Two new infinite families of arc-transitive antipodal distance-regular graphs of diameter three with $\lambda = \mu$ related to groups $Sz(q)$ and ${}^2G_2(q)$

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Abstract In this paper, we construct two new infinite families of arc-transitive distance-regular graphs, related to Suzuki groups $Sz(q)$ and Ree groups ${}^2G_2(q)$, where $q > 3$. They are antipodal r -covers of complete graphs on $q^2 + 1$ or $q^3 + 1$ vertices, respectively, with $\lambda = \mu$ and $r > 1$ being an arbitrary odd divisor of $q - 1$. We also find that the graph on the set of involutions of $Sz(q)$ with $q > 3$, whose edges are the pairs of involutions $\{u, v\}$ such that $|uv| = 5$, is distance-regular.

Keywords Antipodal graph · Arc-transitive graph · Automorphism group · Distance-regular graph · r -Fold covering graph

Mathematics Subject Classification 05E18 · 05E30

1 Introduction

Let Γ be an antipodal distance-regular graph of diameter three. Then Γ is an antipodal r -cover of the complete graph K_{k+1} , Γ has intersection array $\{k, \mu(r-1), 1; 1, \mu, k\}$, and any two adjacent vertices of Γ have exactly $\lambda = k - 1 - \mu(r-1)$ common neighbours, where parameter k is the valency of Γ and μ denotes the number of common neighbours for any two vertices of Γ at distance 2 (for notation, definition, and further theory of distance-regular graphs, see [2, 12]). For more background on antipodal distance-regular covers of complete graphs, we refer the reader to [4]. This subclass of imprimitive distance-regular graphs has been studied by many researchers for over the last twenty years. A very informative overview of the state of the art on such

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covers can be found in the recent paper [6]. Even though the problem of classification of antipodal distance-regular covers of complete graphs seems to be hopeless, a study of some of their subclasses with prescribed automorphism groups or combinatorial properties may provide further understanding of their structure.

Antipodal distance-transitive covers of complete graphs were classified in [5]. A more general problem is to describe all arc-transitive antipodal distance-regular covers of complete graphs (recall that a graph is called *arc-transitive* if its automorphism group acts transitively on ordered pairs of adjacent vertices).

An arc-transitive antipodal distance-regular graph Γ of diameter three possesses an automorphism group which acts 2-transitively on the set of fibres. It follows from the fact that any pair of distinct fibres (F_1, E_1) in Γ can be mapped to any other pair of distinct fibres (F_2, E_2) by an automorphism which maps an arc (x_1, y_1) to an arc (x_2, y_2) with $x_i \in F_i, y_i \in E_i, i = 1, 2$, see [5, Lemma 2.6]. The assumption of distance-transitivity involves another 2-transitive action (which, in general, is not necessary in the arc-transitive case), namely, 2-transitive action of the global stabilizer of an arbitrary fibre F in the full automorphism group on F . These observations together with the classification of finite 2-transitive permutation groups (for example, see [5, Theorem 2.9]) lead to some strong restrictions for the automorphism groups of such graphs.

In [8], a description of arc-transitive antipodal distance-regular graphs of diameter three with $\lambda = \mu$ is given. In particular, in [8] the authors found some necessary conditions characterizing three new potential families of such graphs corresponding to groups $U_3(q)$, $Sz(q)$ and ${}^2G_2(q)$. The existence of arc-transitive candidates for these families is provided by the coset graph construction. However, it was not known whether they are distance-regular or not (except for some distance-regular examples for small values of q which were constructed by the authors of [8] in GAP).

In this paper, we show that in several cases these arc-transitive graphs are indeed distance-regular, namely, we find two new infinite families of distance-regular graphs related to Suzuki groups $Sz(q)$ and Ree groups ${}^2G_2(q)$, where $q = 2^{2n+1} > 2$ or $q = 3^{2n+1} > 3$, respectively. A technique of proving their distance-regularity seems to be new and uses a canonical form of elements in groups $Sz(q)$ and ${}^2G_2(q)$, which in turn determines the structure of matchings in these graphs. In Sect. 2, we have collected some auxiliary results, and Sect. 3 contains the main results and their proofs.

2 Preliminaries

In this section, we provide some necessary fundamental results on arc-transitive graphs, coverings and groups $Sz(q)$ and ${}^2G_2(q)$. Our notation and terminology from group theory are standard and can be found in [1, 3].

Proposition 1 [5, Lemma 2.7] *Suppose that a non-normal subgroup H of a group G and an element $g \in G$ are given. Let $\Gamma(G, H, HgH)$ denote the graph with vertex set $V(G, H) = \{Hx \mid x \in G\}$ whose edges are the pairs $\{Hx, Hy\}$ such that $xy^{-1} \in HgH$.*

- (1) *Assume that G acts faithfully on $V(G, H)$, $g^2 \in H$ and $G = \langle H, g \rangle$. Then $\Gamma(G, H, HgH)$ is a simple, undirected, and connected graph which admits the*

group G acting (by right multiplication) faithfully and transitively both on vertices and on arcs.

- (2) Suppose G acts arc-transitively on a connected graph X , H is the stabilizer of a vertex x in G , and $g \in G$ is a 2-element interchanging x with some vertex adjacent to x in X . Then $X \simeq \Gamma(G, H, HgH)$, and also $g^2 \in H$ and $G = \langle H, g \rangle$.

Recall the notion of a covering graph according to [4]. Let Γ be a simple, undirected graph and suppose there is a partition Σ of its vertices into cells satisfying the following conditions: (1) each cell is an independent set; (2) between any two cells either there are no edges, or there is a matching. Let Γ/Σ be the graph with the cells of Σ as vertices and with two vertices adjacent if and only if there is a matching between them. Then Γ is said to be a *covering graph* of Γ/Σ . The elements of Σ are called *fibres* and if all of them have the same size r , then Γ is called *r -fold covering graph* of Γ/Σ .

Further in the paper for a fixed graph Γ and for any vertex a , we will denote by $[a]$ the set of neighbours of a in Γ .

Proposition 2 *Let H and S be non-normal subgroups of a group G , let $M = N_G(S)$ and let g be an element of G such that $S \leq H < M < G$, $g^2 \in H$, $G = \langle H, g \rangle$, $G = M \cup MgS$, $H^g \cap M \leq H$, $M^g \cap S = 1$ and $\Gamma = \Gamma(G, H, HgH)$. Suppose in addition that $|G : M| = k + 1$, $|M : H| = r$, $\gcd(r, k) = 1$, $|S| = k = |H : H^g \cap H|$. Then Γ is an arc-transitive r -fold covering graph of $\Gamma/\Sigma \simeq K_{k+1}$ with $\Sigma = \{Hm(gs)^e \mid m \in M\} \mid s \in S, e \in \{0, 1\}$.*

Proof Proposition 2 is a modified form of Lemma 2.7 of [5]. For completeness, we provide its proof. Let us consider the following four H -invariant sets $\Gamma_0 = \{H\}$, $\Gamma_1 = \{Hgh \mid h \in H\}$, $\Gamma_2 = \{Hx \mid x \notin gH \cup M\}$ and $\Gamma_3 = \{Hm \mid m \in M - H\}$. Clearly, only Γ_2 or Γ_3 might not be H -orbits. Since $g^2 \in H$, $HH^g = Hg^2g^{-1}Hg = HgHg$. We have $H \leq M$, and then the inclusion $H^g \cap M \leq H$ can be multiplied on the left and right by H . Thus

$$H(H^g \cap M) = (H(g^{-1}Hg)) \cap M = HgHg \cap M \subseteq H, \quad (1)$$

and

$$((H(g^{-1}Hg)) \cap M)H = HgHgH \cap M \subseteq H. \quad (2)$$

Further, the right H -cosets at distance at most 2 from H in Γ are precisely those in $HgHgH \cup HgH$. Suppose that $m \in M - H$ and vertices Hm and H are at distance less than 3. Then $m \in HgHgH \cup HgH$. If $m \in HgHgH$, then by (2) $m \in H$, a contradiction. If $m \in HgH$, then $g \in M$, but $G = \langle H, g \rangle$, a contradiction. Consequently, the set $\Gamma_3 = \{Hm \mid m \in M - H\}$ consists of vertices at distance at least 3 from H in Γ .

As the inclusion (1) can be multiplied on the right by g , we see that $(HgHg \cap M)g = HgH \cap Mg \subseteq Hg$. This implies that there is only edge in Γ between H and $\{Hmg \mid m \in M\}$, that is the edge $\{H, Hg\}$. Let $m_1, m_2 \in M - H$. If $Hm_1 \in [Hm_2]$ then $m_1m_2^{-1} \in HgH$ and $g \in M$, but $G = \langle H, g \rangle$, a contradiction. Suppose that the vertices Hm_1, Hm_2 are at distance 2 in Γ . Then $m_1m_2^{-1} \in HgHgH$. By (2), we have that $m_1m_2^{-1} \in H$, which implies $Hm_1 = Hm_2$, a contradiction.

The valency of Γ is equal to the number of right H -cosets in the double coset HgH , that is to $|H : H^g \cap H| = |S| = k$. Thus every vertex from Γ_3 is adjacent precisely to k vertices from Γ_2 . As $H^g \cap S = 1$, S acts regularly on $[H]$, and G acts arc-transitively on Γ . This implies that there exists a constant λ such that any two adjacent vertices have exactly λ common neighbours. The stabilizer of the arc (H, Hg) in G is precisely $H^g \cap H$.

As given, the group M acts transitively on $\{Mx \mid x \in G - M\}$, $|G : M| = k + 1$, $|M : H| = r$ and $\gcd(r, k) = 1$. We have $|M : M^g \cap M| = k$ and $r = |M^g \cap M : H^g \cap H|$.

This implies that $M^g \cap M$ -orbit on the arcs of Γ , containing the arc (H, Hg) , has size r . Hence there is a matching between $\{Hm \mid m \in M\}$ and $\{Hmg \mid m \in M\}$ in Γ which is $M^g \cap M$ -invariant, and $M^g \cap M$ acts transitively on $\{Hm \mid m \in M\}$.

Since $M = N_G(S)$, it follows that every element of S fixes $\Gamma_0 \cup \Gamma_3$ pointwise.

Let Σ be the partition of vertices of Γ into $k + 1$ cells of type $\{Hm(gs)^e \mid m \in M\}$, where $s \in S$ and $e \in \{0, 1\}$. It follows that Σ forms an imprimitivity system of G on the vertices of Γ , such that each of its blocks corresponds to a right M -coset of G with appropriate representative $(gs)^e$.

Since $\Gamma_0 \cup \Gamma_3$ consists of vertices at pairwise distance at least 3 in Γ , for each element $s \in S$, the set $(\Gamma_0 \cup \Gamma_3)gs$ also consists of vertices at pairwise distance at least 3 in Γ , and there is a matching between $\{Hm \mid m \in M\}$ and $\{Hmgs \mid m \in M\}$ in Γ . Thus each cell of Σ is an independent set of size r . Note also, that S acts regularly on the neighbourhood of any vertex from $\Gamma_0 \cup \Gamma_3$. Therefore, G acts 2-transitively on Σ , and there is a matching between any two cells of Σ . Hence Γ is an r -fold covering graph of K_{k+1} . This completes the proof. $\square$

Proposition 3 [9, 10] *Let $G = Sz(q)$, where $q > 2$ is an odd power of 2, $S \in Syl_2(G)$, $M = N_G(S)$, and let g be an involution contained in $G - S$. Then $G = \langle S, g \rangle$ and the following statements hold.*

- (1) $M = S : K$, where $K = M \cap M^g$, $1 = S \cap M^g$, $K \simeq Z_{q-1}$ and $|S| = q^2$.
- (2) $G = M \cup MgS$ and each element in $G - M$ can be written uniquely in the form xgy , where $x \in M$ and $y \in S$.
- (3) $K \langle g \rangle \simeq D_{2(q-1)}$.
- (4) S contains exactly $q - 1$ involutions. All involutions of S generate $Z(S)$ and $|Z(S)| = q$.
- (5) There is a single class of conjugate involutions in G and $C_G(a) = S$ for any involution $a \in S$.

Proposition 4 [7, 11] *Let $G = {}^2G_2(q)$, where $q > 3$ is an odd power of 3, $S \in Syl_3(G)$, $M = N_G(S)$, and let g be an involution contained in $G - M$. Then $G = \langle S, g \rangle$ and the following statements hold.*

- (1) $M = S : K$, where $K = M \cap M^g$, $1 = S \cap M^g$, $K \simeq Z_{q-1}$ and $|S| = q^3$.
- (2) $G = M \cup MgS$ and each element in $G - M$ can be written uniquely in the form xgy , where $x \in M$ and $y \in S$.
- (3) $K = A_0 \times \langle h \rangle$, where h is the unique involution of K , $(q - 1)/2$ is odd, and $N_G(A_0) \simeq D_{2(q-1)}$.
- (4) There is a single class of conjugate involutions in G .

- (5) $C_S(h) = C_{S_1}(h)$ is an elementary abelian group of order q and $C_S(h) \cap Z(S) = 1$ with $S_1 = S^{(1)}$.
 (6) $C_S(x) = 1$ for each $x \in A_0$.

3 Main results

Theorem 1 Let $G \in \{Sz(q), {}^2G_2(q)\}$, where $q = p^{2n+1} > 3$, $S \in \text{Syl}_p(G)$ with $p = 2$ if $G = Sz(q)$, and $p = 3$ if $G = {}^2G_2(q)$. Let g be an involution of G not contained in $N_G(S)$, let $\langle h \rangle$ be a subgroup of $N_G(S) \cap N_G(S)^g$ of odd index $r > 1$ and $H = S\langle h \rangle$. Then $\Gamma(G, H, HgH)$ is an arc-transitive antipodal distance-regular graph with intersection array $\{q^2, (q^2-1)(r-1)/r, 1; 1, (q^2-1)/r, q^2\}$ if $G = Sz(q)$, or $\{q^3, (q^3-1)(r-1)/r, 1; 1, (q^3-1)/r, q^3\}$ if $G = {}^2G_2(q)$, and $\Gamma(G, H, HgH)$ does not depend (up to isomorphism) on the choice of involution $g \in G - M$.

Proof We put $M = N_G(S)$, and let r be the maximal odd divisor of $|M \cap M^g|$. We have $M \cap M^g = \langle h \rangle \times A$, where $|A| = r$ and $|h| = (q-1)/r$. Let $H = S\langle h \rangle$, $\Gamma = \Gamma(G, H, HgH)$, $A = \langle f \rangle$, and $L = \langle f \rangle \langle g \rangle$. Then $L \cong D_{2r}$.

Further, let $\Sigma = \{Hm(gs)^e \mid m \in M, s \in S, e \in \{0, 1\}\}$. By Propositions 3, 4 and 2, it follows that Γ is an arc-transitive r -fold covering graph of $\Gamma/\Sigma \cong K_{|S|+1}$. Now, we show that there exists a constant μ such that any two vertices from different fibres of Γ have exactly μ common neighbours. In order to do this, we will consider the action of L on the set of vertices of Γ .

By $F(Hx)$, we will denote the fibre of Γ which contains the vertex Hx . We have $F(H) = \{Hf^i\}_{i=1}^r$ and $F(Hg) = \{Hgf^i\}_{i=1}^r$.

For convenience, we will also use the following notation for the vertices of fibres $F(H)$ and $F(Hg)$:

$$\bar{a}_1 = H, \bar{a}_2 = Hf, \dots, \bar{a}_r = Hf^{r-1}$$

and

$$\bar{b}_1 = Hg, \bar{b}_2 = Hgf, \dots, \bar{b}_r = Hgf^{r-1}.$$

It is clear that $\bar{a}_i \in [\bar{b}_i]$ for all $i \in \{1, \dots, r\}$. For a fibre $F \in \Sigma$, an element $y \in G$, and a vertex $Hx = \bar{a} \in F$, we set $\bar{a}y = Hxy$ and $Fy = F(\bar{a}y)$. If Δ is a subset of vertices of Γ , then by Δg we denote the set of images of vertices of Δ under the action of g .

Recall that the stabilizer of any three points in G in its doubly transitive representation is trivial if $G = Sz(q)$, see [9] or [10], or has a unique nontrivial element fixing more than two points (namely, it fixes exactly $q+1$ points) if $G = {}^2G_2(q)$, see [11] or [7]. Since any two involutions are conjugate in G , it follows that the element g fixes (pointwise) exactly w fibres in Γ with $w = 1$ if $G = Sz(q)$ and $w = q+1$ if $G = {}^2G_2(q)$. $\square$

Lemma 1 *The number of $\langle f \rangle$ -orbits on $\Sigma - \{F(H), F(Hg)\}$ is equal to $(|S| - 1)/r$. There is only one L -orbit of size 2 on Σ , namely $\{F(H), F(Hg)\}$. There are exactly w L -orbits of size r on Σ , and the size of any other L -orbit on Σ is $2r$.*

Proof This follows from Propositions 3 and 4. $\square$

Lemma 2 *Suppose L_r is an L -orbit on Σ of size r and define Δ to be the set of all vertices which lie in the union of fibres from L_r . Then, for all $i, j \in \{1, \dots, r\}$, each vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_i] \cap \Delta$.*

Proof Let us consider the action of L on Δ . The element g fixes only one fibre in L_r , which is denoted further by F_1 . Every $\langle f \rangle$ -orbit on Δ contains only one vertex from any fibre in Δ , and the neighbourhood of any vertex from $F(H)$ contains only one vertex from any $\langle f \rangle$ -orbit on Δ . The element g fixes each $\langle f \rangle$ -orbit on Δ setwise by fixing just one vertex (which lies in F_1) in it.

Let $F_1 = F(Hgs) = \{Hgf^j s\}_{j=1}^r$ for some $s \in S$. Then $Hgf^j s g = Hgf^j s$ for all $j \in \{1, \dots, r\}$. We have

$$(\bar{a}_{j+1}g, \bar{b}_{j+1}g) = (Hf^j g, Hgf^j g) = (Hgf^{-j}, Hf^{-j}) = (\bar{b}_{r-j+1}, \bar{a}_{r-j+1})$$

for all $1 \leq j < r$ and $(\bar{a}_1g, \bar{b}_1g) = (\bar{b}_1, \bar{a}_1)$. Hence

$$Hgf^j s = Hgf^j s g \in [Hf^j] \cap [Hf^j g] = [\bar{a}_{j+1}] \cap [\bar{b}_{r-j+1}]$$

for all $1 \leq j < r$. As $\bar{a}_{j+1} = \bar{a}_1 f^j$ and $\bar{b}_{r-j+1} = \bar{b}_1 f^{-j}$, we have

$$Hgf^j s f^{-j} \in [\bar{a}_{j+1} f^{-j}] \cap [\bar{b}_{r-j+1} f^{-j}] = [\bar{a}_1] \cap [\bar{b}_1 f^{-2j}]$$

for all $1 \leq j < r$. Assume that $\bar{b}_1 f^{-2j_1} = \bar{b}_1 f^{-2j_2}$ for some $j_1, j_2 \in \{1, \dots, r\}$ such that $j_1 < j_2$. Then $2(j_2 - j_1) = lr$ for some integer l , hence $l = 0$, a contradiction.

Hence for all $j \in \{1, \dots, r\}$, each vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_1] \cap \Delta$. By a similar argument as above, we obtain that for all $i, j \in \{1, \dots, r\}$, each vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_i] \cap \Delta$. $\square$

Lemma 3 *Suppose L_{2r} is an L -orbit on Σ of size $2r$ and define Δ to be the set of all vertices which lie in the union of fibres from the same $\langle f \rangle$ -orbit on L_{2r} . Then, for all $i, j \in \{1, \dots, r\}$, each vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_i] \cap \Delta$ and to only one vertex in $[\bar{a}_i] \cap \Delta g$.*

Proof Let $D = \Delta \cup \Delta g$. Now consider the action of L on D . We choose two vertices $Hgs, Hgs_1 \in D$, where elements $s, s_1 \in S$ are such that fibres $F(Hgs), F(Hgs_1)$ belong to distinct $\langle f \rangle$ -orbits on Σ . There is an involution gf^α , where $\alpha \in \{1, \dots, r\}$, in L such that $F(Hgsgf^\alpha) = F(Hgs_1)$. Clearly $F(Hgf^\alpha) = F(Hg)$.

Consider the structure of the matching between fibres $F(Hg)$ and $F(Hgs_1)$. We have $Hgf^i s g f^\alpha \in [Hf^i g f^\alpha]$ for all $i \in \{1, \dots, r\}$, $\{Hgf^j s_1\}_{j=1}^r = \{Hgf^j s g f^\alpha\}_{j=1}^r$. Furthermore, $Hgs_1 = Hgf^\beta s g f^\alpha$ for some $\beta \in \{1, \dots, r\}$. The element $gsg \in G - M$

can be written uniquely in the form $mg s_0$ with $m \in M, s_0 \in S$. Assume that $m = s_2 f^\delta h^\gamma$, where $s_2 \in S, \delta \in \{1, \dots, r\}, \gamma \in \{1, \dots, (q-1)/r\}$. Then

$$\begin{aligned} H g f^\beta s g f^\alpha &= H f^{-\beta} g s g f^\alpha = H f^{-\beta} m g s_0 f^\alpha = H f^{-\beta} s_2 f^\delta h^\gamma g s_0 f^\alpha \\ &= H f^{-\beta} f^\delta g f^\alpha \tilde{s}_0 = H f^{\delta-\beta-\alpha} g \tilde{s}_0 = H g s_1 \end{aligned}$$

for some $\tilde{s}_0 \in S$. It follows that $\tilde{s}_0 = s_1$ and $f^{\delta-\beta-\alpha} \in H$. Hence $f^\alpha s_1 = s_0 f^\alpha$ and r divides $\delta - \beta - \alpha$.

Further,

$$\begin{aligned} H g f^t s g f^\alpha &= H f^{-t} g s g f^\alpha = H f^{-t} m g s_0 f^\alpha = H f^{-t} s_2 f^\delta h^\gamma g f^\alpha s_1 \\ &= H f^{\delta-\alpha-t} g s_1 \end{aligned}$$

for any $t \in \{1, \dots, r\}$. Then

$$H g f^i s_1 = H g f^j s g f^\alpha$$

for some $i, j \in \{1, \dots, r\}$ if and only if

$$H f^{\delta-\alpha-j} g s_1 = H f^{-i} g s_1$$

if and only if r divides $\delta - \alpha - j + i - (\delta - \beta - \alpha) = \beta + i - j$. Hence

$$H g f^i s_1 (= H g f^{\beta+i} s g f^\alpha) \in [H f^{\beta+i} g f^\alpha] \cap [H f^i] = [\bar{b}_1 f^{\alpha-\beta-i}] \cap [\bar{a}_1 f^i]$$

and

$$H g f^{\beta+i} s \in [H f^{\beta+i}] \cap [H f^i g f^\alpha] = [\bar{a}_1 f^{\beta+i}] \cap [\bar{b}_1 f^{\alpha-i}]$$

for all $i \in \{1, \dots, r\}$. Then for all $i \in \{1, \dots, r\}$, the following inclusions hold

$$H g f^i s_1 f^{-i} \in [\bar{b}_1 f^{\alpha-\beta-2i}] \cap [\bar{a}_1]$$

and

$$H g f^{\beta+i} s f^{-(\beta+i)} \in [\bar{a}_1] \cap [\bar{b}_1 f^{\alpha-\beta-2i}].$$

Suppose that $\bar{b}_1 f^{\alpha-\beta-2i_1} = \bar{b}_1 f^{\alpha-\beta-2i_2}$ for some $i_1, i_2 \in \{1, \dots, r\}$ such that $i_1 < i_2$. Then $\alpha - \beta - 2i_1 = \alpha - \beta - 2i_2 + lr$ for an integer l , hence $2(i_2 - i_1) = lr$ and $l = 0$, a contradiction. Consequently, for all $j \in \{1, \dots, r\}$ every vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_1] \cap \Delta$ and to only one vertex in $[\bar{a}_1] \cap \Delta g$. By an argument as above, we have that for all $i, j \in \{1, \dots, r\}$ every vertex $\bar{b}_j$ is adjacent to only one vertex in $[\bar{a}_i] \cap \Delta$ and to only one vertex in $[\bar{a}_i] \cap \Delta g$. $\square$

By Lemmas 1–3 for all $i, j \in \{1, \dots, r\}$, we have $|\bar{a}_i \cap \bar{b}_j| = (|S| - 1)/r$. By 2-transitivity of G on Σ , we conclude that any two vertices of Γ from different fibres have exactly $\mu = (|S| - 1)/r$ common neighbours. Thus Γ is an arc-transitive distance-regular graph with intersection array $\{|S|, (|S| - 1)(r - 1)/r, 1; 1, (|S| - 1)/r, |S|\}$. Now, we show that Γ admits distance-regular quotients.

Lemma 4 *Let $H_0 = H \langle f^t \rangle$, where t is a proper divisor of r . Then $\Gamma(G, H_0, H_0 g H_0)$ is an arc-transitive distance-regular graph with intersection array $\{|S|, (|S| - 1)(t - 1)/t, 1; 1, (|S| - 1)/t, |S|\}$.*

Proof It is clear that $\langle f^t \rangle$ -orbit on $F(H)$ containing the vertex H is an imprimitivity block of the group G on the set of vertices of Γ . Hence the set $\pi = \{\{Hxm(gs)^e \mid x \in H_0\} \mid m \in M, s \in S, e \in \{0, 1\}\}$ forms an imprimitivity system of G on the vertices of Γ . Moreover, the partition π is equitable. By [4, Theorem 6.2], we have that $\Gamma' = \Gamma/\pi$ is an antipodal distance-regular graph with intersection array $\{r\mu + 1, (t - 1)r\mu/t, 1; 1, r\mu/t, r\mu + 1\}$. Since H acts transitively on blocks of π , which contain the neighbours of H in Γ , it follows that G acts transitively on arcs of Γ' , and by Proposition 1 $\Gamma' \simeq \Gamma(G, H_0, H_0 g H_0)$.

As the mapping $Hy \mapsto Hy^{f^t}$ is an isomorphism between the graphs $\Gamma(G, H, HgH)$ and $\Gamma(G, H, Hgf^{2t}H)$ for each $t \in \{1, \dots, r\}$, and $\langle f \rangle = \langle f^2 \rangle$, we see that Γ does not depend (up to isomorphism) on the choice of involution $g \in G - M$. The same also holds for $\Gamma(G, H_0, H_0 g H_0)$. This completes the proof of Theorem 1. $\square$

Corollary 1 *Let $G = Sz(q)$, where $q = 2^{2a+1} > 2$. Suppose that Φ is the graph whose vertices are the involutions of G , and two vertices are adjacent if and only if the order of the product of corresponding involutions in G is 5. Then Φ is an arc-transitive antipodal distance-regular graph with intersection array $\{q^2, q^2 - q - 2, 1; 1, q + 1, q^2\}$.*

Proof Note that the graph Φ is connected. Moreover, G acts transitively (by conjugation) on arcs of Φ , see [10]. Let x be a vertex of Φ and put $S = C_G(x)$. Clearly the stabilizer of x in G is precisely S , and $S \in \text{Syl}_2(G)$. Let g be an involution of G such that $|xg| = 5$. Then x and xg^8 are adjacent, and by Proposition 1 $\Phi \simeq \Gamma(G, S, SgS)$. $\square$

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