Combinatorial model for the cluster categories of type E

Lisa Lamberti

Received: 18 July 2014 / Accepted: 9 October 2014 / Published online: 22 October 2014
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Abstract In this paper, we give a geometric-combinatorial description of the cluster categories of type E . In particular, we give an explicit geometric description of all cluster-tilting objects in the cluster category of type E_6 . The model we propose here arises from combining two polygons, and it generalizes the description of the cluster category of type A and D .

Keywords Cluster categories · Type E · Triangulated Calabi–Yau categories · Polygon dissection · Cluster algebra · Auslander–Reiten quiver

1 Introduction

Caldero–Chapoton–Schiffler defined in [5] categories arising from homotopy classes of paths between two vertices of a regular $(n+3)$ -sided polygon. Independently, Buan–Marsh–Reiten–Reineke–Todorov defined cluster categories as certain orbit categories of the bounded derived category of hereditary algebras, see [3]. The latter are algebras arising from oriented graphs Q with no oriented cycles. When Q is an orientation of a Dynkin graph of type A_n , the category constructed in [5] coincides with the one of [3]. In this paper we model a number of different orbit categories arising from orientations of tree graphs $T_{r,s,t}$ in geometric terms. The description we propose here is based on the idea of doubling the set of oriented diagonals in a given regular polygon and combine the dynamics of these two sets in an appropriate way.

Among the additive categories, we can model in this way we find cluster categories of type $T_{r,s,t}$, as well as other triangulated categories of various Calabi–Yau dimension. All these categories arise as mesh categories of a translation quiver whose vertices

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L. Lamberti (✉)
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK
e-mail: Lisa.Lamberti@maths.ox.ac.uk

are single-coloured oriented and paired diagonals in a polygon Π . We will see that polygons of different sizes yield different categories and these are Calabi–Yau of dimension two only in few exceptional cases. The advantage of our approach is that we can model the combinatorics of these various categories in terms of configurations of diagonals in Π .

For a tree diagram $T_{r,s,t}$, let $\mathcal{C}_{T_{r,s,t}}$ be cluster category of type $T_{r,s,t}$ defined as the orbit category $\mathcal{D}^b(\text{mod } kT_{r,s,t})/\tau^{-1}\Sigma$. Then for $n \geq \max\{r+t+1, r+s+1\}$ and a regular $(n+3)$ -gon Π , we construct an additive category $\mathcal{C}_{r,s,t}^{n+3}$ of coloured oriented single and paired diagonals of Π .

With this notation our first main result can be stated as follows.

Theorem (4.3) *We have the following equivalences of additive categories:*

$$\begin{aligned}\mathcal{C}_{1,2,2}^7 &\rightarrow \mathcal{C}_{E_6} \\ \mathcal{C}_{1,2,3}^{10} &\rightarrow \mathcal{C}_{E_7} \\ \mathcal{C}_{1,2,4}^{16} &\rightarrow \mathcal{C}_{E_8}.\end{aligned}$$

Letting the number of sides of the polygon Π tend to infinity, we obtain an equivalence from the category $\mathcal{C}_{r,s,t}^\infty$ associated to the infinite-sided polygon Π^∞ to the full subcategory of the cluster category $\mathcal{C}_{T_{r,s,t}}$ with indecomposable objects in the transjective component of the AR-quiver of $\mathcal{C}_{T_{r,s,t}}$.

These equivalences enable us to investigate the combinatorics of the cluster category in geometric terms. More precisely, we are able to describe all 833 cluster-tilting sets of the cluster category of type E_6 , $\mathcal{C}_{E_6}$, as cluster configuration of six single-coloured oriented and paired diagonals in a heptagon. The strategy will be to first determine two fundamental families of cluster configurations $\mathcal{F}_1$ and $\mathcal{F}_2$ and deduce the remaining cluster-tilting sets using the rotation inside Π , induced from the Auslander–Reiten translation τ in $\mathcal{C}_{E_6}$, as well as a symmetry σ of our model.

Theorem (5.10) *In a heptagon*

- 350 different cluster configurations have one long paired diagonal, and they arise from $\mathcal{F}_1$ through τ .
- 483 other cluster configurations arise from $\mathcal{F}_2$ through σ and τ .

This classification allows us to deduce the following result.

Theorem (5.12) *All 833 cluster-tilting sets of $\mathcal{C}_{E_6}$ can be expressed as configurations of six non-crossing coloured oriented single and paired diagonals inside two heptagons.*

The previous results enable us to deduce geometrical moves describing the mutation process between cluster-tilting objects in $\mathcal{C}_{E_6}$. The geometrical moves we find extend the mutation process of cluster categories of type A , as described by Caldero–Chapoton–Schiffler in [5], to the setting of coloured oriented diagonals. We will see that in many cases mutations inside $\mathcal{C}_{E_6}$ correspond to flips of coloured oriented diagonals in Π . In the final part of the paper we used the previous results to categorify geometrically cluster algebras of type F_4 , providing a geometric description of all clusters and of the mutation rule.

1.1 Relation to previous work

Geometrical models for cluster categories of other types have been investigated also in [2, 18, 20, ...]. Moreover, with an appropriate pairing of coloured oriented single and paired diagonals in an even sided polygon we recover the categorified geometric realization of cluster algebras of type D given by Fomin-Zelevinsky in [8].

The number of clusters in a cluster algebra of finite type was first computed in [8]. Under the bijection of [3] one deduces the number of cluster tilting sets in the corresponding cluster categories. An explicit complete description of the cluster-tilting objects in the cluster category of type E_6 and F_4 however is new.

In addition, Fomin-Pylyavskyy used in [9] polygons to describe the cluster algebra structure in certain rings of $SL(V)$ -invariants. More precisely, they construct invariants determined by tensor graphs associated to diagonals of polygons. Using a heptagon, Fomin-Pylyavskyy also model the cluster algebra structure of the homogeneous coordinate ring $\mathbb{C}[Gr_{3,7}]$ of the affine cone over the Grassmannian $Gr_{3,7}$ of three dimensional subspaces in a seven-dimensional complex vector space. By a result of Scott, [19], it is known that the ring $\mathbb{C}[Gr_{3,7}]$ is a cluster algebra type of E_6 . The approach of [9] however is different than the one we propose here, as it relies on relations satisfied by tensor graphs, called skein relations of tensor graphs.

1.2 Organization of the article

In Sect. 2, we state some preliminary results and definitions. In particular, we recollect the fundamental properties of orbit categories and we remind the action of the shift functor on the Auslander-Reiten quiver of $\mathcal{D}^b(\text{mod } Q)$, for Q an orientation of a simply-laced Dynkin diagram.

In Sect. 3, we construct the additive category $\mathcal{C}_{r,s,t}^{n+3}$ associated to a regular $(n+3)$ -gon Π , where $n = \max\{r+t+1, r+s+1\}$. The objects of $\mathcal{C}_{r,s,t}^{n+3}$ will be single-coloured oriented and paired diagonals of Π . The morphism spaces are generated by minimal rotations in Π , modulo certain equivalence relations.

In Sect. 4, we prove the equivalences of additive categories stated in Theorem 4.3. Further equivalence of additive categories will also be discussed. In Proposition 4.8, we show that there is an equivalence between the category $\mathcal{C}_{r,s,t}^\infty$ associated to an infinite-sided polygon and the full subcategory of the cluster category $\mathcal{C}_{T_{r,s,t}}$, whose indecomposable objects belong to the transjective component of the AR-quiver of $\mathcal{C}_{T_{r,s,t}}$.

In Sect. 5, we describe the combinatorics of $\mathcal{C}_{E_6}$ geometrically inside a heptagon Π . In Theorems 5.10 and 5.12, we describe all cluster-tilting sets of $\mathcal{C}_{E_6}$ in terms of cluster configurations of coloured oriented diagonals of Π . In Proposition 5.3, we also describe all Ext-spaces of $\mathcal{C}_{E_6}$ using curves of coloured oriented diagonals of Π . Finally, results concerning the mutation process of cluster configurations will be stated, see Proposition 5.15.

In Sect. 6, we point out further applications of our work. In particular we deduce a geometric additive categorification of cluster algebras of type F_4 . Moreover, we describe how our construction can be used to understand cluster-tilting sets inside the

cluster categories of type E_7 and E_8 , as well as cluster-tilting sets in the transjective component of the AR-quiver of cluster categories associated to more general tree diagrams $T_{r,s,t}$.

2 Preliminaries

Let k be an algebraically closed field and let Q be an acyclic quiver. Let $\text{mod } kQ$ be the abelian category of k -finite dimensional right-modules over the path algebra kQ . Let $\mathcal{D} := \mathcal{D}_Q := \mathcal{D}^b(\text{mod } kQ)$ be the bounded derived category of $\text{mod } kQ$ endowed with the shift functor $\Sigma : \mathcal{D} \rightarrow \mathcal{D}$ and the Auslander–Reiten translation $\tau : \mathcal{D} \rightarrow \mathcal{D}$ characterized by $\text{Hom}_{\mathcal{D}}(X, -)^* \cong \text{Hom}_{\mathcal{D}}(-, \Sigma \tau X)$, for all $X \in \mathcal{D}$.

2.1 Orbit categories of $\mathcal{D}$

We are interested in the orbit categories $\mathcal{C}_Q^p$ of $\mathcal{D}$, $p \in \mathbb{N}$, generated by the action of cyclic group generated by the auto-equivalences $F^p := (\tau^{-1}\Sigma)^p = \tau^{-p}\Sigma^p$. The objects of $\mathcal{C}_Q^p$ are the same as the objects of $\mathcal{D}$ and

$$\text{Hom}_{\mathcal{C}_Q^p}(X, Y) := \bigoplus_{t \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(X, (F^p)^t Y).$$

Morphisms are composed in a natural way.

When $p = 1$, $\mathcal{C}_Q := \mathcal{C}_Q^1$ is the *cluster category of type Q* defined in [3], and independently in [5] in geometric terms for Q of type A_n . In all other cases $\mathcal{C}_Q^p$ is the *p -repetitive cluster category* studied by the author in [14] for Q of type A_n , and introduced by Zhu in [21] for Q an acyclic quiver.

2.2 Fundamental properties of orbit categories

Like $\mathcal{D}$, the categories $\mathcal{C}_Q^p$ are Krull–Schmidt and have finite dimensional Hom-spaces. The categories $\mathcal{C}_Q^p$ are triangulated categories, and the projection functor $\pi_i : \mathcal{D} \rightarrow \mathcal{C}_Q^p$, $i \in \mathbb{N}$ is a triangle functor, see [12, Theorem 1]. The induced shift functor is again denoted by Σ . Moreover, the categories $\mathcal{C}_Q^p$ have AR-triangles and the AR-translation τ is induced from $\mathcal{D}$. The categories $\mathcal{C}_Q^p$ also have the Calabi–Yau property, i.e. $(\tau \Sigma)^m \xrightarrow{\sim} \Sigma^n$ as triangle functors, here we identify $\tau \Sigma$ with the Serre functor of $\mathcal{C}_Q^p$. In particular, in $\mathcal{C}_Q$ we have that $n = 2$ and $m = 1$, hence $\mathcal{C}_Q$ is Calabi–Yau of dimension 2. In $\mathcal{C}_Q^p$ we have that $m = 2$ and $n = p$ in the above isomorphism of triangle functors, thus $\mathcal{C}_Q^p$ is said to be a Calabi–Yau category of fractional dimension $\frac{p}{2}$. Notice that a Calabi–Yau category of fractional dimension $\frac{p}{2}$ is in general not a Calabi–Yau category of dimension 2. In this paper $p \in \{1, 2\}$, moreover we adopt the convention $\text{Ext}_{\mathcal{C}_Q^p}^i(X, Y) := \text{Hom}_{\mathcal{C}_Q^p}(X, \Sigma^i Y)$.

2.3 Auslander–Reiten quiver of a Krull–Schmidt category

A *stable translation quiver* (Γ, τ) in the sense of Riedtmann, [17], is a quiver Γ without loops nor multiple edges, together with a bijective map $\tau : \Gamma \rightarrow \Gamma$ called *translation* such that for all vertices x in Γ the set of starting points of arrows which end in x is equal to the set of end points of arrows which start at $\tau(x)$.

For (Γ, τ) one defines the mesh category as the quotient category of the additive path category of Γ by the mesh ideal, see for example [13]. In particular, the mesh category of (Γ, τ) is an additive category.

In the next result, let $\mathbb{Z}Q$ be the repetitive quiver of Q , see [10, I.5.6] for a reminder on this construction. Let $\tau : \mathbb{Z}Q \rightarrow \mathbb{Z}Q$ be the automorphism defined on the vertices (n, i) of $\mathbb{Z}Q$ by $\tau(n, i) = (n - 1, i)$, for $n \in \mathbb{Z}$, i a vertex of Q .

Theorem 2.1 [10, I.5.5] *Let kQ be a finite dimensional hereditary k -algebra. If Q is an orientation of*

- *a simply-laced Dynkin graph, $AR(\mathcal{D}^b(\text{mod } kQ))$ is isomorphic (as stable translation quiver) to $\mathbb{Z}Q$.*
- *an affine graph, $AR(\mathcal{D}^b(\text{mod } kQ))$ splits into components of the form $\mathbb{Z}Q$ and $\mathbb{Z}A_\infty/r$, for some $r \in \mathbb{N}$.*
- *a wild graph, the components of $AR(\mathcal{D}^b(\text{mod } kQ))$ are of the form $\mathbb{Z}Q$ and $\mathbb{Z}A_\infty$.*

Let $\text{ind } \mathcal{D}$ be the full subcategory of $\mathcal{D}$ of indecomposable objects.

Theorem 2.2 [10, I.5.6] *Let Q be an orientation of a simply-laced Dynkin graph. The mesh category of $(\mathbb{Z}Q, \tau)$ is equivalent to $\text{ind } \mathcal{D}$.*

A first important consequence of this result is that the AR-quiver of $\mathcal{D}$ is independent of the orientation of Q .

2.4 Induced action of Σ on $\mathbb{Z}Q$

In this section let Q be an orientation of a simply-laced Dynkin graph. Below we point out some known facts about the induced action of Σ and τ on $\mathbb{Z}Q$ taken from [16, Chap. 4], see also [11, Chap. 4]. These considerations, together with Theorem 2.1, will enable us to determine the precise shape of the AR-quiver of various orbit categories investigated in the sequel.

The induced action of τ on $\mathbb{Z}Q$ is always an horizontal shift to the left. The induced action of Σ on $\mathbb{Z}A_n$ coincides with a shift of $\frac{n+1}{2}$ units to the right, composed with a reflection along the horizontal central line of $\mathbb{Z}A_n$. On $\mathbb{Z}D_n$ the action of Σ agrees with $\tau^{-(n-1)}$ composed with an order two automorphism ρ defined on $\mathbb{Z}D_n$ when n is odd. While on $\mathbb{Z}E_6$ the action of Σ coincides with the action of $\rho\tau^{-6}$ where ρ is an automorphism of order two defined on $\mathbb{Z}E_6$. Moreover, Σ acts as τ^{-9} on $\mathbb{Z}E_7$, and as τ^{-15} on $\mathbb{Z}E_8$.

Fig. 1 The AR-quiver of the repetitive cluster category $\mathcal{C}_{A_4}^2$.

2.5 The repetitive cluster category $\mathcal{C}_{A_n}^p$

The geometrical model for the cluster categories of type E we propose in this paper is motivated from the following idea: glue together two copies of AR-quiver of $\mathcal{C}_{A_n}^2$. Let us remind the reader some facts about this category.

The previous discussion implies that $(AR(\mathcal{C}_{A_n}^p), \tau) \cong (\mathbb{Z}A_n/(\tau^{-1}\Sigma)^p, \tau)$, for $p \in \mathbb{N}$. Moreover, $(AR(\mathcal{C}_{A_n}^p), \tau)$ can be modelled using diagonals in polygons as done in [14]. When $p = 2$, the quiver $(AR(\mathcal{C}_{A_n}^2), \tau)$ can entirely be modelled using oriented diagonals in a regular $(n + 3)$ -gon. In Fig. 1 an illustration of this construction is provided for $p = 2, n = 4$.

3 Single-coloured oriented and paired diagonals in polygons

Throughout the rest of the paper let $T_{r,s,t}$ be an orientation of a finite connected graph with $r + s + t + 1$ vertices and three legs. We assume the legs of $T_{r,s,t}$ to have r , resp. s , resp. t vertices and that one vertex of $T_{r,s,t}$ has three neighbours. We say that a tree $T_{r,s,t}$ is *symmetric* if $s = t$.

Unless specified otherwise, for $n \geq \max\{r + s + 1; r + t + 1\}$ let Π be a regular $(n + 3)$ -gon with vertices numbered in the clockwise order by the group $\mathbb{Z}/(n + 3)\mathbb{Z}$.

For vertices i, j, k of Π we write $i \leq j \leq k$ if j is between i and k in the clockwise order. Moreover, we denote by (i, j) the unoriented diagonal of Π joining the vertices i and j and by $[i, j]$ the oriented diagonal of Π starting at i and ending in j . We do not consider boundary segments as oriented diagonals.

3.1 Single-coloured oriented and paired diagonals of Π

We start describing the geometric construction leading to the modelling of a number of orbit categories of $\mathcal{D}^b(\text{mod } kT_{r,s,t})$ arising from orientations of symmetric trees $T_{r,s,t}$.

To begin the construction, we double the set of oriented diagonals of Π and distinguish each set with colours using subscripts R, B e.g. $[1, 3]_R$ is the red diagonal linking the vertex 1 to 3 of Π . For every vertex i of Π we form the following $(r + 1)$ pairs of coloured oriented diagonals:

$$[i, i + 2]_P = [i + 2, i]_P := \{[i, i + 2]_R, [i + 2, i]_B\}$$

Fig. 2 The coloured oriented diagonals in bijection with the vertices of the graph $T_{2,4,1}$

$$\begin{aligned}
 [i, i+3]_P &= [i+3, i]_P := \{[i, i+3]_R, [i+3, i]_B\} \\
 &\dots \\
 [i, i+r+2]_P &= [i+r+2, i]_P := \{[i, i+r+2]_R, [i+r+2, i]_B\}.
 \end{aligned}$$

When $r = 0$, we assume that there are no paired diagonals. Moreover, let us point out that $[i, j]_P \neq [j, i]_P$.

Next we define a subset of $(r+1)(n+3)$ paired, $s(n+3)$ single red and $t(n+3)$ single blue-oriented diagonals of Π as follows:

$$\begin{aligned}
 \Pi_{r,s,t} := \bigg\{ & [i, i+2]_P, \dots, [i, i+r+2]_P, \\
 & [i, i+r+3]_R, \dots, [i, i+r+s+2]_R, \\
 & [i+r+3, i]_B, \dots, [i+r+t+2, i]_B, \quad i \text{ vertex of } \Pi \bigg\}.
 \end{aligned}$$

Once coloured oriented diagonals are paired, they stop existing as single-coloured oriented diagonals in $\Pi_{r,s,t}$.

Consider the subset of elements of $\Pi_{r,s,t}$ given by $\Pi_{r,s,t}|_1 := \{[1, 3]_P, \dots, [1, r+3]_P, [1, r+4]_R, \dots, [1, r+s+3]_R, [r+4, 1]_B, \dots, [r+t+3, 1]_B\}$. Then on the one side elements of $\Pi_{r,s,t}|_1$ are in bijection with the vertices of $T_{r,s,t}$. On the other side, elements of $\Pi_{r,s,t}|_1$ give rise to a triangulation of a region inside Π homotopic to a regular $(r+s+4)$ -gon, resp. to a $(r+t+4)$ -gon.

In Fig. 2 an illustration of this situation is provided. Coloured oriented diagonals with the same label are identified. The black vertices in the figure represent the vertices of $T_{r,s,t}$. The dotted lines are the edges of $T_{r,s,t}$. Vertices on, and edges between, identified paired diagonals give rise to one vertex, and one edge, in $T_{r,s,t}$.

Fig. 3 The symmetry ρ

3.2 The automorphisms ρ and τ

Our next aim is to define two automorphisms: ρ and τ , acting on the set of coloured oriented single and paired diagonals of Π associated to a tree $T_{r,s,t}$. The first automorphism is induced from the graph automorphism of a symmetric tree, hence only defined on $\Pi_{r,t,t}$. The definition of the second automorphism depends on the parity of the number of sides of Π .

Let $c \in \{R, B, P\}$. Then we define $\rho : \Pi_{r,t,t} \rightarrow \Pi_{r,t,t}$ as the automorphism of order two given by

$$\rho([i, j]_c) := \begin{cases} [j, i]_B & \text{if } c = R, \\ [j, i]_R & \text{if } c = B, \\ [i, j]_P & \text{otherwise.} \end{cases}$$

Moreover, we define the automorphism $\tau : \Pi_{r,s,t} \rightarrow \Pi_{r,s,t}$ as follows: if $s \neq t$, then $\tau([i, j]_c) := [i - 1, j - 1]_c$. If $s = t$, then

$$\tau([i, j]_c) := \begin{cases} \rho^{(n+3)}([i - 1, j - 1]_c) & \text{if } [i, j]_c \in \Pi_{r,t,t}|_1 \\ [i - 1, j - 1]_c & \text{otherwise.} \end{cases}$$

Geometrically, the action of τ is given by the anticlockwise rotation through $\frac{2\pi}{n+3}$ around the centre of Π , on all elements different than the diagonals in $\Pi_{r,t,t}|_1$ associated to a symmetric tree in a polygon with an odd number of sides. On the latter the rotation is followed by the simultaneous change of colour and orientation (Fig. 3).

3.3 Minimal clockwise rotations

Minimal clockwise rotations for unoriented diagonals have been introduced in [5, §2] with the aim of modelling irreducible morphisms in the cluster category $\mathcal{C}_A$. Following the spirit of [5] we now define minimal rotations between diagonals of $\Pi_{r,s,t}$.

Let k, l be non-neighbouring vertices of Π and let $c \in \{R, B, P\}$. Let $\Pi_{r,s,t}$ be the set of coloured oriented single and paired diagonals in Π associated to an asymmetric tree $T_{r,s,t}$. Then the following three operations between diagonals in $\Pi_{r,s,t}$ are called

Fig. 4 The quiver $(\Gamma_{1,1,1}^6, \tau)$ corresponds to the AR-quiver of the orbit category $\mathcal{D}^b(\text{mod } D_4)/\Sigma^2$

minimal clockwise rotation: $[k, l]_c \rightarrow [k, l + 1]_c$ and $[k, l]_c \rightarrow [k + 1, l]_c$ and

$$\begin{array}{ccc} & [k, k + r + 3]_R & \\ [k, k + r + 2]_P & \nearrow & \searrow [k + 1, k + r + 3]_P \\ & [k + r + 3, k]_B & \end{array}$$

Next, let $\Pi_{r,t,t}$ be the set of coloured oriented single and paired diagonals in Π associated to a symmetric tree $T_{r,t,t}$. Then minimal clockwise rotations are defined as before with the following adjustment: when $[k, l]_c, [k, k + r + 2]_P$ are in $\tau(\Pi_{r,t,t}|_1)$ and Π is odd sided, then we also change simultaneously the colour and the orientation.

In Figs. 4 and 5 illustrations of the three minimal rotations described above can be found.

The next remark can be used to model orbit categories arising from orientations of tree graphs $T_{r_1, r_2, \dots, r_m}$ where one vertex has m neighbours.

Remark 3.1 There are three minimal rotations linking the three types of coloured oriented single and paired diagonals of $\Pi_{r,t,t}$, namely the red, the blue and the paired ones. Starting with oriented diagonals coloured in $m - 1$ ways, one can describe the m minimal rotations linking the m types of coloured oriented diagonals with a similar diagram as above.

3.4 Quivers of single-coloured oriented and paired diagonals of $\Pi_{r,s,t}$

Let $\Gamma_{r,s,t}^{n+3}$ be the quiver whose vertices are the elements of $\Pi_{r,s,t}$. An arrow between two vertices of $\Gamma_{r,s,t}^{n+3}$ is drawn whenever there is a minimal clockwise rotation linking them. No arrow is drawn otherwise.

Concerning the shape of the quiver we remark that $\Gamma_{r,s,t}^{n+3}$ always lies on a cylinder, except when Π is odd sided, and $s = t$. Then we say that $\Gamma_{r,t,t}^{n+3}$ lies on a Möbius strip, since the τ -orbits of single-coloured oriented diagonals are twice as long as the τ -orbits of paired diagonals.

In Fig. 4 the quiver $(\Gamma_{1,1,1}^6, \tau)$ of coloured oriented diagonals in a hexagon, as well as the quiver $(\Gamma_{1,2,2}^7, \tau)$ associated to a heptagon are illustrated. In each quiver we indicate in the last slice the identifications occurring. On both quivers the action of τ is always given by a shift to the left.

Fig. 5 The quiver $(\Gamma_{1,2,2}^7, \tau)$ corresponds to the AR-quiver of the orbit category $\mathcal{D}^b(\text{mod } E_6)/\tau^{-1}\Sigma$

4 Equivalences of categories

In this section we show that the construction of $\Gamma_{r,t,t}^{n+3}$ and $\Gamma_{r,s,t}^{n+3}$ allow us to model geometrically properties of a number of additive categories.

Let ρ be the automorphism of $T_{r,t,t}$ and let $n \geq \max\{r + t + 1, r + s + 1\}$. Then we can show the following opening result.

Lemma 4.1 *The quiver $(\Gamma_{r,s,t}^{n+3}, \tau)$ is a stable translation quiver.*

Proof We have to show three things. First, that $\Gamma_{r,s,t}^{n+3}$ is connected, has no loops, and is locally finite. Second, that for every vertex v of $\Gamma_{r,s,t}^{n+3}$ the number of arrows going to v equals the number of arrows leaving v . Third, that the map τ is bijective.

All the above properties follow from the construction. Let us first consider the symmetric case. Then it is not hard to see that when Π is even sided, $\Gamma_{r,t,t}^{n+3}|_R \cong \mathbb{Z}A_{r+t+1}/\tau^{-(n+3)}$ and $\Gamma_{r,t,t}^{n+3}|_B \cong \mathbb{Z}A_{r+t+1}/\tau^{-(n+3)}$ since we consider oriented arcs. Thus, we deduce that $(\Gamma_{r,t,t}^{n+3}|_R, \tau|_R)$ and $(\Gamma_{r,t,t}^{n+3}|_B, \tau|_B)$ are stable translation quivers. Forming pairs of coloured oriented diagonals results in glueing these two quivers along r disjoint τ -orbits, and the above properties are preserved. When Π is odd sided, one can check that the modifications in the definition of minimal clockwise rotations and in the definition of τ are such that the resulting quiver has the claimed properties.

Finally, it is not hard to see that also $\Gamma_{r,s,t}^{n+3}$ associated to an asymmetric tree is a stable translation quiver. $\square$

Theorem 4.2 *Let $\mathcal{T}$ be an additively finite Krull–Schmidt category. Let Γ be a connected component of the AR-quiver of $\mathcal{T}$. Assume that $\Gamma \cong \mathbb{Z}T_{r,s,t}/\tau^{-(n+3)}$, resp. $\Gamma \cong \mathbb{Z}T_{r,t,t}/\tau^{-(n+3)}\rho$. Then there is an isomorphism of stable translation quivers $\Gamma_{r,s,t}^{n+3} \rightarrow \Gamma$, for $\Gamma_{r,s,t}^{n+3}$ associated to a regular $(n+3)$ -gon.*

Proof The claim follows from the proof of Lemma 4.1, since we saw that the following are isomorphisms of stable translation quivers: $\Gamma_{r,s,t}^{n+3} \xrightarrow{\cong} \mathbb{Z}T_{r,s,t}/\tau^{-(n+3)}$, $\Gamma_{r,t,t}^{n+3} \xrightarrow{\cong} \mathbb{Z}T_{r,t,t}/\tau^{-(n+3)}\rho^{(n+3)}$ for all integers r, s, t and n as above.

4.1 Projections

Our next goal is to define a translation quiver $\Gamma_{r,t}^{n+3}$ obtained from $\Gamma_{r,t}^{n+3}$ associated to a symmetric tree $T_{r,t,t}$ after folding $\Gamma_{r,t}^{n+3}$ along its central line.

For this consider again the graph automorphism ρ induced by the map simultaneously changing colour and orientation of the diagonals in $\Pi_{r,t,t}$. Then the vertices of $\Gamma_{r,t}^{n+3}$ are the ρ -orbits of vertices of $\Gamma_{r,t,t}^{n+3}$, i.e. the pairs $\{[i, i + j]_R, [i + j, i]_B\}$, for i, j in vertices of Π . The arrows in $\Gamma_{r,t}^{n+3}$ are always single and coincide with minimal clockwise rotation around a common vertex of Π linking pairs of coloured oriented diagonals. The translation on $\Gamma_{r,t}^{n+3}$ is induced from the translation in $\Gamma_{r,t,t}^{n+3}$ and given by the anticlockwise rotation through $\frac{2\pi}{n+3}$ around the centre of Π .

Clearly, $\Gamma_{r,t,t}^{n+3} \rightarrow \Gamma_{r,t}^{n+3}$ is a surjective map of stable translation quivers.

4.2 Cluster categories of type E_6 , E_7 and E_8

As a corollary of Theorem 4.2, we obtain the geometrical modelling of cluster categories $\mathcal{C}_{T_{r,s,t}}$ where $T_{r,s,t}$ is an arbitrary tree. Since the number of connected components of the AR-quiver of $\mathcal{C}_{T_{r,t,t}}$ varies with the shape of $T_{r,t,t}$, we proceed considering two cases. In this section we focus on tree graphs of Dynkin type, in Sect. 4.4 the general case will be treated.

In view of Theorem 4.3 below let $\mathcal{C}_{r,s,t}^{n+3}$ be the additive category generated by the mesh category of $\Gamma_{r,s,t}^{n+3}$, for $r, s, t, n \in \mathbb{N}$. Then we can show the main result of this section.

Theorem 4.3 *We have the following equivalences of additive categories*

$$\begin{aligned}\mathcal{C}_{1,2,2}^7 &\rightarrow \mathcal{C}_{E_6} \\ \mathcal{C}_{1,2,3}^{10} &\rightarrow \mathcal{C}_{E_7} \\ \mathcal{C}_{1,2,4}^{16} &\rightarrow \mathcal{C}_{E_8}.\end{aligned}$$

Proof Since the full subcategories of indecomposable objects of the orbit categories we consider are equivalent to the mesh category of their AR-quiver, we only have to check that there is an isomorphism of stable translation quivers between the AR-quiver of the various orbit categories and the quivers of coloured oriented single and paired diagonals associated to Π . This isomorphism then induces the claimed equivalences.

To do so, the strategy will be to compare the action of τ on $\Gamma_{r,s,t}^{n+3}$ with the actions of τ and of Σ on $\mathbb{Z}Q$ where Q is an orientation of a simply-laced Dynkin diagram, as described in Sect. 2.4.

Below we treat the case $\mathcal{C}_{E_7}$, the remaining two claims can be deduced with a similar reasoning. From the discussion of Sect. 2.4, it follows that the AR-quiver of $\mathcal{C}_{E_7}$ is isomorphic to the quotient graph $\mathbb{Z}E_7/\tau^{-10}$. On the other side, by definition $\mathcal{C}_{1,5,5}^{10}$ is the mesh category of $\Gamma_{1,5,5}^{10}$ associated to a 10-gon and $\Gamma_{1,5,5}^{10} \cong \mathbb{Z}E_7/(\tau^{-10}) \cong \tau^{-1}\Sigma$.

Corollary 4.4 *We also deduce the following equivalences of additive categories*

$$\begin{aligned}\mathcal{C}_{r,0,0}^{r+4} &\rightarrow \mathcal{C}_{A_{r+1}}^2 \\ \mathcal{C}_{r,s,0}^{r+s+4} &\rightarrow \mathcal{C}_{A_{r+s+1}}^2 \\ \mathcal{C}_{r,0,t}^{r+t+4} &\rightarrow \mathcal{C}_{A_{r+t+1}}^2 \\ \mathcal{C}_{r,1,1}^{r+5} &\rightarrow \mathcal{D}^b(\text{mod } D_{r+3})/\tau^{-3}\Sigma.\end{aligned}$$

Proof We follow the proof of Theorem 4.3 and observe that for the first claim we consider only paired oriented diagonals of Π . Since $[i, j]_P \neq [j, i]_P$ and the map ρ are the identity on paired oriented diagonals, we deduce that $\Gamma_{r,0,0}^{r+4}$ always lies on a cylinder. From Lemma 4.1 we deduce that $\Gamma_{r,0,0}^{r+4} \cong \mathbb{Z}A_{r+1}/\tau^{-(r+3)}$. From the discussion of Sect. 2.4 we deduce that $\mathbb{Z}A_{r+1}/\tau^{-(r+3)} \cong \mathbb{Z}A_{r+1}/\tau^{-2}\Sigma$ which we recognize as the AR-quiver of $\mathcal{D}^b(\text{mod } A_{r+1})/\tau^{-2}\Sigma^2$.

The second and third claim follow in a similar fashion.

For the last claim we observe that for each vertex of Π the quiver $\Gamma_{r,1,1}^{r+5}$ has one red and one blue single- coloured oriented diagonal. These correspond to the exceptional vertices of a Dynkin diagram of type D_{r+3} .

Remark 4.5 The equivalences of Theorem 4.3 and Corollary 4.4 allow us to define a shift functor Σ on the categories associated to Π induced by the shift functor Σ defined on the various orbit categories considered above, see also [15]. In the following however, we will not use the triangulated structure of these categories.

Let $[i, j]_c$, $c \in \{R, B, P\}$, be a coloured oriented single or paired diagonal of a heptagon Π and let (i, j) be the underlying unoriented diagonal. Then we deduce the following useful results.

Corollary 4.6 *There is a dense and full functor*

$$\begin{aligned}\mathcal{C}_{E_6} &\rightarrow \mathcal{C}_{A_4} \\ [i, j]_c &\mapsto (i, j)\end{aligned}$$

from the cluster category $\mathcal{C}_{E_6}$ of type E_6 to the cluster category $\mathcal{C}_{A_4}$ of type A_4 .

Proof Consider the projection $\pi_1 : \Gamma_{1,2,2}^7 \rightarrow \Gamma_{1,2}^7$ defined in Sect. 4.1. Let Γ be the stable translation quiver of unoriented diagonals of Π , as defined in Caldero–Chapoton–Schiffler’s paper [5]. Then there is a projection $\pi_2 : \Gamma_{1,2}^7 \rightarrow \Gamma$, which maps $\{[i, i+j]_R, [i+j, i]_B\}$ to the unoriented diagonal $(i, i+j)$ of Π and two arrows in $\Gamma_{1,2}^7$ to the one corresponding arrow in Γ . We get a surjective map of translation quivers $\pi_2 \circ \pi_1 : \Gamma_{1,2,2}^7 \rightarrow \Gamma$. This map then induces a dense and full functor

$$\mathcal{C}_{E_6} \rightarrow \mathcal{C}_{A_4},$$

after identifying $\mathcal{C}_{E_6}$, resp. $\mathcal{C}_{A_4}$, with the additive category generated by the mesh category of $\Gamma_{1,2,2}^7$, resp. Γ .

In Sect. 5 we will use the functor $\mathcal{C}_{E_6} \rightarrow \mathcal{C}_{A_4}$ to describe all cluster-tilting sets of $\mathcal{C}_{E_6}$ as configurations of coloured oriented diagonals in Π .

Corollary 4.7 *There is a dense and full functor*

$$\begin{aligned} \mathcal{C}_{r,0,0}^{r+4} &\rightarrow \mathcal{C}_{A_{r+1}} \\ [i, j]_P &\mapsto (i, j) \end{aligned}$$

from the category $\mathcal{C}_{r,0,0}^{r+4}$ to the cluster category of type A_{r+1} .

Proof Follows from the proof of Corollary 4.4. $\square$

4.3 Coloured oriented single and paired diagonals and the cluster category of type D_n

The aim of this section is to link our category of coloured oriented single and paired diagonals to the cluster category of type D_n , denoted by $\mathcal{C}_{D_n}$. We recall that $\mathcal{C}_{D_n}$ can be modelled geometrically in two equivalent ways. One approach arises categorifying the model given by Fomin–Zelevinsky in [8, Prop. 3.16] defined in terms of unoriented diameters and pairs of diagonals in a regular $2n$ -gon. The second approach uses tagged arcs in a once punctured disc, as described by Schiffler in [18]. We can recover both descriptions from our model.

For the first one, we proceed as follows. Let $(\Gamma_{n-3,n-1,n-1}^{2n}, \tau)$ be as before and define the quiver Γ_{D_n} having as vertices the (unordered) triples of centrally symmetric diagonals of Π :

$$\begin{aligned} &\{[i, i+2]_P, [i+2+n, i+n]_R, [i+n, i+2+n]_B\}, \\ &\{[i, i+3]_P, [i+3+n, i+n]_R, [i+n, i+3+n]_B\}, \\ &\dots \\ &\{[i, i+n-1]_P, [i+2n-1, i+n]_R, [i+n, i+2n-1]_B\} \end{aligned}$$

together with the oriented single diagonals $[i, n+i]_R, [n+1, i]_B$ of Π , for $1 \leq i \leq 2n$. Notice that $[i, n+i]_R$ and $[n+1, i]_B$ are the central oriented diagonals of Π . The arrows of Γ_{D_n} are induced by the minimal clockwise rotations between diagonals of $\Pi_{n-3,n-1,n-1}$, similarly for the translation map. This allows us to obtain a surjective map of stable translation quivers $\Gamma_{n-3,n-1,n-1}^{2n} \rightarrow \Gamma_{D_n}$. Dropping the orientations of all coloured diagonals and adjusting the colors of the central diagonals we also obtain the surjective map $\Gamma_{D_n} \rightarrow AR(\mathcal{C}_{D_n})$ of stable translation quivers. It follows that there is a dense and full functor

$$\mathcal{C}_{n-3,n-1,n-1}^{2n} \rightarrow \mathcal{C}_{D_n}$$

from the mesh category of $(\Gamma_{n-3,n-1,n-1}^{2n}, \tau)$, $\mathcal{C}_{n-3,n-1,n-1}^{2n}$, to the cluster category $\mathcal{C}_{D_n}$. In addition, we observe that $\Gamma_{D_n} \cong \mathbb{Z}D_n/(\tau^{-1}\Sigma)^2 \cong AR(\mathcal{C}_{D_n}^2)$, where $\mathcal{C}_{D_n}^2$ is the 2-repetitive cluster category of type D_n defined in Section 2.1.

To recover Schiffler’s model we need to use a smaller punctured polygon and allow non-contractible loops. Defining $\Gamma_{n-3,1,1}^n$ and $\mathcal{C}_{n-3,1,1}^n$ as before we obtain an isomorphism

Fig. 6 The quiver $\Gamma_{1,1,1}^4 \cong AR(\mathcal{C}_{D_4})$ in a once punctured square

of stable translation quivers $\Gamma_{n-3,1,1}^n \xrightarrow{\cong} AR(\mathcal{C}_{D_n})$ and the desired equivalence of categories: $\mathcal{C}_{n-3,1,1}^n \xrightarrow{\cong} \mathcal{C}_{D_n}$. In Figure 6 we illustrate the quiver $\Gamma_{1,1,1}^4$ in a regular once punctured square.

4.4 Cluster categories associated to trees not of Dynkin type

When $T_{r,s,t}$ is not an orientation of a simply-laced Dynkin graph the AR-quiver of the cluster category $\mathcal{C}_{T_{r,s,t}}$ splits into various connected components. Our next goal is to obtain an isomorphism between the irregular (transjective) component of the AR-quiver of $\mathcal{C}_{T_{r,s,t}}$ and the quiver of coloured oriented single and paired diagonals.

For this we extend the construction of Sect. 3.1 and consider an infinite-sided polygon Π^∞ . Next, let $\Pi_{r,s,t}^\infty$ be the subset of all coloured oriented single and paired diagonals of Π^∞ consisting of $r+1$ paired, s red and t blue single-coloured oriented diagonals for every vertex of Π^∞ :

$$\Pi_{r,s,t} := \left\{ [i, i+2]_P, \dots, [i, i+r+2]_P, \right. \\ [i, i+r+3]_R, \dots, [i, i+r+s+2]_R, \\ \left. [i+r+3, i]_B, \dots, [i+r+t+2, i]_B, i \text{ vertex of } \Pi^\infty \right\}.$$

Then we extend the definition of minimal clockwise rotations of Sect. 3.3 to this setting. Thus we obtain a quiver $\Gamma_{r,s,t}^\infty$, whose vertices are the elements of $\Pi_{r,s,t}^\infty$ and where we link two vertices with an arrow when there is a minimal clockwise rotation between them. Moreover, we define τ on the elements of $\Pi_{r,s,t}^\infty$ as the anticlockwise rotation around the centre of Π^∞ induced by the rotation through $\frac{2\pi}{n+3}$ in an $(n+3)$ -gon Π letting $n \rightarrow \infty$. In this way we turn $(\Gamma_{r,s,t}^\infty, \tau)$ into a stable translation quiver.

To state the next result, let $\mathcal{P}$ be the full subcategory of $\mathcal{C}_{T_{r,s,t}}$, consisting of τ -shifts of indecomposable projective objects in $\mathcal{C}_{T_{r,s,t}}$. Moreover, denote the additive category generated by the mesh category of $(\Gamma_{r,s,t}^\infty, \tau)$ by $\mathcal{C}_{r,s,t}^\infty$. Then we can show the following result.

Proposition 4.8 *The functor*

$$\varphi : \mathcal{C}_{r,s,t}^\infty \rightarrow \mathcal{P}$$

is an equivalence of additive categories.

Proof First we observe that $\Gamma_{r,s,t}^\infty \cong \mathbb{Z}T_{r,s,t}$.

Next, let $\Pi_{r,s,t}^\infty|_1$ be the subset of $\Pi_{r,s,t}^\infty$ consisting of the $r + s + t + 1$ single and paired diagonals: $\{[1, 3]_P, \dots, [1, r + 3]_P, [1, r + 4]_R, \dots, [1, r + s + 3]_R, [r + 4, 1]_B, \dots, [r + t + 3, 1]_B\}$. Then, we observe that since $\Gamma_{r,s,t}^\infty$ is a connected stable translation quiver φ induces an injective map between $\text{Hom}_{C_{r,s,t}^\infty}(\tau(D_j), D_i)$, for $D_i, D_j \in \Pi_{r,s,t}^\infty$ and $\text{Hom}_{C_{T_{r,s,t}}}(I_j, \Sigma(P_i))$ between the indecomposable injective objects and Σ of the projective objects in $C_{T_{r,s,t}}$. Thus φ is full. Moreover, it is not hard to see that φ is dense and faithful, thus φ is indeed an equivalence of additive categories. $\square$

5 Combinatorics of the cluster category C_{E_6}

Our next aim is to describe the combinatorics of the cluster category $C_{E_6} \cong C_{1,2,2}$ inside a heptagon.

Throughout the chapter let Π be a regular heptagon and let $(\Gamma, \tau) := (\Gamma_{1,2,2}^7, \tau)$. Moreover, let Γ_Π be the stable translation quiver having as vertices the unoriented diagonals of Π and arrows given by minimal clockwise rotations, see [5] for details.

5.1 Extension spaces in C_{E_6}

The support of $\text{Hom}(\tau^{-1}X, -)$ in C_Q is called the *front Ext-hammock of X* . The support of $\text{Hom}(-, \tau X)$ in C_Q is called the *back Ext-hammock of X* . These hammocks can be deduced from the AR-quiver using mesh relations, or using starting and ending functions, see [3, Chap. 8]. For AR-quivers isomorphic to $\mathbb{Z}Q$ with Q an orientation of a Dynkin graph, the support of $\text{Hom}(-, -)$ has been described in detail in [1].

Identify again C_{E_6} , resp. C_{A_4} with the additive categories generated by the mesh categories of Γ , resp. Γ_Π . In the sequel we view the hammocks inside Γ or Γ_Π . Moreover, let $\text{Ext}_\Pi^1(D_X, D_Y) := \text{Ext}_{C_{E_6}}^1(X, Y)$, for coloured oriented diagonals D_X, D_Y in Π and indecomposable objects X and Y in C_{E_6} corresponding to D_X and D_Y by the equivalence of Theorem 4.3.

Remark 5.1 As C_{E_6} is 2 Calabi–Yau: $D\text{Ext}_{C_{E_6}}^1(X, Y) \cong \text{Ext}_{C_{E_6}}^1(X, Y)$, for all objects X, Y in C_{E_6} . Therefore, the back and front Ext-hammocks in C_{E_6} coincide for all objects. On the other hand, in $C_{A_4}^2$ the back and front hammocks are disjoint, as the category is not 2-Calabi–Yau.

5.2 Lift of hammocks

Consider again the projection $\tilde{\pi} := \pi_2 \circ \pi_1 : \Gamma \rightarrow \Gamma_\Pi$ defined by $D_{(i,j)} := \tilde{\pi}(D_X)$ as in Corollary 4.6. For each D_X we define two connected sub-quivers of Γ , $I_1(D_X)$ and $I_2(D_X)$, as follows. The vertices of both $I_1(D_X)$ and $I_2(D_X)$, are the vertices of Γ in the Ext-hammock of D_X in Γ and in the preimage under $\tilde{\pi}$ of the Ext-hammock of $D_{(i,j)}$ in Γ_Π . The arrows of $I_1(D_X)$ and $I_2(D_X)$ coincide with the arrows of Γ .

Fig. 7 The Ext-hammocks of the diagonals with vertices of I_1 and I_2 represented in heptagons with bold boundary

Then $I_1(D_X)$ contains the vertex $\tau^{-1}(D_X)$ and will be called the *front crossing* of D_X , $I_2(D_X)$ contains $\tau(D_X)$ and will be called the *back crossing* of D_X .

Note that for all D_X , the subquivers $I_1(D_X)$ and $I_2(D_X)$ are disjoint. In addition, all coloured oriented diagonals in Π crossing D_X in an interior point of D_X are vertices of $I_1(D_X) \cup \rho(I_1(D_X))$ and $I_2(D_X) \cup \rho(I_2(D_X))$. See Fig. 7, where the vertices of inside the front and back crossings of D_X are in heptagons with bold boundary, for D_X a coloured oriented diagonal in the first slice of Γ .

In the next result, we assume that D_X is a coloured oriented diagonal of Π in the first slice of Γ . This assumption can be dropped using τ -shifts, or renumbering the vertices of Π . Moreover, we write $\partial\Pi$ to indicate the boundary of Π , and for two coloured oriented diagonals D_X and D_Y we say that D_Y *enters* the smaller region bounded by D_X and $\partial\Pi$ if the arrow head of D_Y goes to a vertex of $\partial\Pi$ inside the region and different from the vertices joined by D_X .

Proposition 5.2 *Let D_X and D_Y be coloured oriented diagonals of Π . Assume D_X is in the first slice of Γ , and that D_X crosses D_Y .*

- If D_X is a paired diagonal, then $\dim_k(\text{Ext}_\Pi^1(D_X, D_Y)) = 1$.
- If D_X is a single diagonal, and D_Y enters the smaller region bounded by D_X and $\partial\Pi$, then $\dim_k(\text{Ext}_\Pi^1(D_X, D_Y)) = 1$.

Proof If D_X is paired, $I_1(D_X)$ coincides with $I_1(D_X) \cup \rho(I_1(D_X))$ and $I_2(D_X)$ coincides with $I_2(D_X) \cup \rho(I_2(D_X))$, thus the vertices of $I_1(D_X)$ and $I_2(D_X)$ are all the oriented coloured diagonals of Π crossing D_X .

If D_X is a single-coloured diagonal of Π , we need to distinguish between the diagonals inside $I_i(D_X)$ and $\rho(I_i(D_X))$, $i = 1, 2$. Then we observe that the coloured oriented diagonals in $I_1(D_X)$ and $I_2(D_X)$ are precisely the ones satisfying the assumptions of the proposition. $\square$

5.3 Curves of oriented coloured diagonals

The aim of this section is to divide the Ext-hammocks in $\mathcal{C}_{E_6}$ into curves. The reason why we do this is because for each coloured oriented diagonal D_X we want to find a uniform geometric description of the elements inside the Ext-hammock of D_X . Since the hammocks in $\mathcal{C}_{E_6}$ are very big, this goal seems hopeless. However, dividing the Ext-hammock of D_X into smaller sets allows us to describe the elements of each such set in geometric terms. We will call these sets curves.

Let X be an indecomposable object of $\mathcal{C}_{E_6}$ and let D_X be the corresponding coloured oriented diagonal viewed as a vertex of Γ . For $r \in \{2, 4\}$, the curves $C_1(D_X), \dots, C_r(D_X)$ of D_X in Γ are r collections of oriented coloured diagonals having non-vanishing extensions with D_X . Each collection $C_i(D_X)$ has the shape of a curve in Γ .

We begin defining the curves of D_X , for D_X in the first slice of Γ . For all other vertices D_X of Γ , curves can be defined from the previous ones by τ -shifts. The first curve of $[1, 6]_R$, denoted by $C_1([1, 6]_R)$, is defined as follows:

$$\begin{aligned} C_1([1, 6]_R) := & \{[7, 2+i]_c, 0 \leq i \leq 3, c \in \{R, P\}\} \\ & \cup \{[5, 7+i]_c, 0 \leq i \leq 3, c \in \{R, P\}\} \\ & \cup \{[6, 3]_R\}. \end{aligned}$$

The second curve of $[1, 6]_R$, denoted by $C_2([1, 6]_R)$, is defined as follows:

$$\begin{aligned} C_2([1, 6]_R) := & \{[5-i, 7]_c, 0 \leq i \leq 3, c \in \{B, P\}\} \\ & \cup \{[2, 7+i]_c, 0 \leq i \leq 3, c \in \{B, P\}\} \\ & \cup \{[1, 4]_B\}. \end{aligned}$$

By definition, $C_1([1, 6]_R)$ is obtained by the sequence of minimal clockwise rotations around the vertices 7, 5, 3 of Π starting in $\tau^{-1}([1, 6]_R) = [2, 7]_R$ and ending with $[6, 3]_R$. Dually, $C_2([1, 6]_R)$ is obtained by a sequence of minimal anticlockwise rotations around the vertices 7, 2, 4 starting in $\tau([1, 6]_R) = [5, 7]_B$ and ending in $[1, 4]_B$.

By construction, the Ext-hammock of $[1, 6]_R$ is $C_1([1, 6]_R) \cup C_2([1, 6]_R)$. In Fig. 8a the elements of $C_1([1, 6]_R)$ are drawn in the upper half of Γ , while the elements of $C_2([1, 6]_R)$ are in the lower half.

Next, we are going to associate four curves to $[1, 5]_R$. The first curve of $[1, 5]_R$, $C_1([1, 5]_R)$, is the set containing coloured oriented diagonals of Π obtained by a sequence of minimal clockwise rotations around the vertices 6, 4, 2 starting in $\tau^{-1}([1, 5]_R)$ and ending in $[5, 2]_R$. The third curve $C_3([1, 5]_R)$ is obtained by a sequence of minimal anticlockwise rotations around the vertices 7, 2, 4 starting with $\tau([1, 5]_R)$ and ending in $[1, 4]_B$. Moreover, $C_2([1, 5]_R)$ coincides with $C_1([1, 6]_R)$, and $C_4([1, 5]_R)$ coincides with $C_2([1, 6]_R)$.

For $D_X \in \{[1, 6]_R, [1, 5]_R\}$, the curves of $C_1(\rho(D_X)), \dots, C_r(\rho(D_X))$ of $\rho(D_X)$ are defined by $\rho(C_1(D_X)), \dots, \rho(C_r(D_X))$, $r \in \{2, 4\}$.

Fig. 8 Decomposition into curves of the Ext-hammocks of the vertices in the first slice of Γ

We are left with defining the curves of the paired diagonals $[1, 3]_P$ and $[1, 4]_P$. For $[1, 3]_P$ we have $C_1([1, 3]_P)$ given by the set containing both single and paired coloured oriented diagonals obtained by a sequence of minimal rotations in the clockwise order around the vertices 2, 7, 5 starting in $\tau^{-1}([1, 3]_P)$ and ending in $[5, 1]_P$. Similarly $C_2([1, 3]_P)$ is obtained by a sequence of minimal anticlockwise rotations around the vertices 2, 4, 5 starting in $\tau([1, 3]_P)$ and ending in $[6, 2]_P$.

Next, $C_1([1, 4]_P)$ is obtained by a sequence of minimal clockwise rotations starting in $\tau^{-1}([1, 4]_P)$ and ending in $[5, 1]_P$. $C_3([1, 4]_P)$ is obtained rotating in the anticlockwise order $\tau([1, 4]_P)$ to $[4, 7]_P$. Finally, $C_2([1, 4]_P) = \tau^{-1}(C_1([1, 3]_P))$ and $C_4([1, 4]_P) = C_2([1, 3]_P)$.

In Fig. 8a–f, we represent the various curves of D_X , for D_X be in the first (and last) slice of Γ . The numbers 1, 2, 3, 4 indicate the starting term of the curves $C_1(D_X), \dots, C_4(D_X)$ and the colours of the heptagons indicate the curves they intersect.

5.4 Intersections of curves

Proposition 5.3 *Let D_X, D_Y be vertices of Γ . Let $C_1(D_X), \dots, C_r(D_X)$ be the curves of D_X , $r \in \{2, 4\}$. Then $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y))$ is equal to the number of curves of D_X intersecting with D_Y in Γ (0 up to 3).*

Proof First, the Ext-hammocks in the AR-quiver of $\mathcal{C}_{E_6}$ are invariant under τ -shifts. After changing the image of the projective objects of $\text{mod } kE_6$ in the equivalence of Theorem 4.3, we can assume that X or Y corresponds to a diagonal in the first slice of Γ . By Remark 5.1, we can treat the cases where D_X or D_Y belongs to the first slice of Γ in the same way. Second, the curves of D_X are by construction such that their intersection points coincide with the vertices D_Y in Γ for which $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y)) \geq 1$. From the definition of morphisms in the mesh category of a stable translation quiver, it follows that the number of curves intersecting in D_Y is the dimension of $\text{Ext}_{\mathcal{C}_{E_6}}^1(D_X, D_Y)$. $\square$

Let again D_X be in the first slice of Γ . In Fig. 8a–f, the dimension of the space $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y))$ is expressed by the numbers of colours filling the heptagon containing D_Y . If D_Y is in a white heptagon of Γ then $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y)) = 0$.

More precisely, the two curves represented in Fig. 8a, resp. f never intersect, and $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y)) = 1$ for D_Y in a coloured heptagon, and D_X in the first slice of Γ .

In Fig. 8b, resp. e, the curve $C_1(D_X)$ intersects $C_2(D_X)$ in two vertices. Similarly, for $C_3(D_X)$ and $C_4(D_X)$ the curve $C_2(D_X)$ intersects $C_4(D_X)$ only once. The five heptagons where two curves meet have two colours (boundary and interior of the heptagon). Then $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y)) = 2$ for D_Y corresponding to one of these heptagons.

In Fig. 8c, there are two heptagons where three curves meet. They are drawn with three colours, and hence $\dim_k(\text{Ext}_{\Pi}^1(D_X, D_Y)) = 3$ for D_Y corresponding to one of these two. Moreover, in nine heptagons two curves meet, and they are drawn in two colours.

Finally, in Fig. 8d two curves are drawn, and they intersect only in one vertex of Γ .

5.5 Cluster-tilting objects

Let Q be an orientation of a simply-laced Dynkin graph with n vertices. Let $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$ be a set of pairwise non isomorphic indecomposable objects of $\mathcal{C}_Q$. If $\text{Ext}_{\mathcal{C}_Q}^1(T_i, T_j) = 0$ for all $T_i, T_j \in \mathcal{T}$, then one says that $\mathcal{T}$ is a *cluster-tilting set* of $\mathcal{C}_Q$. A *cluster-tilting object* in $\mathcal{C}_Q$ is the direct sum of all objects of a cluster-tilting set in $\mathcal{C}_Q$. Observe that knowing a cluster-tilting object allows to determine a cluster-tilting set and viceversa. Moreover, given a cluster-tilting set $\mathcal{T}$, one says that $\bar{T} = \bigoplus_{j \neq i} T_j$, $T_j \in \mathcal{T}$ is an *almost complete cluster-tilting object* if there is an indecomposable object T_i^* in $\mathcal{C}_Q$ such that $\bar{T} \oplus T_i^*$ is a cluster tilting object of $\mathcal{C}_Q$. The object T_i^* is called the *complement* of T_i .

The *mutation at i* of a cluster-tilting object $\mathcal{T}$ in $\mathcal{C}_Q$, for $1 \leq i \leq n$, is the operation which replaces the indecomposable summand T_i in $\bigoplus_{j=1}^n T_j$ with the complement T_i^* of T_i in $\bar{T} = \bigoplus_{j \neq i} T_j$.

The statements in the next Theorem are shown in [3].

Theorem 5.4 *Let T be a cluster-tilting object in $\mathcal{C}_Q$.*

- *each almost complete cluster-tilting object $\bar{T}$ in $\mathcal{C}_Q$ has exactly two complements, T and T^* .*
- *If T and T^* are complements of $\bar{T}$ then $\dim_k(\text{Ext}_{\mathcal{C}_Q}^1(T, T^*)) = 1$. On the other side, if $\dim_k(\text{Ext}_{\mathcal{C}_Q}^1(T, T^*)) = 1$, then there is an almost complete cluster-tilting object $\bar{T}$ such that T and T^* are complements of $\bar{T}$.*

After Proposition 3.8 in [8] and [3, Thm. 4.5], we know that there are 833 cluster-tilting sets, hence cluster-tilting objects, in $\mathcal{C}_{E_6}$.

From the work of Caldero–Chapoton–Schiffler, see [5], we know that cluster-tilting objects of $\mathcal{C}_{A_n}$ are in bijection with the formal direct sums of diagonals belonging to a maximal collection of non-crossing diagonals in a regular $(n+3)$ -gon. In this context,

mutations correspond to *flips* of diagonals. More precisely, a flip replaces a diagonal D_i in a given triangulation Δ with the unique other diagonal D_i^* crossing D_i and completing $\Delta \setminus D_i$ to a new triangulation of the regular $(n + 3)$ -gon.

5.6 First fundamental family of cluster configurations of Π

Our next aim is to describe cluster-tilting sets of $\mathcal{C}_{E_6}$ as configurations of single and paired coloured oriented diagonals in Π .

Definition 5.5 A *cluster configuration* is a family of pairwise different coloured oriented diagonals of Π , $\mathcal{T} = \{D_1, D_2, \dots, D_6\}$, with the property that $\text{Ext}_{\Pi}^1(D_i, D_j) = 0$ for all $D_i, D_j \in \mathcal{T}$. A coloured oriented diagonal D_i^* is called *complement* of D_i in $\mathcal{T}$ if $D_i^* \neq D_i$ and $\mathcal{T}'$ obtained from $\mathcal{T}$ after replacing D_i by D_i^* is a cluster configuration of Π .

Consider two heptagons and a long paired diagonal $L_P := [i, i + 3]_P, i \in \mathbb{Z}/7\mathbb{Z}$ of Π . Our next goal is to complete L_P to a set of coloured oriented diagonals inside the two heptagons, giving rise to a cluster configuration of Π . For this we remark that L_P divides each Π into the quadrilateral Π_4 with boundary vertices $\{i, i + 1, i + 2, i + 3\}$, and the pentagon Π_5 with boundary vertices $\{i, i + 3, i + 4, i + 5, i + 6\}$, $i \in \mathbb{Z}/7\mathbb{Z}$.

In Lemma 5.6 below we will see that triangulating Π_4 with a short paired diagonal, and each Π_5 with single diagonals of the appropriate colour gives rise to cluster configurations.

Lemma 5.6 Let $L_P := [i, i + 3]_P, i \in \mathbb{Z}/7\mathbb{Z}$.

- For $i \neq 1$, triangulating each Π_5 with single diagonals of the same colour, and Π_4 with a short paired diagonal gives a cluster configuration $\mathcal{T}_{L_P}$ of Π .
- All cluster configurations of Π containing $[j, j + 3]_P$ arise as $\tau^k(\mathcal{T}_{L_P})$, $1 \leq j, k \leq 7$.

Proof Since $i \neq 1$, we can assume that the region in Γ outside the Ext-hammock of L_P has only blue diagonals below L_P and only red diagonals above L_P . Observe that the diagonals outside the Ext-hammock are precisely the diagonals involved in triangulations of the two copies of Π_5 and Π_4 .

Then choose a short paired diagonal S_P triangulating Π_4 with a short paired diagonal. Then triangulating a copy of Π_5 with only single red diagonals, and triangulating the second copy of Π_5 with only single blue diagonals yields a cluster configuration. With Proposition 5.2, we deduce that the arcs obtained in this way have no extension in each region above and below L in Γ . One can then check that the Ext-hammocks in one region do not pass through the other region, nor through S_P . Thus, to each red triangulation one can choose a blue triangulation of Π_5 , and all choices are possible. Similarly, one can complete $\{L_P, S_P^*\}$ to a cluster configuration, where S_P^* is the other short paired diagonal triangulating Π_4 . Notice that there are no other possibilities to complete L_P to a cluster configuration of Π . Next, there are 7 choices for L_P in Γ . For each choice of L_P , the associated cluster configurations are obtained from the previous by rotation through τ . Adjustment of the colour-orientations of the single diagonals triangulating Π_5 are needed if L_P is the first slice of Γ .

Notice that the two triangulations of Π_5 can be different, and the colour is uniquely determined by the position of L_P in Π , resp. in Γ .

In the following, we call the cluster configurations given by a long paired diagonal and coloured oriented single and paired diagonals triangulating two copies of Π_5 and Π_4 , as describe in the first part of Lemma 5.6, the *first fundamental family* of cluster configurations. We denote this family by $\mathcal{F}_1$.

5.7 Second fundamental family of cluster configurations of Π

We saw in Lemma 5.6 that many cluster configurations correspond to two triangulations of Π . Our next goal is to define a second family of cluster configurations describing the remaining cluster-tilting set of $\mathcal{C}_{E_6}$. For this the following general observation is needed.

For $i \in \mathbb{Z}/7\mathbb{Z}$, consider the long single red diagonal $L = [i, i + 4]_R$ of Π . Then L divides Π into the quadrilateral $\Pi_4 := \{i + 4, i + 5, i + 6, i\}$, and the pentagon $\Pi_5 := \{i, i + 1, i + 2, i + 3, i + 4\}$. Let $\mathcal{T}_L$ be a cluster configuration of Π containing L . Then $\mathcal{T}_L$ necessarily also contains one of the two short single diagonals triangulating Π_4 , neighbouring L in Γ . Similarly for $\rho(L) = [i + 4, i]_B$. More precisely,

Lemma 5.7 *Let $i \in \mathbb{Z}/7\mathbb{Z}$, $L = [i, i + 4]_R$ in Π , and $\mathcal{T}_L$ be a cluster configuration containing L .*

- *If $i \neq 1$, exactly one of $\{[i, i + 5]_R, [i + 6, i + 4]_R\}$ is in $\mathcal{T}_L$.*
- *If $i = 1$, exactly one of $\{[i, i + 5]_R, \rho([i + 6, i + 4]_R)\}$ is in $\mathcal{T}_L$.*

Similarly for $\rho(L)$. Moreover, in each case the two diagonals are complements to each other.

Proof Let $i \neq 1$ and consider the Ext-hammock of L in Γ . Since L is not in the first slice of Γ , the diagonals triangulating Π_4 have the same colour as L . Then one can check that all Ext-hammocks of objects outside the Ext-hammock of L , which are different from $[i, i - 2]_R$ and $[i - 1, i - 3]_R$, never contain single diagonals inside the quadrilateral Π_4 in Π . Thus, by maximality we deduce that all cluster-tilting sets containing L necessarily also contain one of the diagonals inside Π_4 . Taking one diagonal triangulating Π_4 rules out the other, thus the two single diagonals triangulating Π_4 are complements to each other. For $i = 1$, L is in the first slice of Γ . Then one can proceed as before adjusting the colour of the diagonal triangulating Π_4 .

In view of the next result, we point out that the short single diagonals of Lemma 5.7, triangulating Π_4 and neighbouring L in Γ , are displayed in filled light grey heptagons in Fig. 9a–n.

Lemma 5.8 *Every six-tuple of diagonals of Fig. 9a–n determines a cluster configuration of Π .*

Proof For each choice of a short single diagonal of Lemma 5.7 triangulating Π_4 and neighbouring L in Γ , the claim can be verified by checking that the diagonals in the highlighted heptagons have no extension among each other. $\square$

Fig. 9 The second fundamental family of cluster configurations of Π

In the following, we refer to the collection of cluster configurations of Fig. 9a–n as the *second fundamental family* of cluster configurations of Π , and we denote this family by $\mathcal{F}_2$.

5.8 Symmetries in Π leading to cluster configurations

We determine two symmetries in Π leading to cluster configurations. One symmetry simply switches colours and orientations of the coloured oriented diagonals of a given

Fig. 10 σ -symmetric cluster configurations of Π

cluster configuration. The second one arises from a left-right symmetry of Γ and corresponds to a reflection in Π .

There are two reasons why these symmetries are important. First, using these symmetries we can deduce all cluster configurations starting from the sets in $\mathcal{F}_2$. Second, knowing how a cluster configuration behaves under mutation, allows to understand how the symmetric ones behave.

Let $c \in \{R, B\}$. For $i \in \mathbb{Z}/7\mathbb{Z}$ let h_i be the line in Π passing through i and the middle point of $i + 3, i + 4$. On all coloured oriented diagonals different then $[i \pm 1, i \mp 1]_c$, let $\sigma_i : \Pi \rightarrow \Pi$ be the reflection in Π along h_i followed by a switch of orientation. Otherwise, $\sigma_i([i \pm 1, i \mp 1]_c) := \rho([i \pm 1, i \mp 1]_c)$.

In Fig. 10, we illustrate on the right the cluster configuration obtained after applying σ_6 to the cluster configuration represented on the left.

Lemma 5.9 *Let $\mathcal{T}$ be a cluster configuration belonging to $\mathcal{F}_2$. Then $\rho(\mathcal{T})$ and $\sigma_6(\mathcal{T})$ are also cluster configurations.*

Proof Apply the map ρ , resp. σ_6 to the cluster configuration of Lemma 5.8. Because of the shape of the Ext-hammocks one indeed produces cluster configurations.

Notice that the set $\rho(\mathcal{T})$ of a cluster configuration $\mathcal{T}$ is an elements of the τ -orbit of $\mathcal{T}$, while $\sigma_i(\mathcal{T})$ does not belong to any τ -orbit of a cluster configuration of $\mathcal{F}_2$ (nor of $\mathcal{F}_1$).

5.9 Classification of cluster-tilting sets of $\mathcal{C}_{E_6}$

From [8, Prop. 3.8] and [3, Thm. 4.5], we know that there are 833 cluster-tilting sets in $\mathcal{C}_{E_6}$. In the next result, we give a complete geometric classification of all cluster-tilting sets in $\mathcal{C}_{E_6}$ in terms of cluster configurations of Π .

Theorem 5.10 *In Π*

- 350 different cluster configurations have one long paired diagonal, and they arise from $\mathcal{F}_1$ through τ .
- 483 other cluster configurations arise from $\mathcal{F}_2$ through σ and τ .

Corollary 5.11 *In Π :*

- 224 cluster configurations have precisely one short paired diagonal,
- 175 cluster configurations have precisely two short paired diagonals,
- 84 cluster configurations have no paired diagonals of Π .

All these cluster configurations are different.

Proofs of Theorem 5.10 and Corollary 5.11 The first part of the claim follows from Lemma 5.6. In fact, we saw that for each long paired diagonal L_p there are 25 ways to triangulate one of the two pentagons Π_5 with single-coloured diagonals. Moreover, there are two ways to triangulate Π_4 with short paired diagonals. Thus, each L_p gives rise to 50 different cluster configurations. Since there are 7 choices for L_p in Π , the first claim follows.

For the second part of the claim, the idea is to consider different cases, depending on the number of short paired diagonals leading to cluster configurations. First case: the only paired diagonal of $\mathcal{T}$ is a short one. Then $\mathcal{T}$ arises from the collection highlighted in (a),(b) or (c) in Fig. 9, up to τ -shifts and the σ -symmetry of Lemma 5.9. Moreover, after Lemma 5.7 for each coloured oriented diagonal $[i, i - 3]_c, c \in \{R, B\}$ in $\mathcal{T}$ there two possible choices of neighbouring short single diagonals in $\mathcal{T}$. Consequently, up to τ -shifts, there are 4 different cluster configurations arising from a collection of diagonals as in (a). Similarly for (b). The collection in (c) gives rise to 8 different cluster configurations up to τ -shifts, as there are 4 choices for short single diagonals, and further 4 arise by taking the σ -symmetric case.

Summing up, the cluster configurations in (a), (b), (c) give rise to 224 different cluster configurations.

Second case: $\mathcal{T}$ has exactly two short paired diagonals. Then one distinguishes further into (d),(e) and (f) which have single-coloured oriented diagonals of the form $[i, i + 3]_c, c \in \{R, B\}$. While the cluster configurations in (g), (i), (h) and (j) only contain short single diagonals of the form $[i, i + 2]_c, c \in \{R, B\}$. Proceeding as before, taking into account the symmetry σ of Lemma 5.9 one obtains the claimed number.

In the third case we count the cluster configurations arising from τ -shifts of the cluster configurations with no paired diagonals in (k) and (l). As before we deduce that there are 28 the cluster configurations arising from τ -shifts of (k) and 56 arising from τ -shifts of the 4 cluster configurations in (l). Together this gives 84 cluster configurations without paired diagonals. $\square$

5.10 The geometry of cluster configurations

In Sect. 5.6, we saw that cluster configurations in $\mathcal{F}_1$ correspond to triangulations with coloured oriented diagonals of two copies of Π . Cluster configurations of $\mathcal{F}_2$ are not as simple to describe. In Theorem 5.12 below, we can give a general statement concerning the geometry of cluster configurations in Π .

Theorem 5.12 *All 833 cluster-tilting sets of $\mathcal{C}_{E_6}$ can be expressed as configurations of six non-crossing coloured oriented single and paired diagonals inside two heptagons.*

Proof First, given a cluster configurations in $\mathcal{F}_1 \cup \mathcal{F}_2$ we divide the coloured oriented diagonals inside two heptagons by colour. Paired diagonals appear in both heptagons.

Then we observe that all cluster configurations in $\mathcal{F}_1$ and $\mathcal{F}_2$ are crossing free. Moreover, the symmetry σ produces new configurations of diagonals which are again crossing free. Taking τ -shifts only rotates the entire configurations inside the two heptagons, occasionally switching colours and orientation according to the action of

τ inside Π . Hence the crossing free property is preserved under τ -shifts and the claim follows. $\square$

In Fig. 11 below some cluster configurations of Theorem 5.12 are represented.

The converse statement of Theorem 5.12 is not true, as configurations of non-crossing coloured diagonals different then cluster configurations of Π are not cluster-tilting sets of $\mathcal{C}_{E_6}$.

5.11 Mutations of cluster-tilting objects in $\mathcal{C}_{E_6}$

In this section we will see that in many cases it is possible to deduce the mutation process in $\mathcal{C}_{E_6}$ from the mutations process inside $\mathcal{C}_{A_4}$. Moreover, we can deduce the mutation process for the families in $\mathcal{F}_1$ and $\mathcal{F}_2$ and taking τ -shifts extend it for the remaining cluster configurations.

In the next definitions we indicate by $\overline{D}$ the unoriented single diagonal corresponding to a coloured oriented diagonal D of Π .

Definition 5.13 Let D_P be a paired oriented diagonal and let $\mathcal{T}$ be a cluster configuration containing D_P . The paired diagonal D_P^* is the flip-complement of D_P , if $\overline{D_P}$ and $\overline{D_P^*}$ are related by a flip.

Definition 5.14 Let D_S be a single-coloured oriented diagonal and let $\mathcal{T}$ be a cluster configuration containing D_S . The single-coloured oriented diagonal D_S^* is the flip-complement of D_S in $\mathcal{T}$, if $\overline{D_S}$ and $\overline{D_S^*}$ are related by a flip and $\mathcal{T} \setminus D_S \cup D_S^*$ is a cluster configuration.

Notice that if a flip-complement exists then the colour and orientation is uniquely determined by Lemmas 5.6 and 5.7.

Proposition 5.15 Let D_L be a long coloured oriented diagonal of Π giving rise to a cluster configuration.

- If D_L is paired: single-coloured diagonals triangulating Π_5 , and paired diagonals triangulating Π_4 have a flip-complement.
- If D_L is single: single diagonals triangulating Π_4 have a flip-complement.

Proof Let D_L be a long coloured oriented diagonal dividing Π into the quadrilateral Π_4 and the pentagon Π_5 . In Lemma 5.6 we saw that if D_L is paired, triangulating Π_4 with a paired diagonal, and two copies of Π_5 with single oriented diagonals, always gives a cluster configuration. Hence removing a single diagonal of a copy of Π_5 or a paired diagonal triangulating Π_4 can only be completed to a cluster configuration in two ways, namely with diagonals being flip-complement of each other. If D_L is single, the claim follows from Lemma 5.7. $\square$

Further instances of the mutation process in $\mathcal{C}_{E_6}$ can be described by flips of coloured oriented diagonals in Π , but not all mutations allow a description of this type. This is unsurprising, as for example not all mutations in the cluster algebra $\mathbb{C}[Gr_{3,7}]$ can be described through Plücker relations, see [19].

With Fig. 9 it is not hard to deduce all the remaining mutations occurring. For example, the cluster configuration in (a) can be mutated to the flip of a diagonal inside one light grey-coloured heptagon as well as to (b), (c), (e), (k) and (j) resp. $\sigma_6(b)$. Similarly, we have a list for the other families. Some instances of the more complicated geometric exchanges can be found on the upper pentagon of Fig. 11.

5.12 Exchange graph

In Fig. 11 we display a part of the *exchange graph* of $\mathcal{C}_{E_6}$. The vertices of the graph correspond to cluster configurations, hence to cluster-tilting sets of $\mathcal{C}_{E_6}$. Edges are drawn when two cluster configurations are related by a single mutation. In the two central heptagons of Fig. 11 the configuration of $\mathcal{T} = \{[5, 3]_R, [5, 2]_R, [5, 1]_P, [5, 6]_P, [3, 5]_B, [2, 5]_B\}$ is displayed. Paired diagonals appear in both heptagons and have labels. The 8 neighbouring configurations are placed on the vertices of the two central pentagons sharing the vertex corresponding to $\mathcal{T}$. These 8 sets are obtained from $\mathcal{T}$ through repeated flips of single diagonals, as described in Proposition 5.15. The vertices of the left pentagon are obtained after mutating $[6, 2]_P$ in $\mathcal{T}$.

5.13 Cluster-tilted algebras

Let $T = T_1 \oplus \dots \oplus T_n$ be a cluster-tilting object of $\mathcal{C}_Q$, then $\text{End}_{\mathcal{C}_Q}(T)$ is the *cluster-tilted algebra* of type Q . The quiver Q_T of $\text{End}_{\mathcal{C}_Q}(T)$ has no loops nor 2-cycles and it encodes precisely the exchange matrix of the cluster associated to T , see [4] and [6]. In $\mathcal{C}_{A_n}$ the quiver Q_T can be read off from the triangulation T , see [5]. The vertices of Q_T are the diagonals of the triangulation and an arrow between D_i and D_j is drawn, whenever D_i and D_j bound a common triangle. The orientation of the arrow is $D_i \rightarrow D_j$, if D_j is linked to D_i by an anticlockwise rotation around the common vertex.

Extending the definition of Q_T to the case at hand, the quivers corresponding to the cluster-tilting sets of Fig. 11 can be deduced. One could also read off the quivers, and relations, directly from Γ by determining the spaces $\text{Hom}_{\mathcal{C}_{E_6}}(T, T)$.

6 Applications and further directions

6.1 Symmetric cluster configurations and the cluster algebra of type F_4

Cluster categories arising from valued quivers have been first categorified algebraically in [7]. The aim of this section is to use our model for the cluster category $\mathcal{C}_{E_6}$ to categorify geometrically the cluster algebra of type F_4 , $\mathcal{A}_{F_4}$. This approach leads to a geometric description of the mutation rule and of all the clusters of $\mathcal{A}_{F_4}$.

Consider again the map ρ given by a simultaneous change of colour and orientation of coloured oriented diagonals of Π .

Definition 6.1 A cluster configuration $\mathcal{T}$ in Π is ρ -*symmetric* if $\mathcal{T} = \rho(\mathcal{T})$.

Fig. 11 Part of the exchange graph for a cluster category of type E_6 . At each vertex of the graph, the *diagonals* with the same labels are identified

Fig. 12 ρ -symmetric cluster configurations of type **L** and **C**

Since we know all cluster configurations of Π , see Theorem 5.10, we can deduce that there are only three types of ρ -symmetric cluster configurations in Π . First, cluster configurations of Π projecting to triangulations of Π consisting of unoriented arcs through $\mathcal{C}_{E_6} \rightarrow \mathcal{C}_{A_4}$. These arise from $\mathcal{F}_1$ and from τ -shifts of the configurations in Fig. 9d. Second, τ -shifts of both, the cluster configuration in Fig. 9g and the σ -symmetric configuration of Fig. 9g. Third, τ -shifts of the configuration in Fig. 9h. We call the first ρ -symmetric cluster configurations of type **T**, the second of type **C** and the third of type **L**. Moreover, we refer to the double short diagonals in a configuration of type **C** as middle diagonals.

In Fig. 12 configurations of type **L** and **C** are illustrated. The blue diagonals have opposite orientation with respect to the ones shown in the figure and are omitted. The two ρ -symmetric cluster configurations of type **C** on the right side of the figure are related through the action of σ_6 , already defined in Sect. 5.8.

Notice that for ρ -symmetric cluster configurations the quiver Q_T of $\text{End}_{\mathcal{C}_{E_6}}(T)$ is symmetric.

In the next result we show that mutations of cluster-tilting objects in $\mathcal{C}_{E_6}$ preserves the ρ -symmetry of the cluster configuration.

Let $\mathcal{T}_\rho$ be a ρ -symmetric cluster configuration of Π and denote by D^* the unique complement in $\mathcal{C}_{E_6}$ of D in $\mathcal{T}_\rho$. Then we observe that ρ -symmetric cluster configurations always have two ρ -orbits consisting of paired diagonals and two consist of single diagonals of opposite colour and orientation, thus we can assume $\mathcal{T}_\rho := \{D_{P_1}, D_{P_2}, D_{S_1}, \rho(D_{S_1}), D_{S_2}, \rho(D_{S_2})\}$. Keeping the notation as above, we have the following result.

Proposition 6.2 *Let $1 \leq k, l \leq 2$, then*

$$\mathcal{T}_\rho \setminus D_{P_k} \cup D_{P_k}^* \text{ and } \mathcal{T}_\rho \setminus \{D_{S_l}, \rho(D_{S_l})\} \cup \{D_{S_l}^*, \rho(D_{S_l}^*)\}$$

are ρ -symmetric cluster configurations of Π .

Proof The claim follows from the mutation rule of $\mathcal{C}_{E_6}$ and the symmetry of the configurations. Moreover, since there are always two paired diagonals and two ρ -orbits of single diagonals in $\mathcal{T}_\rho$ we deduce that paired diagonals are exchanged with paired diagonals, and single diagonals with single diagonals.

We call the ρ -symmetric cluster configuration obtained from $\mathcal{T}_\rho$ of Proposition 6.2 the *mutation of $\mathcal{T}_\rho$ at D_{P_k} , resp. at $\{D_{S_l}, \rho(D_{S_l})\}$, $1 \leq k, l \leq 2$* . We refer to it by $\mathcal{T}_\rho^*$.

In the next result, let $\mathcal{A}_{F_4}$ be again the cluster algebra associated to a root system of type F_4 . Then we are able to prove the claimed result.

Proposition 6.3 *There is a bijection*

$$\{\rho\text{-symmetric cluster configurations in } \Pi\} \rightarrow \{\text{clusters in } \mathcal{A}_{F_4}\}$$

compatible with mutations.

Proof Clearly there is a bijection between the ρ -orbits of coloured oriented diagonals in Γ and the 24 cluster variables of $\mathcal{A}_{F_4}$.

Proceeding as in the proof of Theorem 5.10, we deduce that there are 105 ρ -symmetric cluster configurations in Π : 84 are of type **T**, 14 of type **C** and 7 of type **L**. From [8, Prop. 3.8] we know that this number coincides with the number of clusters in $\mathcal{A}_{F_4}$.

In addition, the mutation rule in $\mathcal{C}_{E_6}$ induces a unique mutation rule for ρ -symmetric cluster configurations of Π . This can simply be seen performing the mutation case by case. The mutation in $\mathcal{C}_{E_6}$ on ρ -symmetric cluster configurations then agrees with the mutation of the cluster algebra $\mathcal{A}_{F_4}$, since the exchange graph associated to $\mathcal{A}_{F_4}$ is regular of degree 4, see [8, Thm. 1.15]. Thus there is essentially just one possible mutation. $\square$

The 84 ρ -symmetric cluster configurations of type **T** in $\mathcal{C}_{E_6}$ are in 2:1 correspondence with the triangulations of Π , thus with the cluster-tilting sets of the cluster category of type $\mathcal{C}_{A_4}$.

Geometrically the mutations of ρ -symmetric cluster configurations in Π are described by the following three moves. Let $c \in \{R, B\}$ and let i a vertex of Π . We call a triangle bounded by coloured oriented diagonals in Π *internal* if all its edges are different then boundary edges.

L-C: In a configuration of type **L** the diagonals $\{[i+3, i+1]_c, \rho([i+3, i+1]_c)\}$ exchange with $\{[i-1, i-3]_c, \rho([i-1, i-3]_c)\}$ in a configuration of type **C**. Similarly $[i+1, i+3]_P \leftrightarrow [i-3, i-1]_P$.

C-T: In a configuration of type **C** the middle diagonals $\{[i+2, i]_c, \rho([i+2, i]_c)\}$ exchange with $\{[i, i-3]_c, \rho([i, i-3]_c)\}$ bounding an internal triangle in a configuration of type **T**. Similarly $[i-2, i]_P \leftrightarrow [i, i+3]_P$.

T-T: All diagonals in a configuration of type **T** not bounding an internal triangle, are exchanged with the usual flip rule. The orientation of the new diagonal is uniquely determined by the type of diagonal one exchanges. More precisely, the orientation is such that paired diagonals are exchanges with paired diagonals, and single with single.

In Fig. 13 we illustrate a mutation between a ρ -symmetric cluster configuration of type **L** and of type **C**, as well as a mutation between a configuration of type **C** and type **T**. The diagonals in dotted lines are complements of each other and as before, paired diagonals are labelled.

Fig. 13 Mutations of ρ -symmetric cluster-tilting configurations

Fig. 14 A cluster-tilting set of $\mathcal{C}_{E_7}$

6.2 Cluster-tilting sets in $\mathcal{C}_{T_{r,s,t}}$

The construction of $\Gamma_{r,s,t}^+$ and $\Gamma_{r,s,t}^-$ of Sect. 3.4 was motivated by the following idea: glue two copies of the AR-quiver of $\mathcal{C}_{A_{r+t+1}}^2$ along two disjoint τ -orbits. This resulted in pairing diagonals (as in Sect. 3.1). In this context one can ask.

Problem 6.4 *How do categorical properties of the original category behave under this glueing operation? How do cluster-tilting sets behave under this operation?*

The cluster categories $\mathcal{C}_{E_7}$ and $\mathcal{C}_{E_8}$ have finitely many cluster-tilting set and they can be described as cluster configurations of coloured oriented single and paired diagonals inside a 10-gon, resp. a 16-gon, in a similar way as we did in Sect. 5.9. From these configurations, we can identify again a family (denoted by $\mathcal{F}_1$ previously) of cluster configurations giving rise to triangulations of regions homotopic to a heptagon and a octagon, resp. a heptagon and a nonagon, compare with Sect. 5.6 and Lemma 5.6. In Fig. 14 a cluster-tilting set of $\mathcal{C}_{E_7}$ arising from a projection of the projective modules in $\text{mod } E_7$ is represented. Similarly, in Fig. 15 a cluster-tilting set of $\mathcal{C}_{E_8}$ is represented.

When $T_{r,s,t}$ is not of Dynkin type, $\mathcal{C}_{T_{r,s,t}}$ has infinitely many cluster-tilting sets. With our geometric approach, a number of cluster-tilting sets of $\mathcal{C}_{T_{r,s,t}}$ can be expressed as configurations of coloured oriented single and paired diagonals inside the $(n+3)$ -gon Π , where $n \geq \max\{r+t+1, r+s+1\}$. Moreover, if one considers the projections $\mathbb{Z}T_{r,s,t} \rightarrow \mathbb{Z}A_{r+t+1}$ and $\mathbb{Z}T_{r,s,t} \rightarrow \mathbb{Z}A_{s+t+1}$ one can describe the Ext-hammocks in $\mathbb{Z}T_{r,s,t}$ using the Ext-hammocks in $\mathbb{Z}A_{r+t+1}$, resp. $\mathbb{Z}A_{s+t+1}$, as we did in Sect. 5.2. This enable us to express a number of cluster configurations of $\mathcal{C}_{T_{r,s,t}}$ as triangulations of a $(s+t+4)$ - and a $(r+t+4)$ -gon.

Finally, since in type A all quivers obtained through Fomin–Zelevinsky quiver mutations are known, we expect that our model can be used to understand the quivers in the mutation class of an orientation of $T_{r,s,t}$, as well as parts of the exchange graph of $\mathcal{C}_{T_{r,s,t}}$.

Fig. 15 A cluster-tilting set of $\mathcal{C}_{E_8}$

6.3 Almost positive roots

Consider the set of almost positive roots associated to a root system Φ . This set consists of all positive roots together with all negative simple roots of Φ and is denoted by $\Phi_{\geq -1}$.

For an initial choice of a triangulation of a regular polygon, an explicit bijection between all (unoriented) diagonals of the polygon and the set $\Phi_{\geq -1}$ of type A was given in [5].

From Theorem 4.3, together with [3, Prop. 4.1] it follows that there is a bijection between the vertices of $\Gamma_{r,s,t}^{\pm}$ and the set of almost positive roots $\Phi_{\geq -1}$ of the root system of type E_6 , E_7 and E_8 .

Problem 6.5 Describe geometrically the bijection between the coloured oriented single and paired diagonals in a 7-, 10-, resp. 16-gon and the almost positive roots of the root system of type E_6 , E_7 , resp. E_8 .

Acknowledgments I would like to thank Karin Baur, Giovanni Felder and Robert Marsh for all the inspiring discussions we had and for the many very helpful comments. I would also like to thank an anonymous referee for helpful suggestions. The author was partially supported by the Swiss National Science Foundation Grant Number PDFMP2127430.

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