Symplectic spreads, planar functions, and mutually unbiased bases

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Abstract In this paper we give explicit descriptions of complete sets of mutually unbiased bases (MUBs) and orthogonal decompositions of special Lie algebras $sl_n(\mathbb{C})$ obtained from commutative and symplectic semifields, and from some other non-semifield symplectic spreads. Relations between various constructions are also studied. We show that the automorphism group of a complete set of MUBs is isomorphic to the automorphism group of the corresponding orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$. In the case of symplectic spreads this automorphism group is determined by the automorphism group of the spread. By using the new notion of pseudo-planar functions over fields of characteristic two we give new explicit constructions of complete sets of MUBs.

Keywords Mutually unbiased bases · Symplectic spreads · Finite semifields · Orthogonal decompositions of Lie algebras · Planar functions · Pseudo-planar functions · Automorphism groups

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1 Introduction

Mutually unbiased bases (MUBs) were first studied by Schwinger [43] in 1960, but the notion itself was defined by Wootters and Fields [48] almost 30 years later, when

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they also presented examples. A set of MUBs in the Hilbert space $\mathbb{C}^n$ is defined as a set of orthonormal bases $\{B_0, B_1, \dots, B_r\}$ of the space such that the square of the absolute value of the inner product $|(x, y)|^2$ is equal to $1/n$ for any two vectors x, y from distinct bases. The notion of MUBs is one of the basic concepts of quantum information theory and plays an important role in quantum tomography and state reconstruction [41, 48]. It is also valuable for quantum key distribution: the famous Benneth–Brassard secure quantum key exchange protocol BB84 and its developments are based on MUBs [9]. Sequences with low correlations are known to be extremely useful in the design of radar and communication systems. In 1980 Alltop [5] presented examples of complex sequences with low periodic correlations. Later it was noted that these sequences are examples of MUBs. On the other hand, in the 1980s Kostrikin et al. defined and studied orthogonal decompositions of complex Lie algebras. The notion of orthogonal decompositions originated in the pioneering work of Thompson, who discovered an orthogonal decomposition of the Lie algebra of type E_8 and used it for construction of a sporadic finite simple group, nowadays called the Thompson group. Orthogonal decompositions turn out to be interesting not only for their inner geometric structures, but also for their interconnections with other areas of mathematics [37]. In 2007 Boykin et al. [11] discovered a connection between MUBs and orthogonal decompositions of Lie algebras. It was found that the existence of a complete set of MUBs is equivalent to finding an orthogonal decomposition of the complex Lie algebra $sl_n(\mathbb{C})$. Recently it was discovered that MUBs are very closely related or even equivalent to other problems in various parts of mathematics, such as algebraic combinatorics, finite geometry, discrete mathematics, coding theory, metric geometry, sequences, and spherical codes.

There is no general classification of MUBs. The main open problem in this area is to construct a maximal number of MUBs for any given n . It is known that the maximal set of MUBs of $\mathbb{C}^n$ consists of at most $n + 1$ bases, and sets attaining this bound are called complete sets of MUBs. Constructions of complete sets are known only for prime power dimensions. Even for the smallest non-prime power dimension six the problem of finding a maximal set of MUBs is extremely hard [37] and remains open after more than 30 years.

Several kinds of constructions of MUBs are available [5, 8, 24, 26, 34, 35, 48]. As a matter of fact, essentially there are only three types of constructions up to now: constructions associated with symplectic spreads, using planar functions over fields of odd characteristic, and Gow's construction in [24] of a unitary matrix (although it seems it is isomorphic to the classical one).

On the other hand one can consider MUBs in the Euclidean space $\mathbb{R}^n$ (real MUBs), and in this case the upper bound for a maximal set is $n/2 + 1$. Constructions of real MUBs are strictly connected to algebraic coding theory (optimal Kerdock type codes) and the notion of extremal line-sets in Euclidean spaces [4, 13]. LeCompte et al. [38] characterized collections of real MUBs in terms of association schemes. In [4] connections of real MUBs with binary and quaternary Kerdock and Preparata codes [25, 39], and association schemes were studied. We also note that one can come to Kerdock and Preparata codes through integral lattices associated with orthogonal decompositions of Lie algebras $sl_n(\mathbb{C})$ [2, 3].

In this paper we give explicit descriptions of complete sets of MUBs and orthogonal decompositions of special Lie algebras $sl_n(\mathbb{C})$ obtained from commutative and symplectic semifields, and from some other non-semifield symplectic spreads. We provide direct formulas to construct MUBs from semifields. We show that automorphism groups of complete sets of MUBs and corresponding orthogonal decompositions of Lie algebras $sl_n(\mathbb{C})$ are isomorphic, and in the case of symplectic spreads these automorphism groups are determined by the automorphism groups of those spreads. Automorphism groups are important invariants, therefore they can be used to show inequivalence of the various known types of MUBs. Planar functions over fields of odd characteristics also lead to constructions of MUBs. There are no planar functions over fields of characteristic two. However, Zhou [51] proposed a new notion of “planar” functions over fields of characteristic two. Based on this notion, we present new constructions of MUBs (but we prefer to call these functions pseudo-planar). We also propose a generalization of the notion of pseudo-planar functions for arbitrary characteristic.

MUBs can be constructed using different objects. We study their mutual relations. Connections between them are given in the following road map:

This map shows that starting from a commutative presemifield one can construct consecutively (pseudo-)planar functions and then MUBs. On the other hand, one can start from a commutative presemifield and then construct consecutively symplectic presemifield, symplectic spread, orthogonal decomposition of Lie algebra and finally MUBs. We will show that in fact our diagram is “commutative”: we will not get new MUBs moving from one construction to others. Note that there are planar functions not coming from presemifields, symplectic spreads not related to semifields [7, 27–29, 33, 46] and orthogonal decompositions not related to symplectic spreads.

2 Automorphism groups

The Lie algebra $L = sl_n(\mathbb{C})$ is the algebra of $n \times n$ traceless matrices over $\mathbb{C}$, where the operation of multiplication is given by the commutator of matrices: $[A, B] = AB - BA$. A subalgebra H of a Lie algebra L is called a Cartan subalgebra if it is nilpotent and equal to its normalizer, which is the set of those elements X in L

such that $[X, H] \subseteq H$. In case of $L = sl_n(\mathbb{C})$ Cartan subalgebras are maximal abelian subalgebras and they are conjugate under automorphisms of the Lie algebra. In particular, they are all conjugate to the standard Cartan subalgebra, consisting of all traceless diagonal matrices. A decomposition of a simple Lie algebra L into a direct sum of Cartan subalgebras

$$L = H_0 \oplus H_1 \oplus \cdots \oplus H_n$$

is called an orthogonal decomposition [37], if the subalgebras H_i are pairwise orthogonal with respect to the Killing form $K(A, B)$ on L . Recall that the Killing form on a Lie algebra L is defined by $K(A, B) = \text{Tr}(\text{ad}A \cdot \text{ad}B)$, where the operator $\text{ad}A : L \rightarrow L$ is given by $\text{ad}A(C) = [A, C]$ and Tr is the trace. The Killing form is symmetric and non-degenerate on L . In the case of $L = sl_n(\mathbb{C})$ we have

$$K(A, B) = 2n\text{Tr}(AB).$$

The adjoint operation $*$ on the set of $n \times n$ complex matrices is given by $A^* = \overline{A}^t$ (conjugate transpose). A Cartan subalgebra is called closed under the adjoint operation if $H^* = H$.

Theorem 1 [11, Theorem 5.2] *Complete sets of MUBs in $\mathbb{C}^n$ are in one-to-one correspondence with orthogonal decompositions of Lie algebra $sl_n(\mathbb{C})$ such that all Cartan subalgebras in this decomposition are closed under the adjoint operation.*

This correspondence is given in the following way. If $\mathcal{B} = \{B_0, B_1, \dots, B_n\}$ is a complete set of MUBs then the associated Cartan subalgebra H_i consists of all traceless matrices that are diagonal with respect to the basis B_i (in other words, vectors of the basis B_i are common eigenvectors of all matrices in H_i).

The automorphism group $\text{Aut}(\mathcal{D})$ [37] of an orthogonal decomposition $\mathcal{D}$ of a Lie algebra L consists of all automorphisms of L preserving $\mathcal{D}$:

$$\text{Aut}(\mathcal{D}) = \{\varphi \in \text{Aut}(L) \mid (\forall i)(\exists j) \varphi(H_i) = H_j\}.$$

Recall that $\text{Aut}(L) = \text{Inn}(L) \cdot \langle T \rangle$, where $\text{Inn}(L) \cong PSL_n(\mathbb{C})$ is the group of all inner automorphisms of L and T is the outer automorphism $A \mapsto -A^t$.

Let $\mathcal{B} = \{B_0, B_1, \dots, B_n\}$ be a complete set of MUBs of $\mathbb{C}^n$. With any orthonormal basis B_i we can associate an orthoframe $\text{Ort}(B_i)$ of 1-spaces generated by vectors of B_i . We note that if we consider the representations of all bases B_i in some fixed standard basis, then the conjugation map

$$\tau(x_1, \dots, x_n) = \overline{x} = (\overline{x_1}, \dots, \overline{x_n})$$

sends one orthoframe $\text{Ort}(B_i)$ to another orthoframe. Define

$$PGL_n(\mathbb{C})^+ = PGL_n(\mathbb{C})\langle \tau \rangle,$$

$$\text{Aut}(\mathcal{B}) = \{\psi \in PGL_n(\mathbb{C})^+ \mid (\forall i)(\exists j) \psi(\text{Ort}(B_i)) = \text{Ort}(B_j)\}.$$

Theorem 2 Let $\mathcal{D}$ be an orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$ and $\mathcal{B}$ be the corresponding complete set of MUBs. Then

$$\text{Aut}(\mathcal{D}) \cong \text{Aut}(\mathcal{B}).$$

Proof Let $\varphi \in \text{Aut}(\mathcal{D})$. For any automorphism φ of the Lie algebra $sl_n(\mathbb{C})$ we have $\varphi = \varphi_X$ or $\varphi = \varphi_X T$, where

$$\varphi_X(A) = XAX^{-1}$$

for some matrix $X \in SL_n(\mathbb{C})$, and

$$T(A) = -A^t.$$

Let H be a Cartan subalgebra from $\mathcal{D}$ and $B = \{e_1, \dots, e_n\}$ be the corresponding basis, so for any $h \in H$ we have $he_i = \alpha_i(h)e_i$ for some linear map $\alpha_i : H \rightarrow \mathbb{C}$. Assume that $\varphi = \varphi_X$. Then the vectors $f_i = Xe_i$ generate the orthoframe associated with the Cartan subalgebra $\varphi(H)$:

$$XhX^{-1}f_i = XhX^{-1}Xe_i = X\alpha_i(h)e_i = \alpha_i(h)f_i.$$

Therefore, the matrix X generates an element from $\text{Aut}(\mathcal{B})$.

Let $\varphi = \varphi_X T$. Note that for $h \in H$ one has $\overline{h^t} = h^* \in H$ by Theorem 1. Then the vectors $f_i = X\overline{e_i}$ determine a basis corresponding to the Cartan subalgebra $\varphi(H)$:

$$\begin{aligned} (\varphi_X Th)f_i &= -Xh^tX^{-1}f_i = -X\overline{h^*}X^{-1}f_i = -X\overline{h^*}X^{-1}X\overline{e_i} = -X\overline{h^*e_i} \\ &= -X\overline{\alpha_i(h^*)e_i} = -X\overline{\alpha_i(h^*)}\overline{e_i} = -\overline{\alpha_i(h^*)}X\overline{e_i} = -\overline{\alpha_i(h^*)}f_i. \end{aligned}$$

Therefore, $X\tau$ generates an element from $\text{Aut}(\mathcal{B})$.

Conversely, let $\psi \in \text{Aut}(\mathcal{B})$. Suppose first that ψ is generated by the matrix $X \in GL_n(\mathbb{C})$. We can assume that $\det X = 1$ (otherwise we multiply X by an appropriate scalar matrix). Let $B = \{e_i\}$ be a basis from $\mathcal{B}$ and let ψ map the orthoframe $\text{Ort}(B)$ to another orthoframe $\text{Ort}(B')$. Denote $f_i = Xe_i$. Let $H' = \varphi_X(H) = XHX^{-1}$ and $h \in H$. Then

$$\varphi_X(h)f_i = XhX^{-1}Xe_i = Xhe_i = X\alpha_i(h)e_i = \alpha_i(h)Xe_i = \alpha_i(h)f_i.$$

Therefore, the vectors f_i are common eigenvectors for H' , so H' is the Cartan subalgebra associated with the basis B' . Hence $\varphi_X \in \text{Aut}(\mathcal{D})$.

Now assume that ψ is generated by $X\tau$. Then the vectors $f_i = X\overline{e_i}$ from an orthoframe B' are common eigenvectors of matrices of the Cartan subalgebra $\varphi_X T(H)$:

$$\begin{aligned}
(\varphi_X Th) f_i &= -Xh^t X^{-1} X \overline{e_i} = -Xh^t \overline{e_i} = -X \overline{h^* e_i} = -X \overline{h^*} \overline{e_i} \\
&= -X \overline{\alpha_i(h^*) e_i} = -X \overline{\alpha_i(h^*)} \overline{e_i} = -\overline{\alpha_i(h^*)} X \overline{e_i} = -\overline{\alpha_i(h^*)} f_i,
\end{aligned}$$

so $\varphi_X T \in \text{Aut}(\mathcal{D})$. $\square$

All known constructions of complete sets of MUBs were obtained with the help of symplectic spreads and planar functions, and the construction from [24]. Now we recall constructions of orthogonal decompositions of Lie algebras $sl_n(\mathbb{C})$ associated with symplectic spreads [30, 37].

Let $F = \mathbb{F}_q$ be a finite field of order q . Let V be a vector space over F with the usual dot product $u \cdot v$. We can consider $W = V \oplus V$ as a vector space over the prime field $\mathbb{F}_p$ and define an alternating bilinear form on W by

$$\langle (u, v), (u', v') \rangle = \text{tr}(u \cdot v' - v \cdot u'), \quad (1)$$

where tr is a trace function from $\mathbb{F}_q$ to $\mathbb{F}_p$.

Let $n = |V|$ and let $\{e_w\}$ denote the standard basis of $\mathbb{C}^n$, indexed by elements of V . Let $\varepsilon \in \mathbb{C}$ be a primitive p th root of unity. For $u \in V$, the generalized Pauli matrices are defined as the following $n \times n$ matrices:

$$\begin{aligned}
X(u) : e_w &\mapsto e_{u+w} \\
Z(v) : e_w &\mapsto \varepsilon^{\text{tr}(v \cdot w)} e_w
\end{aligned}$$

The matrices

$$D_{u,v} = X(u)Z(v)$$

form a basis of the space of complex square matrices of size $n \times n$. Moreover, the matrices $D_{u,v}$, $(u, v) \neq (0, 0)$, generate the Lie algebra $sl_n(\mathbb{C})$. Note that

$$[D_{u,v}, D_{u',v'}] = \varepsilon^{\text{tr}(v \cdot u')} (1 - \varepsilon^{\langle (u,v), (u',v') \rangle}) D_{u+u', v+v'},$$

so $[D_{u,v}, D_{u',v'}] = 0$ if and only if $\langle (u, v), (u', v') \rangle = 0$.

Let V be an r -dimensional space over $\mathbb{F}_p$ (so $n = p^r$). A symplectic spread of the symplectic $2r$ -dimensional space $W = V \oplus V$ over $\mathbb{F}_p$ is a family of $n + 1$ totally isotropic r -subspaces of W such that every nonzero point of W lies in a unique subspace. Thus such r -subspaces are maximal totally isotropic subspaces. Let $\Sigma = \{W_0, W_1, \dots, W_n\}$ be a symplectic spread. Then

$$L = H_0 \oplus H_1 \oplus \dots \oplus H_n,$$

$$H_i = \langle D_{u,v} \mid (u, v) \in W_i \rangle,$$

gives [30, 37] the corresponding orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$.

We define

$$\begin{aligned}\mathrm{Sp}^\pm(W) &= \{\varphi \in GL(W) \mid (\exists s = \pm 1)(\forall u, v \in W) \langle \varphi(u), \varphi(v) \rangle = s \langle u, v \rangle\}, \\ \mathrm{Aut}(\Sigma) &= \{\varphi \in \mathrm{Sp}^\pm(W) \mid (\forall i)(\exists j) \varphi(W_i) = W_j\}.\end{aligned}$$

Theorem 3 [37, Proposition 1.3.3.] *Let $\mathcal{D}$ be the orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$ corresponding to a symplectic spread Σ . Then*

$$\mathrm{Aut}(\mathcal{D}) = K \cdot \mathrm{Aut}(\Sigma),$$

where K is the set of all automorphisms that fix every line $\langle D_{u,v} \rangle$.

In the case of odd characteristic, K is isomorphic to $W = V^2$ and an embedding of K in $PSL_n(\mathbb{C})$ is given through conjugations by matrices $D_{u,v}$:

$$D_{u,v} D_{a,b} D_{u,v}^{-1} = \varepsilon^{-\langle (u,v), (a,b) \rangle} D_{a,b}.$$

The outer automorphism T is given by:

$$T(D_{a,b}) = -\varepsilon^{-\mathrm{tr}(a \cdot b)} D_{-a,b},$$

therefore in the case of even characteristic, we have $K \cong V^2 \cdot \mathbb{Z}_2$.

Any symplectic spread Σ determines a translation plane $A(\Sigma)$. The collineation group of $A(\Sigma)$ is a semidirect product of the translation group V^2 and the translation complement. However, the group extension in Theorem 3 can be non-split (see Sect. 4.1).

3 Symplectic spreads in odd characteristics and MUBs

In this section we show how to construct a complete set of MUBs directly from symplectic spreads and semifields. Throughout this section $F = \mathbb{F}_q$ denotes a finite field of odd order $q = p^r$. Let $\omega \in \mathbb{C}$ be a primitive p th root of unity.

Lemma 1 *Let Σ be a symplectic spread of $W = V \oplus V$, and let $h : V \rightarrow V$ be a mapping such that $\{(u, h(u)) \mid u \in V\}$ is a maximal totally isotropic subspace. Then $\mathrm{tr}(u \cdot h(w)) = \mathrm{tr}(h(u) \cdot w)$ for all $u \in V, w \in V$.*

Proof $\mathrm{tr}(u \cdot h(w) - h(u) \cdot w) = \langle (u, h(u)), (w, h(w)) \rangle = 0.$ $\square$

Lemma 2 *Let Σ be a symplectic spread of $W = V \oplus V$, and let $h : V \rightarrow V$ be a linear mapping such that $\{(u, h(u)) \mid u \in V\}$ is a maximal totally isotropic subspace. Then for any $v \in V$, the vector*

$$b_{h,v} = \sum_{w \in V} \omega^{\mathrm{tr}(\frac{1}{2} w \cdot h(w) + v \cdot w)} e_w$$

is an eigenvector of $D_{u, h(u)}$ for all $u \in V$.

Proof Indeed,

$$\begin{aligned}
 D_{u,h(u)}(b_{h,v}) &= \sum_{w \in V} \omega^{\text{tr}(\frac{1}{2}w \cdot h(w) + v \cdot w + h(u) \cdot w)} e_{w+u} \\
 &= \sum_{w \in V} \omega^{\text{tr}(\frac{1}{2}(w-u) \cdot h(w-u) + v \cdot (w-u) + h(u) \cdot (w-u))} e_w \\
 &= \sum_{w \in V} \omega^{\text{tr}(\frac{1}{2}w \cdot h(w) + v \cdot w) + \frac{1}{2}\text{tr}(h(u) \cdot w - u \cdot h(w)) - \text{tr}(\frac{1}{2}u \cdot h(u) + v \cdot u)} e_w \\
 &= \omega^{-\text{tr}(\frac{1}{2}u \cdot h(u) + v \cdot u)} b_{h,v}
 \end{aligned}$$

by Lemma 1. $\square$

Lemma 2 allows us to construct the complete set of MUBs corresponding to the orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$ obtained from a symplectic spread.

Symplectic spreads can be constructed using semifields. A finite presemifield is a ring with no zero-divisors, and with left and right distributivity [18]. A presemifield with multiplicative identity is called a semifield. A finite presemifield can be obtained from a finite field $(F, +, \cdot)$ by introducing a new product operation $*$, so it is denoted by $(F, +, *)$. Every presemifield determines a spread Σ consisting of subspaces $(0, F)$ and $\{(x, x * y) \mid x \in F, y \in F\}$. A presemifield is called symplectic if the corresponding spread is symplectic with respect to some alternating form [31] (so the spread might not be symplectic with respect to other forms).

Two presemifields $(F, +, *)$ and $(F, +, \star)$ are called isotopic if there exist three bijective linear mappings $L, M, N : F \rightarrow F$ such that

$$L(x * y) = M(x) \star N(y)$$

for any $x, y \in F$. If $M = N$ then the presemifields are called strongly isotopic. Every presemifield is isotopic to a semifield. Isotopic semifields determine isomorphic planes.

Lemma 2 implies

Theorem 4 *Let $(F, +, \circ)$ be a finite symplectic presemifield of odd characteristic. Then the following set forms a complete set of MUBs:*

$$\begin{aligned}
 B_\infty &= \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F, \\
 b_{m,v} &= \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}(\frac{1}{2}w \cdot (w \circ m) + v \cdot w)} e_w.
 \end{aligned}$$

A function $f : F \rightarrow F$ is called planar if

$$x \mapsto f(x+a) - f(x), \quad (2)$$

is a permutation of F for each $a \in F^*$. Any planar function over F allows us to construct a complete set of MUBs.

Theorem 5 [21,34,41] *Let F be a finite field of odd order q and f be a planar function. Then the following forms a complete set of MUBs:*

$$B_\infty = \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F,$$

$$b_{m,v} = \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}(\frac{1}{2}mf(w)+vw)} e_w.$$

Every commutative presemifield of odd order corresponds to a quadratic planar polynomial, and vice versa. If $(F, +, *)$ is a commutative presemifield then $f(x) = x * x$ is a planar function, and one can use Theorem 5 for constructing MUBs. Up to now, there is only one known type of planar function [15], which is not related to semifields:

$$f(x) = x^{(3^k+1)/2},$$

where $q = 3^r$, $\gcd(k, 2r) = 1$, $k \not\equiv \pm 1 \pmod{2r}$.

On the other hand, from a commutative presemifield one can construct a symplectic presemifield (using Knuth's cubical array method [36]) and then construct consecutively a symplectic spread, an orthogonal decomposition of the Lie algebra $sl_n(\mathbb{C})$ and finally MUBs. Below we show that we will not get new MUBs by these operations (a similar statement is true for even characteristic, see Sect. 4).

Let $f(x) = \sum_{i \leq j} a_{ij} x^{p^i + p^j}$ be a quadratic planar polynomial. Then the corresponding commutative presemifield $(F, +, *)$ is given by:

$$x * y = \frac{1}{2}(f(x+y) - f(x) - f(y)) = \frac{1}{2} \sum_{i \leq j} a_{ij} (x^{p^i} y^{p^j} + x^{p^j} y^{p^i}).$$

It defines a spread Σ consisting of subspaces $(0, F)$ and $\{(x, x * y) \mid x \in F, y \in F\}$. Starting from Σ we will construct a symplectic spread using the Knuth cubical arrays method ([31], Proposition 3.8). The dual spread Σ^d is a spread of the dual space of $F \oplus F$. We can identify that dual space with $F \oplus F$ by using the alternating form (1). For each $y \in F$ we have to find all (u, v) such that $\langle (x, x * y), (u, v) \rangle = 0$ for all $x \in F$. We have

$$\begin{aligned} & \langle (x, x * y), (u, v) \rangle \\ &= \text{tr}(xv - u(x * y)) \\ &= \text{tr} \left(xv - u \frac{1}{2} \sum_{i \leq j} a_{ij} (x^{p^i} y^{p^j} + x^{p^j} y^{p^i}) \right) \\ &= \text{tr} \left(xv - \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-i}} u^{p^{r-i}} x y^{p^{j-i}} - \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-j}} u^{p^{r-j}} x y^{p^{r+i-j}} \right) \\ &= \text{tr} \left(x \left(v - \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-i}} u^{p^{r-i}} y^{p^{j-i}} - \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-j}} u^{p^{r-j}} y^{p^{r+i-j}} \right) \right), \end{aligned}$$

which implies

$$v = \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-i}} u^{p^{r-i}} y^{p^{j-i}} + \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-j}} u^{p^{r-j}} y^{p^{r+i-j}}.$$

Therefore, Σ^d corresponds to the presemifield $(F, +, \bullet)$ defined by

$$u \bullet y = \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-i}} u^{p^{r-i}} y^{p^{j-i}} + \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-j}} u^{p^{r-j}} y^{p^{r+i-j}}.$$

Then the presemifield $(F, +, \circ)$ with multiplication

$$x \circ y = y \bullet x = \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-i}} x^{p^{j-i}} y^{p^{r-i}} + \frac{1}{2} \sum_{i \leq j} a_{ij}^{p^{r-j}} x^{p^{r+i-j}} y^{p^{r-j}}$$

defines a symplectic spread Σ^{d*} . It is straightforward to check directly that the spread Σ^{d*} with subspaces $(0, F)$ and $\{(x, x \circ y) \mid x \in F, y \in F\}$, $y \in F$, is symplectic with respect to the form (1).

Then by Theorem 4 the following bases will form a complete set of MUBs:

$$B_\infty = \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F,$$

$$b_{m,v} = \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}\left(\frac{1}{2}m(w*w)+vw\right)} e_w,$$

since

$$\begin{aligned} \text{tr}\left(\frac{1}{2}w \cdot (w \circ m)\right) &= \text{tr}\left(\frac{1}{4} \sum_{i \leq j} w a_{ij}^{p^{r-i}} w^{p^{j-i}} m^{p^{r-i}} + \frac{1}{4} \sum_{i \leq j} w a_{ij}^{p^{r-j}} w^{p^{r+i-j}} m^{p^{r-j}}\right) \\ &= \text{tr}\left(\frac{1}{4} \sum_{i \leq j} w^{p^i} a_{ij} w^{p^j} m + \frac{1}{4} \sum_{i \leq j} w^{p^j} a_{ij} w^{p^i} m\right) \\ &= \text{tr}\left(\frac{1}{2}m(w * w)\right). \end{aligned}$$

Remark 1 It is easy to see that strongly isotopic commutative semifields provide equivalent MUBs (equivalence under the action of the group $PGL_n(\mathbb{C})^+$). Theorems 2 and 3 can be used to show inequivalence of MUBs obtained from nonisotopic semifields. Practically they distinguish all known types of MUBs.

3.1 Desarguesian spreads

Desarguesian spreads are constructed with the help of finite fields, and the corresponding commutative and symplectic semifields are the same: $x * y = x \circ y = xy$, and the planar function is $f(x) = x^2$.

Godsil and Roy [23] showed that the constructions of Alltop [5], Ivanović [26], Wooters and Fields [48], Klappenecker and Rötteler [35], and Bandyopadhyay, Boykin, Roychowdhury and Vatan [8] are all equivalent to particular cases of this construction. It seems that Gow's construction is equivalent to this case as well.

Theorem 2 and [37] imply that for the automorphism group of the corresponding complete set of MUBs we have

$$\text{Aut}(\mathcal{B}) \cong F^2(SL_2(q)\mathbb{Z}_r)\mathbb{Z}_2,$$

where extensions are split.

3.2 Spreads from Albert's generalized twisted fields

Let ρ be a nontrivial automorphism of a finite field F such that $-1 \notin F^{\rho-1}$. It means that F has odd degree over the fixed field F_ρ of ρ . Let $V = F$. Then the BKLA [6] symplectic spread Σ of $W = V \oplus V$ consists of the subspace $\{(0, y) \mid y \in F\}$ and subspaces $\{(x, mx^{\rho^{-1}} + m^\rho x^\rho) \mid x \in F\}$, $m \in F$. This spread arises from a presemifield given by the operation $x \circ m = mx^{\rho^{-1}} + m^\rho x^\rho$.

If we start from the planar function $f(x) = x^{p^k+1} = x^{\rho+1}$, then the commutative presemifield and corresponding symplectic presemifield are given by

$$\begin{aligned} x * y &= \frac{1}{2}(x^\rho y + xy^\rho), \\ x \circ y &= \frac{1}{2}(x^\rho y + x^{\rho^{-1}}y^{\rho^{-1}}). \end{aligned}$$

3.3 Ball–Bamberg–Lauraauw–Penttila symplectic spread

This non-semifield spread [7] is obtained from the previous one with the help of a technique known as “net replacement”. Let Σ be the BKLA spread. One can change this spread and get a new spread Σ' in the following way. For all $s \in F^*$ the map $\sigma_s(v, w) = (sv, s^{-1}w)$ is an isometry of $V \oplus V$ with respect to the form $\langle \cdot, \cdot \rangle$. Denote the group of all σ_s by G . Let τ be an involution of $V \oplus V$ which switches coordinates: $(v, w) \mapsto (w, v)$. Then Σ is G -invariant. For $W_1 = \{(x, x^\rho + x^{\rho^{-1}}) \mid x \in F\}$ we consider the G -orbit

$$\mathcal{N} = \{\sigma_s(W_1) \mid s \in F^*\}.$$

Note that $\mathcal{N}$ and $\tau(\mathcal{N})$ are G -invariant. Now we define

$$\mathcal{N}' = \tau(\mathcal{N}).$$

Then

$$\Sigma' = (\Sigma \setminus \mathcal{N}) \cup \mathcal{N}'$$

is a symplectic spread with respect to the form (1).

Lemma 3 *Under the action of the group G the spread Σ has the following four orbits: $\{(x, 0) \mid x \in F\}$; $\{(0, y) \mid y \in F\}$; subspaces $\{(x, mx^{\rho^{-1}} + m^{\rho}x^{\rho}) \mid x \in F\}$ with nonsquare elements $m \in F^*$; and subspaces $\{(x, mx^{\rho^{-1}} + m^{\rho}x^{\rho}) \mid x \in F\}$ with square elements $m \in F^*$.*

Proof We have

$$\begin{aligned}\sigma_s(x, mx^{\rho^{-1}} + m^{\rho}x^{\rho}) &= (sx, s^{-1}mx^{\rho^{-1}} + s^{-1}m^{\rho}x^{\rho}) \\ &= (sx, s^{-1-\rho^{-1}}m(sx)^{\rho^{-1}} + s^{-1-\rho}m^{\rho}(sx)^{\rho}) \\ &= (u, s^{-1-\rho^{-1}}mu^{\rho^{-1}} + s^{-1-\rho}m^{\rho}u^{\rho}),\end{aligned}$$

where $u = sx$. Therefore, the isometry σ_s sends the subspace indexed by m to the subspace indexed by $s^{-1-\rho^{-1}}m$. Since $s^{-1-\rho^{-1}} = (s^{\rho+1})^{-\rho^{-1}}$, it remains for us to show that the set $\{s^{\rho+1} \mid s \in F^*\}$ is the set of all squares in F^* . Indeed, assume that the fixed field F_{ρ} of ρ has q_1 elements. Then $\rho = q_1^k$, $|F| = q_1^t$, $\gcd(t, k) = 1$, and t is an odd integer. We have $\gcd(q_1^k + 1, q_1^t - 1) = 2$, so $\{s^{\rho+1} \mid s \in F^*\}$ is the set of all squares in F^* . $\square$

Lemma 4 *Suppose the function $\alpha : F \rightarrow F$ is given by $\alpha(u) = u^{\rho^{-1}} + u^{\rho}$. Let $\beta = \alpha^{-1}$ and let the order of ρ be t . Then*

$$\beta(v) = \frac{1}{2} \left(a_0v + a_1v^{\rho} + \cdots + a_{t-1}v^{\rho^{t-1}} \right),$$

where $a_{4i} = a_{4i+1} = 1$, $a_{4i+2} = a_{4i+3} = -1$, $i \geq 0$, for $t \equiv 1 \pmod{4}$, and $a_{4i} = a_{4i+3} = -1$, $a_{4i+1} = a_{4i+2} = 1$, $i \geq 0$, for $t \equiv 3 \pmod{4}$.

Proof We have

$$\begin{aligned}\beta(v) + \beta(v)^{\rho^2} &= \frac{1}{2} \left(a_0v + \cdots + a_{t-1}v^{\rho^{t-1}} \right) \\ &\quad + \frac{1}{2} \left(a_0v^{\rho^2} + \cdots + a_{t-3}v^{\rho^{t-1}} + a_{t-2}v + a_{t-1}v^{\rho} \right) \\ &= \frac{1}{2} \left((a_0 + a_{t-2})v + (a_1 + a_{t-1})v^{\rho} + \sum_{k=2}^{t-1} (a_k + a_{k-2})v^{\rho^k} \right) = v^{\rho}.\end{aligned}$$

Therefore $\beta(v)^{\rho^{-1}} + \beta(v)^{\rho} = v$. $\square$

We use this function β for the following

Corollary 1 *The spread Σ' consists of the following subspaces: $\{(x, 0) \mid x \in F\}$; $\{(0, y) \mid y \in F\}$; subspaces $\{(x, mx^{\rho^{-1}} + m^{\rho}x^{\rho}) \mid x \in F\}$ with nonsquare elements $m \in F$; and subspaces $\{(x, s\beta(xs)) \mid x \in F\}$ with $s \in F^*/\langle -1 \rangle$.*

Proof The collection $\mathcal{N}'$ consists of elements of the form $(s^{-1}(u^{\rho^{-1}} + u^{\rho}), su)$. We denote $s^{-1}(u^{\rho^{-1}} + u^{\rho}) = x$. Then $u^{\rho^{-1}} + u^{\rho} = xs$, $u = \beta(xs)$ and $su = s\beta(xs)$. Therefore the corresponding spread subspaces are $\{(x, s\beta(xs)) \mid x \in F\}$, $s \in F^*$. Finally we note that s and $-s$ determine the same subspaces. $\square$

From this corollary we get the following complete set of MUBs:

$$B_{\infty} = \{e_w \mid w \in F\}, \quad B_0 = \left\{ \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}(vw)} e_w \mid v \in F \right\},$$

$$B_m = \left\{ \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}(m^{\rho}w^{\rho+1}+vw)} e_w \mid v \in F \right\}, \quad m \in F^*, \quad m \text{ is nonsquare},$$

$$B_s = \left\{ \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{tr}(\frac{1}{2}ws\beta(ws)+vw)} e_w \mid v \in F \right\}, \quad s \in F^*/\langle -1 \rangle.$$

We note that the Ball–Bamberg–Laurauw–Penttila symplectic spread is not a semifield spread. If F is a field of order 27, then the corresponding plane is the Hering plane and its translation complement is isomorphic to $SL_2(13)$. If $|F| = q_1^3$ and $|F_{\rho}| = q_1$ then the corresponding plane is the Suetake [44] plane and the translation component has order $3(q_1 - 1)(q_1^3 - 1)$.

3.4 Other known semifields

In this section we consider examples of known commutative and symplectic semifields. They are presented in pairs, so that symplectic semifields produce symplectic spreads with respect to the alternating form (1).

The Dickson commutative semifield [18, 19] and corresponding Knuth [36] symplectic presemifields are defined by

$$(a, b) * (c, d) = (ac + jb^{\sigma}d^{\sigma}, ad + bc),$$

$$(a, b) \circ (c, d) = (ac + bd, ad + j^{\sigma^{-1}}bc^{\sigma^{-1}}),$$

where q is odd, j be a nonsquare element of $F = \mathbb{F}_q$, $1 \neq \sigma \in \text{Aut}(F)$ and $V = F \oplus F$.

The Cohen–Ganley commutative semifield [14] and the corresponding symplectic Thas–Payne presemifield [45] are defined by

$$(a, b) * (c, d) = (ac + jbd + j^3(bd)^9, ad + bc + j(bd)^3),$$

$$(a, b) \circ (c, d) = (ac + bd, ad + jbc + j^{1/3}bc^{1/9} + j^{1/3}bd^{1/3}),$$

where $q = 3^r \geq 9$ and $j \in F$ is nonsquare.

Ganley commutative semifields [22] and corresponding symplectic presemifields are defined by

$$(a, b) * (c, d) = (ac - b^9d - bd^9, ad + bc + b^3d^3),$$

$$(a, b) \circ (c, d) = (ac + bd, ad + bd^{1/3} - b^{1/9}c^{1/9} - b^9c),$$

where $F = \mathbb{F}_q$, $q = 3^t$, and $t \geq 3$ odd.

The Penttilä–Williams sporadic symplectic semifield [40] $(\mathbb{F}_{3^5} \oplus \mathbb{F}_{3^5}, +, \circ)$ of order 3^{10} and the corresponding commutative semifield are given by

$$(a, b) \circ (c, d) = (ac + bd, ad + bd^9 + bc^{27}),$$

$$(a, b) * (c, d) = (ac + (bd)^9, ad + bc + (bd)^{27}).$$

During recent years some families of quadratic planar functions were discovered [10, 12, 15–17, 20, 47, 49, 50, 52].

4 Symplectic spreads in even characteristics and MUBs

In this section we use symplectic spreads in even characteristic for constructing MUBs. Constructions of MUBs are also given in [34], where related objects are considered over prime fields. We will use Galois rings, so descriptions can be given explicitly (especially in the case of presemifields). Throughout this section $F = \mathbb{F}_q$ denotes a finite field of even order $q = 2^r$. Let $\omega \in \mathbb{C}$ be a primitive 4th root of unity.

To write corresponding MUBs, we need the Galois ring $R = GR(4^r)$ of characteristic 4 and cardinality 4^r . We recall some facts about R [25, 39]. We have $R/2R \cong \mathbb{F}_q$, the unit group $R^* = R \setminus 2R$ contains a cyclic subgroup C of size $2^r - 1$ isomorphic to $\mathbb{F}_q^*$. The set $\mathcal{T} = \{0\} \cup C$ is called the Teichmüller set in R . Every element $x \in R$ can be written uniquely in the form $x = a + 2b$ for $a, b \in \mathcal{T}$. Then

$$\text{Tr}(x) = (a + a^2 + \cdots + a^{2^{r-1}}) + 2(b + b^2 + \cdots + b^{2^{r-1}}).$$

Since $R/2R \cong \mathbb{F}_q$, for every element $u \in \mathbb{F}_q$ there exists a corresponding unique element $\hat{u} \in \mathcal{T}$, called the Teichmüller lift of u . If $x, y, z \in \mathcal{T}$ and $z \equiv x + y \pmod{2R}$ then

$$z = x + y + 2\sqrt{xy}.$$

The last equation can be written in the form

$$\hat{w} = \hat{u} + \hat{v} + 2\sqrt{\hat{u}\hat{v}}$$

for elements $w, u, v \in \mathbb{F}_q$, $w = u + v$.

Let $(F, +, \circ)$ be a presemifield with multiplication

$$x \circ y = \sum a_{ij} x^{2^i} y^{2^j}.$$

Then we extend this multiplication to $\mathcal{T} \times \mathcal{T}$ in the following way:

$$\hat{x} \circ \hat{y} = \sum \widehat{a_{ij}} \hat{x}^{2^i} \hat{y}^{2^j}$$

Lemma 5 *Let $(F, +, \circ)$ be a symplectic presemifield. Then*

1. $\text{tr}(x \cdot (z \circ y)) = \text{tr}(z \cdot (x \circ y))$ for all $x, y, z \in F$.
2. $\text{Tr}(\hat{x} \cdot (\hat{z} \circ \hat{y})) = \text{Tr}(\hat{z} \cdot (\hat{x} \circ \hat{y}))$ for all $x, y, z \in F$.

Proof 1. $\text{tr}(x \cdot (z \circ y) - z \cdot (x \circ y)) = \langle ((x, x \circ y), (z, z \circ y)) \rangle = 0$.

2. We have

$$\begin{aligned} \text{tr}(z \cdot (x \circ y) - x \cdot (z \circ y)) &= \text{tr} \left(z \sum a_{ij} x^{2^i} y^{2^j} - x \sum a_{ij} z^{2^i} y^{2^j} \right) \\ &= \text{tr} \left(z \sum a_{ij} x^{2^i} y^{2^j} - \sum a_{ij}^{2^{-i}} x^{2^{-i}} z y^{2^{j-i}} \right) \\ &= \text{tr} \left(z \left(\sum a_{ij} x^{2^i} y^{2^j} - \sum a_{ij}^{2^{-i}} x^{2^{-i}} y^{2^{j-i}} \right) \right) = 0, \end{aligned}$$

where powers of 2 and indices are considered modulo r for convenience. Therefore,

$$\sum a_{ij} x^{2^i} y^{2^j} = \sum a_{ij}^{2^{-i}} x^{2^{-i}} y^{2^{j-i}} = \sum a_{r-i, j}^{2^i} x^{2^i} y^{2^{j+i}} = \sum a_{r-i, j-i}^{2^i} x^{2^i} y^{2^j}.$$

Hence $a_{ij} = a_{r-i, j-i}^{2^i}$ and $\widehat{a_{ij}} = \widehat{a_{r-i, j-i}}^{2^i}$. Then

$$\begin{aligned} \text{Tr}(\hat{z} \cdot (\hat{x} \circ \hat{y}) - \hat{x} \cdot (\hat{z} \circ \hat{y})) &= \text{Tr} \left(\hat{z} \sum \widehat{a_{ij}} \hat{x}^{2^i} \hat{y}^{2^j} - \hat{x} \sum \widehat{a_{ij}} \hat{z}^{2^i} \hat{y}^{2^j} \right) \\ &= \text{Tr} \left(\hat{z} \left(\sum \widehat{a_{ij}} \hat{x}^{2^i} \hat{y}^{2^j} - \sum \widehat{a_{ij}}^{2^{-i}} \hat{x}^{2^{-i}} \hat{y}^{2^{j-i}} \right) \right) \\ &= \text{Tr} \left(\hat{z} \left(\sum \widehat{a_{ij}} \hat{x}^{2^i} \hat{y}^{2^j} - \sum \widehat{a_{r-i, j}}^{2^i} \hat{x}^{2^i} \hat{y}^{2^{j+i}} \right) \right) \\ &= \text{Tr} \left(\hat{z} \left(\sum \widehat{a_{ij}} \hat{x}^{2^i} \hat{y}^{2^j} - \sum \widehat{a_{r-i, j-i}}^{2^i} \hat{x}^{2^i} \hat{y}^{2^j} \right) \right) = 0. \end{aligned}$$

□

Theorem 6 *Let $(F, +, \circ)$ be a symplectic presemifield. Then the following forms a complete set of MUBs:*

$$B_\infty = \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F,$$

$$b_{m,v} = \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{Tr}(\hat{w} \cdot (\hat{w} \circ \hat{m}) + 2\hat{w}\hat{v})} e_w.$$

Proof We show that for any $m, v \in F$, the vector

$$d_{m,v} = \sum_{w \in F} \omega^{\text{Tr}(\hat{w} \cdot (\hat{w} \circ \hat{m}) + 2\hat{w}\hat{v})} e_w$$

is an eigenvector of $D_{u,u \circ m}$ for all $u \in V$. Indeed,

$$\begin{aligned} D_{u,u \circ m}(d_{m,v}) &= \sum_{w \in F} \omega^{\text{Tr}(\hat{w} \cdot (\hat{w} \circ \hat{m}) + 2\hat{w}\hat{v} + 2\hat{w} \cdot \widehat{u \circ m})} e_{w+u} \\ &= \sum_{w \in F} \omega^{\text{Tr}(\widehat{(w+u)} \cdot (\widehat{(w+u)} \circ \hat{m}) + 2(\hat{w} + \hat{u})\hat{v} + 2(\hat{w} + \hat{u}) \cdot (\hat{u} \circ \hat{m}))} e_w \\ &= \sum_{w \in F} \omega^{\text{Tr}((\hat{w} + \hat{u} + 2\sqrt{\hat{w}\hat{u}}) \cdot ((\hat{w} + \hat{u} + 2\sqrt{\hat{w}\hat{u}}) \circ \hat{m}) + 2(\hat{w} + \hat{u})\hat{v} + 2(\hat{w} + \hat{u}) \cdot (\hat{u} \circ \hat{m}))} e_w \\ &= \sum_{w \in F} \omega^{\text{Tr}(\hat{w} \cdot (\hat{w} \circ \hat{m}) + 2\hat{w}\hat{v} + S + \hat{u}(\hat{u} \circ \hat{m}) + 2\hat{u}\hat{v} + 2\hat{u}(\hat{u} \circ \hat{m}))} e_w \\ &= \omega^{\text{Tr}(2\hat{u}\hat{v} - \hat{u}(\hat{u} \circ \hat{m}))} d_{m,v}, \end{aligned}$$

where

$$\begin{aligned} \text{Tr}(S) &= \text{Tr}((\hat{w} \cdot (\hat{u} \circ \hat{m}) + \hat{u} \cdot (\hat{w} \circ \hat{m}) + 2\hat{w} \cdot (\hat{u} \circ \hat{m})) \\ &\quad + (2\hat{w} \cdot (\sqrt{\hat{w}\hat{u}} \circ \hat{m}) + 2\sqrt{\hat{w}\hat{u}} \cdot (\hat{w} \circ \hat{m})) \\ &\quad + (2\hat{u} \cdot (\sqrt{\hat{w}\hat{u}} \circ \hat{m}) + 2\sqrt{\hat{w}\hat{u}} \cdot (\hat{u} \circ \hat{m}))) = 0 \end{aligned}$$

by Lemma 5. □

Theorem 7 *Let $(F, +, *)$ be a commutative presemifield. Then the following forms a complete set of MUBs:*

$$B_\infty = \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F,$$

$$b_{m,v} = \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{Tr}(\hat{m}(\hat{w} * \hat{w}) + 2\hat{w}\hat{v})} e_w.$$

Proof Let $x * y = \sum a_{ij} x^{2^i} y^{2^j}$. Working as in Sect. 3 we see that the corresponding symplectic presemifield $(F, +, \circ)$ is given by multiplication:

$$x \circ y = \sum a_{ii}^{2^{r-i}} x y^{2^{r-i}} + \sum_{i < j} a_{ij}^{2^{r-i}} x^{2^{j-i}} y^{2^{r-i}} + \sum_{i < j} a_{ij}^{2^{r-j}} x^{2^{r+i-j}} y^{2^{r-j}}.$$

We have

$$\begin{aligned} \text{Tr}(\hat{w} \cdot (\hat{w} \circ \hat{m})) &= \text{Tr} \left(\sum \hat{w} \widehat{a_{ii}}^{2^{r-i}} \hat{w} \hat{m}^{2^{r-i}} + \sum_{i < j} \hat{w} \widehat{a_{ij}}^{2^{r-i}} \hat{w}^{2^{j-i}} \hat{m}^{2^{r-i}} \right. \\ &\quad \left. + \sum_{i < j} \hat{w} \widehat{a_{ij}}^{2^{r-j}} \hat{w}^{2^{r+i-j}} \hat{m}^{2^{r-j}} \right) \\ &= \text{Tr} \left(\sum \hat{w}^{2^i} \widehat{a_{ii}} \hat{w}^{2^i} \hat{m} + \sum_{i < j} \hat{w}^{2^i} \widehat{a_{ij}} \hat{w}^{2^j} \hat{m} + \sum_{i < j} \hat{w}^{2^j} \widehat{a_{ij}} \hat{w}^{2^i} \hat{m} \right) \\ &= \text{Tr}(\hat{m}(\hat{w} * \hat{w})). \end{aligned}$$

Now, the statement of theorem follows from Theorem 6. $\square$

The above theorems allow us to construct complete sets of MUBs directly from presemifields (in particular, from Kantor–Williams [32] semifields).

4.1 Desarguesian spreads

Desarguesian spreads are constructed with the help of finite fields, and the corresponding commutative and symplectic semifields are the same: $x * y = x \circ y = xy$. Theorem 2 and [37] imply that for the automorphism group of the corresponding complete set of MUBs we have

$$\text{Aut}(\mathcal{B}) \cong F^2(SL_2(q)\mathbb{Z}_r)\mathbb{Z}_2,$$

where the last factor $\mathbb{Z}_2$ is absent in the case $q = 2$. We note that the extension $F^2.(SL_2(q).\mathbb{Z}_r)$ is non-split for $q \geq 8$ (see [1]), but the automorphism group of the corresponding plane is a semidirect product.

4.2 Lüneburg planes and Suzuki groups

In this subsection we consider non-semifield symplectic spreads related to Lüneburg planes and Suzuki groups [34, 46]. We give an explicit construction of MUBs in terms of Galois rings. Let $F = \mathbb{F}_q$ be a finite field of order $q = 2^{2k+1}$ and let $\sigma : \alpha \rightarrow \alpha^{2^{k+1}}$ be an automorphism of $\mathbb{F}_q$, $V = \mathbb{F}_q \oplus \mathbb{F}_q$.

The symplectic spread Σ of a space $W = V \oplus V$ consists of the subspace $\{(0, y) \mid y \in V\}$ and subspaces $\{(x, xM_c) \mid x \in V\}$, $c \in V$, where

$$c = (\alpha, \beta) \in V, \quad M_c = \begin{pmatrix} \alpha & \alpha^{\sigma^{-1}} + \beta^{1+\sigma^{-1}} \\ \alpha^{\sigma^{-1}} + \beta^{1+\sigma^{-1}} & \beta \end{pmatrix}.$$

The automorphism group of this symplectic spread is

$$\text{Aut}(\Sigma) = \text{Aut}(Sz(q)) = Sz(q)\mathbb{Z}_{2k+1},$$

where $Sz(q) = {}^2B_2(q)$ is the Suzuki twisted simple group.

The corresponding orthogonal decomposition of the Lie algebra $L = sl_{q^2}(\mathbb{C})$ is given by

$$L = H_\infty \oplus_{c \in V} H_c,$$

$$H_\infty = \langle D_{0,y} \mid y \in V, y \neq 0 \rangle, \quad H_c = \langle D_{x,xM_c} \mid x \in V, x \neq 0 \rangle.$$

Let $R = GR(4^{2k+1})$ be a Galois ring of characteristic 4 and cardinality 4^{2k+1} . For $v = (\alpha, \beta) \in V$ we define the Teichmüller lift of v as $(\hat{\alpha}, \hat{\beta})$, and for the matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ we define the Teichmüller lift of M as $\hat{M} = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}$. Then the following bases form a complete set of MUBs:

$$B_\infty = \{e_w \mid w \in V\}, \quad B_c = \{b_{c,v} \mid v \in V\}, \quad c \in V,$$

$$b_{c,v} = \frac{1}{q} \sum_{w \in V} \omega^{\text{Tr}(\hat{w} \cdot \hat{w} \hat{M}_c + 2\hat{v} \cdot \hat{w})} e_w.$$

Indeed,

$$\begin{aligned} D_{u,uM_c}(b_{c,v}) &= \frac{1}{q} \sum_{w \in V} \omega^{\text{Tr}(\hat{w} \cdot \hat{w} \hat{M}_c + 2\hat{v} \cdot \hat{w} + 2\hat{u} \hat{M}_c \cdot \hat{w})} e_{w+u} \\ &= \frac{1}{q} \sum_{w \in V} \omega^{\text{Tr}(\widehat{(w+u)} \cdot \widehat{(w+u)} \hat{M}_c + 2\hat{v} \cdot \widehat{(w+u)} + 2\hat{u} \hat{M}_c \cdot \widehat{(w+u)})} e_w. \end{aligned}$$

Now we set $w = (w_1, w_2)$, $u = (u_1, u_2)$. Then

$$(\widehat{w+u}) = (\widehat{w}_1 + \widehat{u}_1 + 2\sqrt{\widehat{w}_1 \widehat{u}_1}, \widehat{w}_2 + \widehat{u}_2 + 2\sqrt{\widehat{w}_2 \widehat{u}_2}) = \hat{w} + \hat{u} + 2\hat{z},$$

where $\hat{z} = (\sqrt{\widehat{w}_1 \widehat{u}_1}, \sqrt{\widehat{w}_2 \widehat{u}_2})$. Therefore,

$$\begin{aligned} D_{u,uM_c}(b_{c,v}) &= \frac{1}{q} \sum_{w \in V} \omega^{\text{Tr}((\hat{w} + \hat{u} + 2\hat{z}) \cdot (\hat{w} + \hat{u} + 2\hat{z}) \hat{M}_c + 2\hat{v} \cdot (\hat{w} + \hat{u}) + 2\hat{u} \hat{M}_c \cdot (\hat{w} + \hat{u}))} e_w \\ &= \frac{1}{q} \sum_{w \in V} \omega^{\text{Tr}[(\hat{w} \cdot \hat{w} \hat{M}_c + 2\hat{v} \cdot \hat{w}) + (3\hat{u} \cdot \hat{u} \hat{M}_c + 2\hat{v} \cdot \hat{u})]} e_w \\ &= \omega^{\text{Tr}(2\hat{v} \cdot \hat{u} - \hat{u} \cdot \hat{u} \hat{M}_c)} b_{c,v}, \end{aligned}$$

which shows that the vectors $b_{c,v}$ are common eigenvectors for the Cartan subalgebra H_c .

5 Pseudo-planar functions and MUBs

Let $F = \mathbb{F}_q$ be a finite field of even order q . There are no planar functions over fields of even characteristic. Recently Zhou [42, 51] introduced the notion of “planar” functions in even characteristic, however we adopt another term and call a function $f : F \rightarrow F$ pseudo-planar if

$$x \mapsto f(x+a) + f(x) + ax$$

is a permutation of F for each $a \in F^*$. Using the Teichmüller lift, we can also consider f as a function $f : \mathcal{T} \rightarrow \mathcal{T}$.

Theorem 8 *Let $F = \mathbb{F}_q$ be a finite field of even characteristic and f be a pseudo-planar function. Then the following forms a complete set of MUBs:*

$$B_\infty = \{e_w \mid w \in F\}, \quad B_m = \{b_{m,v} \mid v \in F\}, \quad m \in F,$$

$$b_{m,v} = \frac{1}{\sqrt{q}} \sum_{w \in F} \omega^{\text{Tr}(\hat{m}(\hat{w}^2 + 2f(\hat{w})) + 2\hat{v}\hat{w})} e_w,$$

Proof If $m = m_1$ then

$$(b_{m,v}, b_{m_1,v_1}) = \frac{1}{q} \sum_{w \in F} \omega^{\text{Tr}(2(\hat{v} - \hat{v}_1)\hat{w})} = \begin{cases} 1 & \text{if } v = v' \\ 0 & \text{if } v \neq v'. \end{cases}$$

If $m \neq m_1$ then

$$\begin{aligned} (b_{m,v}, b_{m_1,v_1}) &= \frac{1}{q} \sum_{w \in F} \omega^{\text{Tr}((\hat{m} - \hat{m}_1)(\hat{w}^2 + 2f(\hat{w})) + 2(\hat{v} - \hat{v}_1)\hat{w})} \\ &= \frac{1}{q} \sum_{w \in F} \omega^{\text{Tr}(M(\hat{w}^2 + 2f(\hat{w})) + 2\hat{u}\hat{w})}, \end{aligned}$$

where $M = \hat{m} - \hat{m}_1 \notin 2R$, $\hat{u} = \widehat{v - v_1}$. Then

$$\begin{aligned} |(b_{m,v}, b_{m_1,v_1})|^2 &= \frac{1}{q^2} \sum_{w, w_1 \in F} \omega^{\text{Tr}(M(\hat{w}^2 + 2f(\hat{w})) + 2\hat{u}\hat{w} - M((\widehat{w_1})^2 + 2f(\widehat{w_1})) - 2\hat{u}\widehat{w_1})} \\ &= \frac{1}{q^2} \sum_{w, w_1 \in F} \omega^{\text{Tr}(M(\hat{w}^2 - (\widehat{w_1})^2 + 2f(\hat{w}) - 2f(\widehat{w_1})) + 2\hat{u}(\hat{w} - \widehat{w_1}))}. \end{aligned}$$

Let $(\widehat{w_1})^2 \equiv \hat{w}^2 + \hat{a}^2 \pmod{2R}$, $a \in F$. Then $(\widehat{w_1})^2 = \hat{w}^2 + \hat{a}^2 + 2\hat{w}\hat{a}$. Therefore,

$$\begin{aligned} |(b_{m,v}, b_{m_1,v_1})|^2 &= \frac{1}{q^2} \sum_{w,a \in F} \omega^{\text{Tr}(M(-\hat{a}^2 - 2\hat{w}\hat{a} + 2f(\hat{w}) - 2f(\widehat{w+a})) - 2\hat{u}\hat{a})} \\ &= \frac{1}{q^2} \sum_{a \in F} \omega^{\text{Tr}(-M\hat{a}^2 - 2\hat{u}\hat{a})} \sum_{w \in F} \omega^{\text{Tr}(2M(f(\widehat{w+a}) + f(\hat{w}) + \hat{w}\hat{a}))}. \end{aligned}$$

Since $f(w)$ is a pseudo-planar function, we have

$$\sum_{w \in F} \omega^{\text{Tr}(2M(f(\widehat{w+a}) + f(\hat{w}) + \hat{w}\hat{a}))} = \begin{cases} q & \text{if } a = 0 \\ 0 & \text{if } a \neq 0. \end{cases}$$

Hence,

$$|(b_{m,v}, b_{m_1,v_1})|^2 = \frac{1}{q^2} \cdot q = \frac{1}{q}.$$

□

The following theorem shows connections between pseudo-planar functions and commutative presemifields.

Theorem 9 *Let F be a finite field of characteristic two.*

1. *If $(F, +, *)$ is a commutative presemifield with multiplication given by*

$$x * y = xy + \sum_{i < j} a_{ij} (x^{2^i} y^{2^j} + x^{2^j} y^{2^i})$$

then $f(x) = \sum_{i < j} a_{ij} x^{2^i + 2^j}$ is a pseudo-planar function and $x * y = xy + f(x + y) + f(x) + f(y)$.

2. *If $(F, +, *)$ is a commutative presemifield then there exist a strongly isotopic commutative presemifield $(F, +, \star)$ and a pseudo-planar function f such that $x \star y = xy + f(x + y) + f(x) + f(y)$. Therefore, up to isotopism, any commutative semifield can be described by pseudo-planar functions.*
3. *Let f be a pseudo-planar function. Then $(F, +, *)$ with multiplication $x * y = xy + f(x + y) + f(x) + f(y)$ is a presemifield if and only if f is a quadratic function.*

Proof 1. If $x * y = xy + f(x + y) + f(x) + f(y)$ determines a presemifield then the map $x \mapsto xy + f(x + y) + f(x) + f(y)$ is a permutation for any $y \in F^*$, therefore the map $x \mapsto xy + f(x + y) + f(x)$ is a permutation for any $y \in F^*$.

2. Assume that the presemifield $(F, +, *)$ is given by

$$x * y = \sum_{i < j} a_{ij} x^{2^i} y^{2^j} + \sum_{i < j} a_{ij} (x^{2^i} y^{2^j} + x^{2^j} y^{2^i}).$$

Denote $g(x) = \sum a_i x^{2^i}$. Then $x * x = g(x^2)$. We need only to prove that the linear function g is invertible and then we can take $x \star y = g^{-1}(x * y)$ and apply part 1.

Suppose that g is not invertible. Then there exists $a \in F^*$ such that $g(a^2) = 0$. Then

$$(x + a) * (x + a) = g((x + a)^2) = g(x^2 + a^2) = g(x^2) = x * x,$$

which means $x * x + x * a + a * x + a * a = x * x$ and $a * a = 0$, a contradiction.

3. The condition $(x + z) * y = x * y + z * y$ is equivalent to

$$\begin{aligned} (x + z)y + f(x + z + y) + f(x + z) + f(y) \\ = xy + f(x + y) + f(x) + f(y) + zy + f(z + y) + f(z) + f(y). \end{aligned}$$

Therefore,

$$f(x + y + z) + f(x + y) + f(x + z) + f(y + z) + f(x) + f(y) + f(z) = 0,$$

which means that f is a quadratic function. $\square$

Remark 2 The notion of pseudo-planar functions can be defined over a field F of arbitrary characteristic. We call a function $f : F \rightarrow F$ pseudo-planar if the map $x \mapsto f(x+a) - f(x) + ax$ is a permutation of F for each $a \in F^*$. Such functions carry similar properties as pseudo-planar functions in even characteristic (including MUBs constructions). In particular, if f is a quadratic pseudo-planar function then the product $x * y = xy + f(x + y) - f(x) - f(y)$ defines a commutative presemifield $(F, +, *)$. Probably such a definition of pseudo-planar functions provides unified approach to all characteristics, at least it provides a bridge between the even and odd characteristic cases. Note that in the case of odd characteristic, a function f is pseudo-planar if and only if the function $x^2 + 2f(x)$ is planar, and formulas in Theorems 5 and 8 have a unified look in the language of pseudo-planar functions.

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