Elementary abelian regular coverings of Platonic maps

Case I: ordinary representations

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Abstract We classify those orientably regular maps which are elementary abelian regular branched coverings of Platonic maps $\mathcal{M}$, in the case where the covering group and the rotation group G of $\mathcal{M}$ have coprime orders. The method involves studying the representations of G on certain homology groups of the sphere, punctured at the branch-points. We give a complete classification for branching over faces (or, dually, vertices) of $\mathcal{M}$, and we outline how the method extends to other branching patterns.

Keywords Orientably regular map · Platonic map · Elementary abelian covering · Automorphism group

Mathematics Subject Classification 05C10 · 20B25

1 Introduction

This paper addresses the problem of classifying orientably regular maps, those maps $\mathcal{N}$ on surfaces (always assumed to be compact and orientable) for which the orientation-preserving automorphism group $\text{Aut}^+ \mathcal{N}$ acts transitively on directed edges. Such maps are often formed as regular coverings of a simpler orientably regular map $\mathcal{M}$, in which case they correspond to normal subgroups of the fundamental group $\pi_1 \mathcal{S}$ of the underlying surface $\mathcal{S}$ of $\mathcal{M}$, punctured at any branch-points, which are invariant under the induced action of the orientation-preserving automorphism group $G := \text{Aut}^+ \mathcal{M}$ of $\mathcal{M}$ on $\pi_1 \mathcal{S}$. In the case of abelian coverings, one can abelianise $\pi_1 \mathcal{S}$, and instead look for G -invariant submodules of the first homology group $H_1(\mathcal{S}; \mathbb{Z})$: for instance,

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Surowski and Jones [13] have used this method to classify the cyclic regular coverings of the Platonic maps $\mathcal{M}$, branched over the vertices, edges or faces; this work has recently been extended by Hu [9] to include branching over vertices and faces (see also [10]). When considering coverings by elementary abelian p -groups, one can go further and reduce the homology module mod p , so that G acts on the vector space $H_1(\mathcal{S}; \mathbb{F}_p)$: this idea has been used by Kazaz [14–16] to classify the elementary abelian unbranched regular coverings of the orientably regular hypermaps of genus 2.

Here we combine these two approaches to study regular branched coverings of the Platonic maps, the orientably regular maps of genus 0, by elementary abelian p -groups, as a first step towards a more general classification of the abelian coverings of these maps (indeed, it is hoped that the techniques used here can also be applied to maps on other surfaces than the sphere). The representation theory of the corresponding Platonic groups G plays an important role in this work. It is rather easier to apply when the prime p does not divide $|G|$, as we shall generally assume here, so that Maschke's Theorem applies, and ordinary representation theory can be used. The remaining cases, where p divides $|G|$ and modular representation theory is required, will be considered in a separate paper.

The action of automorphism groups on homology has been used, either explicitly or implicitly, to study abelian coverings in other contexts. For instance, it is a classical technique for Riemann surfaces, with Farkas and Kra [7, Sect. V.3] describing the background theory and Cohen [2] applying it to Klein's quartic curve. In recent years it has also been applied, often using the language of voltage assignments, to abelian coverings of graphs. For instance Malnič et al. [19] have developed the theory and applied it to elementary abelian coverings of dipoles and of the Heawood graph, while Kwak and Oh [17] and Conder and Ma [4, 5] have respectively considered elementary abelian coverings of the octahedral graph and abelian coverings of various cubic graphs. In fact, the present work could be restated in terms of coverings of the Platonic graphs, restricted to those coverings which respect the surface-embeddings.

The main result is the following theorem, which uses the notation $\{n, m\}$ of Coxeter and Moser [6] for the m -valent Platonic map with n -gonal faces:

Theorem 1 *The orientably regular covering maps $\mathcal{N}$ of the Platonic maps $\mathcal{M}$, branched over the faces, with elementary abelian p -groups of rank $c \geq 1$ as covering groups for primes p not dividing the order of the group $G = \text{Aut}^+ \mathcal{M}$, are (up to orientation-preserving isomorphism) as follows:*

- if $\mathcal{M}$ is the tetrahedron $\{3, 3\}$ there is one map $\mathcal{N}$ for each $p > 3$, with $c = 3$;
- if $\mathcal{M}$ is the cube $\{4, 3\}$ there are three maps $\mathcal{N}$ for each $p > 3$, with $c = 2, 3$ and 5;
- if $\mathcal{M}$ is the octahedron $\{3, 4\}$ there are seven maps $\mathcal{N}$ for each $p > 3$, with $c = 1, 3, 3, 4, 4, 6$ and 7;
- if $\mathcal{M}$ is the dodecahedron $\{5, 3\}$ there are seven maps $\mathcal{N}$ for each $p \equiv \pm 1 \pmod{5}$, with $c = 3, 3, 5, 6, 8, 8$ and 11, and three when $3 < p \equiv \pm 2 \pmod{5}$, with $c = 5, 6$ and 11;

- if $\mathcal{M}$ is the icosahedron $\{3, 5\}$ there are $8p + 23$ maps $\mathcal{N}$ for each $p \equiv \pm 1 \pmod{5}$, with $c \in \{3, 4, \dots, 16, 19\}$, and $4p + 11$ when $3 < p \equiv \pm 2 \pmod{5}$, with $c \in \{4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 19\}$;
- if $\mathcal{M}$ is the dihedron $\{n, 2\}$ there is one map $\mathcal{N}$ for each p coprime to $2n$, with $c = 1$;
- if $\mathcal{M}$ is the hosohedron $\{2, n\}$ there are $2^v - 1$ maps $\mathcal{N}$ for each p coprime to $2n$, where v is the number of orbits $\Delta \neq \{1\}$ of the group generated by the Frobenius automorphism and inversion on the set of n -th roots of 1 in the algebraic closure $\overline{\mathbb{F}}_p$ of the field $\mathbb{F}_p$.

For each $\mathcal{M} = \{n, m\}$ and prime p these maps $\mathcal{N}$ have type $\{np, m\}$ and genus

$$\left(\frac{f}{2} - 1\right)p^c - \frac{f}{2}p^{c-1} + 1,$$

where f is the number of faces of $\mathcal{M}$; they are all quotients of a single covering map with $c = f - 1$. The maps $\mathcal{N}$ are all regular, except when $\mathcal{M}$ is the icosahedron, in which case there are $4(p - 1)$ or $2(p - 1)$ chiral pairs as $p \equiv \pm 1$ or $\pm 2 \pmod{5}$, the other 31 or 15 being regular. In all cases except when $\mathcal{M}$ is the icosahedron, the $2^v - 1$ maps $\mathcal{N}$ are the joins formed from a set of v irreducible coverings of $\mathcal{M}$.

Comments

(1) The orientation-preserving automorphism group $\text{Aut}^+ \mathcal{N}$ of each of these maps is an extension of the covering group, an elementary abelian normal subgroup K of order p^c , by G . Since G and K have coprime orders, the Schur–Zassenhaus Theorem implies that this extension splits. As a G -module under conjugation, K is isomorphic to a quotient of $H_1(S; \mathbb{F}_p)$, identified specifically in Sects. 3–9 for the individual maps $\mathcal{M}$.

(2) An orientably regular map is called regular or chiral as it is or is not isomorphic to its mirror image. When $\mathcal{M}$ is the icosahedron, the chiral covering maps arise from the fact that its full automorphism group $\text{Aut } \mathcal{M}$ has fewer orbits than G on ordered pairs of faces of $\mathcal{M}$ (see Sects. 2.6 and 7). The fact that in this case the number of coverings is unbounded as a function of p is related to the fact that the permutation character of G on faces of $\mathcal{M}$ is not multiplicity-free (see Sects. 2.5 and 7).

(3) In the case of the hosohedron, the method for finding the ranks c of the covering groups K , too complicated to state here, is described in Sect. 9.2.

(4) The conclusions of Theorem 1 also apply in a few cases where p divides $|G|$, such as the tetrahedron and octahedron with $p = 3$, and the hosohedron with n odd and $p = 2$; these cases will be studied in a later paper.

The paper is organised as follows. In Sect. 2 some useful techniques are outlined. These are applied in Sects. 3–9 to enumerate and describe the coverings in Theorem 1. The individual cases are fairly straightforward, except for that involving the hosohedron $\mathcal{M} = \{n, 2\}$, which depends on the reduction mod p of cyclotomic polynomials. Branching over vertices is easily dealt with by duality. In Sect. 10 we consider

simultaneous branching over vertices and faces, concentrating on the tetrahedron as a typical example. In Sects. 11 and 12 we briefly consider the hypermaps arising from branching over edges, and over vertices, edges and faces. The results for these different branching patterns are similar to those in Theorem 1.

2 Techniques

In this section, several techniques from algebraic map theory, topology and representation theory are used to reduce the classification problem for maps to a number of mainly group- and number-theoretic calculations. No originality is claimed for the results quoted, which are all standard in their areas. To help the reader through what is a rather long narrative, some important observations, often used implicitly in later calculations, are italicised.

2.1 Maps

First, we briefly sketch the connections between maps and triangle groups; for background, see [11]. Maps are always assumed to be oriented, that is, on orientable surfaces with a chosen orientation. The aim is to classify them up to orientation-preserving isomorphism.

We define a triangle group to be a group of the form

$$\Delta(p, q, r) = \langle x, y, z \mid x^p = y^q = z^r = xyz = 1 \rangle$$

where $p, q, r \in \mathbb{N} \cup \{\infty\}$ and we ignore any relation $g^\infty = 1$. Any m -valent map $\mathcal{N}$ corresponds to a subgroup N of the triangle group

$$\Delta := \Delta(m, 2, \infty) = \langle x, y, z \mid x^m = y^2 = xyz = 1 \rangle,$$

with vertices, edges and faces corresponding to the cycles of x , y and z on the cosets of N .

The map $\mathcal{N}$ is orientably regular if and only if N is normal in Δ , in which case the orientation-preserving automorphism group $\text{Aut}^+ \mathcal{N}$ is isomorphic to Δ/N ; we will assume this throughout.

In particular, the Platonic map $\mathcal{M} = \{n, m\}$ corresponds to the normal closure M of z^n in Δ , and

$$\text{Aut}^+ \mathcal{M} \cong \Delta/M \cong \Delta(m, 2, n).$$

A map $\mathcal{N}$ is a d -sheeted covering of $\mathcal{M}$ if and only if N is a subgroup of index d in M , in which case $\mathcal{N} \rightarrow \mathcal{M}$ is a regular covering and the group of covering transformations is M/N .

Thus the orientably regular m -valent maps which are d -sheeted abelian coverings of $\mathcal{M}$ are (up to orientation-preserving isomorphism) in bijective correspondence with the subgroups N of M which are normal in Δ , with M/N abelian of order d .

If M/N has exponent e , such subgroups N contain the commutator subgroup M' and the subgroup M^e generated by the e -th powers in M , so they correspond to subgroups $\bar{N} = N/M'M^e$ of $\bar{M} = M/M'M^e$, with $M/N \cong \bar{M}/\bar{N}$. The action of Δ by conjugation on the normal subgroup M preserves its characteristic subgroups M' and M^e , giving an induced action of Δ on $\bar{M}$; since M acts trivially on $\bar{M}$, we have an action of the group

$$G := \text{Aut}^+ \mathcal{M} \cong \Delta/M$$

on $\bar{M}$, which is a module for G over the ring $\mathbb{Z}_e := \mathbb{Z}/e\mathbb{Z}$.

It follows that a subgroup N of M , containing $M'M^e$, is normal in Δ if and only if $\bar{N}$ is a G -submodule of $\bar{M}$.

The orientation-preserving automorphism group $\tilde{G} := \text{Aut}^+ \mathcal{N} \cong \Delta/N$ of $\mathcal{N}$ has an abelian normal subgroup K , corresponding to M/N , with $\tilde{G}/K \cong G$; under the induced action of G on K by conjugation, K is isomorphic as a G -module to $\bar{M}/\bar{N}$. If e is coprime to $|G|$, then the Schur–Zassenhaus Theorem implies that $\tilde{G}$ is a semidirect product of K by G .

2.2 Homology modules

Let $S = S^2 \setminus \{c_1, \dots, c_f\}$, where $c_1, \dots, c_f$ are the centres of the f faces of $\mathcal{M}$. Then M can be identified with the fundamental group $\pi_1(S)$ of S , a free group of rank $f - 1$ generated by the homotopy classes g_i of loops around the punctures c_i , with a single defining relation $g_1 \dots g_f = 1$. It follows that the group $M^{\text{ab}} = M/M'$ can be identified with the first integer homology group $H_1(S; \mathbb{Z}) = \pi_1(S)^{\text{ab}}$ of S , a free abelian group of rank $f - 1$ generated by the homology classes $[g_i]$, with defining relation $[g_1] + \dots + [g_f] = 0$. By the Universal Coefficient Theorem, $\bar{M} = M/M'M^e$ is identified with the mod e homology group $H_1(S; \mathbb{Z}_e) = H_1(S; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_e \cong \mathbb{Z}_e^{f-1}$.

Under these identifications, the actions of G induced by conjugation in Δ and by homeomorphisms of S are the same, so our problem is to find the G -submodules of $H_1(S; \mathbb{Z}_e)$.

The prime power factorisation $\prod_p p^{i_p}$ (p prime, $i_p \geq 1$) of e induces a G -invariant decomposition of $H_1(S; \mathbb{Z}_e)$ as a direct sum of its Sylow p -subgroups $H_1(S; \mathbb{Z}_{p^{i_p}})$, so it is sufficient to restrict attention to the case where e is a power of a prime p . For each prime p we have an infinite descending series

$$M^{\text{ab}} = M/M' > M'M^p/M' > M'M^{p^2}/M' > \dots > M'M^{p^i}/M' > \dots,$$

of characteristic subgroups of M^{ab} , corresponding to a descending series

$$H_1(S; \mathbb{Z}) = H^0 > H^1 > \dots > H^i > \dots,$$

of G -submodules of $H_1(S; \mathbb{Z})$, where each H^i is the kernel of the reduction $(\text{mod } p^i) : H_1(S; \mathbb{Z}) \rightarrow H_1(S; \mathbb{Z}_{p^i})$. This induces a finite descending series

$$H_1(S; \mathbb{Z}_{p^i}) = H^0/H^i > H^1/H^i > \dots > H^i/H^i = 0$$

of G -submodules of $H_1(S; \mathbb{Z}_{p^i})$. Successive quotients

$$(H^j/H^i)/(H^{j+1}/H^i) \cong H^j/H^{j+1} \cong M'M^{p^j}/M'M^{p^{j+1}}$$

in this series are G -isomorphic to the module

$$H_1(S; \mathbb{F}_p) = H_1(S; \mathbb{Z}_p) = H^0/H^1 \cong M_p := M/M'M^p,$$

where $\mathbb{F}_p$ is the field of p elements. This G -module $H_1(S; \mathbb{F}_p)$, along with all the other quotients in the series, is a vector space over $\mathbb{F}_p$ of dimension $f - 1$ affording a representation $\rho_p : G \rightarrow \text{Aut}(H_1(S; \mathbb{F}_p)) \cong \text{GL}_{f-1}(\mathbb{F}_p)$ of G .

Our main problem therefore is to determine the structure of this G -module $H_1(S; \mathbb{F}_p)$ for each prime p ; in particular, this will immediately determine the elementary abelian regular coverings of $\mathcal{M}$.

A G -submodule L of $H_1(S; \mathbb{F}_p)$ of codimension c corresponds to a normal subgroup N of Δ contained in M , with the group $K := M/N$ elementary abelian of order p^c , and hence to a regular p^c -sheeted covering $\mathcal{N}$ of $\mathcal{M}$, branched over its face-centres, with covering group K . The image in K of a generator g_i of M in K is called a *monodromy generator* of K , representing the permutation of the sheets of the covering induced by traversing a small loop in S around the corresponding branch-point.

This covering $\mathcal{N}$ is an orientably regular map of type $\{np, m\}$. There are p^{c-1} points above each of the f face-centres of $\mathcal{M}$, so the total order of branching of the covering $\mathcal{N} \rightarrow \mathcal{M}$ is $(p^c - p^{c-1})f$. By the Riemann-Hurwitz formula, $\mathcal{N}$ therefore has genus

$$g = 1 - p^c + \frac{1}{2}p^{c-1}(p-1)f = \left(\frac{f}{2} - 1\right)p^c - \frac{f}{2}p^{c-1} + 1.$$

The orientation-preserving automorphism group $\tilde{G} = \text{Aut}^+\mathcal{N}$ of $\mathcal{N}$ has an elementary abelian normal subgroup K of order p^c , isomorphic to $H_1(S; \mathbb{F}_p)/L$ as a G -module over $\mathbb{F}_p$.

For notational convenience, we let $E_p(\mathcal{M})$ denote the set of (orientation-preserving isomorphism classes of) such regular covers $\mathcal{N}$ of $\mathcal{M}$ with elementary abelian p -groups as covering groups, and for each $\mathcal{N} \in E_p(\mathcal{M})$ we let $\dim(\mathcal{N}) = c$ where the covering $\mathcal{N} \rightarrow \mathcal{M}$ has p^c sheets. Thus $\dim(\mathcal{N})$ is simply the dimension of the corresponding G -module $H_1(S; \mathbb{F}_p)/L$, or equivalently the codimension of L in $H_1(S; \mathbb{F}_p)$. We will denote the representation of G on $H_1(S; \mathbb{F}_p)/L$ and the corresponding character by $\rho_{\mathcal{N}}$ and $\chi_{\mathcal{N}}$, and we will say that they are both *afforded* by $\mathcal{N}$; this is because $H_1(S; \mathbb{F}_p)/L$ is isomorphic, as a G -module, to M/N and hence to K .

The representation $\rho_{\mathcal{N}}$ can therefore be regarded as the induced action of G , by conjugation, on the covering group K associated with $\mathcal{N}$.

If p is coprime to $|G|$ then the representation theory of G over fields of characteristic p is essentially the same as that over fields of characteristic 0 (ordinary representation theory): for instance Maschke's Theorem applies in both cases. In this situation, the decomposition of $H_1(S; \mathbb{F}_p)$ as a G -module can be obtained from that of the corresponding homology module over $\mathbb{C}$ (or indeed any algebraically closed field of characteristic 0, such as the field $\overline{\mathbb{Q}}$ of algebraic numbers). Moreover, if p is coprime to $|G|$ then $\tilde{G}$ is a semidirect product of K by G , by the Schur–Zassenhaus Theorem.

If, on the other hand, p divides $|G|$ then this extension splits if and only if its Sylow p -subgroups split over K . Moreover, in this situation, Maschke's Theorem does not apply, and the submodule structure of $H_1(S; \mathbb{F}_p)$ is less transparent, requiring modular rather than ordinary representation theory. In this paper we therefore concentrate on the ordinary case, avoiding the finitely many primes p dividing $|G|$; they will be considered in a separate paper.

2.3 Permutation modules

Let Φ be the set of faces of $\mathcal{M}$, and let $\mathbb{C}\Phi$ be the corresponding permutation module for G ; this is an f -dimensional complex vector space with basis Φ permuted naturally by G . The homology module $H_1(S; \mathbb{C})$ is isomorphic to the quotient of $\mathbb{C}\Phi$ by the 1-dimensional G -submodule spanned by the element $\sum_{\phi \in \Phi} \phi$.

The G -module decomposition of $\mathbb{C}\Phi$ corresponds to that of the corresponding permutation character π , where $\pi(g)$ is the number of faces $\phi \in \Phi$ invariant under each $g \in G$.

If H denotes the subgroup $\langle z \rangle \cong C_n$ of G leaving invariant a face, then since G acts transitively on Φ we have $\pi = 1_H^G$, the character obtained by inducing the principal character 1_H of H up to G . By Frobenius reciprocity, the multiplicity (χ_i, π) of any irreducible complex character χ_i of G in π is given by

$$(\chi_i, \pi) = (\chi_i|_H, 1_H) = |H|^{-1} \sum_{g \in H} \chi_i(g),$$

the multiplicity of 1_H in the restriction of χ_i to H .

Equivalently, the multiplicity (χ_i, π) of an irreducible character χ_i of G in π is the multiplicity of 1 as an eigenvalue of z in the representation ρ_i of G corresponding to χ_i .

Given these multiplicities, we know the direct sum decomposition of $\mathbb{C}\Phi$, and hence that of $H_1(S; \mathbb{C})$. In particular, this module affords the character

$$\chi = \pi - \chi_1,$$

where χ_1 is the principal character 1_G of G , given by $\chi_1(g) = 1$ for all $g \in G$.

The same decompositions apply if $\mathbb{C}$ is replaced with the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ for any prime p not dividing $|G|$. To find the corresponding decompositions over $\mathbb{F}_p$, various algebraically conjugate summands must be merged to give summands realised

over $\mathbb{F}_p$. For the various Platonic maps $\mathcal{M}$, how this happens depends on whether p satisfies certain congruences.

2.4 Finding submodules

Some submodules L of $H_1(S; \mathbb{F}_p)$ arise naturally from the action of G on Φ , whether or not p divides $|G|$. Let P denote the permutation module $\mathbb{F}_p\Phi$, a G -module over $\mathbb{F}_p$. Given any subset $\Psi \subseteq \Phi$, let $\underline{\Psi}$ denote the element $\sum_{\phi \in \Psi} \phi$ of P . The G -module $H_1(S; \mathbb{F}_p)$ can then be identified with the quotient $Q := P/P_1$ of P by the 1-dimensional submodule P_1 spanned by $\underline{\Phi}$.

2.4.1 The submodule $L = 0$ of Q

There is a regular covering $\mathcal{M}_0 \in E_p(\mathcal{M})$, with $\dim(\mathcal{M}_0) = f - 1$, corresponding to the submodule $L = 0$ of $Q = H_1(S; \mathbb{F}_p)$, i.e. to the submodule P_1 of P ; all other regular coverings $\mathcal{N}$ of $\mathcal{M}$ in $E_p(\mathcal{M})$ are proper quotients of $\mathcal{M}_0$, so they satisfy $\dim(\mathcal{N}) \leq f - 2$.

2.4.2 The submodule $L = Q^1$ of Q

There is a G -submodule P^1 of codimension 1 in P , consisting of the elements $\sum_{\phi \in \Phi} a_\phi \phi$ with coordinate-sum $\sum_{\phi \in \Phi} a_\phi = 0$. If p divides f then $P_1 \leq P^1$, giving a G -submodule $L = Q^1 = P^1/P_1$ of codimension 1 in Q , and hence a regular covering $\mathcal{M}^1 \in E_p(\mathcal{M})$ with $c = \dim(\mathcal{M}^1) = 1$. This is therefore a cyclic covering of $\mathcal{M}$ (meaning that the covering group K is cyclic), and since $L/Q^1 \cong P/P^1$ affords the principal representation of G it is also a central covering, in the sense that K is in the centre of $\tilde{G}$. The cyclic coverings of the Platonic hypermaps were classified by the author and Surowski in [13] (see also [20]), where Proposition 3 implies that $\tilde{G}$ is a split extension of G by K , giving $\tilde{G} = G \times K \cong G \times C_p$, if and only if p divides $2m$ but not n . In the cases where p does not divide f we have $P = P_1 \oplus P^1$, so $Q \cong P^1$ and hence P^1 does not correspond to a proper covering of $\mathcal{M}$.

2.4.3 The submodules $L = Q_\sim$ of Q

If G acts imprimitively on Φ , preserving a non-trivial equivalence relation $\sim$ (equivalently, if there is a subgroup $H_\sim$ of G such that $H < H_\sim < G$), then as Ψ ranges over the $k = |G : H_\sim|$ equivalence classes, the elements $\underline{\Psi}$ form a basis for a k -dimensional G -submodule $P_\sim$ of P , containing P_1 (and contained in P^1 if and only if p divides the size $|H_\sim : H| = f/k$ of each class). The G -submodule $L = Q_\sim = P_\sim/P_1$ of Q therefore corresponds to a regular covering $\mathcal{M}_\sim \in E_p(\mathcal{M})$ of dimension $\dim(\mathcal{M}_\sim) = f - k$, affording the character $\chi_{\mathcal{M}_\sim} = \pi - \pi_\sim$ of G where $\pi_\sim$ is the permutation character of G on the equivalence classes of $\sim$, i.e. on the cosets of $H_\sim$ in G . If $\approx$ is a G -invariant equivalence relation which refines $\sim$ (i.e. $\phi_1 \approx \phi_2$ implies $\phi_1 \sim \phi_2$), then the inclusion $Q_\sim \leq Q_\approx$ induces a covering $\mathcal{M}_\sim \rightarrow \mathcal{M}_\approx$.

For instance, suppose that $\mathcal{M}$ has an antipodal automorphism, so that f is even. Then P has a G -submodule $P_a \geq P_1$ of dimension $k = f/2$ with basis elements $\underline{\psi}$ corresponding to the antipodal pairs $\Psi \subseteq \Phi$. This gives a G -submodule $L = Q_a = P_a/P_1$ of codimension $f - k = f/2$ in Q , so we obtain a regular covering $\mathcal{M}_a \in E_p(\mathcal{M})$ with $\dim(\mathcal{M}_a) = f/2$. If $p > 2$ there is a second G -submodule $P_{a'}$ of dimension $f/2$ in P , with a basis element $\phi - \phi'$ for each antipodal pair $\{\phi, \phi'\} \subseteq \Phi$ (so $P_{a'} \leq P^1$). The G -submodule $L = Q_{a'} = (P_{a'} \oplus P_1)/P_1$ of codimension $(f - 2)/2$ in Q corresponds to a regular covering $\mathcal{M}_{a'} \in E_p(\mathcal{M})$ with $\dim(\mathcal{M}_{a'}) = (f - 2)/2$. A simple calculation shows that $P = P_a \oplus P_{a'}$, so $Q = Q_a \oplus Q_{a'}$. Since P_a affords the permutation character π_a of G on the antipodal pairs of faces, it follows that Q_a and hence $Q/Q_{a'}$ afford $\pi_a - \chi_1$, while $Q_{a'}$ and Q/Q_a afford $\chi - \pi_a$.

2.4.4 The case $p = 2$

If $p = 2$ one can identify P with the power set $\mathcal{P}(\Phi)$ of Φ , a group under symmetric difference, with the natural induced action of G , so that $P^1 = \{\Psi \subseteq \Phi \mid |\Psi| \text{ is even}\}$ and $P_1 = \{\emptyset, \Phi\}$. In this case, Q can be identified with the set of complementary pairs $\{\Psi, \Phi \setminus \Psi\}$ of subsets of Φ .

2.4.5 The natural representation ρ_n

For each $\mathcal{M}$, the group G has a natural 3-dimensional complex representation ρ_n , obtained by extending its natural real representation, as the rotation group of $\mathcal{M}$, by linearity to $\mathbb{C}$. This representation is irreducible, except when $\mathcal{M}$ is the dihedron or hosohedron, in which case it splits as a direct sum of 1- and 2-dimensional representations. When $\mathcal{M}$ is one of the five Platonic solids, by sending each basis element $\underline{\phi}$ ($\phi \in \Phi$) of $\mathbb{C}\Phi$ to the position-vector in $\mathbb{R}^3 \subset \mathbb{C}^3$ of the face-centre of ϕ , we see that ρ_n is a quotient, and hence a direct summand, of the representation of G on $\mathbb{C}\Phi$. Frobenius reciprocity implies that ρ_n has multiplicity 1 in this representation, and in the cases of the dodecahedron and icosahedron, the same applies to the Galois conjugate representation ρ_{n*} . By realising ρ_n over the ring of integers of an appropriate algebraic number field (namely $\mathbb{Q}$ for the tetrahedron, cube or octahedron, $\mathbb{Q}(\sqrt{5})$ for the dodecahedron or icosahedron, and a cyclotomic field $\mathbb{Q}(\zeta_n)$ for the dihedron or hosohedron, where $\zeta_n = e^{2\pi i/n}$), and taking quotients modulo some prime ideal containing p , we find that the mod p reduction of ρ_n or of $\rho_n \oplus \rho_{n*}$ is a summand of the representation of G on P , and hence on Q .

2.5 Diagonal submodules

We need to deal with those cases where Q has summands of multiplicity greater than one. If A_1 and A_2 are isomorphic FG -modules for some field F and group G , then a *diagonal submodule* of $A_1 \oplus A_2$ is a submodule of the form $\{(a, a') \mid a \in A_1\}$ for some isomorphism $A_1 \rightarrow A_2, a \mapsto a'$.

Lemma 1 *Let U be a submodule of a module $V = V_1 \oplus V_2$, let $U_i = U \cap V_i$, and let $\overline{V}_i = V_i/U_i$ for $i = 1, 2$. Then U is the inverse image in V of a diagonal submodule of $\overline{W}_1 \oplus \overline{W}_2$ for some submodules W_i such that $U_i \leq W_i \leq V_i$ for $i = 1, 2$.*

Proof The image of U in $\overline{V}_1 \oplus \overline{V}_2$ intersects each summand $\overline{V}_i$ trivially, so it is isomorphic to its projection $\overline{W}_i$ in each $\overline{V}_i$, and is therefore a diagonal submodule of $\overline{W}_1 \oplus \overline{W}_2$. $\square$

Thus, if the submodule structures of each V_i are known, so is that of V .

Lemma 2 *If V_1 and V_2 are isomorphic irreducible FG -modules, then the only proper submodules of $V_1 \oplus V_2$ are 0 , V_1 , V_2 and the diagonal submodules; given any isomorphism $V_1 \rightarrow V_2$, $v \mapsto v'$, each of the latter has the form $\{(v, \lambda v') \mid v \in V_1\}$ for some $\lambda \in F^*$.*

Proof Let U be a submodule of $V = V_1 \oplus V_2$. In the notation of Lemma 1, for each $i = 1, 2$ the irreducibility of V_i implies that $U_i = 0$ or V_i . If $U_i = V_i$ for some i then $U \geq V_i$, so the irreducibility of V/V_i implies that $U = V_i$ or V . Otherwise, $U_1 = U_2 = 0$, so in Lemma 1 we have either $W_i = 0$ for some i , giving $U = 0$, or $W_i = V_i$ for each i , in which case U is a diagonal submodule of V . In the latter case, Schur's Lemma implies that the isomorphisms $V_1 \rightarrow V_2$ are the functions $v \mapsto \lambda v'$ for $\lambda \in F^*$. $\square$

This shows that if $V_1 = V_2$ is irreducible and $F = \mathbb{F}_q$, then $V = V_1 \oplus V_2 = V_1 \oplus V_1$ has $q + 1$ non-trivial proper submodules, all isomorphic to V_1 . We can write them as

$$V(\lambda) = \{(\alpha v, \beta v) \mid v \in V_1, \alpha/\beta = \lambda\}$$

for $\lambda \in \mathbb{P}^1(\mathbb{F}_q) = \mathbb{F}_q \cup \{\infty\}$; here $V(\infty)$ and $V(0)$ are the first and second direct summands, with $\beta = 0$ and $\alpha = 0$, respectively.

2.6 Regularity and chirality

Each of the covering maps $\mathcal{N}$ constructed here is orientably regular, since it corresponds to a normal subgroup N of Δ . It is regular (i.e. it also admits an orientation-reversing automorphism) if and only if N is normal in the extended triangle group $\Gamma = \Delta[m, 2, \infty]$, which contains Δ as a subgroup of index 2. This is equivalent to the corresponding submodule L of $H_1(S; \mathbb{Z}_e)$ being invariant, not just under the orientation-preserving automorphism group $G = \text{Aut}^+ \mathcal{M} \cong \Delta/N$ of $\mathcal{M}$, but under its full automorphism group $A = \text{Aut} \mathcal{M} \cong \Gamma/N$. When $e = p$, it is clear from their construction that the maps $\mathcal{M}_0, \mathcal{M}^1, \mathcal{M}_a$ and $\mathcal{M}_{a'}$ (where they exist) are all regular.

One can find the structure of $H_1(S; \mathbb{F}_p)$ as an A -module by treating it as the quotient $Q = P/P_1$ of P , where P is now regarded as the permutation module for A on Φ .

In particular, the permutation character $\pi|_A$ of A on P can be found as for $\pi|_G$ (see Sect. 2.3), taking H to be the stabiliser $A_\phi \cong D_n$ of a face $\phi \in \Phi$ in A instead of its stabiliser $G_\phi \cong C_n$ in G .

Lemma 3 *If $\mathcal{M}$ is not the icosahedron, and if p is a prime not dividing $|G|$, then all the maps $\mathcal{N} \in E_p(\mathcal{M})$ are regular.*

Proof It is sufficient to show that the G -submodules of $H_1(S; \mathbb{C})$ are all A -invariant, i.e. that the A - and G -module direct sum decompositions of $H_1(S; \mathbb{C})$ are identical. If this is not the case, then at least one irreducible summand of this A -module must split into a direct sum of two or more irreducible summands of the G -module, so that the permutation character π on Φ satisfies $(\pi|_A, \pi|_A)_A < (\pi|_G, \pi|_G)_G$. But the two sides of this inequality are equal to the ranks of A and G as permutation groups on Φ , i.e. the number of orbits of A and G on Φ^2 , or equivalently the number of orbits of their face-stabilisers A_ϕ and G_ϕ on Φ . Now $A_\phi \cong D_n$ and $G_\phi \cong C_n$, and for all $\mathcal{M}$ except the icosahedron their orbits on Φ are either trivial (fixing ϕ and its antipodal face, if it exists), or natural orbits of length n . Thus A_ϕ and G_ϕ have the same number of orbits on Φ , so A and G have the same rank, as required. $\square$

This argument fails for the icosahedron, with A and G having ranks 6 and 8 on Φ , and A_ϕ having two regular orbits, of length $2n = 6$; in Sect. 7 we will give explicit examples of chiral maps in $E_p(\mathcal{M})$ in this case.

2.7 Maximal submodules and irreducible covers

If, as we generally assume here, p does not divide $|G|$, then Maschke's Theorem implies that $H_1(S; \mathbb{F}_p)$ is a direct sum of irreducible G -submodules.

It follows that each map $\mathcal{N} \in E_p(\mathcal{M})$ is the join (minimal common cover) of a finite set of irreducible covering maps $\mathcal{N}_i \in E_p(\mathcal{M})$.

Here we define any $\mathcal{N}_i \in E_p(\mathcal{M})$ to be an *irreducible* covering map if the corresponding submodule L_i of $H_1(S; \mathbb{F}_p)$ is maximal, or equivalently the covering group $K_i \cong H_1(S; \mathbb{F}_p)/L_i \cong \overline{N}_i$ is an irreducible G -module, so that $\mathcal{N}_i$ does not cover any other map in $E_p(\mathcal{M})$. The covering group K corresponding to a map $\mathcal{N} \in E_p(\mathcal{M})$ is then the direct sum of the covering groups K_i corresponding to a set of irreducible covering maps $\mathcal{N}_i$. In order to understand the covering maps $\mathcal{N} \in E_p(\mathcal{M})$ of a Platonic map $\mathcal{M}$ it is therefore sufficient to understand its irreducible covering maps, so we will pay particular attention to these when dealing with the individual maps $\mathcal{M}$.

3 The tetrahedron

Let $\mathcal{M}$ be the tetrahedral map $\{3, 3\}$, with $f = 4$ faces and $G = \text{Aut}^+ \mathcal{M} \cong A_4$. The group A_4 has four conjugacy classes, consisting of the identity, the three double transpositions, and two mutually inverse classes of four 3-cycles. Its character table is shown in Table 1; here χ_1 is the principle character, χ_2 and χ_3 are the faithful characters of $A_4/V_4 \cong C_3$ where V_4 is the normal Klein four-group, and χ_4 is the character of the natural representation ρ_n as the rotation group of the tetrahedron, extended from $\mathbb{R}^3$ to $\mathbb{C}^3$.

Since G acts doubly transitively on Φ , as the natural representation of A_4 , it follows that $\pi = \chi_1 + \chi_4$, so $H_1(S; \mathbb{C})$ is an irreducible G -module affording the character χ_4 .

Table 1 The character table of A_4

	1	$(..)(..)$	$(...)^+$	$(...)^-$
χ_1	1	1	1	1
χ_2	1	1	ω	$\bar{\omega}$
χ_3	1	1	$\bar{\omega}$	ω
χ_4	3	-1	0	0

The primes dividing $|G|$ are 2 and 3, so for primes $p > 3$ we can regard Table 1 as the character table of A_4 over $\mathbb{F}_p$ by suitable reduction mod p , with $\omega^2 + \omega + 1 = 0$. In particular, the G -module $Q = H_1(S; \mathbb{F}_p)$ is irreducible, affording the character $\chi = \chi_4$ of G . It follows that in this case $E_p(\mathcal{M})$ consists of a single regular map $\mathcal{M}_0$ (see Sect. 2.4.1), with $\dim(\mathcal{M}_0) = 3$. (If $p = 3$ then Q is again irreducible, and the result is the same.) The map $\mathcal{M}_0$ has type $\{3p, 3\}$ and genus $p^3 - 2p^2 + 1$. Its orientation-preserving automorphism group $\text{Aut}^+ \mathcal{M}_0$ has an elementary abelian normal subgroup $K \cong (C_p)^3$; the quotient group, isomorphic to G , has an induced action by conjugation on K , so that K is a G -module over $\mathbb{F}_p$, isomorphic to Q . The full automorphism group $\text{Aut} \mathcal{M}_0$ is an extension of K by $A = \text{Aut} \mathcal{M} \cong S_4$, with $K \cong Q = P/P_1$ where we now regard P as the natural permutation module for S_4 ; the character of S_4 on K is that denoted by χ_4 in Table 2 (see Sect. 4). When $p = 3$ or 5 the map $\mathcal{M}_0$ has genus 10 or 76, and appears as the dual of the map R10.1 or R76.2 in Conder's computer-generated list of regular orientable maps [3].

Here it is useful to mention a related result, obtained by similar methods, namely the classification by Conder and Ma [4] of the arc-transitive cubic graphs which arise as regular abelian coverings of the tetrahedral graph K_4 (among several other cubic graphs).

4 The cube

Next we consider coverings of the cube $\mathcal{M} = \{4, 3\}$ branched over its face-centres (or, dually, of the octahedron $\{3, 4\}$, branched over its vertices). In this case $G = \Delta(3, 2, 4) \cong S_4$, acting on the set Φ of six faces of $\mathcal{M}$, and the stabilisers of faces are the three subgroups $H \cong C_4$ of G .

In S_4 there are five conjugacy classes: the identity, six transpositions, three double transpositions, eight 3-cycles, and six 4-cycles. Hence there are five irreducible characters, shown in Table 2. In addition to the principal character χ_1 and the alternating character $\chi_2(g) = \text{sgn}(g)$, χ_3 and χ_4 are the non-principal irreducible characters obtained from the doubly transitive representations of S_4 of degrees 3 (via the epimorphism $S_4 \rightarrow S_3$) and 4 (the natural action), while χ_5 is the character of the natural representation ρ_n of G as the rotation group of the cube or octahedron.

The subgroup H consists of the identity element, one double transposition and two 4-cycles. By averaging their values over H , one therefore sees that the characters χ_i have multiplicities 1, 0, 1, 0 and 1 in the permutation character π on P , that is,

$$\pi = \chi_1 + \chi_3 + \chi_5.$$

Table 2 The character table of S_4

	1	(..)	(..)(..)	(...)	(...)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

It follows that the representation of G on $H_1(S; \mathbb{C})$ has character

$$\chi = \chi_3 + \chi_5,$$

so $H_1(S; \mathbb{C})$ is a direct sum of two irreducible submodules of dimensions 2 and 3.

The primes dividing $|G|$ are 2 and 3, so we can use this decomposition to find the decomposition of $H_1(S; \mathbb{F}_p)$ for any prime $p > 3$. In this case $H_1(S; \mathbb{F}_p)$ is also a direct sum of two irreducible submodules of dimensions 2 and 3, so $E_p(\mathcal{M})$ consists of three covering maps $\mathcal{N}$, with $\dim(\mathcal{N}) = 2, 3$ and 5: the first two are irreducible covers of $\mathcal{M}$, and the third is their join. Since the cube has an antipodal symmetry, with the $f = 6$ faces permuted in three antipodal pairs, the first two coverings $\mathcal{N}$ are the maps $\mathcal{M}_{a'}$ and $\mathcal{M}_a$ corresponding to the summands $L = Q_{a'}$ and Q_a in the G -module decomposition $Q = Q_a \oplus Q_{a'}$ (see Sect. 2.4.3), while the third is $\mathcal{M}_0$, corresponding to $L = 0$ (see Sect. 2.4.1). These are regular maps of type $\{4p, 3\}$ and genus $2p^c - 3p^{c-1} + 1$ where $c = \dim(\mathcal{N}) = 2, 3$ or 5, affording the character χ_3, χ_5 or $\chi_3 + \chi_5$ respectively; for instance, if $p = 5$ or 7 then the maps $\mathcal{M}_{a'}$ are the duals of the regular maps R36.3 and R78.1 in [3]. The inclusions $0 < Q_{a'}$ and $0 < Q_a$ of G -submodules induce regular coverings $\mathcal{M}_0 \rightarrow \mathcal{M}_{a'}$ and $\mathcal{M}_0 \rightarrow \mathcal{M}_a$, with covering groups $Q_{a'}$ and Q_a .

Using similar methods, Conder and Ma [4] have classified the arc-transitive graphs which are abelian covers of the 3-cube graph Q_3 , the 1-skeleton of $\mathcal{M}$.

5 The octahedron

We now consider coverings of the octahedron $\mathcal{M} = \{3, 4\}$ branched over its face-centres (or, dually, of the cube $\{4, 3\}$, branched over its vertices). As in the case of the cube we have $G \cong S_4$, but now acting on the set Φ of eight faces of $\mathcal{M}$, so the stabilisers of faces are the four Sylow 3-subgroups $H \cong C_3$ of G . The conjugacy classes and character table of G are as before, but now H consists of the identity and two 3-cycles. We find that

$$\pi = \chi_1 + \chi_2 + \chi_4 + \chi_5,$$

so

$$\chi = \chi_2 + \chi_4 + \chi_5.$$

Thus $H_1(S; \mathbb{C})$ is a direct sum of three irreducible G -submodules of dimensions 1, 3 and 3. For primes $p > 3$ we therefore obtain seven coverings $\mathcal{N} \in E_p(\mathcal{M})$ with $c =$

$\dim(\mathcal{N}) = 1, 3, 3, 4, 4, 6$ and 7 , corresponding to the seven proper G -submodules L of $\mathcal{Q}$; the first three are irreducible covers, and the other four are their joins, two or three at a time.

These are regular maps of type $\{3p, 4\}$ and genus $g = 3p^c - 4p^{c-1} + 1$. When $c = 1$ we have a regular map $\mathcal{N}$ of genus $3(p - 1)$, affording the character χ_2 of G . The Riemann surfaces underlying these cyclic covers are examples of the well-known surfaces with $8(g + 3)$ automorphisms discovered by Accola [1] and Maclachlan [18]; for instance, taking $p = 5, 7, 11, 13, 17, 19, 23, 29$ and 31 we obtain the duals of the regular maps R12.1, R18.1, R30.1, R36.5, R48.1, R54.1, R66.3, R84.1 and R90.1 in [3]. When $c = 3$ or $c = 4$ we obtain a pair of p^c -sheeted coverings $\mathcal{N}$ of $\mathcal{M}$; not only are they non-isomorphic, but their automorphism groups are non-isomorphic since the corresponding pairs of normal subgroups K of $\tilde{G}$ are non-isomorphic as G -modules, affording characters χ_4 and χ_5 when $c = 3$, or $\chi_2 + \chi_4$ and $\chi_2 + \chi_5$ when $c = 4$. The remaining coverings, with $c = 6$ and 7 , afford the characters $\chi_4 + \chi_5$ and $\chi_2 + \chi_4 + \chi_5$.

These G -submodules $L \leq \mathcal{Q}$, and the corresponding maps $\mathcal{N} \in E_p(\mathcal{M})$, all have combinatorial interpretations. The octahedron has an antipodal symmetry, so for each prime p there is a G -submodule $\mathcal{Q}_a$ affording the character $\pi_a - \chi_1 (= \chi_4$ if $p > 3$), giving a regular map $\mathcal{N} = M_a \in E_p(\mathcal{M})$ with $\dim(\mathcal{N}) = 4$ and $K \cong \mathcal{Q}/\mathcal{Q}_a$ affording $\chi - \pi_a + \chi_1 (= \chi_2 + \chi_5$ if $p > 3$). If $p > 2$ there is a G -invariant complement $\mathcal{Q}_{a'}$ for $\mathcal{Q}_a$, giving a regular map $\mathcal{N} = M_{a'} \in E_p(\mathcal{M})$ with $\dim(\mathcal{N}) = 3$ and $K \cong \mathcal{Q}/\mathcal{Q}_{a'}$ affording $\pi_a - \chi_1 (= \chi_4$ if $p > 3$).

Since the octahedron is 2-face-colourable (i.e. its dual, the cube, embeds a bipartite graph), there is a 2-dimensional G -submodule $P_b > P_1$ spanned by the sums of the two monochrome subsets $\Psi \subset \Phi$, and a 5-dimensional G -submodule $P_a + P_b < P^1$; these lead, via G -submodules $\mathcal{Q}_b = P_b/P_1$ and $\mathcal{Q}_a \oplus \mathcal{Q}_b$ of $\mathcal{Q}$, to coverings $\mathcal{M}_b$ and $\mathcal{M}_{a,b}$ in $E_p(\mathcal{M})$ of dimensions $c = 6$ and 3 respectively. If $p > 3$ then since P_b affords the character $\chi_1 + \chi_2$ of G , these two coverings afford $\chi_4 + \chi_5$ and χ_5 respectively. There is also a 7-dimensional G -submodule $P_{b'} < P$ consisting of the elements $\sum_{\phi \in \Phi} a_\phi \phi$ such that $\sum_{\phi \in \Psi} a_\phi = \sum_{\phi \in \Psi'} a_\phi$ where Ψ and Ψ' are the monochrome subsets; it contains P_1 , so its image $\mathcal{Q}_{b'}$ in $\mathcal{Q}$ is a 6-dimensional G -submodule leading to a cyclic covering $\mathcal{M}_{b'}$ affording χ_2 . Finally, $\mathcal{Q}_{a'} \cap \mathcal{Q}_{b'}$ is a 3-dimensional G -submodule of $\mathcal{Q}$ (the reduction mod p of the natural representation ρ_n of G) giving a 4-dimensional covering $\mathcal{M}_{a',b'}$ affording $\chi_2 + \chi_4$. Together with $\mathcal{M}_0$ this accounts for all seven maps in $E_p(\mathcal{M})$ when $p > 3$.

The direct sum decomposition $\mathcal{Q} = \mathcal{Q}_b \oplus \mathcal{Q}_a \oplus (\mathcal{Q}_{a'} \cap \mathcal{Q}_{b'})$, with direct factors affording the irreducible representations ρ_2, ρ_4 and ρ_5 corresponding to the characters χ_2, χ_4 and χ_5 , is also valid when $p = 3$. (The character χ_3 , which reduces mod 3 to $\chi_1 + \chi_2$, is not a summand of χ ; the remaining four characters in Table 2 correspond to irreducible characters over $\mathbb{F}_3$.) Thus $E_3(\mathcal{M})$ consists of the seven regular maps described above.

The dimensions $c = 1, 3, 4, 6$ and 7 have also been obtained by Kwak and Oh [17] in their classification of arc-transitive elementary abelian coverings of the octahedral graph: they find $p + 4$ such covering for each prime $p > 2$ (and 14 when $p = 2$). Of course, these include the 1-skeletons of the maps in $E_p(\mathcal{M})$.

Table 3 The character table of A_5

	1	(..)(..)	(...)	(.....) ⁺	(.....) ⁻
χ_1	1	1	1	1	1
χ_2	3	-1	0	λ	μ
χ_3	3	-1	0	μ	λ
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

6 The dodecahedron

We now consider coverings of the dodecahedron $\mathcal{M} = \{5, 3\}$ branched over its face-centres (or, dually, of the icosahedron $\{3, 5\}$, branched over its vertices). In this case $G = \Delta(3, 2, 5) \cong A_5$, acting on the set Φ of twelve faces of $\mathcal{M}$, so the stabilisers of faces are the Sylow 5-subgroups $H \cong C_5$ of G . The maps $\mathcal{N} \in E_p(\mathcal{M})$ have type $\{5p, 3\}$ and genus $5p^c - 6p^{c-1} + 1$ where $c = \dim(\mathcal{N})$.

There are five conjugacy classes in A_5 , the identity, fifteen double transpositions, twenty 3-cycles, and two classes of twelve 5-cycles, so there are five irreducible characters. The character table is as shown in Table 3, with $\lambda, \mu = (1 \pm \sqrt{5})/2$.

In addition to the principal character χ_1 , there are algebraically conjugate characters χ_2 and χ_3 obtained from the natural representation ρ_n of G as a rotation group, while the irreducible characters χ_4 and χ_5 are the non-principal summands of the permutation characters corresponding to the doubly transitive natural permutations representations of G as A_5 and as $\text{PSL}_2(5)$.

The subgroup H consists of the identity element, and two elements each from the two conjugacy classes of 5-cycles. By averaging their values over H , one sees that the characters χ_i have multiplicities 1, 1, 1, 0 and 1 in the permutation character π on P , that is,

$$\pi = \chi_1 + \chi_2 + \chi_3 + \chi_5.$$

Thus

$$\chi = \chi_2 + \chi_3 + \chi_5,$$

so $H_1(S; \mathbb{C})$ is a direct sum of three irreducible submodules of dimensions 3, 3 and 5.

The primes dividing $|G|$ are 2, 3 and 5, so we can use this decomposition to find the decomposition of $H_1(S; \mathbb{F}_p)$ for any prime $p > 5$. If $p \equiv \pm 1 \pmod{5}$, so that 5 has a square root in $\mathbb{F}_p$, then $H_1(S; \mathbb{F}_p)$ is also a direct sum of three irreducible submodules of dimensions 3, 3 and 5; in this case we obtain seven covering maps $\mathcal{N} \in E_p(\mathcal{M})$, of dimensions $c = 3, 3, 5, 6, 8, 8$ and 11, with the first three irreducible. If $p \equiv \pm 2 \pmod{5}$, on the other hand, 5 has no square root in $\mathbb{F}_p$; in this case $H_1(S; \mathbb{F}_p)$ is a direct sum of irreducible submodules of dimensions 5 and 6, and there are three covering maps $\mathcal{N} \in E_p(\mathcal{M})$, with $c = 5, 6$ and 11; the first two are irreducible. In either case, the maps in $E_p(\mathcal{M})$ are all regular; those with $c = 5$ and 6 are $\mathcal{M}_{a'}$

and $\mathcal{M}_a$, arising from the antipodal symmetry of the dodecahedron, while $\mathcal{M}_0$ has $c = 11$; these coverings afford the characters χ_5 , $\chi_2 + \chi_3$ and χ respectively. When $p \equiv \pm 1 \pmod{5}$ there are two 3-dimensional submodules Q_n and Q_{n*} , obtained from the natural representation ρ_n and its algebraic conjugate ρ_{n*} ; these lead to maps $\mathcal{M}_n$ and $\mathcal{M}_{n*}$ in $E_p(\mathcal{M})$ with $c = 8$, affording $\chi_2 + \chi_5$ and $\chi_3 + \chi_5$. Finally, in this case there are also two 8-dimensional submodules $Q_a \oplus Q_n$ and $Q_a \oplus Q_{n*}$, leading to irreducible 3-dimensional coverings $\mathcal{M}_{a,n}$ and $\mathcal{M}_{a,n*}$ affording χ_3 and χ_2 .

The direct sum decomposition $Q = Q_a \oplus Q_{a'}$, with direct factors affording the characters χ_5 and $\chi_2 + \chi_3$, is in fact also valid when $p = 3$ or 5 . Over $\mathbb{F}_3$, the character χ_5 splits as $\chi_1 + \chi_4$, while $\chi_2 + \chi_3$ is irreducible (but splits into two algebraically conjugate characters over $\mathbb{F}_9$).

7 The icosahedron

We now consider coverings of the icosahedron $\mathcal{M} = \{3, 5\}$ branched over its face-centres (or, dually, of the dodecahedron $\{5, 3\}$, branched over its vertices). The maps $\mathcal{N} \in E_p(\mathcal{M})$ have type $\{3p, 5\}$ and genus $9p^c - 10p^{c-1} + 1$, where $c = \dim(\mathcal{N})$.

As in the case of the dodecahedron we have $G \cong A_5$, but now acting on the set Φ of twenty faces of $\mathcal{M}$, so the stabilisers of faces are the ten Sylow 3-subgroups $H \cong C_3$ of G . The conjugacy classes and character table of G are as before, but now H consists of the identity and two 3-cycles. In this case we find that

$$\pi = \chi_1 + \chi_2 + \chi_3 + 2\chi_4 + \chi_5,$$

so

$$\chi = \chi_2 + \chi_3 + 2\chi_4 + \chi_5.$$

Thus $H_1(S; \mathbb{C})$ is a direct sum of irreducible submodules of dimensions 3, 3, 4 (with multiplicity 2) and 5. In contrast with the previous examples, the existence of a summand with multiplicity greater than 1 implies that the direct sum decompositions of $\mathbb{C}\Phi$ and of $H_1(S; \mathbb{C})$ are not unique: in each case the 8-dimensional submodule affording the character $2\chi_4$ contains infinitely many mutually isomorphic 4-dimensional irreducible submodules affording χ_4 (see Sect. 2.5).

As in the case of the dodecahedron, we can use this to obtain the decomposition of $H_1(S; \mathbb{F}_p)$ for primes $p > 5$, but first we need to consider this lack of uniqueness of 4-dimensional submodules.

There is a chiral pair of G -invariant equivalence relations $\sim_b$ and $\sim_c$ on Φ , each with five classes of four faces, corresponding to the inclusion of H in two subgroups of G isomorphic to A_4 . (These are the well-known five-colourings of the faces of $\mathcal{M}$, often used to demonstrate the isomorphism of G with A_5 .) Their equivalence classes Ψ provide basis elements $\underline{\psi}$ for a pair of 5-dimensional submodules $P_b, P_c > P_1$ of P , and hence a pair of 4-dimensional submodules Q_b and Q_c of Q . There is an isomorphism $Q_b \rightarrow Q_c$, $q \mapsto q'$, induced by the antipodal automorphism i of $\mathcal{M}$, and these submodules each afford the character χ_4 which has multiplicity 2 in the

decomposition of χ . They generate an 8-dimensional submodule $Q_{b,c} = Q_b \oplus Q_c$ which contains $p - 1$ other submodules isomorphic to them, each of the form $Q(\lambda) = \{q + \lambda q' \mid q \in Q_b\}$ for some $\lambda \in \mathbb{F}_p^*$ (see Sect. 2.5). Now i transposes $\sim_b$ and $\sim_c$, and hence also transposes $Q(0) := Q_b$ and $Q(\infty) := Q_c$, so it permutes these $p + 1$ submodules $Q(\lambda)$ ($\lambda \in \mathbb{P}^1(p) = \mathbb{F}_p \cup \{\infty\}$), leaving $Q(1)$ and $Q(-1)$ (if $p > 2$) invariant and otherwise transposing pairs $Q(\lambda^{\pm 1})$. Since $A = G \times \langle i \rangle$, the G -submodules $Q(\pm 1)$ are actually A -submodules, affording the irreducible characters $\chi_4 \otimes \varphi_{\pm 1}$ of A , where φ_1 and φ_{-1} are the principal and non-principal characters of $\langle i \rangle \cong C_2$.

If $p \equiv \pm 1 \pmod{5}$ then $H_1(S; \mathbb{F}_p)$ is a direct sum of irreducible submodules of the same dimensions and multiplicities as those for $H_1(S; \mathbb{C})$. The two irreducible submodules Q_n and Q_{n*} of dimension 3 are obtained from the natural representation of G , those of dimension 4 are the submodules $Q(\lambda) < Q_{b,c}$ defined above, and there is also one irreducible submodule Q_d of dimension 5. (This is the image in Q of the 6-dimensional submodule of P spanned by the elements $\underline{\psi}$, where ψ consists of the ten faces of $\mathcal{M}$ incident with an antipodal pair of vertices; these six elements are permuted by G as it permutes the diagonals of $\mathcal{M}$, or equivalently as $\text{PSL}_2(5)$ permutes $\mathbb{P}^1(5)$.) There are eight submodules intersecting $Q_{b,c}$ trivially, namely the various sums of Q_n , Q_{n*} and Q_d , with codimensions $c = 8, 11, 11, 13, 14, 16, 16$ and 19 ; by taking the direct sum of one of these with a submodule $Q_{b,c}$ or $Q(\lambda)$ we obtain eight more submodules with codimensions $0, 3, 3, 5, 6, 8, 8$ and 11 , or $8(p + 1)$ with codimensions $4, 7, 7, 9, 10, 12, 12$ and 15 . Excluding the submodule Q of codimension 0, this gives a total of $8p + 23$ proper submodules, and hence the same number of covering maps $\mathcal{N} \in E_p(\mathcal{M})$, with $\dim(\mathcal{N}) = 3, 4, \dots, 16$ and 19 . The irreducible coverings are the two, $p + 1$ and one of dimensions $3, 4$ and 5 . Of the maps in $E_p(\mathcal{N})$, $8(p - 1)$ are chiral, namely those corresponding to submodules containing a single submodule $Q(\lambda)$, with $\lambda \neq \pm 1$, while the remaining $8 + 16 + 7 = 31$ are regular.

If $3 < p \equiv \pm 2 \pmod{5}$ then $H_1(S; \mathbb{F}_p)$ can be decomposed as the direct sum of four irreducible submodules: two are isomorphic 4-dimensional submodules $Q(\lambda) < Q_{b,c}$, affording χ_4 , and the other two are Q_d of dimension 5, affording χ_5 , and $Q_e := Q_n \oplus Q_{n*}$ of dimension 6, affording $\chi_2 + \chi_3$. There are four submodules intersecting $Q_{b,c}$ trivially, namely $Q_d \oplus Q_e$, Q_e , Q_d and 0 with codimensions $8, 13, 14$ and 19 ; by taking the direct sums of these with a submodule $Q_{b,c}$ or $Q(\lambda)$ we obtain four more submodules with codimensions $0, 5, 6$ and 11 , or $4(p + 1)$ with codimensions $4, 9, 10$ and 15 . Excluding Q as before, we obtain $4p + 11$ proper submodules, and the same number of covering maps $\mathcal{N} \in E_p(\mathcal{M})$, with $\dim(\mathcal{N}) = 4, 5, 6, 8, 9, 10, 11, 13, 14, 15$ and 19 . There are $p + 1$ irreducible coverings of dimension 4, and one each of dimensions 5 and 6. There are $4(p - 1)$ chiral maps in $E_p(\mathcal{M})$, corresponding to submodules containing a single $Q(\lambda)$, with $\lambda \neq \pm 1$, while the other 15 maps are regular.

In either case, the covering maps $\mathcal{N} = \mathcal{M}_a$ and $\mathcal{M}_{a'}$ obtained from the antipodal symmetry of $\mathcal{M}$ arise as above from the submodules $Q_a = Q_d \oplus Q(1)$ and $Q_{a'} = Q_e \oplus Q(-1)$.

8 The dihedron

If $\mathcal{M}$ is the dihedron $\{n, 2\}$, then

$$G = \Delta(2, 2, n) = \langle x, y, z \mid x^2 = y^2 = z^n = xyz = 1 \rangle.$$

This can be identified with the dihedral group

$$D_n = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle,$$

where $x = b$ and $y = ab$ are rotations through π , fixing a vertex and edge-centre respectively and transposing the two faces, and $z = a$ is a rotation through $2\pi/n$, preserving the faces, so that $H = \langle a \rangle \cong C_n$. Then $H_1(S; \mathbb{Z})$ is an infinite cyclic group, with a acting on it as the identity and b inverting every element. Thus

$$\pi = \chi_1 + \chi_2$$

and hence

$$\chi = \chi_2,$$

where χ_1 is the principal character of G and $\chi_2(g) = (-1)^k$ for each $g = a^j b^k \in G$. It follows that for each integer $e \geq 1$ there is a single regular covering $\mathcal{N}$ of exponent e , with a cyclic covering group, corresponding to the unique subgroup of index e in $H_1(S; \mathbb{Z})$. This map is the dihedron $\{en, 2\}$, a regular map of genus 0 with $G \cong D_{en}$ and $A = G \times \langle c \rangle \cong D_{en} \times C_2$, where c is the reflection fixing all the vertices and edges of $\mathcal{N}$ and transposing its two faces.

9 The hosohedron

If $\mathcal{M}$ is the hosohedron $\{2, n\}$, then for each prime p the maps $\mathcal{N} \in E_p(\mathcal{M})$ are all regular, of type $\{2p, n\}$ and genus

$$1 + \frac{p^{c-1}}{2}(n(p-1) - 2p),$$

where $c = \dim(\mathcal{N})$. As in the case of the dihedron, G is isomorphic to D_n , but now acting naturally with degree n on Φ , so that the subgroup stabilising a face is $H = \langle b \rangle \cong C_2$. This makes the analysis of the coverings more complicated than before.

9.1 Conjugacy classes and characters of D_n

First suppose that n is odd. Then apart from the conjugacy class $\{1\}$, there are $(n-1)/2$ conjugacy classes $\{a^{\pm j}\}$ for $j = 1, 2, \dots, (n-1)/2$, and a single conjugacy class $\{a^j b \mid j = 0, 1, \dots, n-1\}$ consisting of the involutions in G .

Table 4 The character table of D_n for odd n

	1	$a^{\pm j}$	$a^j b$
χ_1	1	1	1
χ_2	1	1	-1
ξ_k	2	α_{jk}	0

Table 5 The character table of D_n for even n

	1	$a^{\pm j}$	$a^j b$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	1	$(-1)^j$	$(-1)^j$
χ_4	1	$(-1)^j$	$(-1)^{j+1}$
ξ_k	2	α_{jk}	0

In addition to the characters χ_1 and χ_2 of degree 1 defined in the preceding section, G has $(n-1)/2$ irreducible characters ξ_k of degree 2, for $k = 1, 2, \dots, (n-1)/2$, where $\xi_k(a^{\pm j}) = \alpha_{jk} := \zeta_n^{jk} + \zeta_n^{-jk}$ with $\zeta_n = \exp(2\pi i/n)$. The character table of G is as in Table 4, where the column headed $a^{\pm j}$ and the row labelled ξ_k represent $(n-1)/2$ conjugacy classes and characters respectively, as j and k each range over $\{1, 2, \dots, (n-1)/2\}$. For each k , the kernel of the representation ρ_k corresponding to ξ_k is the unique subgroup $\langle a^m \rangle$ of order $\gcd(k, n)$ in $G_0 := \langle a \rangle$, where $m = n/\gcd(k, n) \geq 3$, so ρ_k can be regarded as a faithful representation of $G/\langle a^m \rangle \cong D_m$.

By computing the inner product of each irreducible character with π we find that

$$\pi = \chi_1 + \sum_{k=1}^{(n-1)/2} \xi_k$$

and hence

$$\chi = \sum_{k=1}^{(n-1)/2} \xi_k.$$

If n is even the conjugacy classes of G are $\{1\}$, $\{a^{n/2}\}$, $\{a^{\pm j}\}$ for $j = 1, 2, \dots, (n-2)/2$, $\{a^j b \mid j \text{ is even}\}$ and $\{a^j b \mid j \text{ is odd}\}$. In addition to χ_1 and χ_2 , defined as above for n odd, there are two more characters χ_3 and χ_4 of degree 1, together with irreducible characters ξ_k of degree 2 for $k = 1, 2, \dots, (n-2)/2$. The character table is as in Table 5, with the columns headed $a^{\pm j}$ and $a^j b$ representing $1 + (n-2)/2 = n/2$ and 2 conjugacy classes respectively, and the row labelled ξ_k representing $(n-2)/2$ characters. We can take $H = \langle b \rangle$, so that

$$\pi = \chi_1 + \chi_3 + \sum_{k=1}^{(n-2)/2} \xi_k$$

and hence

$$\chi = \chi_3 + \sum_{k=1}^{(n-2)/2} \xi_k.$$

9.2 Submodule structure

Instead of considering the reduction mod p of the characters appearing in π , as in the preceding cases, it is more convenient to use the fact that the group $G = D_n$ has a subgroup $G_0 = \langle a \rangle$ acting regularly on Φ . This allows a more explicit description of the G -submodules of P and Q , and hence of the maps $\mathcal{N} \in E_p(\mathcal{M})$. We will assume that p is coprime to n . As in Sect. 2, certain key observations are italicised in the preliminary explanation of this method.

In order to find the G -module decompositions of $P = \mathbb{F}_p\Phi$, and hence of $H_1(S; \mathbb{F}_p) \cong Q = P/P_1$, we first consider them as G_0 -modules. Since G_0 acts regularly on Φ , one can identify P with the group algebra $\mathbb{F}_p G_0$ of G_0 over $\mathbb{F}_p$, or equivalently with the polynomial algebra $\mathbb{F}_p[x]/(x^n - 1)$, so that the elements $a^j \in G_0$ correspond to the images of the powers x^j of x .

The G_0 -submodules are then the ideals of $\mathbb{F}_p[x]/(x^n - 1)$, each generated by the image of the ideal in $\mathbb{F}_p[x]$ generated by a polynomial dividing $x^n - 1$.

To find these polynomials we need to determine the irreducible factors of $x^n - 1$ in $\mathbb{F}_p[x]$. In $\mathbb{Z}[x]$ we have a factorisation

$$x^n - 1 = \prod_{m|n} \Phi_m(x), \quad (1)$$

where $\Phi_m(x)$ is the m -th cyclotomic polynomial, the minimal polynomial of the primitive m -th roots of 1 in $\mathbb{C}$. In fact, equation (1), together with the equation $\Phi_1(x) = x - 1$, can be regarded as a recursive definition of $\Phi_n(x)$ in terms of the polynomials $\Phi_m(x)$ for the proper divisors m of n .

Each cyclotomic polynomial is irreducible in $\mathbb{Z}[x]$ (in fact, in $\mathbb{Q}[x]$), but in almost all cases it factorises when reduced mod p , that is, when regarded as a polynomial in $\mathbb{F}_p[x]$. We are assuming that n is coprime to p , so the set $\Omega = \Omega_n$ of n -th roots of 1 in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ contains n elements. This set is partitioned into disjoint subsets Π_m ($m|n$) consisting of the $\varphi(m)$ primitive m -th roots of 1 in $\overline{\mathbb{F}}_p$. For each divisor m of n , let $e = e_m$ denote the multiplicative order of p as a unit mod m , so that m divides $p^e - 1$ but not $p^f - 1$ for any $f < e$. Then each $\omega \in \Pi_m$ generates the subfield $\mathbb{F}_{p^e}$ of $\overline{\mathbb{F}}_p$. The restriction to $\mathbb{F}_{p^e}$ of the Frobenius automorphism $\phi : t \mapsto t^p$ of $\overline{\mathbb{F}}_p$ generates the Galois group $C = \text{Gal } \mathbb{F}_{p^e} \cong C_e$ of this subfield, which partitions Π_m into $\varphi(m)/e$ orbits

$$\Gamma = \{\omega, \omega^p, \dots, \omega^{p^{e-1}}\}$$

of length e . Each such orbit Γ is the set of roots ω^{p^i} of an irreducible monic factor $f^\Gamma(x)$ of degree e of $\Phi_m(x)$ in $\mathbb{F}_p[x]$.

As Γ ranges over the orbits of the Galois group C on Ω , these polynomials $f^\Gamma(x)$ give all the irreducible factors of $x^n - 1$, and the ideals P^Γ they generate give all the maximal G_0 -submodules of P .

By taking the ideals generated by all products of these irreducible factors we obtain all the G_0 -submodules of P as intersections of these maximal submodules. In particular, P is the direct sum of the irreducible G_0 -submodules $P_\Gamma \cong P/P^\Gamma$ generated by the complementary factors $f_\Gamma(x) = (x^n - 1)/f^\Gamma(x)$ of $x^n - 1$, with the generator a of G_0 having minimal polynomial $f^\Gamma(x)$ on P_Γ . For instance, the orbit $\Gamma = \Pi_1 = \{1\}$ corresponds to the submodule P_1 fixed by G_0 ; this is generated by

$$\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x + 1,$$

the product of all the irreducible factors except $\Phi_1(x) = x - 1$. More generally, if m divides $\gcd(n, p - 1)$ then Π_m consists of $\varphi(m)$ singleton orbits $\Gamma = \{\omega\}$, each giving a linear factor $f^\Gamma(x) = x - \omega$ of $\Phi_n(x)$ and corresponding to a 1-dimensional direct summand P_Γ of P on which a has eigenvalue ω .

To understand the G -module structures of P and Q , we need to consider the action of b , acting by conjugation on G_0 as inversion $\iota : a^j \mapsto a^{-j}$.

Given any orbit $\Gamma \subseteq \Pi_m$ of the Galois group $C = \langle \phi \rangle$ on Ω , the inverses of its elements also form such an orbit Γ^* , so that $\Delta = \Gamma \cup \Gamma^*$ is an orbit of $B := \langle \phi, \iota \rangle$. The orbit Γ containing ω coincides with Γ^* if and only if

$$\omega^{p^f} = \omega^{-1}, \text{ that is, } p^f \equiv -1 \pmod{m},$$

for some integer f . (This is always true if $m = 1$ or 2 , but otherwise it implies that e is even and $f \equiv e/2 \pmod{e}$.) This condition depends only on p and m , so for a given m , either all of the orbits $\Gamma \subseteq \Pi_m$ of C are self-inverse, each forming an orbit Δ of B , or none of them are, so that each orbit Δ of B is a union of two mutually inverse orbits Γ and Γ^* of C . In the first case, the G_0 -submodules P^Γ and P_{Γ^*} of P are also G -modules, with $P_\Gamma \cong P/P^\Gamma$ irreducible and e -dimensional, whereas in the second case $P_\Gamma \oplus P_{\Gamma^*}$ is an irreducible $2e$ -dimensional G -submodule, isomorphic to $P/(P^\Gamma \cap P^{\Gamma^*})$, with b transposing the two direct factors. For instance, any 1-dimensional G_0 -submodules P_Γ are of the second type, apart those with $m = 1$ or 2 . In either case, we will denote this irreducible submodule by R_Δ , and we let $d = e$ or $2e$ denote its dimension. The kernel of the representation of G on R_Δ is the normal subgroup $\langle a^m \rangle \cong C_{n/m}$, so that G acts on R_Δ as $G/\langle a^m \rangle \cong D_m$.

These G -submodules R_Δ are the irreducible summands in the direct sum decomposition of the G -module $P = P_1 \oplus P^1$; deleting the 1-dimensional submodule P_1 gives the corresponding decomposition of $Q = P/P_1 \cong P^1$.

We therefore know all the proper G -submodules L of Q and hence all the maps $\mathcal{N}$ in $E_p(\mathcal{M})$. The possible values of $c = \dim(\mathcal{N})$ range from 1 (corresponding to $L = P^\Delta/P_1$ where n is even and $\Delta = \Pi_2 = \{-1\}$) to $n - 1$ (for $\mathcal{N} = \mathcal{M}_0$, corresponding to $L = 0$). The number ν of irreducible summands R_Δ in the direct

sum decomposition of the G -module Q is equal to the number of orbits $\Delta \neq \{1\}$ of B on Ω . Since these summands are mutually non-isomorphic, the total number of G -submodules of Q is 2^ν ; thus Q has $2^\nu - 1$ proper G -submodules and hence $|E_p(\mathcal{M})| = 2^\nu - 1$.

This completes the proof of Theorem 1. The rest of this section consists of examples illustrating the methods applied to the hosohedron, for various n .

9.3 Examples

To illustrate the preceding arguments, let $n = 95$ and $p = 7$, so $G \cong D_{95}$. To decompose P and hence Q as direct sums, we consider the divisors $m = 1, 5, 19$ and 95 of n in turn.

The divisor $m = 1$ leads to a single orbit $\Delta = \Pi_1 = \{1\}$ of B , with the corresponding 1-dimensional direct summand $R_\Delta = P_1$ of P , generated by $(x^{95} - 1)/(x - 1)$ and affording the principal representation of G .

Next we consider $m = 5$. The prime $p = 7$ has order $e = 4$ in $\mathbb{Z}_5^*$; since $\varphi(5) = 4$, the Galois group $C = \text{Gal } \mathbb{F}_{7^4} \cong C_4$ has $\varphi(5)/e = 1$ orbit Γ on the set $\Pi_5 \subset \mathbb{F}_{7^4} \subset \overline{\mathbb{F}}_7$ of primitive 5th roots of 1. The cyclotomic polynomial $\Phi_5(x) = f^\Gamma(x) = x^4 + x^3 + x^2 + x + 1$ is therefore irreducible in $\mathbb{F}_7[x]$, giving a 4-dimensional irreducible G_0 -submodule $P_\Gamma = (x^{95} - 1)/(f^\Gamma(x))$ of P . Since $-1 \in \langle 7 \rangle$ in $\mathbb{Z}_5^*$, this is also a 4-dimensional irreducible G -submodule R_Δ , with G acting on it as D_5 . When we factor out P_1 , this appears as a direct summand of $Q = H_1(S; \mathbb{F}_7)$.

Now let $m = 19$. Since 7 has order $e = 3$ in $\mathbb{Z}_{19}^*$, and there are $\varphi(19) = 18$ primitive 19th roots of 1 in $\mathbb{F}_{7^3}$, forming six orbits Γ under the Galois group $C \cong C_3$ of this field, we obtain six 3-dimensional irreducible G_0 -submodules P_Γ as summands of P , corresponding to six irreducible cubic factors $f^\Gamma(x)$ of $\Phi_{19}(x)$ in $\mathbb{F}_7[x]$. Since $-1 \notin \langle 7 \rangle$ in $\mathbb{Z}_{19}^*$, these six orbits form three mutually inverse pairs Γ and Γ^* , so they merge to give three orbits $\Delta = \Gamma \cup \Gamma^*$ of B . These correspond to three 6-dimensional irreducible G -submodules $R_\Delta = P_\Gamma \oplus P_{\Gamma^*}$ of P , and hence also of Q , with G acting on each as D_{19} .

Finally let $m = 95$. Since $\mathbb{Z}_{95} \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{19}$ we see that 7 has order $e = \text{lcm}\{4, 3\} = 12$ in $\mathbb{Z}_{95}^*$, so $\Phi_{95}(x)$ is a product of $\varphi(95)/12 = 6$ irreducible polynomials $f^\Gamma(x)$ of degree 12 in $\mathbb{F}_7[x]$, giving six 12-dimensional irreducible G_0 -submodules P_Γ as summands of P . Since $-1 \notin \langle 7 \rangle$ in $\mathbb{Z}_{19}^*$ we have $-1 \notin \langle 7 \rangle$ in $\mathbb{Z}_{95}^*$, so these give three 24-dimensional irreducible G -submodules $R_\Delta = P_\Gamma \oplus P_{\Gamma^*}$ of P , and hence of Q , with G acting faithfully on each.

Adding up the dimensions of these irreducible submodules R_Δ , we have

$$1 + 4 + 3 \times 6 + 3 \times 24 = 95 = \dim P,$$

so they give the complete direct sum decomposition $\oplus_\Delta R_\Delta$ of P as a G -module; deleting the 1-dimensional submodule P_1 gives the corresponding decomposition of Q . Since Q has $\nu = 7$ mutually non-isomorphic irreducible summands, we obtain $2^7 - 1 = 127$ proper G -submodules $L < Q$. Thus $E_7(\mathcal{M})$ consists of 127 regular

maps $\mathcal{N}$, with $\dim(\mathcal{N}) = 4$ (one map), 6 (three maps), and so on, up to 94 (one map, $\mathcal{M}_0$).

Examples with even n can be treated similarly, except that now there is always at least one 1-dimensional summand R_Δ , corresponding to the orbit $\Delta = \Gamma = \Pi_2 = \{-1\}$ of B , with $f^\Gamma(x) = \Phi_2(x) = x + 1$. This corresponds to a cyclic covering $\mathcal{N}$ on which a has eigenvalue -1 (see [13, Sect. 9(d)]).

For instance, if $n = 4$ then $E_p(\mathcal{M})$ consists of regular maps of type $\{2p, 4\}$ and genus $g = 1 + p^{e-1}(p - 2)$. If $p \neq 2$ then B has orbits $\Delta = \{1\}$, $\{-1\}$ and $\{\zeta_4^{\pm 1}\}$ on Ω_4 , the last two providing irreducible G -submodules of Q of dimensions 1 and 2, affording the characters χ_3 and ξ_1 of G . Thus $E_p(\mathcal{M})$ consists of three maps $\mathcal{N}$ with $c = \dim(\mathcal{N}) = 1, 2$ and 3, of genus $g = p - 1$, $(p - 1)^2$ and $(p - 1)(p^2 - p - 1)$, affording the characters χ_3 , ξ_1 and $\chi_3 + \xi_1$. The maps with $c = 1$ lie on the Accola-Maclachlan surfaces with $8(g + 1)$ automorphisms [1, 18]; for $p = 3, 5, 7$ and 11 they are the duals of the regular maps R2.2, R4.4, R6.4 and R10.11 in [3]. For the same primes, the maps with $c = 2$ (the maps $\{2p, 4\}_4$ in [6, Sect. 8.6 and Table 8]) are the duals of R4.3, R16.3, R36.4 and R100.5 in [3].

9.4 Prime n

If n is prime then the only divisors m of n are $m = 1$, giving rise to the G -submodule $P_1 \leq P$, and $m = n$. If $p \neq n$ then $\Phi_n(x)$ factorises in $\mathbb{F}_p[x]$ as a product of $(n - 1)/e$ irreducible polynomials $f^\Gamma(x)$ of degree e equal to the order of p in the group $\mathbb{Z}_n^* \cong C_{n-1}$. As a G_0 -module, Q is a direct sum of $(n - 1)/e$ corresponding irreducible submodules Q_Γ of dimension e . If $n = 2$ or if e is even then $-1 \in \langle p \rangle$ in $\mathbb{Z}_n^*$, so these are the $v = (n - 1)/e$ irreducible summands R_Δ of dimension $d = e$, for $\Delta = \Gamma$, in the direct sum decomposition of Q as a G -module. If $n > 2$ and e is odd then $-1 \notin \langle p \rangle$ in $\mathbb{Z}_n^*$, so we obtain $v = (n - 1)/2e$ direct summands $R_\Delta = P_\Gamma \oplus P_{\Gamma^*}$ of dimension $d = 2e$ in this decomposition. In either case, there are $2^v - 1$ maps $\mathcal{N} \in E_p(\mathcal{M})$, specifically $\binom{v}{i}$ with $c = \dim(\mathcal{N}) = id$ for each $i = 1, 2, \dots, v$, affording the sum of i distinct irreducible characters of degree d . In particular, if $n > 2$ and $p \equiv \pm 1 \pmod{n}$ then $d = 2$, so $v = (n - 1)/2$, with the irreducible coverings affording the characters ξ_k of G for $k = 1, 2, \dots, (n - 1)/2$.

(If $p = n$ then $Q \cong \mathbb{F}_p[x]/((x - 1)^{n-1}) = \mathbb{F}_n[x]/((x - 1)^{n-1})$, with proper G -submodules of codimensions $c = 1, 2, \dots, n - 1$, each generated by the image of $(x - 1)^c$. It follows that $E_p(\mathcal{M})$ consist of $n - 1$ coverings, one for each of these dimensions c .)

In the next two subsections we give some simple illustrative examples.

9.4.1 Example: $n = 3$

The first value of n not covered by Sect. 8 (on the dihedron) is $n = 3$. In this case the maps $\mathcal{N} \in E_p(\mathcal{M})$ are regular, of type $\{2p, 3\}$ and genus $1 + \frac{1}{2}p^{c-1}(p - 3)$ where $c = \dim(\mathcal{N})$. If $p \neq 3$ then p has order $e = 1$ or 2 in $\mathbb{Z}_3^*$ as $p \equiv 1$ or $-1 \pmod{3}$, so as a G_0 -module Q is respectively a direct sum of two 1-dimensional submodules, or is 2-dimensional and irreducible. In either case B has a single orbit $\Delta \neq \{1\}$ on

Ω_3 , namely $\{\zeta_3^{\pm 1}\}$ where ζ_3 is a primitive cube root of 1, so Q is an irreducible 2-dimensional G -module; hence $E_p(\mathcal{M})$ consists of a single map $\mathcal{N} = \mathcal{M}_0$ of genus $(p-1)(p-2)/2$, affording the character ξ_1 of G (this is the map $\{2p, 3\}_6$ in [6, Sect. 8.6 and Table 8]). For instance, if $p = 2$ then $\mathcal{N}$ is the cube $\{4, 3\}$, with $\mathcal{M} = \mathcal{N}/V_4$, and for $p = 5, 7, 11$ or 13 it is the dual of the regular map R6.1, R15.2, R45.3 or R66.1 in [3].

9.4.2 Example: $n = 13$

If $n = 13$ the maps in $E_p(\mathcal{M})$ have type $\{2p, 13\}$ and genus $1 + \frac{1}{2}p^{c-1}(11p-13)$. If $p \equiv \pm 1 \pmod{13}$ then $e = 1$ or 2 ; in either case $d = 2$, so there are $v = 12/2 = 6$ irreducible coverings, affording the characters $\xi_1, \dots, \xi_6$, and a total of $2^6 - 1 = 63$ coverings $\mathcal{N} \in E_p(\mathcal{M})$, with $\dim(\mathcal{N}) = 2, 4, \dots, 12$. If $p \equiv 3$ or $9 \pmod{13}$ then $e = 3$; this is odd, so $d = 6$, and hence there are $v = 2$ irreducible coverings of dimension 6; thus $E_p(\mathcal{M})$ consists of these two irreducible coverings and $\mathcal{M}_0$ of dimension 12. The same applies if $p \equiv \pm 4 \pmod{13}$, so that $d = e = 6$. If $p \equiv \pm 5 \pmod{13}$ then $e = 4$, so $d = 4$ also; thus $v = 3$, so $E_p(\mathcal{M})$ consists of seven coverings of dimensions 4, 8 and 12. If $p \equiv \pm 2$ or $\pm 6 \pmod{13}$ then $d = e = 12$, giving one irreducible covering $\mathcal{M}_0$ of dimension 12.

10 Branching over faces and vertices

It is now easy to find the elementary abelian coverings of a Platonic map $\mathcal{M}$, where the branching is over its vertices rather than its faces: simply use the preceding results to find the corresponding coverings of the dual map $\mathcal{M}^*$ of $\mathcal{M}$, branched over its faces, and then take the duals of the resulting maps.

One can also use similar methods to determine the elementary abelian coverings, where the branching is over both the faces and the vertices (see also [9]). In this case, we replace S with the sphere S' punctured at the set $\Phi' = \Phi \cup \Phi^*$, where Φ is the set of face-centres of $\mathcal{M}$ and Φ^* is its set of vertices (i.e. the face-centres of $\mathcal{M}^*$). The resulting homology module $H_1(S'; \mathbb{F}_p)$ for G on S' is the quotient $Q' = P'/P'_1$, where P' is the permutation module $\mathbb{F}_p\Phi' = \mathbb{F}_p\Phi \oplus \mathbb{F}_p\Phi^*$ for G on Φ' , and P'_1 is the 1-dimensional G -submodule of P' spanned by $\underline{\Phi}' = \underline{\Phi} + \underline{\Phi}^*$. The representation and character of G on P' are $\rho' = \rho \oplus \rho^*$ and $\pi' = \pi + \pi^*$, where ρ and ρ^* are the representations on the permutation modules $P = \mathbb{F}_p\Phi$ and $P^* = \mathbb{F}_p\Phi^*$, and π and π^* are their associated characters. Since G has two orbits on Φ' , the principal character χ_1 of G has multiplicity 2 in π' , afforded by the 1-dimensional fixed submodules P_1 and P_1^* of P and P^* , spanned by $\underline{\Phi}$ and $\underline{\Phi}^*$; deleting one copy of χ_1 gives the character

$$\chi' = \pi + \pi^* - \chi_1 = \chi_1 + \chi + \chi^*$$

of G on Q' , where $\chi = \pi - \chi_1$ and $\chi^* = \pi^* - \chi_1$ are the characters on $Q = P/P_1$ and $Q^* = P^*/P_1^*$. We have already determined the G -module structures of P and of P^* (the latter in the context of $\mathcal{M}^*$ rather than $\mathcal{M}$), so we can immediately deduce the structure of $P' = P \oplus P^*$, using Lemma 2.1, and hence that of its quotient Q' .

This gives us information about the set $E'_p(\mathcal{M})$ of coverings of $\mathcal{M}$ by elementary abelian p -groups, branched over faces and vertices, since these correspond to the proper G -submodules of Q' .

In all such cases, Q' is an extension of a fixed G -submodule

$$Q'_1 = (P_1 \oplus P_1^*)/P'_1 \cong P_1 (\cong P_1^*)$$

by

$$P/(P_1 \oplus P_1^*) = (P \oplus P_1^*)/(P_1 \oplus P_1^*) \oplus (P_1 \oplus P^*)/(P_1 \oplus P_1^*) \cong Q \oplus Q^*.$$

If the modules P and P^* both split over their fixed submodules P_1 and P_1^* (equivalently, if both $f = |\Phi|$ and $f^* = |\Phi^*|$ are coprime to p), then Q' also splits over Q'_1 : we have $P = P_1 \oplus P^1$ and $P^* = P_1^* \oplus P^{*1}$, so

$$P' = P_1 \oplus P_1^* \oplus P^1 \oplus P^{*1};$$

since P'_1 is a submodule of $P_1 \oplus P_1^*$ there is a complement

$$Q'^1 = (P'_1 \oplus P^1 \oplus P^{*1})/P'_1 \cong P^1 \oplus P^{*1} \cong Q \oplus Q^*$$

for Q'_1 on Q' , giving

$$Q' = Q'_1 \oplus Q'^1 \cong P_1 \oplus Q \oplus Q^*.$$

Instead of dealing with all cases in detail, we will consider one simple example, the tetrahedron, and just summarise the results in the other cases.

10.1 The tetrahedron

If $\mathcal{M}$ is the tetrahedron then $\mathcal{M}^* \cong \mathcal{M}$, and so $P \cong P^*$. We saw in Sect. 3 that if $p \neq 2$ then $P = P_1 \oplus P^1$, affording the representation $\rho_1 \oplus \rho_4$ of G with character $\chi_1 + \chi_4$. It follows that

$$Q' = Q'_1 \oplus Q \oplus Q^*,$$

affording the representation $\rho_1 \oplus \rho_4 \oplus \rho_4$ with character

$$\chi' = \chi_1 + 2\chi_4;$$

here Q'_1 affords ρ_1 , while Q and Q^* , the images in Q' of P^1 and P^{*1} , both afford ρ_4 . Now $Q \cong Q^*$, so $Q \oplus Q^*$ contains $p + 1$ submodules $Q(\lambda) \cong Q$, where $\lambda \in \mathbb{P}^1(p)$ (see Sect. 2.5, and also Sect. 7 for a similar phenomenon concerning the icosahedron). Thus Q' has the following proper submodules L : there is one submodule $Q \oplus Q^*$ of codimension $c = 1$; there are $p + 1$ submodules $Q'_1 \oplus Q(\lambda)$ of codimension 3, and

$p + 1$ submodules $Q(\lambda)$ of codimension 4; there is one submodule Q'_1 of codimension 6, and one submodule 0 of codimension 7. Thus there are $|E'_p(\mathcal{M})| = 2p + 5$ coverings $\mathcal{N}$ of $\mathcal{M}$, of these dimensions c , affording the representations $\rho_1, \rho_4, \rho_1 \oplus \rho_4, \rho_4 \oplus \rho_4$, and $\rho_1 \oplus \rho_4 \oplus \rho_4$ respectively. In contrast with the case of the icosahedron, orientation-reversing automorphisms of $\mathcal{M}$ leave all submodules of Q' invariant, including the submodules $Q(\lambda)$, so the maps $\mathcal{N} \in E'_p(\mathcal{M})$ are all regular.

The coverings of dimension 3 corresponding to $L = Q'_1 \oplus Q^*$ and $Q'_1 \oplus Q$ are, respectively, branched over only the faces and only the vertices; the first of these is the covering $\mathcal{M}_0$ of type $\{3p, 3\}$ and genus $p^3 - 2p^2 + 1$ described in Sect. 3, and the second is its dual. The remaining $2p + 3$ coverings have branching of order $p^c - p^{c-1}$ at the four vertices and the four faces of $\mathcal{M}$, so they have type $\{3p, 3p\}$ and genus $3p^c - 4p^{c-1} + 1$. For example, when $L = Q \oplus Q^*$ we have a cyclic covering of genus $3(p - 1)$, obtained by assigning mutually inverse monodromy permutations in $K = C_p$ to the vertices and face-centres of $\mathcal{M}$; for instance, if $p = 3, 5, 7, 11$ these are the maps R6.9, R12.8, R18.10, R30.10 in [3]. The $p - 1$ coverings with $L = Q'_1 \oplus Q(\lambda)$ and $\lambda \neq 0, \infty$ have genus $3p^3 - 4p^2 + 1$; when $p = 3$ these are the maps R46.27 of type $\{9, 9\}_4$ and R46.28 of type $\{9, 9\}_{12}$ in [3].

10.2 The other Platonic maps

Here we simply give the homology character χ' for each of the remaining Platonic maps, in terms of the irreducible characters in the tables given earlier, together with a few comments. The G -module structure of Q' and hence the properties of the elementary abelian coverings $\mathcal{N}$ of $\mathcal{M}$ can be deduced by methods similar to those used for the tetrahedron.

If $\mathcal{M}$ is the cube or octahedron then

$$\chi' = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5.$$

The one-dimensional rational summands χ_1 and χ_2 lead to both central and non-central cyclic coverings $\mathcal{N}$ of $\mathcal{M}$. As in the case of the tetrahedron, the coefficient 2 of χ_5 means that the number of coverings is unbounded as a function of p : in fact, since $\rho_5 \oplus \rho_5$ has $p + 1$ submodules isomorphic to ρ_5 we have $|E'_p(\mathcal{M})| = 2^4(p + 3) - 1 = 16p + 47$ for $p > 3$. The resulting maps are all regular, since all G -submodules of Q and Q^* , and hence of Q' , are invariant under orientation-reversing automorphisms of $\mathcal{M}$ (see Sects. 4, 5).

If $\mathcal{M}$ is the dodecahedron or icosahedron then

$$\chi' = \chi_1 + 2\chi_2 + 2\chi_3 + 2\chi_4 + 2\chi_5.$$

The character χ_1 again leads to central cyclic coverings, but now there are no non-central cyclic coverings. The characters χ_4 and χ_5 are both rational, so the representations ρ_4 and ρ_5 are realised over $\mathbb{F}_p$, as are ρ_2 and ρ_3 if $p \equiv \pm 1 \pmod{5}$; in this case counting proper G -submodules gives $|E'_p(\mathcal{M})| = 2(p + 3)^4 - 1$. If $p \equiv \pm 2 \pmod{5}$, however, we obtain a 6-dimensional irreducible representation over $\mathbb{F}_p$ with

character $\chi_2 + \chi_3$ and multiplicity 2, so that $|E'_p(\mathcal{M})| = 2(p+3)^3 - 1$. In either case the summand $2\chi_4$ of χ' is afforded by an 8-dimensional submodule $Q_{a,b}$ (defined in Sect. 7), inherited from the action of G on the faces of an icosahedron, or equivalently the vertices of a dodecahedron; since the antipodal automorphism of $\mathcal{M}$ acts non-trivially on the 4-dimensional submodules $Q(\lambda)$ of $Q_{a,b}$, we see that $E'_p(\mathcal{M})$ contains $(p+3)^3(p-1)$ or $(p+3)^2(p-1)$ chiral pairs of maps as $p \equiv \pm 1$ or $\pm 2 \pmod{5}$.

If $\mathcal{M}$ is the dihedron $\{n, 2\}$ or the hosohedron $\{2, n\}$ then

$$\chi' = \chi_1 + \chi_2 + \sum_{k=1}^{(n-1)/2} \xi_k$$

if n is odd, and

$$\chi' = \chi_1 + \chi_2 + \chi_3 + \sum_{k=1}^{(n-2)/2} \xi_k$$

if n is even. The 1-dimensional rational characters χ_k give rise to cyclic coverings, central when $k = 1$ and non-central when $k > 1$. The fact that χ' is multiplicity-free means that $|E'_p(\mathcal{M})|$ is bounded above as a function of p , by $2^{(n+3)/2} - 1$ or $2^{(n+4)/2} - 1$ as n is odd or even. These bounds are attained when $p \equiv 1 \pmod{n}$, so that the representations corresponding to the characters ξ_k all reduce mod p to 2-dimensional representations over $\mathbb{F}_p$. For each n there are infinitely many such primes p , by Dirichlet's theorem on primes in arithmetic progressions. As in Sect. 8 and Sect. 9, the maps in $E'_p(\mathcal{M})$ are regular for all p .

11 Branching over edges

Hypermaps (Grothendieck's *dessins d'enfants*, see [8, 12]) are generalisations of maps, corresponding to triangle groups $\Delta(p, q, r)$ with q not necessarily equal to 2. These arise if we extend the investigation of coverings of the Platonic maps to allow branching over their edges. The general principles are the same as for branching over the faces, except that Φ is now replaced with the set $\Phi^\dagger$ of edges of $\mathcal{M}$, the sphere is punctured at their midpoints, and H is the subgroup $\langle y \rangle \cong C_2$ of G leaving invariant an edge. Dual pairs of maps $\mathcal{M}$ and $\mathcal{M}^*$ can be treated together, since duality induces G -isomorphisms between their edge sets. As in Sect. 10, we will study the resulting set $E_p^\dagger(\mathcal{M})$ of coverings of $\mathcal{M}$ by concentrating on the tetrahedron as a typical example.

The decompositions of the character $\chi^\dagger$ of G on the homology module are as follows, with the notation for irreducible characters as before. For the tetrahedron,

$$\chi^\dagger = \chi_2 + \chi_3 + \chi_4.$$

For the cube and the octahedron,

$$\chi^\dagger = \chi_3 + 2\chi_4 + \chi_5.$$

For the dodecahedron and the icosahedron,

$$\chi^\dagger = \chi_2 + \chi_3 + 2\chi_4 + 3\chi_5.$$

For the dihedron and the hosohedron, if n is odd then

$$\chi^\dagger = \sum_{k=1}^{(n-1)/2} \xi_k,$$

the same decomposition as for branching over the faces of the hosohedron, by the duality between edges and faces of this map; if n is even then taking $H = \langle ab \rangle$ as the subgroup stabilising an edge (since we took $\langle b \rangle$ in Sect. 9.1 as the stabiliser of a face), we have

$$\chi^\dagger = \chi_1 + \chi_4 + \sum_{k=1}^{(n-2)/2} \xi_k.$$

11.1 The tetrahedron

Let $\mathcal{M}$ be the tetrahedron, so that the coverings $\mathcal{N}$ are hypermaps of type $(3, 2p, 3)$ and genus $2p^c - 3p^{c-1} + 1$. If $p \equiv 1 \pmod{3}$ then $\chi^\dagger$ decomposes over $\mathbb{F}_p$ as above, with irreducible summands χ_2 , χ_3 and χ_4 of degrees 1, 1 and 3, so we obtain seven coverings $\mathcal{N}$, with $c = 1, 1, 2, 3, 4, 4$ and 5. Since χ_2 and χ_3 are transposed by orientation-reversing automorphisms of $\mathcal{M}$, those with $c = 1$ or 4 occur in chiral pairs, whereas the other three are regular. For instance, if $p = 7$ then the two cyclic coverings (with $c = 1$) are duals of the chiral pair of hypermaps CH12.1 of genus 12 and type $(3, 3, 14)$ in [3]. If $2 < p \equiv 2 \pmod{3}$ then $\chi^\dagger$ decomposes as a sum of two irreducible characters $\chi_2 + \chi_3$ and χ_4 of degrees 2 and 3, so in this case there are three coverings $\mathcal{N}$ with $c = 2, 3$ and 5, all of them regular hypermaps. For instance, if $p = 5$ then the covering with $c = 2$ is a dual of the hypermap RPH36.3 in [3].

11.2 The other Platonic maps

In the cases of the cube and the octahedron, there is one irreducible covering with $c = 2$, affording ρ_3 , and there are $p + 2$ with $c = 3$, of which $p + 1$ afford ρ_4 and one affords ρ_5 ; thus $|E_p^\dagger(\mathcal{M})| = 4p + 11$ and there are no cyclic coverings. Since G has rank 7 as a permutation group on $\Phi^\dagger$, whereas A has rank 5, some of the hypermaps in $E_p^\dagger(\mathcal{M})$ are chiral; in fact, two of the irreducible G -submodules of $Q^\dagger$ affording ρ_4 are non-isomorphic A -submodules, while the other $p - 1$ are transposed in pairs by

orientation-reversing automorphisms, so there are 15 regular hypermaps in $E_p^\dagger(\mathcal{M})$ and $2(p-1)$ chiral pairs.

If $\mathcal{M}$ is the dodecahedron or the icosahedron, then the presence of the conjugate pair of summands χ_2 and χ_3 in $\chi^\dagger$ means that the decomposition of $Q^\dagger$, and hence the number and nature of the coverings, again depends on whether $p \equiv \pm 1$ or $\pm 2 \pmod{5}$, as in Sects. 6, 7 and 10.2. As in the case of the cube and the octahedron, there are chiral pairs but no cyclic coverings.

Hypermap coverings branched over the edges of a hosohedron $\mathcal{M} = \{2, n\}$ (or its dual dihedron) can be dealt with as in Sect. 9 by considering the map coverings branched over the faces of the hosohedron obtained by rotating $\mathcal{M}$ through π/n about its two vertices.

12 Branching over vertices, edges and faces

Just as we considered branching over the vertices and faces in Sect. 10, one can also consider coverings which are branched over any combination of vertices, edges and faces. In the most general case, where we allow branching over all three sets, G has three orbits on the punctures of the sphere; the permutation module P'' is therefore the direct sum of P , P^* and a third permutation module $P^\dagger$ with $\Phi^\dagger$ as its basis, and the corresponding homology character is

$$\chi'' = \pi + \pi^* + \pi^\dagger - \chi_1 = 2\chi_1 + \chi + \chi^* + \chi^\dagger.$$

The corresponding coverings of each $\mathcal{M}$ can now be found by using the techniques described earlier. For instance, in all cases the character χ_1 has multiplicity 2 in χ'' , yielding $p+1$ central cyclic coverings of $\mathcal{M}$ for each prime p not dividing $|G|$: three are branched over just two from the three sets of vertices, edges and faces, and $p-2$ are branched over all three sets. These are all quotients of a 2-dimensional central covering.

12.1 The tetrahedron

Taking the tetrahedron again as a typical example, we have

$$\chi'' = 2\chi_1 + \chi_2 + \chi_3 + 3\chi_4.$$

If $p \equiv 1 \pmod{3}$ then this gives the direct sum decomposition of Q'' , with the principal submodule of multiplicity 2, two conjugate non-principal 1-dimensional submodules of multiplicity 1, and a 3-dimensional irreducible submodule of multiplicity 3. Counting proper submodules, we find $8(p+3)(p^2+p+2)-1$ coverings, of which $p+1$ are central and cyclic, and two are non-central and cyclic. As in Sect. 11.1, the appearance of the characters χ_2 and χ_3 , transposed by orientation-reversing automorphisms, gives rise to chiral hypermaps, this time forming $2(p+3)(p^2+p+2)$ pairs.

Even in the smallest case, when $p = 7$, there are far too many coverings to describe them all, but we can at least describe the central cyclic coverings. Branching over edges and either faces or vertices yields the regular hypermap RPH24.3 in [3], of genus 24 and type $(3, 14, 21)$, or its dual of type $(21, 14, 3)$; in each case the same monodromy element of $K = C_7$ is assigned to each of the ten branch-points. Branching over the vertices and faces yields the regular map R18.10, as in Sect. 10.1. Branching over the vertices, edges and faces yields five regular hypermaps of genus 36 and type $(21, 14, 21)$; these are duals of RPH36.62, RPH36.63 and RPH36.64. The two non-central cyclic coverings form a chiral pair of genus 12, branched over the edges and described in Sect. 11.1.

If $2 < p \equiv 2 \pmod{3}$ then in the above decomposition of Q'' the two non-principal 1-dimensional summands are replaced with one irreducible summand of dimension 2, which affords the character $\chi_2 + \chi_3$. In this case there are $4(p+3)(p^2+p+2) - 1$ coverings in $E_p''(\mathcal{M})$; the $p+1$ cyclic coverings are all central, and there are no chiral pairs.

If $p = 5$, for instance, the cyclic coverings are as follows. Branching over edges and either faces or vertices yields the regular hypermap RPH16.8, of genus 16 and type $(3, 10, 15)$, or its dual of type $(15, 10, 3)$; in each case the same monodromy element of $K = C_5$ is assigned to each of the ten branch-points. Branching over the vertices and faces yields the regular map R12.8, as in Sect. 10.1. Branching over the vertices, edges and faces yields three regular hypermaps of genus 24 and type $(15, 10, 15)$; these are duals of RPH24.37 and RPH24.38.

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