/\lambda_{p_i}})$, it follows that $a^m = g_1^{mn_{p_1}} g_2^{mn_{p_2}} \cdots g_t^{mn_{p_t}}$ is of order λ . So the order $o(a) = m\lambda$. Similarly, b^n is of order λ , and the order $o(b) = n\lambda$. Thus, $G = \langle a \rangle \langle b \rangle = \langle ba \rangle$, and this completes the proof. $\square$

We now give a coset graph representation of $\mathbf{K}_{m,n}^{(\lambda)}$.

Definition 3.2 Let $G = \langle a \rangle \langle b \rangle$ be a group such that $\langle a \rangle \cong \mathbb{Z}_{m\lambda}$, $\langle b \rangle \cong \mathbb{Z}_{n\lambda}$, and $\langle a \rangle \cap \langle b \rangle \cong \mathbb{Z}_\lambda$. Define a bipartite multi-graph Γ to have biparts U and W , and edge set E as follows:

$$\begin{aligned} U &= [G : \langle a \rangle] = \{\langle a \rangle, \langle a \rangle b, \dots, \langle a \rangle b^{n-1}\}, \\ W &= [G : \langle b \rangle] = \{\langle b \rangle, \langle b \rangle a, \dots, \langle b \rangle a^{m-1}\}, \\ E &= \{[\langle a \rangle x, z, \langle b \rangle y] \mid x, y \in G, yx^{-1} \in \langle b \rangle \langle a \rangle, z \in \langle a \rangle x \cap \langle b \rangle y\}. \end{aligned}$$

This graph is called a *coset multi-graph*, and denoted by $\text{Cos}(G, \langle a \rangle, \langle b \rangle, \langle a \rangle \cap \langle b \rangle)$.

Lemma 3.3 A coset multi-graph $\text{Cos}(G, \langle a \rangle, \langle b \rangle, \langle a \rangle \cap \langle b \rangle)$ defined in Definition 3.2 is isomorphic to a complete bipartite graph $\mathbf{K}_{m,n}^{(\lambda)}$, and is G -edge-regular.

Proof By definition, all edges lie between the vertices of U and W , and hence Γ is bipartite with biparts U and W , where $|U| = n$ and $|W| = m$. Moreover, each vertex $\langle a \rangle x \in U$ is adjacent to all vertices $\langle b \rangle y \in W$ where $x, y \in G$, so Γ is a complete bipartite multi-graph.

Since $\langle a \rangle x$ and $\langle b \rangle y$ are adjacent, $yx^{-1} = b^k a^l \in \langle b \rangle \langle a \rangle$ for some integers k, l by definition, it follows that

$$\begin{aligned} |\langle a \rangle x \cap \langle b \rangle y| &= |(\langle a \rangle \cap \langle b \rangle yx^{-1})x| = |(\langle a \rangle \cap \langle b \rangle b^k a^l)x| \\ &= |(\langle a \rangle \cap \langle b \rangle) a^l x| = |\langle a \rangle \cap \langle b \rangle| = \lambda, \end{aligned}$$

Thus there are exactly λ edges between any two adjacent vertices of Γ , namely, the edge-multiplicity of Γ is λ , that is, $\Gamma \cong \mathbf{K}_{m,n}^{(\lambda)}$.

For any edge $[\langle a \rangle x, z, \langle b \rangle y]$, by definition, $z \in \langle a \rangle x \cap \langle b \rangle y$, we have $z = a^i x = b^j y$ for some integers i and j . Thus the right multiplication of z is such that

$$[\langle a \rangle, 1, \langle b \rangle]^z = [\langle a \rangle(a^i x), 1z, \langle b \rangle(b^j y)] = [\langle a \rangle x, z, \langle b \rangle y],$$

and so the group G induces a transitive action on the edge set of Γ . Furthermore, it is easily shown that every element of G is an automorphism of Γ by right multiplication. As Γ has $mn\lambda$ edges, which equals the order $|G|$, thus G is regular on the edge set of Γ . $\square$

The following lemma shows that a complete bipartite graph with certain property has an edge-transitive single-face embedding. Since $\gcd(m, n) = 1$ and $\mathbf{K}_{m,n}^{(\lambda)} \cong \mathbf{K}_{n,m}^{(\lambda)}$, we may assume that $m < n$. The proof of the lemma provides an explicit simple construction of orientable surfaces by a single disc.

Lemma 3.4 *Let $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ be such that $\gcd(m, n) = 1$ and mn is even whenever λ is even. Then Γ has an edge-transitive uniface embedding.*

Proof Let $G = \mathbb{Z}_{mn\lambda}$, and $\ell = mn\lambda$. By Lemma 3.1, there exists $a, b \in G$ such that $o(a) = m\lambda$, $o(b) = n\lambda$, and $G = \langle a \rangle \langle b \rangle = \langle ba \rangle$. Then by Lemma 3.3, $\text{Cos}(G, \langle a \rangle, \langle b \rangle, \langle a \rangle \cap \langle b \rangle) = \Gamma = \mathbf{K}_{m,n}^{(\lambda)}$, and G is edge-regular on Γ . Let $g = ba$, and let

$$a = g^{n_1}, \quad b = ga^{-1} = g^{1-n_1},$$

where n_1 is a positive integer such that $|g^{n_1}| = |g^n|$ and $|g^{1-n_1}| = |g^m|$. Then for each j with $0 \leq j \leq \ell - 1$, we have $ag^j = g^{n_1+j}$. Identifying edges of Γ with elements of G , the following sequence of edges

$$\mathbf{C}_{2\ell} : 1, a, g, ag, g^2, \dots, g^j, ag^j, g^{j+1}, ag^{j+1}, \dots, g^{\ell-1}, ag^{\ell-1} \quad (2)$$

forms a closed walk such that each edge of Γ appears precisely two times. Now we construct an embedding of Γ : $\square$

Construction 3.5 Set $\ell = mn\lambda$.

- (i) Take a regular 2ℓ -gon $\mathbf{P}_{2\ell}$, which has 2ℓ sides and is homomorphic to an open disc.
- (ii) Label the 2ℓ sides of $\mathbf{P}_{2\ell}$ by elements in the sequence (2) anticlockwise; the order of edges in the sequence naturally assigns an orientation to each side edge of $\mathbf{P}_{2\ell}$.
- (iii) For each j with $0 \leq j \leq \ell - 1$, glue the two edges with labels ag^j and g^{n_1+j} in opposite orientation. This converts the polygon $\mathbf{P}_{2\ell}$ into a topological space $\widehat{\mathbf{P}}_{2\ell}$.

If e, e' are two side edges of $\mathbf{P}_{2\ell}$ which are incident, let

$$e \bigwedge e'$$

be the *corner* of $\mathbf{P}_{2\ell}$ surrounded by these two edges. Then the 2ℓ corners of $\mathbf{P}_{2\ell}$ have the following forms:

$$\begin{array}{ll} \text{type I: } g^i \bigwedge ag^i, & \text{where } 0 \leq i \leq \ell - 1, \\ \text{type II: } ag^i \bigwedge g^{i+1}, & \text{where } 0 \leq i \leq \ell - 1. \end{array}$$

Since the edge 1 is incident in the graph Γ with vertex $\langle a \rangle$, the corner vertex of $1 \bigwedge a$ is $\langle a \rangle$. As the edge 1 is also incident with $\langle b \rangle$ and $b^{-1} = ag^{-1}$, there is a corner formed by b^{-1} and 1, and the corner vertex of $b^{-1} \bigwedge 1$ is $\langle b \rangle$. Further, for positive integers i , the corner vertex of $g^i \bigwedge ag^i$ is the vertex $\langle a \rangle g^i$ of Γ , while the corner vertex of $ag^i \bigwedge g^{i+1}$ is the vertex $\langle b \rangle g^{i+1}$ of Γ .

Claim The topological space $\widehat{\mathbf{P}}_{2\ell}$ defined in Construction 3.5 is an orientable surface, and Γ is embedded in $\widehat{\mathbf{P}}_{2\ell}$ with a single face $\mathbf{P}_{2\ell}$.

Proof of the claim We first consider the corners of type I. Since $a = g^{n_1}$, for any given i with $0 \leq i \leq n - 1$, we have $ag^{i+jn_1} = g^{i+(j+1)n_1}$. Thus, in the sequence of the following $m\lambda$ corners, any two consecutive ones share a common edge:

$$g^i \bigwedge ag^i, g^{i+n_1} \bigwedge ag^{i+n_1}, \dots, g^{i+(m\lambda-1)n_1} \bigwedge ag^{i+(m\lambda-1)n_1}.$$

In the graph Γ , the $m\lambda$ corner vertices are the vertex $\langle a \rangle g^i$, and the side edges are

$$g^i, g^{i+n_1}, \dots, g^{i+(m\lambda-1)n_1}.$$

By Construction 3.5, these $m\lambda$ corners are glued together into a disc D in $\widehat{\mathbf{P}}_{2\ell}$, namely, the disc D is formed by the $m\lambda$ sectors. Moreover, the $m\lambda$ corner vertices are identified with a single interior point of D , and any sufficient small neighbourhood of each point in the $m\lambda$ corners is an open disc.

We now consider the corners of type II. Since $b = g^{m_1}$ where $m_1 = 1 - n_1$, we have

$$g^{i+jm_1} = g^{-jm_1} g^{i+(j+1)m_1} = b^{-j} g^{i+(j+1)m_1} = ag^{(i-1)+(j+1)m_1},$$

where $1 \leq i \leq m$, and $0 \leq j \leq n\lambda - 1$. Thus in the sequence of the $n\lambda$ corners:

$$ag^{i-1} \bigwedge g^i, ag^{i-1+m_1} \bigwedge g^{i+m_1}, \dots, ag^{i-1+(n\lambda-1)m_1} \bigwedge g^{i+(n\lambda-1)m_1},$$

any two consecutive corners share a common edge. By Construction 3.5, these $n\lambda$ corners (as sectors) are glued together into a disc D' such that any sufficient small neighbourhood of each point of the $n\lambda$ sectors is an open disc.

Since each point of $\mathbf{P}_{2\ell}$ is contained in some corner (sector) $\widehat{g^i} \bigwedge ag^i$ or $ag^{i-1} \bigwedge g^i$, each point of $\mathbf{P}_{2\ell}$ is an interior point of some open disc of $\widehat{\mathbf{P}}_{2\ell}$. Therefore, the topological space $\widehat{\mathbf{P}}_{2\ell}$ is a surface. Now the graph Γ is embedded in the surface $\widehat{\mathbf{P}}_{2\ell}$ with a single face $\mathbf{P}_{2\ell}$. $\square$

We are now ready to prove our main theorems.

Proof of Theorem 1.1 Let Γ be a bipartite graph which has an orientably edge-regular uniface embedding $\mathcal{M}$. By Lemmas 2.1–2.4, $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ such that $\gcd(m, n) = 1$, and mn is even whenever λ is even.

Conversely, assume that $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ such that $\gcd(m, n) = 1$, and mn is even whenever λ is even. By Lemmas 3.1, 3.3, and 3.4, the graph $\Gamma = \mathbf{K}_{m,n}^{(\lambda)}$ has an orientably edge-regular uniface embedding. $\square$

Proof of Theorem 1.2 For a uniface map $\mathcal{K}_{m,n}^{(\lambda)} = (V, E, F)$, we have

$$|E| = mn\lambda, |V| = m + n \text{ and } |F| = 1.$$

Since $\mathcal{K}_{m,n}^{(\lambda)}$ is orientable, by Euler's formula, the genus of $\mathcal{K}_{m,n}^{(\lambda)}$ is equal to

$$\frac{1}{2}(|E| - |V| - |F| + 2) = \frac{1}{2}(mn\lambda - (m + n) + 1).$$

Furthermore, if $\lambda = 1$, then $\mathcal{K}_{m,n}$ has genus $\frac{1}{2}(mn - (m + n) + 1) = \frac{1}{2}(m - 1)(n - 1)$. $\square$

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