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Lit-only sigma-game on nondegenerate graphs

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Abstract A configuration of the lit-only σ-game on a graph Γ is an assignment of one of two states, *on* or *off*, to each vertex of Γ. Given a configuration, a move of the lit-only σ-game on Γ allows the player to choose an *on* vertex s of Γ and change the states of all neighbors of s. Given an integer k, the underlying graph Γ is said to be k-lit if for any configuration, the number of *on* vertices can be reduced to at most k by a finite sequence of moves. We give a description of the orbits of the lit-only σ-game on nondegenerate graphs Γ which are not line graphs. We show that these graphs Γ are 2-lit and provide a linear algebraic criterion for Γ to be 1-lit.

Keywords Group action · Lit-only σ-game · Nondegenerate graph

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1 Introduction

The notion of the σ-*game* on finite graphs Γ was first introduced by Sutner [17,18] around 1989. A *configuration* of the σ-game on Γ is an assignment of one of two states, *on* or *off*, to each vertex of Γ. Given a configuration, a *move* consists of choosing a vertex of Γ, followed by changing the states of all of its neighbors. If only *on* vertices can be chosen in each move, we come to the variation: *lit-only* σ *-game*. Starting from an initial configuration, the goal of the lit-only σ-game on Γ is to minimize the number of *on* vertices of Γ, or to reach an assigned configuration by a finite sequence of moves.

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Given an integer k, the underlying graph Γ is said to be *k-lit* if for any configuration, the number of *on* vertices can be reduced to at most k by a finite sequence of moves. More precisely, we are interested in the orbits of the lit-only σ-game on Γ and the smallest integer k, the *minimum light number* of Γ [19], for which Γ is k-lit. The notion of lit-only σ-games occurred implicitly in the study of equivalence classes of Vogan diagrams. The Borel-de Siebenthal theorem [2] showed that every Vogan diagram is equivalent to one with a single-painted vertex, which implies that each simply-laced Dynkin diagram is 1-lit. The equivalence classes of Vogan diagrams were described by Chuah and Hu [7]. A conjecture made by Chang [5,6] that any tree with k leaves is $\lceil k/2 \rceil$-lit was confirmed by Wang and Wu [19], where the name "lit-only σ-game" was coined.

The lit-only σ-game on a simple graph Γ is simply the natural action of a certain subgroup H_Γ of the general linear group over $\mathbb{F}_2$ [19]. Under the assumption that Γ is the line graph of a simple graph G, Wu [21] described the orbits of the lit-only σ-game on Γ and gave a characterization for the minimum light number of Γ. Moreover, if G is a tree of order $n \geq 3$, Wu showed that H_Γ is isomorphic to the symmetric group on n letters. Weng and the author [13] determined the structure of H_Γ without any assumption on G. The lit-only σ-game on a simple graph Γ can also be considered as a representation κ_Γ of the simply-laced Coxeter group W_Γ over $\mathbb{F}_2$ [12]. The dual representation of κ_Γ preserves a certain symplectic form B_Γ. The two representations are equivalent whenever the form B_Γ is nondegenerate. From this viewpoint it is natural to partition simple connected graphs into two classes according as B_Γ is degenerate or nondegenerate.

In this paper, we treat nondegenerate graphs Γ which are not line graphs. We show that H_Γ is isomorphic to an orthogonal group, followed by a description of the orbits of lit-only σ-game on Γ (Theorem 3.1). Moreover, we show that these graphs Γ are 2-lit and provide a linear algebraic criterion for Γ to be 1-lit (Theorem 3.2). Combining Theorem 3.1, Theorem 3.2, and those in [13,21], the study of the lit-only σ-game on nondegenerate graphs is quite completed, and the focus for further research is on degenerate graphs.

2 Preliminaries

From now on, let $\Gamma = (S, R)$ denote a finite simple connected graph with vertex set S and edge set R. Let $\mathbb{F}_2$ denote the two-element field $\{0, 1\}$. Let V denote an $\mathbb{F}_2$-vector space that has a basis $\{\alpha_s \mid s \in S\}$ in one-to-one correspondence with S. Let V^* denote the dual space of V. For each $s \in S$, we define $f_s \in V^*$ by

$$f_s(\alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{else} \end{cases} \tag{1}$$

for all $t \in S$. The set $\{f_s \mid s \in S\}$ forms a basis of V^* and is called the *basis of V^* dual to* $\{\alpha_s \mid s \in S\}$. Each configuration f of the lit-only σ-game on Γ is interpreted as the vector

$$\sum_{\text{on vertices } s} f_s \in V^*. \tag{2}$$

If all vertices of Γ are assigned the *off* state by f, we interpret (2) as the zero vector of V^*. Given $s \in S$ and $f \in V^*$ observe that $f(\alpha_s) = 1$ (resp. 0) if and only if the vertex s is assigned the *on* (resp. *off*) state by f.

For each $s \in S$ define a linear transformation $\kappa_s : V^* \to V^*$ by

$$\kappa_s f = f + f(\alpha_s) \sum_{st \in R} f_t \qquad \text{for all f} \in V^*. \tag{3}$$

Fix a vertex s of Γ. Given any $f \in V^*$, if the state of s is *on*, then $\kappa_s f$ is obtained from f by changing the states of all neighbors of s, and $\kappa_s f = f$ otherwise. Therefore, we may view κ_s as the move of the lit-only σ-game on Γ for which we choose the vertex s and change the states of all neighbors of s if the state of s is *on*. In particular $\kappa_s^2 = 1$. For any vector space U, let $\mathrm{GL}(U)$ denote the general linear group of U. Then $\kappa_s \in \mathrm{GL}(V^*)$ for all $s \in S$. The subgroup $H = H_\Gamma$ of $\mathrm{GL}(V^*)$ generated by the κ_s for all $s \in S$ was first mentioned by Wu [19], which is called the *flipping group* of Γ in [12] and the *lit-only group* of Γ in [21].

The lit-only groups are closely related to the simply-laced Coxeter groups in the following way. Recall that the simply-laced Coxeter group $W = W_\Gamma$ associated with $\Gamma = (S, R)$ is the group generated by all elements $s \in S$ subject to the relations

$$\begin{aligned} s^2 &= 1, \\ (st)^2 &= 1 \qquad \text{if } st \notin R, \\ (st)^3 &= 1 \qquad \text{if } st \in R \end{aligned}$$

for all $s, t \in S$. By [12, Theorem 3.2], there exists a unique representation $\kappa = \kappa_\Gamma : W \to \mathrm{GL}(V^*)$ such that $\kappa(s) = \kappa_s$ for all $s \in S$. Clearly $\kappa(W) = H$. Given any $f, g \in V^*$ observe that g can be obtained from f by a finite sequence of moves of the lit-only σ-game on Γ if and only if there exists $w \in W$ such that $g = \kappa(w) f$. Given an integer k, the underlying graph Γ is k-lit if and only if for each $\kappa(W)$-orbit O on V^*, there exists a subset K of S with size at most k such that $\sum_{s \in K} f_s \in O$.

We now give the definitions of degenerate and nondegenerate graphs. Let $B = B_\Gamma$ denote the symplectic form on V defined by

$$B(\alpha_s, \alpha_t) = \begin{cases} 1 & \text{if } st \in R, \\ 0 & \text{else} \end{cases} \tag{4}$$

for all $s, t \in S$ [16]. The *radical* of V (relative to B) is the subspace of V consisting of the vectors α that satisfy $B(\alpha, \beta) = 0$ for all $\beta \in V$. The form B is said to be *degenerate* whenever the radical of V is nonzero and *nondegenerate* otherwise. The graph Γ is said to be *degenerate* whenever the form B is degenerate, and *nondegenerate* otherwise. The form B induces a linear map $\theta : V \to V^*$ given by

$$\theta(\alpha)\beta = B(\alpha, \beta) \qquad \text{for all } \alpha, \beta \in V. \tag{5}$$

Since the kernel of θ is the radical of V and the matrix representing B with respect to the basis $\{\alpha_s \mid s \in S\}$ is the adjacency matrix of Γ over $\mathbb{F}_2$, the following lemma is straightforward.

Lemma 2.1 *Let A denote the adjacency matrix of Γ over $\mathbb{F}_2$. Then the following are equivalent:*

(i) *Γ is a nondegenerate graph.*
(ii) *θ is an isomorphism of vector spaces.*
(iii) *A is invertible.*

Recall that given a simple graph G, the *line graph of G* is a simple graph that has a vertex for each edge of G, and two of these vertices are adjacent whenever the corresponding edges in G have a common vertex. The purpose of this paper is to investigate the lit-only σ-game on nondegenerate graphs which are not line graphs. Thus, it is natural to ask how to determine if a nondegenerate graph is a line graph. We will give two characterizations of nondegenerate line graphs as Proposition 2.4 below.

Lemma 2.2 *Let G denote a finite simple connected graph of order n. Assume that Γ is the line graph of G. Then $\theta(V)$ has dimension $n-1$ if n is odd and has dimension $n-2$ if n is even.*

Proof Let U denote the vertex space of G over $\mathbb{F}_2$. Define a linear map $\mu : V \to U$ by

$$\mu(\alpha_s) = u + v \qquad \text{for all } s \in S,$$

where u and v are the two endpoints of s in G. Since G is connected, the image of μ is the subspace of U consisting of these vectors each of which equals the sum of an even number of vertices of U. Define a linear map $\lambda : U \to V^*$ by

$$\lambda(u)\alpha_s = \begin{cases} 1 & \text{if } u \text{ is incident to } s \text{ in } G, \\ 0 & \text{else} \end{cases}$$

for all $u \in U$ and for all $s \in S$. There is only one nonzero vector, the sum of all vertices of G, in the kernel of λ. Since $\theta = \lambda \circ \mu$ and by the above comments, the result follows. □

A *claw* is a tree with one internal vertex and three leaves. A simple graph is said to be *claw-free* if it does not contain a claw as an induced subgraph. A *cut-vertex* of Γ is a vertex of Γ whose deletion increases the number of components. A *block* of Γ is a maximal connected subgraph of Γ without cut-vertices. A *block graph* is a simple connected graph in which every block is a complete graph.

Lemma 2.3 *[10, Theorem 8.5]. Let Γ denote a simple connected graph. Then Γ is the line graph of a tree if and only if Γ is a claw-free block graph.*

The following proposition follows by combining Lemmas 2.1–2.3.

Proposition 2.4 *Let* Γ *denote a simple connected graph. Then the following are equivalent:*

(i) Γ *is a nondegenerate line graph.*
(ii) Γ *is the line graph of an odd-order tree.*
(iii) Γ *is a claw-free block graph of even order.*

3 Main results

A *quadratic form* Q *on* V is a function $Q : V \to \mathbb{F}_2$ satisfying

$$Q(\alpha + \beta) = Q(\alpha) + Q(\beta) + B(\alpha, \beta) \qquad \text{for all } \alpha, \beta \in V. \tag{6}$$

Given a quadratic form Q on V, the *orthogonal group* with respect to Q is the subgroup of $\mathrm{GL}(V)$ consisting of all $\sigma \in \mathrm{GL}(V)$ such that $Q(\sigma\alpha) = Q(\alpha)$ for all $\alpha \in V$. Given a basis P of V we define Q_P to be the unique quadratic form on V with $Q_P(\alpha) = 1$ for all $\alpha \in P$.

For the rest of this paper, the form B is assumed to be nondegenerate. Moreover, let $Q = Q_P$ where $P = \{\alpha_s \mid s \in S\}$ and let $O(V)$ denote the orthogonal group with respect to Q. By (6), for any $T \subseteq S$ a combinatorial interpretation of $Q\big(\sum_{s\in T} \alpha_s\big)$ is the parity of the number of vertices and edges on the subgraph of Γ induced by T.

We now can state the main results of this paper, which are Theorem 3.1, Theorem 3.2, and Corollary 3.3.

Theorem 3.1 *Assume that* Γ *is a nondegenerate graph, but not a line graph. Then* $\kappa(W)$ *is isomorphic to* $O(V)$*. Moreover, the* $\kappa(W)$*-orbits on* V^* *are*

$$\{0\}, \quad \theta(Q^{-1}(0)\backslash\{0\}), \quad \theta(Q^{-1}(1)).$$

Under the assumption that B is nondegenerate, the number $|S| = 2m$ is even and there exists a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of V such that $B(\beta_i, \beta_j) = 0,\ B(\gamma_i, \gamma_j) = 0$ and

$$B(\beta_i, \gamma_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else} \end{cases}$$

for all $1 \le i, j \le m$. Such a basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of V is called a *symplectic basis* of V. The *Arf invariant* of Q is defined to be

$$\mathrm{Arf}(Q) = \sum_{i=1}^{m} Q(\beta_i) Q(\gamma_i),$$

which is independent of the choice of the symplectic basis $\{\beta_1, \gamma_1, \ldots, \beta_m, \gamma_m\}$ of V (for example see [1] or [9, Theorem 13.13]). Any two quadratic forms over $\mathbb{F}_2$ are equivalent if and only if they have the same Arf invariant and the underlying spaces have the same dimension (for example see [1] or [9, Proposition 13.14]). The order of

$O(V)$ and the sizes of nontrivial $O(V)$-orbits on V are as follows (cf. [9, Chapter 14]). If $\mathrm{Arf}(Q) = 0$ then

$$\begin{aligned} \left|O(V)\right| &= 2^{m^2-m+1}(2^m-1)(2^2-1)(2^4-1)\cdots(2^{2m-2}-1), \\ \left|Q^{-1}(1)\right| &= 2^{2m-1}-2^{m-1}, \\ \left|Q^{-1}(0)\backslash\{0\}\right| &= 2^{2m-1}+2^{m-1}-1. \end{aligned}$$

If $\mathrm{Arf}(Q) = 1$ then

$$\begin{aligned} \left|O(V)\right| &= 2^{m^2-m+1}(2^m+1)(2^2-1)(2^4-1)\cdots(2^{2m-2}-1), \\ \left|Q^{-1}(1)\right| &= 2^{2m-1}+2^{m-1}, \\ \left|Q^{-1}(0)\backslash\{0\}\right| &= 2^{2m-1}-2^{m-1}-1. \end{aligned}$$

For each $s \in S$, there exists $\alpha_s^\vee \in V$ such that

$$B(\alpha_s^\vee, \alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{else} \end{cases} \tag{7}$$

for all $t \in S$. The set $\{\alpha_s^\vee | s \in S\}$ forms a basis of V and is called *the basis of V dual to $\{\alpha_s \mid s \in S\}$ (with respect to B)*.

Theorem 3.2 *Assume that $\Gamma = (S, R)$ is a nondegenerate graph, but not a line graph. Then Γ is 2-lit. Moreover, the following are equivalent:*

(i) *Γ is 1-lit.*
(ii) *The restriction of Q to $\{\alpha_s^\vee \mid s \in S\}$ is surjective.*

When the nondegenerate graph Γ is bipartite, Theorem 3.2 can be improved as follows.

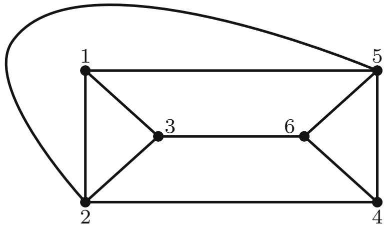
Corollary 3.3 *Assume that Γ is a nondegenerate bipartite graph. Then Γ is 2-lit. Moreover, the following are equivalent:*

(i) Γ is 1-lit
(ii) Γ contains a vertex with even degree or Γ is a single edge.

As consequences of Corollary 3.3, we obtain two families of 1-lit graphs as follows.

- A tree is nondegenerate if and only if it has a perfect matching. By [11, Lemma 2.4], a tree with a perfect matching satisfies Corollary 3.3(ii) and is therefore 1-lit (cf. [14, Theorem 1.1]). This result gives a partial affirmative answer for [20, Conjecture 7].
- For any two positive integers m and n, the $m \times n$ grid is nondegenerate if and only if $m + 1$ and $n + 1$ are coprime [18]. By Corollary 3.3 any such $m \times n$ grid is 1-lit. This result partially improves [8, Theorem 26].

The following example shows that Corollary 3.3 is no longer true if the assumption of Γ is the same as that of Theorem 3.2. Consider the graph $\Gamma = (S, R)$ as below.



The graph $\Gamma = (S, R)$ is nondegenerate and not a block graph. Therefore Γ is not a line graph by Proposition 2.4. The basis $\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_6^\vee\}$ of V dual to $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ can be expressed as follows.

$$\begin{aligned} \alpha_1^\vee &= \alpha_2 + \alpha_6, & \alpha_4^\vee &= \alpha_3 + \alpha_5, \\ \alpha_2^\vee &= \alpha_1 + \alpha_3 + \alpha_5 + \alpha_6, & \alpha_5^\vee &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6, \\ \alpha_3^\vee &= \alpha_2 + \alpha_4 + \alpha_5, & \alpha_6^\vee &= \alpha_1 + \alpha_2 + \alpha_5. \end{aligned}$$

A direct computation shows that $Q(\alpha_s^\vee) = 0$ for all $s \in S$. Therefore Γ is not 1-lit by Theorem 3.2, but the vertices 2,5 have even degree in Γ.

4 Proof of Theorem 3.1

To prove Theorem 3.1, we consider a family of linear transformations on V defined as follows. For $\alpha \in V$, the *transvection on V with direction α* is a linear transformation $\tau_\alpha : V \to V$ defined by

$$\tau_\alpha \beta = \beta + B(\beta, \alpha)\alpha \qquad \text{for all } \beta \in V.$$

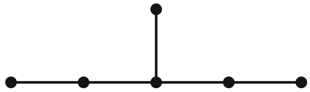
Observe that τ_α preserves the form B and that $\tau_\alpha \in \mathrm{GL}(V)$ since $\tau_\alpha^2 = 1$.

For a subset P of V define $Tv(P)$ to be the subgroup of $\mathrm{GL}(V)$ generated by τ_α for $\alpha \in P$, and define $G(P)$ to be the simple graph whose vertex set is P and where α, β in P form an edge if and only if $B(\alpha, \beta) = 1$. For any two linearly independent sets P and P' of V, we say that P' is *elementary t-equivalent* to P whenever there exist $\alpha, \beta \in P$ such that P' is obtained from P by changing β to $\tau_\alpha \beta$. The equivalence relation generated by the elementary t-equivalence relation is called the *t-equivalence relation* [3].

Lemma 4.1 [3, Theorem 3.3]. *Let P denote a linearly independent set of V. Assume that $G(P)$ is a connected graph. Then there exists P' in t-equivalence class of P for which $G(P')$ is a tree.*

Lemma 4.2 [15, Lemma 3.7]. *Let P denote a linearly independent set of V. Assume that $G(P)$ is the line graph of a tree. Then, for each P' in the t-equivalence class of P, the graph $G(P')$ is the line graph of a tree.*

A basis P of V is said to have *orthogonal type* [4] if P is t-equivalent to some P' for which $G(P')$ is a tree containing the graph



as a subgraph.

Lemma 4.3 *Assume that P is a basis of V for which $G(P)$ is a tree, but not a path. Then P is of orthogonal type.*

Proof Since $G(P)$ is not a path it contains a vertex α with degree at least three. If any two neighbors of α, say β and γ, are leaves of $G(P)$, then $\beta + \gamma$ lies in the radical of V, which contradicts that B is nondegenerate. Therefore, at most one neighbor of α is a leaf in $G(P)$ and so P is of orthogonal type.

Lemma 4.4 [4, Section 10]. *Let P denote a basis of V which is of orthogonal type. Then $Tv(P)$ is the orthogonal group with respect to Q_P. Moreover, the $Tv(P)$-orbits on V are*

$$\{0\}, \quad Q_P^{-1}(0)\backslash\{0\}, \quad Q_P^{-1}(1).$$

Proof of Theorem 3.1. For each $s \in S$, let τ_s denote the transvection on V with direction α_s. By [16, Section 5], there exists a unique representation $\tau = \tau_\Gamma : W \to \mathrm{GL}(V)$ such that $\tau(s) = \tau_s$ for all $s \in S$. For each $w \in W$ the transpose of $\tau(w^{-1})$ is equal to $\kappa(w)$. Therefore κ is the dual representation of τ. Since τ preserves the form B we have

$$\theta \circ \tau(w) = \kappa(w) \circ \theta \qquad \text{for all } w \in W. \tag{8}$$

Let $P = \{\alpha_s \mid s \in S\}$. Clearly $Tv(P) = \tau(W)$ and $G(P)$ is (isomorphic to) Γ. By Lemma 4.1 there exists P' in t-equivalence class of P for which $G(P')$ is a tree. Since $G(P)$ is not a line graph, the tree $G(P')$ is not a path by Lemma 4.2. By Lemma 4.3 the basis P' of V, as well as P, is of orthogonal type. By Lemma 4.4, the group $\tau(W) = O(V)$ and the $\tau(W)$-orbits on V are $\{0\}$, $Q^{-1}(0)\backslash\{0\}$, and $Q^{-1}(1)$. Applying (8) and since θ is an isomorphism by Lemma 2.1, the result follows. □

5 Proof of Theorem 3.2 and Corollary 3.3

Recall the basis $\{\alpha_s^\vee \mid s \in S\}$ of V from (7). To prove Theorem 3.2 and Corollary 3.3, we introduce a simple graph which includes the information of the values $B(\alpha_s^\vee, \alpha_t^\vee)$ for all $s, t \in S$ as follows.

Define $R^\vee$ to be the set consisting of all two-element subsets $\{s, t\}$ of S with $B(\alpha_s^\vee, \alpha_t^\vee) = 1$. Define $\Gamma^\vee$ to be the simple graph with vertex set S and edge set $R^\vee$. We will refer to $\Gamma^\vee$ as the *dual graph* of Γ. Note that the notion of dual graphs defined above is different from the usual ones in graph theory. The following lemma suggests why the graph $\Gamma^\vee$ is of interest.

Lemma 5.1 *For each $s \in S$ we have $\theta(\alpha_s^\vee) = f_s$.*

Proof Let $s, t \in S$ be given. Using (5) and (7), we have $\theta(\alpha_s^\vee)\alpha_t = 1$ whenever $s = t$ and otherwise $\theta(\alpha_s^\vee)\alpha_t = 0$. Comparing this with (1) the result follows. □

Lemma 5.2 *For each $s \in S$ we have*

$$\alpha_s = \sum_{st \in R} \alpha_t^\vee.$$

Proof Fix $s \in S$. By (1), (4), and (5), the vector $\theta(\alpha_s)$ is equal to

$$\sum_{st \in R} f_t.$$

By Lemma 5.1 the above is equal to

$$\theta\left(\sum_{st \in R} \alpha_t^\vee\right).$$

Now, by Lemma 2.1(ii) this lemma follows. □

Observe that $B_{\Gamma^\vee}$ is equivalent to B. Therefore $\Gamma^\vee$ is a nondegenerate graph. Since $\{\alpha_s \mid s \in S\}$ is the basis of V dual to $\{\alpha_s^\vee \mid s \in S\}$, the graph Γ is the dual graph of $\Gamma^\vee$. By duality Lemma 5.2 implies that

Lemma 5.3 *For each $s \in S$ we have*

$$\alpha_s^\vee = \sum_{st \in R^\vee} \alpha_t.$$

Lemma 5.4 *Let A and $A^\vee$ denote the adjacency matrices of Γ and $\Gamma^\vee$ over $\mathbb{F}_2$, respectively. Then A and $A^\vee$ are inverses of each other.*

Proof We show that $A^\vee A$ is equal to the identity matrix. Let $s, t \in S$ be given. By the comment below Lemma 5.1 the (s, t)-entry of A (resp. $A^\vee$) is equal to $B(\alpha_s, \alpha_t)$ (resp. $B(\alpha_s^\vee, \alpha_t^\vee)$). By the definition of $\Gamma^\vee$ the (s, t)-entry of $A^\vee A$ is equal to

$$B\left(\sum_{su \in R^\vee} \alpha_u, \alpha_t\right). \tag{9}$$

By Lemma 5.3 the vector in the first coordinate of (9) is equal to $\alpha_s^\vee$. Therefore (9) is equal to 1 if and only if $s = t$ by (7). The result follows. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2 In Lemma 5.1 we saw that $\theta(\alpha_s^\vee) = f_s$ for all $s \in S$. Therefore (i) and (ii) are equivalent by Theorem 3.1. To show that Γ is 2-lit, it is now enough to consider the two cases: (a) $Q(\alpha_s^\vee) = 0$ for all $s \in S$; (b) $Q(\alpha_s^\vee) = 1$ for all $s \in S$.

(a) It suffices to show that there exist $s, t \in S$ such that $Q(\alpha_s^\vee + \alpha_t^\vee) = 1$. Since the form B is nontrivial there exist $s, t \in S$ such that $B(\alpha_s^\vee, \alpha_t^\vee) = 1$. Then the s and t are the desired elements in S.

(b) It suffices to show that there exist two distinct $s, t \in S$ such that $Q(\alpha_s^\vee + \alpha_t^\vee) = 0$. By our assumption, the graph Γ is not a complete graph. Using Lemma 5.4, we deduce that $\Gamma^\vee$ is not a complete graph. Therefore there exist two distinct $s, t \in S$ such that $B(\alpha_s^\vee, \alpha_t^\vee) = 0$. Such s and t are the desired elements in S. □

To prove Corollary 3.3, we give a sufficient condition for Theorem 3.2(ii).

Lemma 5.5 *Let $\Gamma = (S, R)$ denote a nondegenerate graph. Assume that there exists $s \in S$ with even degree in Γ such that*

$$\sum_{\substack{\{u,v\}\subseteq S \\ su,sv\in R}} B(\alpha_u^\vee, \alpha_v^\vee) = 0, \tag{10}$$

where the sum is over all two-element subsets $\{u, v\}$ of S with $su, sv \in R$. Then the restriction of Q to $\{\alpha_t^\vee \mid st \in R\}$ is surjective.

Proof Apply Q to either side of the equation in Lemma 5.2. Using (6), (10) and $Q(\alpha_s) = 1$ to evaluate the resulting equation, we obtain that

$$\sum_{st\in R} Q(\alpha_t^\vee) = 1. \tag{11}$$

By (11) there exists a neighbor u of s for which $Q(\alpha_u^\vee) = 1$. Since s has even degree in Γ there exists a neighbor v of s for which $Q(\alpha_v^\vee) = 0$. The result follows. □

Proof of Corollary 3.3. By Proposition 2.4 a nondegenerate bipartite graph Γ is a line graph if and only if Γ is a path of even order. Since every path is 1-lit, this corollary holds for Γ as a line graph. We thus assume that Γ is not a line graph. By Theorem 3.2 the graph Γ is 2-lit. By Lemma 5.4 we deduce that the graph $\Gamma^\vee$ is bipartite with the same bipartition that of Γ. We use this to show that (i) and (ii) are equivalent.

(ii) ⇒ (i): Let s denote a vertex of Γ with even degree. Since Γ and $\Gamma^\vee$ are bipartite graphs with same bipartition, we deduce that $B(\alpha_u^\vee, \alpha_v^\vee) = 0$ for any neighbors u, v of s in Γ. Therefore (10) holds. By Lemma 5.5 the restriction of Q on $\{\alpha_t^\vee \mid st \in R\}$ is onto. Therefore Γ is 1-lit by Theorem 3.2.

(i) ⇒ (ii): Suppose on the contrary that each vertex of Γ has odd degree. Using Lemma 5.4, we deduce that each vertex of $\Gamma^\vee$ has odd degree. Let s denote any element of S. By Lemma 5.3, $Q(\alpha_s^\vee)$ is equal to

$$Q\left(\sum_{st\in R^\vee} \alpha_t\right). \tag{12}$$

Since the bipartite graphs Γ and $\Gamma^\vee$ have the same bipartition, we deduce that $B(\alpha_u, \alpha_v) = 0$ for any neighbors u, v of s in $\Gamma^\vee$. By (6), the summation in (12) can be moved out front. Since $Q(\alpha_s) = 1$ for all $s \in S$, it follows that (12) is equal to 1, contradicting Theorem 3.2(ii). □

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