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Permutonestohedra

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Abstract There are several real spherical models associated with a root arrangement, depending on the choice of a *building set*. The connected components of these models are manifolds with corners which can be glued together to obtain the corresponding real De Concini–Procesi models. In this paper, starting from any root system Φ with finite Coxeter group W and any W-invariant building set, we describe an explicit realization of the real spherical model as a union of polytopes (nestohedra) that lie inside the chambers of the arrangement. The main point of this realization is that the convex hull of these nestohedra is a larger polytope, a *permutonestohedron*, equipped with an action of W or also, depending on the building set, of $Aut(\Phi)$. The permutonestohedra are natural generalizations of Kapranov's permutoassociahedra.

Keywords Models of arrangements · Coxeter groups · Nestohedra

1 Introduction

Let V be a Euclidean vector space of dimension n, let $\Phi \subset V$ be a root system which spans V and has finite Coxeter group W and let $\mathcal{G}$ be a W-invariant building set associated with Φ (see Sect. 2 for a definition of building set).

The main goal of this paper is to provide an explicit linear realization of the *permutonestohedron* $P_{\mathcal{G}}(\Phi)$, a polytope whose face poset was introduced in [20]; this polytope is linked with the geometry of the wonderful model of the arrangement and is equipped with an action of W or also, depending on the building set, of $Aut(\Phi)$.

Let us briefly describe the framework from which the permutonestohedra arise.

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Given a building set $\mathcal{G}$ one can construct as in [7] a real and a complex De Concini–Procesi model (resp. $Y_{\mathcal{G}}(\mathbb{R})$ and $Y_{\mathcal{G}}$), or their 'spherical' version, a real model with corners $CY_{\mathcal{G}}$ (see [19]).

These models play a relevant role in several fields of mathematical research: subspace and toric arrangements, toric varieties (see for instance [9,17,31]), moduli spaces of curves and configuration spaces (see for instance [13,26]), box splines and index theory (see the exposition in [8]) and discrete geometry (see [14] for further references).

From the point of view of discrete geometry and combinatorics, the spherical model $CY_{\mathcal{G}}$ is a particularly interesting object. In fact $CY_{\mathcal{G}}$ has as many connected components as the number of chambers and these connected components are nonlinear realizations of some polytopes that belong to the family of nestohedra. The nestohedra have been defined and studied in [28,29,35], and successively in several other papers, due to the interest of nested sets complexes in combinatorics and geometry (see for instance [16] for applications to tropical geometry and [4] for applications to the study of Dahmen-Micchelli modules).

Moreover, we remark that a suitable gluing of the connected components of $CY_{\mathcal{G}}$ produces the model $Y_{\mathcal{G}}(\mathbb{R})$, which is, therefore, presented as a CW complex. If $\mathcal{G}$ is W-invariant, $Y_{\mathcal{G}}(\mathbb{R})$ (as well as $Y_{\mathcal{G}}$) inherits an action of W which gives rise to geometric representations in cohomology.

The permutonestohedron $P_{\mathcal{G}}(\Phi)$ arises in the middle of this rich algebraic and geometric picture. Its name comes from the following remarks:

- some of its facets lie inside the chambers of the arrangement; they are nestohedra, and their union is a W-invariant linear realization of $CY_{\mathcal{G}}$; their convex hull is the full permutonestohedron;
- if $\Phi = A_n$ and $\mathcal{G}$ is the minimal building set associated to Φ, the permutonestohedron is a realization of Kapranov's permutoassociahedron (see [25]).

We remark that a nonlinear realization in V of $CY_{\mathcal{G}}$ (and of a convex body whose face poset is the same as the face poset of $P_{\mathcal{G}}(\Phi)$) was provided in [20].

As we explain in Sect. 3.3, the linear realization of $CY_{\mathcal{G}}$ that we describe in this paper is inspired by (and generalizes in a way) Stasheff and Shnider's construction of the associahedron in the Appendix B of [33].

The main point in our choice of the defining hyperplanes is that the every nestohedron in $CY_{\mathcal{G}}$ lies inside a chamber of the arrangement; furthermore, each of these hyperplanes is invariant with respect to the action of a parabolic subgroup. These are the reasons why, in the global construction, when we consider the convex hull of all the nestohedra that lie in the chambers, the extra facets of the permutonestohedron $P_{\mathcal{G}}(\Phi)$ appear. These turn out to be combinatorially equivalent to the product of a nestohedron with some smaller permutonestohedra (see Theorem 6.1).

We notice that when $\Phi = A_1^n$, as the building set varies, the nestohedra we obtain inside every orthant are all the nestohedra in the 'interval simplex-permutohedron' (see [27]). In particular, in the case of the associahedron, our construction coincides with the one in [33], while for the other nestohedra it is analogue to the constructions in [12] and [3,11] that are remarkable generalizations of Stasheff and Shnider's construction.

In general, once Φ is fixed, the nestohedron which lies inside a chamber depends on $\mathcal{G}$. For instance, the minimal building set associated to Φ is the building set made by the *irreducible* subspaces, i.e. the subspaces spanned by the irreducible root subsystems, while the maximal building set is made by all the subspaces spanned by some of the roots. Now, if Φ is A_n, B_n or C_n and $\mathcal{G}$ is the minimal building set, the nestohedron an $(n-1)$-dimensional Stasheff's associahedron; if Φ is D_n and $\mathcal{G}$ is the minimal building set, the nestohedron is a graph associahedron of type D. For any n-dimensional root system, if $\mathcal{G}$ is the maximal building set the nestohedron is an $(n-1)$-dimensional permutohedron (so in this case our permutonestohedron could also be called 'permutopermutohedron'). When the root system is of type A_n, B_n, C_n, D_n, the full poset of invariant building sets has been described in [21].

As mentioned before, we obtain the permutonestohedron $P_{\mathcal{G}}(\Phi)$ by taking the convex hull of the nestohedra that lie in the chambers and whose union realizes $CY_{\mathcal{G}}$. From this point of view, the construction of the permutonestohedron is inspired by Reiner and Ziegler construction of *Coxeter associahedra* in [32], since these are obtained as the convex hull of some Stasheff's associahedra that lie inside the chambers. Anyway we remark that only when $\Phi = A_n$ and $\mathcal{G}$ is the minimal building set the permutonestohedron $P_{\mathcal{G}}(\Phi)$ is combinatorially equivalent to a Coxeter associahedron (i.e. in the case of Kapranov's permutoassociahedron). For instance, in the B_n case, the Coxeter associahedron is the convex hull of some $(n-2)$-dimensional Stasheff's associahedra, while our minimal permutonestohedron is the convex hull of some $(n-1)$-dimensional Stasheff's associahedra; furthermore, nonminimal permutonestohedra are convex hulls of different nestohedra.

We will also focus on the group of isometries of the permutonestohedron $P_{\mathcal{G}}(\Phi)$: it contains W or even, depending on the 'symmetry' of $\mathcal{G}$, the group $Aut(\Phi)$, and we will describe some conditions which are sufficient to ensure that it is equal to $Aut(\Phi)$. We will also point out that a subposet of the minimal permutonestohedron of type A_{n-1} is equipped with an 'extended' action of S_{n+1}.

Finally, here is a short outline of this paper. In Sect. 2 we briefly recall the basic properties of building sets, nested sets and spherical models, while Sect. 3 contains the definition of permutonestohedra and the statement of main theorems, Theorems 3.1 and 3.2, whose proofs can be found, respectively, in Sects. 4 and 5.

Section 6 is devoted to a presentation of the face poset of the permutonestohedron $P_{\mathcal{G}}(\Phi)$; this points out the action of W, since the faces of $P_{\mathcal{G}}(\Phi)$ are indexed by pairs of the form (coset of W, labelled nested set). Furthermore, Theorem 6.1 shows that every facet that does not lie inside a chamber is combinatorially equivalent to the product of a nestohedron with some smaller permutonestohedra. We also show (Corollary 6.2) that $P_{\mathcal{G}}(\Phi)$ is a simple polytope if and only if $\mathcal{G}$ is the maximal building set associated to Φ.

The full group of isometries that leave $P_{\mathcal{G}}(\Phi)$ invariant is explored in Sect. 7: if $\mathcal{G}$ is invariant with respect to $Aut(\Phi)$ this group contains $Aut(\Phi)$, and we describe some natural conditions that are sufficient to ensure that it is equal to $Aut(\Phi)$.

In Sect. 8 we show how the well-known 'extended' S_{n+1} action on the minimal De Concini–Procesi model of type A_{n-1} can be lifted to a subposet of the minimal permutonestohedron of type A_{n-1}, providing geometrical realizations of the representations $Ind_G^{S_{n+1}} Id$, where G is any subgroup of the cyclic group of order $n+1$.

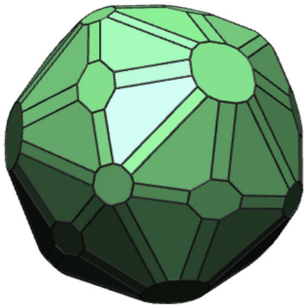


Fig. 1 The maximal permutonestohedron of type B_3: *inside* each chamber of the arrangement there is a nestohedron (in this case a hexagon, i.e. a two-dimensional permutohedron), and the permutonestohedron is the convex hull of all these nestohedra

Finally, in Sect. 9 we show some examples and pictures of permutonestohedra and, as an example of face counting, we compute the f-vectors of the minimal and of the maximal permutonestohedron of type A_n.

All the pictures of three-dimensional polytopes (like the one in Fig. 1) have been realized using the mathematical software package *polymake* ([22]).

2 Building sets, nested sets, nestohedra and spherical models

Let us recall from [6,7] the definitions of nested set and building set of subspaces; more precisely, we specialize these definitions to the case we are interested in, i.e. the case when we deal with a central hyperplane arrangement.

Let $\mathcal{A}$ be a central line arrangement in an Euclidean space V. We denote by $\mathcal{C}_{\mathcal{A}}$ the closure under the sum of $\mathcal{A}$ and by $\mathcal{A}^{\perp}$ the hyperplane arrangement

$$\mathcal{A}^{\perp} = \{A^{\perp} \mid A \in \mathcal{A}\}.$$

Definition 2.1 The collection of subspaces $\mathcal{G} \subset \mathcal{C}_{\mathcal{A}}$ is called *building set associated to* $\mathcal{A}$ if $\mathcal{A} \subset \mathcal{G}$ and every element C of $\mathcal{C}_{\mathcal{A}}$ is the direct sum $C = G_1 \oplus G_2 \oplus \cdots \oplus G_k$ of the maximal elements $G_1, G_2, \ldots, G_k$ of $\mathcal{G}$ contained in C. This is called the $\mathcal{G}$-decomposition of C.

Given a central line arrangement $\mathcal{A}$, there are several building sets associated to it. Among these there are always a maximum and a minimum (with respect to inclusion). The maximum is $\mathcal{C}_{\mathcal{A}}$, the minimum is the building set of *irreducibles*. In the case of a root arrangement, the building set of irreducibles is the set of subspaces spanned by the irreducible root subsystems of the given root system (see [34]).

Definition 2.2 Let $\mathcal{G}$ be a building set associated to $\mathcal{A}$. A subset $\mathcal{S} \subset \mathcal{G}$ is called ($\mathcal{G}$-) nested, if given a subset $\{U_1, \ldots, U_h\}$ (with $h > 1$) of pairwise noncomparable elements in $\mathcal{S}$, then $U_1 \oplus \cdots \oplus U_h \notin \mathcal{G}$.

After De Concini and Procesi's paper [7], nested sets and building sets appeared in the literature, connected with several combinatorial problems. One can see for instance [15], where building sets and nested sets were defined in the more general context of meet semilattices, and [10], where the connection with Dowling lattices was investigated. Other purely combinatorial definitions were used to give rise to the polytopes that were named *nestohedra* in [29] and turned out to play a relevant role in discrete mathematics and tropical geometry. Presentations of nestohedra can be found for instance in [2,3,16,27–29,35].

Here we recall, for the convenience of the reader, the combinatorial definitions of building sets and nested sets as they appear in [28,29]. One can refer to Sect. 2 of [27] for a short comparison among various definitions and notations in the literature.

Definition 2.3 A building set $\mathcal{B}$ is a set of subsets of $\{1, 2, \dots, n\}$, such that

(a) If $A, B \in \mathcal{B}$ have nonempty intersection, then $A \cup B \in \mathcal{B}$.
(b) The set $\{i\}$ belongs to $\mathcal{B}$ for every $i \in \{1, 2, \dots, n\}$.

Definition 2.4 A subset $\mathcal{S}$ of a building set $\mathcal{B}$ is a nested set if and only if the following three conditions hold:

(a) For any $I, J \in \mathcal{S}$ we have that either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.
(b) Given elements $\{J_1, \dots, J_k\}$ $(k \geq 2)$ of $\mathcal{S}$ pairwise not comparable with respect to inclusion, their union is not in $\mathcal{B}$.
(c) $\mathcal{S}$ contains all the sets of $\mathcal{B}$ that are maximal with respect to inclusion.

The nested set complex $\mathcal{N}(\mathcal{B})$ is the poset of all the nested sets of $\mathcal{B}$ ordered by inclusion. A nestohedron $P_{\mathcal{B}}$ is a polytope whose face poset, ordered by reverse inclusion, is isomorphic to the nested set complex $\mathcal{N}(\mathcal{B})$.

Let us now consider a 'geometric' building set $\mathcal{G}$ of subspaces associated with a root arrangement, according to Definition 2.1, and let us suppose that $V \in \mathcal{G}$.

Definition 2.5 We will denote by $\mathcal{G}_{fund}$ the building set made by the subspaces in $\mathcal{G}$ that are spanned by some subset of the set of simple roots.

Now $\mathcal{G}_{fund}$ gives rise to a building set in the sense of Definition 2.3 in the following way: we associate to a subspace $A \in \mathcal{G}_{fund}$ the set of indices of the simple roots contained in it.

Having established this correspondence, the nested sets in the sense of De Concini and Procesi are nested sets in the sense of Definition 2.4 provided that they contain V, the maximal subspace in $\mathcal{G}_{fund}$. These nested sets form a nested set complex denoted by $\mathcal{N}(\mathcal{G}_{fund})$. The nestohedra that will appear in our construction, as facets of permutonestohedra, are the nestohedra $P_{\mathcal{G}_{fund}}$ associated with the nested set complexes $\mathcal{N}(\mathcal{G}_{fund})$.

Now let $\mathcal{A}$ be, as above, a central line arrangement in a Euclidean space V, and let $\mathcal{M}(\mathcal{A}^\perp)$ be the complement in V to $\mathcal{A}^\perp$. In [19] the following compactifications of $\widehat{\mathcal{M}}(\mathcal{A}^\perp) = \mathcal{M}(\mathcal{A}^\perp)/\mathbb{R}^+$ were defined, in the spirit of De Concini–Procesi construction of wonderful models.

Let us denote by $S(V)$ the $(n-1)$-th dimensional unit sphere in V, and, for every subspace $A \subset V$, let $S(A) = A \cap S(V)$. Let $\mathcal{G}$ be a building set associated to $\mathcal{A}$, and let us consider the compact manifold

$$K = S(V) \times \prod_{A \in \mathcal{G}} S(A)$$

There is an open embedding $\phi : \widehat{\mathcal{M}}(\mathcal{A}^{\perp}) \longrightarrow K$ which is obtained as a composition of the section $s : \widehat{\mathcal{M}}(\mathcal{A}^{\perp}) \mapsto \mathcal{M}(\mathcal{A}^{\perp})$

$$s([p]) = \frac{p}{|p|} \in S(V) \cap \mathcal{M}(\mathcal{A}^{\perp})$$

with the map

$$\mathcal{M}(\mathcal{A}^{\perp}) \mapsto S(V) \times \prod_{A \in \mathcal{G}} S(A)$$

that is well defined on each factor.

Definition 2.6 We denote by $CY_{\mathcal{G}}$ the closure in K of $\phi(\widehat{\mathcal{M}}(\mathcal{A}^{\perp}))$.

It turns out (see [19]) that $CY_{\mathcal{G}}$ is a smooth manifold with corners. Its (not connected) boundary components are in correspondence with the elements of the building set $\mathcal{G}$, and the intersection of some boundary components is nonempty if and only if these components correspond to a nested set. We notice that $CY_{\mathcal{G}}$ has as many connected components as the number of chambers of $\mathcal{M}(\mathcal{A}^{\perp})$.

In [20] these connected components were realized inside the chambers, as the complements of a suitable set of tubular neighborhoods of the subspaces in $\mathcal{G}^{\perp}$, giving rise to some nonlinear realizations of the nestohedra $P_{\mathcal{G}_{fund}}$.

3 The construction of permutonestohedra

3.1 Selected hyperplanes

The goal of this section is to give the equations of the hyperplanes that will be used in the definition of the permutonestohedra.

Let V be as before a Euclidean vector space of dimension n with scalar product $(\, , \,)$. Let us consider a root system Φ in V with finite Coxeter group W, and a basis of *simple roots* $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for Φ. If Φ is irreducible, we consider in the open fundamental chamber $Ch(\Delta)$ the vector

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{n} \omega_i,$$

where Φ^+ is the set of positive roots and the simple weights ω_i are defined by

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}$$

(see [1,24] as general references on root systems and finite Coxeter groups).

If Φ is not irreducible and splits into the irreducible subsystems $\Phi_1, \Phi_2, \ldots, \Phi_s$ we put $\delta = \sum \delta_{\Phi_i}$, where for every i, δ_{Φ_i} is half the sum of all the positive roots of Φ_i.

Let us denote by $\mathcal{C}_\Phi$ the building set of all the subspaces that can be generated as the span of some of the roots in Φ and by $\mathcal{F}_\Phi$ the building set of all the irreducible subspaces in $\mathcal{C}_\Phi$. As we recalled in Sect. 2, $\mathcal{F}_\Phi$ is made by all the subspaces that are spanned by the irreducible root subsystems of Φ.

Let $\mathcal{G}$ be a building set associated to the root system Φ which contains V and is W-invariant; when the root system is of type A_n, B_n, C_n, D_n these building sets have been classified in [21].

Given $A \in \mathcal{G}_{fund} - \{V\}$, we will denote by W_A the parabolic subgroup generated by the reflections s_α with $\alpha \in \Phi \cap A$. We denote by δ_A the orthogonal projection of δ to $A^\perp$; we have

$$\delta_A = \frac{1}{|W_A|} \sum_{\sigma \in W_A} \sigma(\delta)$$

and we also write $\delta_A = \delta - \pi_A$, where π_A belongs to A.

Remark 3.1 If $A \cap \Phi$ is an irreducible root subsystem, we notice that π_A is half the sum of all its positive roots. If $A \cap \Phi$ is not irreducible, and splits into the irreducible subsystems $\Phi_1, \dots, \Phi_s$, then $\pi_A = \sum \pi_{\Phi_i}$ where π_{Φ_i} is half the sum of all the positive roots of Φ_i.

Let I be the set of indices $i \in \{1, 2, \dots, n\}$ such that $\alpha_i \in A$ and let J be the set of indices, in the complement of I, such that $j \in J$ iff $(\alpha_j, \alpha_i) < 0$ for some $i \in I$. Then

$$\pi_A = \sum_{i \ s.t. \ \alpha_i \in A} \omega_i - \sum_{j \in J} c_j \omega_j$$

with all $c_j > 0$. Therefore,

$$\delta_A = \sum_{s \in \{1,2,\dots,n\} - I} b_s \omega_s$$

with all $b_s \geq 1$.

Remark 3.2 We put $\delta_V = \pi_V = \delta$. Despite appearances, this notation will not be confusing.

Let us consider two subspaces $B \subset A$ in $\mathcal{G}_{fund}$ of dimension $j < i$, respectively, and write π_A and π_B as nonnegative linear combinations of the simple roots. We denote by a the maximum coefficient of π_A and by b the minimum coefficient of π_B and put $R_B^A = \frac{a}{b}$.

We then define R_j^i as the maximum among all the R_B^A with A, B as above.

Definition 3.1 A list of positive real numbers $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_{n-1} < \epsilon_n = a$ will be *suitable* if $\epsilon_i > 2R_{i-1}^i \epsilon_{i-1}$ for every $i = 2, \dots, n$.

We are now in position to define a set of selected hyperplanes that depend on the choice of a suitable list $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_{n-1} < \epsilon_n = a$ and are indexed by the elements of $\mathcal{C}_{\Phi_{fund}}$. These hyperplanes, together with their images via the W action, will be the defining hyperplanes of the permutonestohedron $P_{\mathcal{G}}(\Phi)$.

The motivation for this definition of suitable list will be pointed out in Sect. 3.3.

We start with the following definition.

Definition 3.2 Let us denote by H_V the hyperplane

$$H_V = \{x \in V \mid (x, \delta) = a\}$$

and by HS_V the closed half-space that contains the origin and whose boundary is H_V.

For every $i = 1, 2, \ldots, n$ we call v_i the intersection $H_V \cap < \omega_i >$; all the vectors v_i lie on the hyperplane H_V and their convex hull, as it is well known, is an $(n-1)$-simplex.

Definition 3.3 For every $A \in \mathcal{G}_{fund} - \{V\}$, we define the hyperplane H_A as

$$H_A = \{x \in V \mid (x, \delta_A) = a - \epsilon_{dim\ A}\}$$

and we denote by HS_A the closed half-space that contains the origin and whose boundary is H_A.

Now we have to define the hyperplanes indexed by the elements of $\mathcal{C}_{\Phi_{fund}} - \mathcal{G}_{fund}$:

Definition 3.4 Let B be a subspace in $\mathcal{C}_{\Phi_{fund}} - \mathcal{G}_{fund}$, i.e. a subspace which does not belong to $\mathcal{G}_{fund}$ and is generated by some of the simple roots, and let $B = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, where $A_1, A_2, \ldots, A_k$ is its $\mathcal{G}_{fund}$-decomposition. Then we put

$$\overline{H}_B = \{x \in V \mid (x, \delta_B) = a - \epsilon_{dim\ A_1} - \epsilon_{dim\ A_2} - \cdots - \epsilon_{dim\ A_k}\},$$

where $\delta_B = \delta - \sum_{i=1}^{k} \pi_{A_i}$. We denote by $\overline{HS}_B$ the closed half-space that contains the origin and whose boundary is $\overline{H}_B$.

3.2 Definition of the permutonestohedra

This section starts with the definition of the permutonestohedron $P_{\mathcal{G}}(\Phi)$ as the intersection of closed half-spaces. We then state two theorems that show that the permutonestohedron is a convex hull of nestohedra and explicitly determine its vertices.

Definition 3.5 The *permutonestohedron* $P_{\mathcal{G}}(\Phi)$ is the polytope given by the intersection of the half-spaces σHS_A and of the half-spaces $\sigma \overline{HS}_B$, for all $\sigma \in W$, $A \in \mathcal{G}_{fund}$ and $B \in \mathcal{C}_{\Phi_{fund}} - \mathcal{G}_{fund}$.

Remark 3.3 We are in fact defining infinite permutonestohedra, depending on the choice of a suitable list $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_{n-1} < \epsilon_n = a$, so we should write

$P_{\mathcal{G}}(\Phi, \epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ instead of $P_{\mathcal{G}}(\Phi)$. We use the shorter notation $P_{\mathcal{G}}(\Phi)$ since in this paper the dependence on the suitable list will be relevant only when we will deal with the automorphism group in Sect. 7, where we will take care of avoiding ambiguities.

Now, given a subset $\mathcal{T}$ of $\mathcal{G}_{fund}$ containing V and of cardinality n, with the property that the vectors $\{\delta_A \mid A \in \mathcal{T}\}$ are linearly independent, we denote by $v_{\mathcal{T}}$ the vector defined by

$$v_{\mathcal{T}} = \bigcap_{A \in \mathcal{T}} H_A.$$

As we will observe in Proposition 4.2, every maximal nested set $\mathcal{S}$ of $\mathcal{G}_{fund}$ has the above mentioned property.

The following theorems will be proven in Sects. 4 and 5.

Theorem 3.1 *The intersection of H_V with all the half-spaces HS_A, for $A \in \mathcal{G}_{fund} - \{V\}$, is a realization of the nestohedron $P_{\mathcal{G}_{fund}}$ which lies in the open fundamental chamber. Its vertices are the vectors $v_{\mathcal{S}}$, where $\mathcal{S}$ ranges among the maximal nested sets in $\mathcal{G}_{fund}$.*

Theorem 3.2 *The permutonestohedron $P_{\mathcal{G}}(\Phi)$ coincides with the convex hull of all the nestohedra $\sigma P_{\mathcal{G}_{fund}}$ ($\sigma \in W$) that lie inside the open chambers. Its vertices are $\sigma v_{\mathcal{S}}$, where $\sigma \in W$ and $\mathcal{S}$ ranges among the maximal nested sets in $\mathcal{G}_{fund}$.*

3.3 Motivations for the choice of suitable lists

In the proofs of Theorems 3.1 and 3.2 the properties of suitable lists will play a crucial role. This is why we devote this section to suitable lists: we will prove a key lemma and we will show how these lists generalize Stasheff and Shnider's choice of parameters in their construction of the associahedron in [33].

The following lemma introduces an important inequality that is satisfied by suitable lists and is tied to the combinatorics of root systems.

Lemma 3.1 *Let $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_{n-1} < \epsilon_n = a$ be a suitable list of positive numbers. Let B be a subspace in $\mathcal{G}_{fund}$ that can be expressed as a sum of some subspaces $B_1, B_2, \ldots, B_r$ in $\mathcal{G}_{fund}$ ($r > 1$), and let this sum be nonredundant, i.e. if we remove anyone of the subspaces B_i, the sum of the others is strictly included in B. Then, we have*

$$\epsilon_{dim\ B} > \sum_{i=1}^{r} R_{dim\ B_i}^{dim\ B} \epsilon_{dim\ B_i}$$

Proof Let $m = dim\ B$. We notice that by definition of the numbers R_j^i we have

$$\sum_{i=1}^{r} R_{dim\ B_i}^{m} \pi_{B_i} \geqslant \pi_B,$$

where $\alpha \gtrdot \beta$ means that the difference $\alpha - \beta$ can be expressed as a nonnegative linear combination of the simple roots. Let $m-k$ be the maximum among the dimensions of the B_i's. Then $r \leq k+1$ because of the nonredundancy of the B_i's: anyone of the B_i's contains at least a simple root which is not contained in the others. Now $\epsilon_m > 2R^m_{m-1}\epsilon_{m-1}$ by definition, and then recursively we obtain

$$\epsilon_m > 2^k R^m_{m-1} R^{m-1}_{m-2} \cdots R^{m-k+1}_{m-k} \epsilon_{m-k}.$$

Since $R^m_{m-1} R^{m-1}_{m-2} \cdots R^{m-k+1}_{m-k} \geq R^m_{m-k}$ and $2^k \geq k+1$ (when $k \geq 1$), this implies

$$\epsilon_m > 2^k R^m_{m-k}\epsilon_{m-k} \geq (k+1) R^m_{m-k}\epsilon_{m-k} \geq r R^m_{m-k}\epsilon_{m-k}.$$

The final inequality

$$r R^m_{m-k}\epsilon_{m-k} \geq \sum_{i=1}^{r} R^m_{dim\ B_i}\epsilon_{dim\ B_i}$$

is now straightforward. □

Depending on Φ and on the building set $\mathcal{G}$, there may be lists of numbers ϵ_i that are not suitable but can be used to construct permutonestohedra, since they ensure that the claim of the above lemma is true.

More generally, we could construct permutonestohedra using the following *suitable functions*: first, for every set of subspaces $B, B_1, B_2, \ldots, B_r$ as in the statement of the Lemma 3.1, let us choose numbers $R^B_{B_i}$ such that

$$\sum_{i=1}^{r} R^B_{B_i}\pi_{B_i} \gtrdot \pi_B.$$

Definition 3.6 A function $\epsilon : \mathcal{G} \mapsto \mathbb{R}^+$ is suitable if, for every set of subspaces $B, B_1, B_2, \ldots, B_r$ in $\mathcal{G}_{fund}$ as above and for every $w \in W$, it satisfies

$$\epsilon(wB) > \sum_{i=1}^{r} R^B_{B_i}\epsilon(wB_i).$$

This is the essential property we need in our construction of permutonestohedra. As it is shown by Lemma 3.1, given a suitable list of numbers, one obtains a suitable function by putting $\epsilon(C) = \epsilon_{dim\ C}$ for every $C \in \mathcal{G}$.

We chose to use suitable lists to make our construction more concrete and to obtain more symmetry: since the associated suitable function depends only on the dimension of the subspaces, if $\mathcal{G}$ is $Aut(\Phi)$-invariant the automorphism group of the permutonestohedron includes $Aut(\Phi)$, as we will show in Sect. 7. Anyway, the definition of the hyperplanes H_A, $\overline{H}_B$, and the proofs of Theorems 3.1 and 3.2 in the next sections could be repeated almost verbatim using a suitable function instead of a suitable list.

A further interesting aspect of suitable lists and suitable functions is illustrated by the example of the root system A_1^n that corresponds to the boolean arrangement.

In this case our definition of suitable list consists in the condition $\epsilon_i > 2\epsilon_{i-1}$ for every $i > 1$. As for suitable functions, in their definition we can choose all the numbers R_C^B equal to 1. Now let us number from 1 to n the positive roots of A_1^n. Then let us denote by $\mathcal{G}'$ the $W = \mathbb{Z}_2^n$-invariant building set made by the subspaces that are spanned by a set of positive roots whose associated numbers are an *interval* in $[1, \dots, n]$. For this $\mathcal{G}'$ we have $\mathcal{G}' = \mathcal{G}'_{fund}$ and the corresponding nestohedron $P_{\mathcal{G}'_{fund}}$ that we obtain in the fundamental chamber is a Stasheff's associahedron. In fact in this case, one immediately checks that our suitable functions are the same as the suitable functions used by Stasheff and Shnider in their construction of the associahedron in Appendix B of [33].

In Sect. 9 of [12], Došen and Petrić describe a generalization of Stasheff and Shnider's construction to all the nestohedra that are in the 'interval simplex-permutohedron'. These nestohedra are exactly all the nestohedra obtained as $\mathcal{G}$ varies among the $\mathbb{Z}_2^n$-invariant building sets associated to A_1^n and our suitable functions for this root system are compatible with the ones described in [12].

4 The nestohedron $P_{\mathcal{G}_{fund}}$

This section is devoted to the proof of Theorem 3.1. We will give a self-contained proof that the hyperplanes H_A for $A \in \mathcal{G}_{fund}$ define a realization of the nestohedron $P_{\mathcal{G}_{fund}}$; as we have remarked in Sect. 3.3, our construction is a generalization of Stasheff and Shnider's construction of the associahedron in Appendix B of [33]. We notice that other constructions of nestohedra can be found for instance in [2,3,12,27–29,35].

Our choice of the hyperplanes ensures that the resulting nestohedron lies inside a chamber of the arrangement. Furthermore, for every $A \in \mathcal{G}_{fund}$ the hyperplane H_A is fixed by the action of W_A. Thanks to these properties when we pass to the global construction, and consider the convex hull of all the nestohedra which lie in the chambers, the extra facets of the permutonestohedron $P_{\mathcal{G}}(\Phi)$ appear.

Proposition 4.1 *Let us consider a subset $\mathcal{T}$ of $\mathcal{G}_{fund}$ containing V and of cardinality n such that the vectors $\{\delta_A \mid A \in \mathcal{T}\}$ are linearly independent: if $\mathcal{T}$ is not nested the vector $v_{\mathcal{T}}$ does not belong to $P_{\mathcal{G}_{fund}}$.*

Proof If $v_{\mathcal{T}}$ does not belong to the open fundamental chamber, it does not belong to $P_{\mathcal{G}_{fund}}$. In fact let us write a vector $x \in P_{\mathcal{G}_{fund}}$ in terms of the basis ω_i, $x = \sum b_i\omega_i$; since x belongs to $HS_{<\alpha_i>}$ for every simple root $\alpha_i \in \Delta$, we have $(x, \pi_{<\alpha_i>}) \geq \epsilon_1 > 0$, which implies $b_i > 0$ for every i.[1]

Let us then consider the case when $v_{\mathcal{T}}$ lies in the open fundamental chamber.

[1] This has the following interpretation: after truncating the starting simplex (the one generated in H_V by the vectors v_i, see Sect. 3.2) by the hyperplanes $H_{<\alpha_i>}$ we have a polytope which lies in the fundamental chamber and which, after other truncations, will become $P_{\mathcal{G}_{fund}}$.

Let B be a subspace in $\mathcal{G}_{fund}$ that can be expressed as a sum of some (more than one) subspaces in $\mathcal{T}$. Such B exists since $\mathcal{T}$ is not nested.

Now let $B = B_1 + \cdots + B_r$ with $r > 1$, $\{B_1, B_2, \ldots, B_r\} \subset \mathcal{T}$ be a nonredundant sum (see the statement of Lemma 3.1).

First we show that $B \notin \mathcal{T}$. Since $v_{\mathcal{T}}$ is in the open fundamental chamber, it can be written as $v_{\mathcal{T}} = \sum_{i=1,\ldots,n} b_i \omega_i$ with all the coefficients $b_i > 0$. Then if $B \in \mathcal{T}$ we have $(v_{\mathcal{T}}, \pi_B) = \epsilon_{dim\ B}$ and $(v_{\mathcal{T}}, \pi_{B_i}) = \epsilon_{dim\ B_i}$. According to the definition of the numbers R^i_j we have that

$$\sum_{i=1,\ldots,r} R^{dim\ B}_{dim\ B_i} \pi_{B_i} \gtrdot \pi_B.$$

Then we deduce that $(v_{\mathcal{T}}, \sum_{i=1,\ldots,r} R^{dim\ B}_{dim\ B_i} \pi_{B_i}) \geq (v_{\mathcal{T}}, \pi_B)$, which is a contradiction since $\epsilon_{dim\ B} > \sum_{i=1,\ldots,r} R^{dim\ B}_{dim\ B_i} \epsilon_{dim\ B_i}$ by Lemma 3.1.

So we can assume $B \notin \mathcal{T}$. We will show that $v_{\mathcal{T}} \notin HS_B$.

We notice that

$$\begin{aligned} HS_B \cap H_V &= \{x \in H_V \mid (x, \delta_B) \leq a - \epsilon_{dim\ B}\} = \\ &= \{x \in H_V \mid (x, \delta - \pi_B) \leq a - \epsilon_{dim\ B}\} = \{x \in H_V \mid (x, \pi_B) \geq \epsilon_{dim\ B}\} \end{aligned}$$

Let us then check if $v_{\mathcal{T}}$ belongs to HS_B. As we observed before we have

$$(v_{\mathcal{T}}, \pi_B) \leq (v_{\mathcal{T}}, \sum_{j=1,\ldots,r} R^{dim\ B}_{dim\ B_j} \pi_{B_j}) = \sum_{j=1,\ldots,r} R^{dim\ B}_{dim\ B_j} \epsilon_{dim\ B_j}$$

Now Lemma 3.1 implies $(v_{\mathcal{T}}, \pi_B) < \epsilon_{dim\ B}$, which proves that $v_{\mathcal{T}}$ does not belong to HS_B (so it does not belong to $P_{\mathcal{G}_{fund}}$). □

Proposition 4.2 *If $\mathcal{S}$ is a nested set of $\mathcal{G}_{fund}$, the vectors $\{\delta_A \mid A \in \mathcal{S}\}$ are linearly independent. If $\mathcal{S}$ is a maximal nested set, the vectors $\{\delta_A \mid A \in \mathcal{S}\}$ are a basis of V and the vectors $v_{\mathcal{S}}$ lie inside the fundamental chamber.*

Proof It is sufficient to prove the linear independence for maximal nested sets, since every nested set can be completed to a maximal nested set.

Let then $\mathcal{S}$ be a maximal nested set (therefore, it contains V). As we have already observed, the vectors $\{\delta_A \mid A \in \mathcal{S}\}$ are linearly independent if and only if the vectors $\{\pi_A \mid A \in \mathcal{S}\}$ are linearly independent. If these vectors are not linearly independent, since for every A the vector π_A belongs to A, we can find a minimal $C \in \mathcal{S}$ such that the vectors $\{\pi_D \mid D \in \mathcal{S}, D \subseteq C\}$ are not linearly independent. Therefore, we have, by nestedness of $\mathcal{S}$ and by minimality of C, a relation of the form

$$\pi_C = \sum_{D \in \mathcal{S}, D \subsetneq C} \gamma_D \pi_D$$

This is a contradiction, since C contains a simple root α_i which is not contained in any $D \in \mathcal{S}$, $D \subsetneq C$ [2]: when π_C is written in terms of the simple roots α_j, its $i-th$ coefficient is >0, while when we write π_D ($D \in \mathcal{S}$, $D \subsetneq C$) in terms of the simple roots the $i-th$ coefficient is equal to 0.

This proves the linear independence, and therefore, $v_{\mathcal{S}}$ is well defined.

Let us now prove that $v_{\mathcal{S}}$ lies in the fundamental chamber. Let us consider the graph associated to $\mathcal{S}$. This graph is a tree and coincides with the Hasse diagram of the poset induced by the inclusion relation. Therefore, it can be considered as an oriented rooted tree: the root is V and the orientation goes from the root to the leaves that are the minimal subspaces of $\mathcal{S}$. We observe that we can partition the set of vertices of the tree into *levels*: level 0 is made by the leaves, and in general, level k is made by the vertices v such that the maximal length of an oriented path that connects v to a leaf is k.

Let $v_{\mathcal{S}} = \sum_{i=1}^{n} b_i \omega_i$. Since $\mathcal{S}$ is maximal, it contains at least a subspace of dimension 1. In particular, all the minimal subspaces, i.e. the leaves of the graph, have dimension 1. Let $< \alpha_{i_1} >, \ldots, < \alpha_{i_r} >$ be the subspaces of dimension 1 in $\mathcal{S}$. From the relation $(v_{\mathcal{S}}, \pi_{<\alpha_{i_j}>}) = \epsilon_1$ we deduce that $b_{i_j} > 0$ for every $j = 1, \ldots, r$. Now if A is a subspace which in the graph is in level 1, then it contains some of the leaves, say $< \alpha_{i_1} >, \ldots, < \alpha_{i_q} >$. By the maximality of $\mathcal{S}$ we deduce that $dim\ A = q+1$ and we can write $A =< \alpha_h, \alpha_{i_1}, \ldots, \alpha_{i_q} >$, where α_h is the only simple root which belongs to A but does not belong to the leaves of the graph. Then

$$\pi_A = c_h \alpha_h + \sum_{j=1}^{q} a_j \pi_{<\alpha_{i_j}>}$$

with $c_h > 0$ and $a_j \leq R_1^{q+1}$ for every $j = 1, \ldots, q$. Therefore,

$$(v_{\mathcal{S}}, \pi_A) = \epsilon_{q+1} = c_h(v_{\mathcal{S}}, \alpha_h) + \sum_{j=1}^{q} a_j (v_{\mathcal{S}}, \pi_{<\alpha_{i_j}>}) \leq c_h(v_{\mathcal{S}}, \alpha_h) + q R_1^{q+1} \epsilon_1.$$

From Lemma 3.1 we know that

$$\epsilon_{q+1} > (q+1) R_1^{q+1} \epsilon_1$$

Then $c_h(v_{\mathcal{S}}, \alpha_h)$ must be > 0 and from this we deduce, given that $c_h > 0$ and $(v_{\mathcal{S}}, \alpha_h)$ is a positive multiple of b_h that $b_h > 0$. In a similar way, by induction on the level, we prove that all the coefficients b_i are > 0. □

Proposition 4.3 *Let us consider a maximal nested set $\mathcal{S}$ of $\mathcal{G}_{fund}$. For every $A \in \mathcal{G}_{fund} - \mathcal{S}$ the vector $v_{\mathcal{S}}$ belongs to the open part of HS_A, therefore, $v_{\mathcal{S}}$ is a vertex of $P_{\mathcal{G}_{fund}}$.*

[2] Otherwise C would be equal to the sum of the subspaces D, and this would contradict the nestedness of $\mathcal{S}$.

Proof We know by definition that $v_{\mathcal{S}}$ belongs to the hyperplanes H_Γ for every $\Gamma \in \mathcal{S}$, therefore, to prove the claim it is sufficient to show that for every $A \in \mathcal{G}_{fund} - \mathcal{S}$, the vector $v_{\mathcal{S}}$ belongs to the open part of HS_A.

The set $\{A\} \cup \mathcal{S}$ is not nested since $\mathcal{S}$ is a maximal nested set and A does not belong to $\mathcal{S}$.

Let C be the minimal element in $\mathcal{S}$ which strictly contains A (it could be $C = V$).

We observe that there is one (and only one, by the maximality of $\mathcal{S}$) simple root α_i which belongs to C but does not belong to any $K \in \mathcal{S}$ such that $K \subsetneq C$. Then α_i must belong to A: if $\alpha_i \notin A$, we have $A \subseteq T = \sum_{K \in \mathcal{S}, K \subsetneq C, K \cap A \neq \{0\}} K$. We notice that T strictly includes the subspaces K such that $K \in \mathcal{S}$, $K \subsetneq C$, $K \cap A \neq \{0\}$ because of the minimality of C. Now, since $A + T \in \mathcal{G}_{fund}$[3] this implies that $T \in \mathcal{G}_{fund}$ which contradicts the nestedness of $\mathcal{S}$.

Since $\alpha_i \in A$ we can find a subset $\mathcal{I}$ of $\{K \in \mathcal{S} \mid K \subsetneq C\}$ such that

$$C = A + \sum_{K \in \mathcal{I}} K$$

and the sum is nonredundant. Then we have

$$R^{dim\ C}_{dim\ A} \pi_A + \sum_{K \in \mathcal{I}} R^{dim\ C}_{dim\ K} \pi_K > \pi_C$$

which implies

$$\pi_A > \frac{1}{R^{dim\ C}_{dim\ A}} \left(\pi_C - \sum_{K \in \mathcal{I}} R^{dim\ C}_{dim\ K} \pi_K \right).$$

Now since $v_{\mathcal{S}}$ belongs to the open fundamental chamber (Proposition 4.2) we have

$$(v_{\mathcal{S}}, \pi_A) \geq \frac{1}{R^{dim\ C}_{dim\ A}} \left(\epsilon_{dim\ C} - \sum_{K \in \mathcal{I}} R^{dim\ C}_{dim\ K} \epsilon_{dim\ K} \right).$$

We observe that $v_{\mathcal{S}}$ belongs to the open part of HS_A if and only $(v_{\mathcal{S}}, \pi_A) > \epsilon_{dim\ A}$; this inequality is verified since, by Lemma 3.1 we have

$$\epsilon_{dim\ C} > R^{dim\ C}_{dim\ A} \epsilon_{dim\ A} + \sum_{K \in \mathcal{I}} R^{dim\ C}_{dim\ K} \epsilon_{dim\ K}.$$

□

The proof of Theorem 3.1 is an immediate consequence of Propositions 4.1, 4.2, 4.3: the faces of dimension i are in bijection with the nested sets of cardinality $n - i$ containing V.

[3] For any K in the sum, $A + K \in \mathcal{G}_{fund}$ since $A, K \in \mathcal{G}_{fund}$ and $A \cap K \neq \{0\}$; for the same reason, if we take $K_1 \neq K$ in the sum we have $A + K + K_1 \in \mathcal{G}_{fund}$, and so on..

5 From nestohedra to the permutonestohedron

This section is devoted to the proof of Theorem 3.2. We will split the proof in two steps, given by following propositions:

Proposition 5.1 *For every* $\sigma \in W$*, for every* $A \in \mathcal{G}_{fund} - \{V\}$ *and for every maximal nested set* $\mathcal{S}$ *of* $\mathcal{G}_{fund}$*, we have*

$$(\delta_A, \sigma v_{\mathcal{S}}) \leq (\delta_A, v_{\mathcal{S}})$$

and the equality holds if and only if $\sigma \in W_A$*. This means that* $\sigma v_{\mathcal{S}}$ *belongs to* HS_A*, and it lies in* H_A *if and only if* $A \in \mathcal{S}$ *and* $\sigma \in W_A$*; more precisely,* H_A *is the affine span of such vectors. Furthermore, we have*

$$(\delta, \sigma v_{\mathcal{S}}) < (\delta, v_{\mathcal{S}})$$

for every $\sigma \in W$ *different from the identity.*

Proof Let us consider $A \in \mathcal{G}_{fund} - \{V\}$ (in the case $A = V$ the proof is similar). Let $I = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}\}$ be the set of simple roots that belong to A. Then $\delta_A = \sum_{i \in \Delta - I} b_i \omega_i$ with all the $b_i > 0$ (Remark 3.1).

Since $v_{\mathcal{S}}$ lies inside the open fundamental chamber, we can write $v_{\mathcal{S}} = \sum_{j \in \Delta} a_j \omega_j$ with all the $a_j > 0$. As it is well known (see for instance [1,24]),

$$\sigma(\omega_i) = \omega_i - \sum_{t=1}^{s_i} \beta_{it},$$

where the β_{it} are positive roots and $s_i \in \mathbb{N}$: as a notation, when $\sigma(\omega_i) = \omega_i$ we put $s_i = 0$ and the sum is empty. Then we have

$$(\delta_A, \sigma v_{\mathcal{S}}) = \left(\sum_{i \in \Delta - I} b_i \omega_i, \sum_{j \in \Delta} a_j (\omega_j - \sum_{t=1}^{s_j} \beta_{jt}) \right) =$$

$$= (\delta_A, v_{\mathcal{S}}) - \left(\sum_{i \in \Delta - I} b_i \omega_i, \sum_{j,t} \beta_{jt} \right).$$

The scalar product

$$\left(\sum_{i \in \Delta - I} b_i \omega_i, \sum_{j,t} \beta_{jt} \right)$$

is easily seen to be ≥ 0 since $(\omega_i, \beta_{jt}) \geq 0$.

If $\sigma \in W_A$ then all the roots β_{jt} belong to A, therefore,

$$\left(\sum_{i \in \Delta - I} b_i \omega_i, \sum_{j,t} \beta_{jt} \right) = 0.$$

Otherwise, at least one of the positive roots, say β_{rs}, does not belong to A,[4] and we have

$$(\sum_{i \in \Delta - I} b_i \omega_i, \beta_{rs}) > 0$$

It remains to prove that H_A is the affine span of the vectors $\sigma v_{\mathcal{S}}$ with $\sigma \in W_A$ and $A \in \mathcal{S}$. The vectors $v_{\mathcal{S}}$ with $A \in \mathcal{S}$ are all the vertices of an $(n-2)$-dimensional face of the nestohedron $P_{\mathcal{G}_{fund}}$. The affine span of this face is the subspace T whose elements satisfy the relations $(x, \delta) = a$ and $(x, \pi_A) = \epsilon_{dim\ A}$. Then $T + A$ coincides with the hyperplane H_A defined by the relation $(x, \delta_A) = a - \epsilon_{dim\ A}$.

Now the vectors $\sigma v_{\mathcal{S}}$ with $A \subset \mathcal{S}$ and $\sigma \in W_A$ lie on $T + A$ and, since for every simple root α_i which belongs to A, we have that $\sigma_{\alpha_i} v_{\mathcal{S}} - v_{\mathcal{S}}$ is a nonzero scalar multiple of α_i, the affine span of these vectors coincides with $T + A = H_A$. □

Lemma 5.1 *Let B be a subspace which does not belong to $\mathcal{G}_{fund}$ and is generated by some of the simple roots and let $B = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ be its $\mathcal{G}_{fund}$-decomposition. Then the subspaces $A_1, A_2, \cdots, A_k$ are pairwise orthogonal. The vector $\delta_B = \delta - \sum_{i=1}^{k} \pi_{A_i}$ of Definition 3.4 can be obtained as*

$$\delta_B = \frac{1}{|W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}|} \sum_{\sigma \in W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}} \sigma \delta$$

Furthermore, if I is the set of indices given by the $i \in \{1, 2, \ldots, n\}$ such that $\alpha_i \in B$, we have

$$\delta_B = \sum_{s \in \{1,2,\ldots,n\} - I} c_s \omega_s$$

with all $c_s > 0$.

[4] In fact let $\sigma = \sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_k}}$ be a reduced expression for σ, and let r be the smallest index such that α_{i_r} does not belong to I. Then for every j the roots β_{jt} which appear in the proof are among the roots:

$$\sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_{k-1}}} \alpha_{i_k}, \quad \sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_{k-2}}} \alpha_{i_{k-1}}, \quad \ldots \quad, \sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_{r-1}}} \alpha_{i_r}, \ldots.$$

In particular, the root $\sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_{r-1}}} \alpha_{i_r}$ is one of the roots $\beta_{r,t}$ and it satisfies

$$(\sigma_{\alpha_{i_1}} \sigma_{\alpha_{i_2}} \cdots \sigma_{\alpha_{i_{r-1}}} \alpha_{i_r}, \omega_{i_r}) = (\alpha_{i_r}, \omega_{i_r}) > 0$$

Proof As for the orthogonality of the subspaces A_i, we notice that if two simple roots α, β satisfy $\alpha \in A_1, \beta \in A_2$ and $(\alpha, \beta) < 0$, then $< \alpha, \beta >$ is an irreducible subspace. Therefore, it belongs to $\mathcal{G}_{fund}$, and this contradicts that $B = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ is a $\mathcal{G}_{fund}$-decomposition. The claims on δ_B easily follow from the orthogonality of the subspaces A_i and from the formula for the vectors π_A in Remark 3.1. □

Proposition 5.2 *Let B be a subspace which does not belong to $\mathcal{G}_{fund}$ and is generated by some of the simple roots and let $B = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ be its $\mathcal{G}_{fund}$-decomposition. Then for every $\sigma \in W$ and for every maximal nested set $\mathcal{S}$ of $\mathcal{G}_{fund}$ the vertices $\sigma v_{\mathcal{S}}$ lie in the half-space $\overline{HS}_B$. They lie on the hyperplane $\overline{H}_B$ if and only if $\{A_1, A_2, \ldots, A_k\} \subset \mathcal{S}$ and $\sigma \in W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}$ and $\overline{H}_B$ is the affine span of all such vertices.*

Proof The vertices $v_{\mathcal{S}}$ with $\{A_1, A_2, \ldots, A_k\} \subset \mathcal{S}$ are all the vertices of a $(n-1-k)$-dimensional face of the nestohedron $P_{\mathcal{G}_{fund}}$. The affine span of this face is the subspace T whose elements satisfy the relations $(x, \delta) = a$ and $(x, \pi_{A_j}) = \epsilon_{dim\ A_j}$ for every $j = 1, \ldots, k$. Then $T + B$ is the hyperplane[5] $\overline{H}_B$ whose elements are subject to the relation $(x, \delta_B) = a - \epsilon_{dim\ A_1} - \epsilon_{dim\ A_2} - \cdots - \epsilon_{dim\ A_k}$.

As shown in Lemma 5.1, δ_B is a scalar multiple of $\sum_{\sigma \in W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}} \sigma\delta$. Therefore, for every $\sigma \in W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}$ we have

$$(\sigma v_{\mathcal{S}}, \delta_B) = (v_{\mathcal{S}}, \sigma^{-1}\delta_B) = (v_{\mathcal{S}}, \delta_B)$$

and the vectors $\sigma v_{\mathcal{S}}$ with σ as above lie on the hyperplane $\overline{H}_B$.

Their affine span is exactly $T + B = \overline{H}_B$ since the vertices $v_{\mathcal{S}}$ span T and, for every simple root α_i which belongs to $B = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, we have that $\sigma_{\alpha_i} v_{\mathcal{S}} - v_{\mathcal{S}}$ is a nonzero scalar multiple of α_i.

Let us now prove that for every $\gamma \in W$ and for every maximal nested set $\mathcal{T}$ in $\mathcal{G}_{fund}$ which does not contain $\{A_1, A_2, \ldots, A_k\}$ we have

$$(\gamma v_{\mathcal{T}}, \delta_B) < a - \epsilon_{dim\ A_1} - \epsilon_{dim\ A_2} - \cdots - \epsilon_{dim\ A_k}$$

This inequality is equivalent to

$$(\gamma v_{\mathcal{T}}, \pi_B) > \epsilon_{dim\ A_1} + \epsilon_{dim\ A_2} + \cdots + \epsilon_{dim\ A_k}$$

where $\pi_B = \delta - \delta_B = \pi_{A_1} + \cdots + \pi_{A_k}$. We first prove this when $\gamma = e$.

For every $i = 1, 2, \ldots, k$ let C_i be the minimal subspace in $\mathcal{T}$ which contains A_i. We notice that at least for one index j we have $C_j \neq A_j$ because $\mathcal{T}$ does not contain $\{A_1, A_2, \ldots, A_k\}$. Then as in the proof of Proposition 4.3 we deduce that $(v_{\mathcal{T}}, \pi_{A_j}) > \epsilon_{dim\ A_j}$ because the list of the numbers ϵ_j is suitable. This easily leads to prove that $(v_{\mathcal{T}}, \pi_B) > \epsilon_{dim\ A_1} + \epsilon_{dim\ A_2} + \cdots + \epsilon_{dim\ A_k}$.

[5] One can check that the dimension of $(T - v_{\mathcal{S}}) + B$ is equal to $n-1$ by checking that the dimension of $(T - v_{\mathcal{S}}) \cap B$ is $dim\ B - k$. In fact a vector in B belongs to $(T - v_{\mathcal{S}})$ if and only if it satisfies the independent equations $(x, \pi_{A_j}) = 0$ for every $j = 1, \ldots, k$ (from Lemma 5.1 one deduces that the equation $(x, \delta) = 0$ is dependent on these for the vectors in B).

When $\gamma \neq e$, since $\delta_B = \sum_{i\ s.t. \alpha_i \notin B} c_i \omega_i$ with all $c_i > 0$ (Lemma 5.1) we can conclude, as in the first part of the proof of Proposition 5.1 that

$$(\gamma v_{\mathcal{T}}, \delta_B) \leq (v_{\mathcal{T}}, \delta_B).$$

It remains to prove the claim when the maximal nested set $\mathcal{T}$ contains $\{A_1, A_2, \ldots, A_k\}$ but $\gamma \notin W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}$; this is essentially the same reasoning as in the second part of the proof of Proposition 5.1. □

Propositions 5.1 and 5.2 determine the vertices of the permutonestohedron and prove Theorem 3.2.

6 The face poset of the permutonestohedron

In this section, we will give a description of the full face poset of the permutonestohedron. We will improve and complete the description that was first sketched in [20], where a nonlinear realization of the permutonestohedron appeared.

The faces of $P_{\mathcal{G}}(\Phi)$ are in bijective correspondence with the pairs $(\sigma H, \mathcal{S})$, where

- $\mathcal{S}$ is a nested set of $\mathcal{G}_{fund}$ that contains V and has labels attached to its minimal elements: if A is a minimal element, its label is either the subgroup W_A of W or the trivial subgroup $\{e\}$. For brevity in the sequel we will omit to write the label $\{e\}$ and we will write $\underline{A}$ to indicate that A is labelled by W_A;
- σH is a coset of W, where H is the subgroup of W given by the direct product of all the labels.

This bijective correspondence is motivated in the following way: to obtain the face represented by $(\sigma H, \mathcal{S})$, one starts from the face F of the nestohedron $P(\mathcal{G}_{fund})$ which is associated with the nested set $\mathcal{S}$. Then one considers all the images of this face under the action of the elements of H and takes their convex hull. This gives a face F' of the permutonestohedron which intersects the fundamental chamber (see Proposition 6.1 below). Then $\sigma F'$ is the face associated with the pair $(\sigma H, \mathcal{S})$.

Proposition 6.1 *Let $\mathcal{S}$ be a nested set of $\mathcal{G}_{fund}$ which contains V and has labels attached to its minimal elements, with at least one nontrivial label. Let $A_1, A_2, \ldots, A_k$ be the subset of its minimal elements that have a nontrivial label and let $H = W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}$. Let us then consider the face F of the nestohedron $P(\mathcal{G}_{fund})$ associated with $\mathcal{S}$, and let F' be the convex hull of all the faces hF with $h \in H$. Then F' is the $(|\mathcal{S}| - k)$-codimensional face of $P_{\mathcal{G}}(\Phi)$ determined by the intersection of the defining hyperplanes associated with $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ and $B + (A_1 \oplus A_2 \oplus \cdots \oplus A_k)$ for every $B \in \mathcal{S} - \{V, A_1, A_2, \ldots, A_k\}$.*

Proof First we observe that for every $B \in \mathcal{S} - \{V, A_1, A_2, \ldots, A_k\}$ the sum $B + (A_1 \oplus A_2 \oplus \cdots \oplus A_k)$ is equal, by nestedness, to $B \oplus A_{i_1} \oplus \cdots \oplus A_{i_r}$ where $\{A_{i_1}, \ldots, A_{i_r}\}$ is a (possibly empty) subset of $\{A_1, A_2, \ldots, A_k\}$; if $\{A_{i_1}, \ldots, A_{i_r}\}$ is not empty then the corresponding space is $\overline{H}_{B \oplus A_{i_1} \oplus \cdots \oplus A_{i_r}}$, otherwise it is H_B. In both cases it is one of the defining hyperplanes of the permutonestohedron.

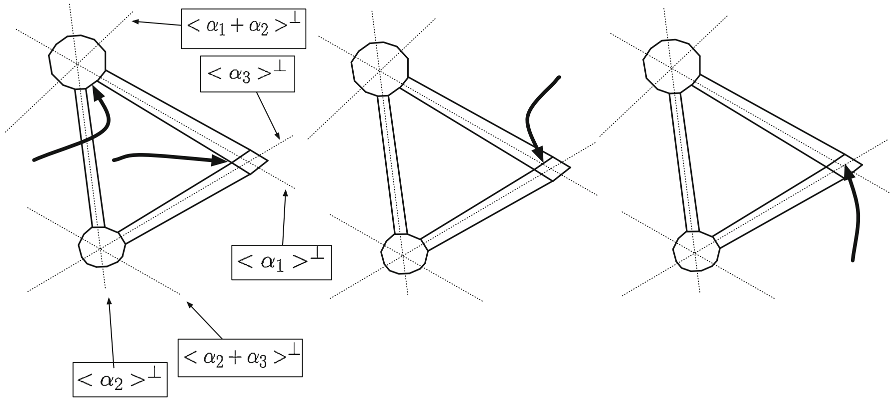


Fig. 2 Some planar pictures of the portion of $P_{\mathcal{F}_{A_3}}(A_3)$ which is around the fundamental chamber: the *dotted lines* represent the hyperplanes which intersect the closed fundamental chamber, as indicated in the picture on the *left*. In the picture on the *left*, the *thick arrows* indicate, respectively, the vertex $(\{e\}, \{V, <\alpha_1>, <\alpha_3>\})$ and the edge $(\{e\}, \{V, <\alpha_1, \alpha_2>\})$. In the picture in the *middle*, the *thick arrow* indicates the edge $(W_{<\alpha_1>}, \{V, \underline{<\alpha_1>}, <\alpha_3>\})$. In the picture on the *right*, the *thick arrow* indicates the facet $(W_{<\alpha_1>} \times W_{<\alpha_3>}, \{V, \underline{<\alpha_1>}, \underline{<\alpha_3>}\})$

Let us then denote by L the face of the permutonestohedron determined by the intersection of the defining hyperplanes mentioned in the claim.

It is straightforward to check, using Propositions 5.1 and 5.2 that the set of vertices of L coincides with the set of all vertices which belong to $\cup_{h \in H} hF$.

Therefore, $L = F'$; now, applying an argument analogue to the one of the proof of Proposition 5.2 one checks that the affine span of the vertices in F' coincides with the subspace $< F > +(A_1 \oplus A_2 \oplus \cdots \oplus A_k)$ that has codimension $|\mathcal{S}| - k$ (here $< F >$ denotes the affine span of F). □

The pictures in Figs. 2 and 3 illustrate, in the case of the permutoassociahedron $P_{\mathcal{F}_{A_3}}(A_3)$, the correspondence between faces and pairs described above.

From now on we will denote a face by its corresponding pair; as an immediate consequence of Proposition 6.1 we have the following statement.

Corollary 6.1 *The dimension of the face* $(\sigma H, \mathcal{S})$ *is given by* $n - |\mathcal{S}| + l$ *where* l *is the number of nontrivial labels.*

Remark 6.1 For instance, $(W, \{\underline{V}\})$ is the full permutonestohedron. The nestohedra inside the chambers correspond to the pairs

$$(\sigma\{e\}, \{V\})$$

for every $\sigma \in W$, while the other facets, the facets that 'cross some of the chambers', correspond to

$$(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\})$$

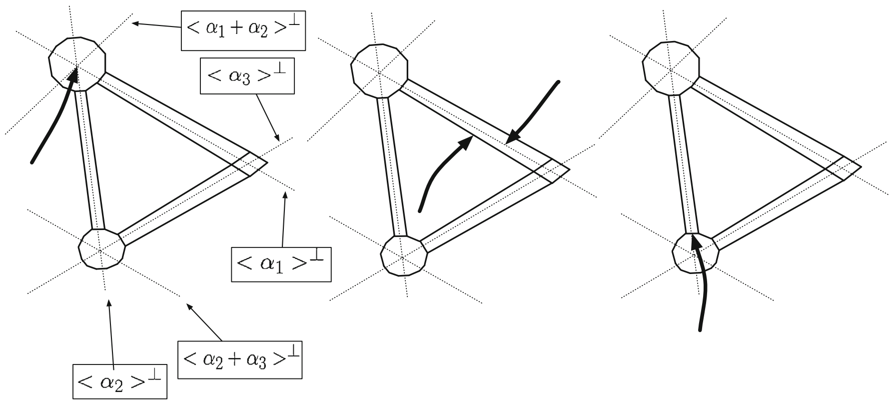


Fig. 3 Again some planar pictures of the portion of $P_{\mathcal{F}_{A_3}}(A_3)$ which is around the fundamental chamber. In the picture on the *left*, the *thick arrow* indicates the facet $(W_{<\alpha_1,\alpha_2>}, \{V, \underline{<\alpha_1, \alpha_2>}\})$. In the picture in the *middle*, the *thick arrows* indicate, respectively, the edge $(\{e\}, \{\overline{V, <\ \alpha_1\ >}\})$ and the facet $(W_{<\alpha_1>}, \{V, \underline{<\alpha_1>}\})$. In the picture on the *right*, the *thick arrow* indicates the edge $(W_{<\alpha_2>}, \{V, \underline{<\alpha_2>}, \overline{<\alpha_2, \alpha_3>}\})$

Here the nested set on the right is made by V and by the proper subspaces $A_1, A_2, \ldots, A_k$ that are all minimal and with nontrivial label.

The following corollary points out that, once Φ is fixed, only the maximal permutonestohedron $P_{\mathcal{C}_\Phi}(\Phi)$ is a simple polytope.

Corollary 6.2 *A vertex $v_{\mathcal{S}}$ in $P_{\mathcal{G}_{fund}}$ belongs to exactly n facets of $P_{\mathcal{G}}(\Phi)$ if and only if $\mathcal{S}$ has only one minimal element. Therefore, the polytope $P_{\mathcal{G}}(\Phi)$ is simple if and only if $\mathcal{G}$ is the maximal building set $\mathcal{C}_\Phi$.*

Proof First we observe that a nested set $\mathcal{S}$ in $\mathcal{G}_{fund}$ has only one minimal element if and only if it is linearly ordered.

Then we notice that Proposition 6.1 can be used to determine all the facets that contain $v_{\mathcal{S}}$. These are the facets

$$(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\}),$$

where $\{V, A_1, A_2, \ldots, A_k\}$ is a nested subset of $\mathcal{S}$ and the A_i is minimal (we are including the case $k = 0$, i.e. the face $(\{e\}, \{V\})$). If $\mathcal{S}$ is not linearly ordered, these facets are more than n, while if $\mathcal{S}$ is linearly ordered they are exactly n.

To finish the proof we remark that if $\mathcal{G}$ is not the maximal building set $\mathcal{C}_\Phi$ there exist nonlinearly ordered $\mathcal{G}$-nested sets: to obtain a nonlinearly ordered nested set made by two elements it is sufficient to take two subspaces $A, B \in \mathcal{G}$ whose sum is direct and does not belong to $\mathcal{G}$. As an immediate consequence, if $\mathcal{G}$ is not the maximal building set associated to Φ there exists a maximal nested set $\mathcal{S}$ in $\mathcal{G}_{fund}$ that is not linearly ordered. □

We now focus on the facets that cross the chambers: they are combinatorially equivalent to a product of a nestohedron with some permutonestohedra. More precisely, let us consider the facet

$$(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\})$$

and denote by $\mathcal{G}^{A_i}$ the subset of $\mathcal{G}$ given by the subspaces which are included in A_i. This is a building set associated with the root system $\Phi \cap A_i$. Furthermore, let us denote $A_1 \oplus A_2 \oplus \cdots \oplus A_k$ by D and consider the building set $\overline{\mathcal{G}}$ in V/D given by

$$\overline{\mathcal{G}} = \{(C + D)/D \mid C \in \mathcal{G}_{fund}\}.$$

According to the notation in Sect. 2 we call $P_{\overline{\mathcal{G}}}$ the nestohedron associated with $\overline{\mathcal{G}}$.

Theorem 6.1 *The facet* $(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\})$ *of* $P_{\mathcal{G}}(\Phi)$ *is combinatorially equivalent to the product*[6]

$$P_{\overline{\mathcal{G}}} \times P_{\mathcal{G}^{A_1}}(\Phi \cap A_1) \times \cdots \times P_{\mathcal{G}^{A_k}}(\Phi \cap A_k).$$

Proof Let us show how to associate to a face $(\sigma H, \mathcal{S})$ of $(\sigma W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}, \{V, \underline{A_1}, \underline{A_2}, \ldots, \underline{A_k}\})$ a face in the product

$$P_{\overline{\mathcal{G}}} \times P_{\mathcal{G}^{A_1}}(\Phi \cap A_1) \times \cdots \times P_{\mathcal{G}^{A_k}}(\Phi \cap A_k).$$

We remark that

(a) $\mathcal{S} \subset \mathcal{G}_{fund}$ is a labelled nested set that contains $V, A_1, A_2, \ldots, A_k$;
(b) $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ with $\sigma_i \in W_{A_i}$;
(c) H is a subgroup of $W_{A_1} \times W_{A_2} \times \cdots \times W_{A_k}$ that can be expressed as a product of $W_{A_{ij}}$ for some minimal subspaces $A_{ij} \in \mathcal{S}$ ($i = 1, \ldots, k$ and, for every i, $A_{ij} \subset A_i$ and the index j ranges from 0 to a natural number s_i, with the convention that $W_{A_{i0}}$ is the trivial group).

Then we associate to $(\sigma H, \mathcal{S})$:

- the face of $P_{\overline{\mathcal{G}}}$ which corresponds to the nested set $\overline{\mathcal{S}} = \{K + D/D \mid K \in \mathcal{S}\}$ of $\overline{\mathcal{G}}$;
- for every $i = 1, \ldots, k$, the face

$$(\sigma_i W_{A_{i1}} \times W_{A_{i2}} \times \cdots \times W_{A_{is_i}}, \mathcal{S}^{A_i})$$

of $P_{\mathcal{G}^{A_i}}(\Phi \cap A_i)$, where $\mathcal{S}^{A_i}$ is the subset of $\mathcal{S}$ given by the subspaces which are included into A_i and the labelled subspaces of $\mathcal{S}$ keep their label in $\mathcal{S}^{A_i}$.

The above described map is easily shown to be bijective, and, using Proposition 6.1, a poset isomorphism. □

As a corollary of Proposition 6.1 and Theorem 6.1, we conclude this section with an explicit description of the order relation on the face poset of $P_{\mathcal{G}}(\Phi)$.

6 Here we are considering the well-defined product of combinatorial polytopes.

Corollary 6.3 *Given two faces $(\sigma' H', \mathcal{S}')$ and $(\sigma H, \mathcal{S})$ in the face poset of $P_{\mathcal{G}}(\Phi)$ we have*

$$(\sigma' H', \mathcal{S}') < (\sigma H, \mathcal{S})$$

if and only if $\sigma' H' \subseteq \sigma H$ and $\mathcal{S}'$ is obtained from $\mathcal{S}$ by a composition of some of the following moves:

- *adding a subspace which is not minimal, i.e. it contains some of the subspaces in $\mathcal{S}$;*
- *adding a subspace A, minimal in $\mathcal{S}'$, with trivial label and with the property that A is not included in any of the minimal subspaces of $\mathcal{S}$, or it is included in a minimal subspace B of $\mathcal{S}$ which is labelled by $\{e\}$; in the latter case B loses its label;*
- *adding some subspaces $A_1, \ldots, A_k$ that are minimal in $\mathcal{S}'$, all with nontrivial label, and all included in a minimal subspace B of $\mathcal{S}$ which was labelled by W_B and loses its label;*
- *changing the nontrivial label of a minimal subspace into the trivial label.*

7 The automorphism group

In this section we study the automorphism group $Aut(P_{\mathcal{G}}(\Phi))$, i.e. the group of the isometries of V that send $P_{\mathcal{G}}(\Phi)$ onto itself. We adopt here the following normalization of the root system Φ: if Φ is made by two or more irreducible components, we impose that all the short roots have the same length[7].

Let $Aut(\Phi)$ be the group of the automorphisms of V that leave Φ invariant. With the above normalization its elements are isometries.

Theorem 7.1 *Let us suppose that $\mathcal{G}$ is $Aut(\Phi)$ invariant. Then $Aut(\Phi)$ is a subgroup of $Aut(P_{\mathcal{G}}(\Phi))$. If $Aut(P_{\mathcal{G}}(\Phi))$ leaves invariant the set of the nestohedra $\{w P_{\mathcal{G}_{fund}} \mid w \in W\}$ then $Aut(\Phi) = Aut(P_{\mathcal{G}}(\Phi))$.*

Remark 7.1 Until now in this paper we have used the notations $wv_{\mathcal{S}}$, wH_A..., without parentheses, to indicate the action of an element w of the Weyl group. We feel that in the following proof this could be confusing, since the product of elements in the automorphism group comes into play, so we decided, for the sake of clarity, to put parentheses here.

Proof To prove the first part of the claim we will show that an element $\varphi \in Aut(\Phi)$ permutes the defining hyperplanes of the permutonestohedron.

Since $Aut(\Phi)$ is the semidirect product of the Weyl group with the automorphism group Γ of the Dynkin diagram of Φ, we can write $\varphi = w\gamma$ where $w \in W$ and $\gamma \in \Gamma$.

It is, therefore, sufficient to show that φ sends the hyperplanes H_V, H_A, $\overline{H}_B$, for any $A \in \mathcal{G}_{fund}$, $B \in \mathcal{C}_{\Phi\, fund} - \mathcal{G}_{fund}$, to other defining hyperplanes of $P_{\mathcal{G}}(\Phi)$.

Now $\gamma(\delta) = \delta$ since γ leaves invariant the set of the positive roots. Then $\varphi(\delta) = w(\delta)$, and therefore, $\varphi(H_V) = w(H_V)$ since the vectors in $\varphi(H_V)$ satisfy the defining equation of $w(H_V)$ which is $(x, w\delta) = a$.

[7] If all the roots of an irreducible component have the same length we consider them short roots.

Then we show that for any $A \in \mathcal{G}_{fund}$ we have $\varphi(H_A) = w(H_{\gamma(A)})$: since $\delta_A = \delta - \pi_A$ we have $\gamma(\delta_A) = \delta - \gamma(\pi_A)$, but $\gamma(\pi_A)$ half the sum of all positive roots contained in $\gamma(A)$, thus it is equal to $\pi_{\gamma(A)}$, therefore, $\gamma(\delta_A) = \delta_{\gamma(A)}$. It is now immediate to prove that $\varphi(H_A)$ satisfies the defining equation of $w(H_{\gamma(A)})$.

The same reasoning applies to prove that, for $B \in \mathcal{C}_{\Phi fund} - \mathcal{G}_{fund}$ we have $\varphi(\overline{H}_B) = w(\overline{H}_{\gamma(B)})$.

For the second part of the claim, let us suppose that $\theta \in Aut(P_{\mathcal{G}}(\Phi))$ sends $P_{\mathcal{G}_{fund}}$ to $w(P_{\mathcal{G}_{fund}})$. It is sufficient to show that $w^{-1}\theta$ belongs to $Aut(\Phi)$. First we observe that since $w^{-1}\theta$ sends H_V onto itself and is an isometry we have $w^{-1}\theta(\delta) = \delta$. Now, let us consider a simple root $\alpha \in \Delta$ and let $\mathcal{S}$ be a nested set which contains $< \alpha >$, so that $v_{\mathcal{S}} \in H_{<\alpha>}$.

We observe that $w^{-1}\theta(v_{\mathcal{S}})$ belongs to $P_{\mathcal{G}_{fund}}$; it also belongs to $H_{<\beta>}$, where β is a simple root. In fact if we denote by F the facet of $P_{\mathcal{G}_{fund}}$ determined by a hyperplane H_A with $A \in \mathcal{G}_{fund}$ (the corresponding pair in the poset is $(\{e\}, \{V, A\})$), and by $\overline{F}$ the facet of the permutonestohedron different from $P_{\mathcal{G}_{fund}}$ which contains F (the corresponding pair in the poset is $(W_A, \{V, \underline{A}\})$), we notice that in $\overline{F}$ there are exactly $|W_A|$ $(n-2)$-dimensional faces which belong to one of the nestohedra $\{w(P_{\mathcal{G}_{fund}}) | w \in W\}$. As a consequence, under our hypothesis, also the facet $w^{-1}\theta(\overline{F})$ has $|W_A|$ $(n-2)$-dimensional faces which belong to one of the nestohedra $\{w(P_{\mathcal{G}_{fund}}) \mid w \in W\}$.

Moreover, we observe that A is the span of a simple root if and only if $|W_A| = 2$.

Therefore, in our case we have $A =< \alpha >$ and $w^{-1}\theta(\overline{F})$ has two $(n-2)$-dimensional faces which belong to one of the nestohedra $\{w(P_{\mathcal{G}_{fund}}) \mid w \in W\}$. This means that $w^{-1}\theta(F)$ is a facet of $P_{\mathcal{G}_{fund}}$ determined by a hyperplane H_B with B equal to the span $< \beta >$ of a simple root β (possibly equal to α).

Since $w^{-1}\theta$ sends $H_{<\alpha>}$ onto $H_{<\beta>}$ and is an isometry we have that $w^{-1}\theta(\delta_{<\alpha>}) = \delta_{<\beta>}$. From this, since $w^{-1}\theta(\delta) = \delta$ and $\delta_{<\alpha>} = \delta - \frac{1}{2}\alpha$, $\delta_{<\beta>} = \delta - \frac{1}{2}\beta$, it follows $w^{-1}\theta(\alpha) = \beta$.

This means that $w^{-1}\theta$ sends Δ to itself.

Now, considering a two-dimensional subspace $D \in \mathcal{G}_{fund}$ and comparing the cardinality of W_D and $W_{w^{-1}\theta(D)}$, we prove that two simple roots α and β are orthogonal if and only if their images $w^{-1}\theta(\alpha)$ and $w^{-1}\theta(\beta)$ are orthogonal. Moreover, $w^{-1}\theta$ preserves root lengths, since sends Δ to itself and is an isometry. These properties imply that $w^{-1}\theta$ is an automorphism of the Dynkin diagram. □

Remark 7.2 If $\Phi = A_n$, any W-invariant building set is also $Aut(\Phi)$ invariant and satisfies the hypothesis of the theorem above. If a building set is not $Aut(\Phi)$ invariant, let G be the maximal subgroup of $Aut(\Phi)$ which leaves $\mathcal{G}$ invariant. Then, the claim of the theorem (and its proof) is still valid with G in place of $Aut(\Phi)$.

As we will see in the next section, we can choose a suitable list $\epsilon_1 < \epsilon_2$ such that $Aut(P_{\mathcal{F}_{A_2}}(A_2; \epsilon_1, \epsilon_2))$ is greater than $Aut\ A_2$. Anyway, we can state the following theorem.

Theorem 7.2 *Let $\mathcal{G}$ be $Aut(\Phi)$ invariant. There are infinite suitable lists $\epsilon_1 < \cdots < \epsilon_n = a$ such that $Aut(\Phi) = Aut(P_{\mathcal{G}}(\Phi))$. More precisely, once a is fixed, for all the* *possible suitable lists whose greatest number is a, except at most for a finite number, we have* $Aut(\Phi) = Aut(P_{\mathcal{G}}(\Phi))$.

Proof This follows from the observation that the elements of $Aut(P_{\mathcal{G}}(\Phi))$ are isometries and, once a is fixed, all the choices, except for a finite number of exceptions, of the other numbers ϵ_i imply that the distance from the origin of H_V is different from the distances from the origin of the other defining hyperplanes. Therefore, $Aut(P_{\mathcal{G}}(\Phi))$ leaves invariant the set of the nestohedra $\{wP_{\mathcal{G}_{fund}} | w \in W\}$ and we can apply Theorem 7.1. □

The following corollary illustrates another sufficient (nonmetric) condition for $Aut(P_{\mathcal{G}}(\Phi)) = Aut(\Phi)$. For any $C \in \mathcal{C}_{\Phi\, fund}$, let G_C be the subgroup of $Aut(P_{\mathcal{G}}(\Phi))$ which leaves the facet determined by the hyperplane H_C (or $\overline{H}_C$) invariant. We observe that $W_C \subset G_C$ by construction.

Corollary 7.1 *Let* $\mathcal{G}$ *be* $Aut(\Phi)$ *invariant. If for every* $C \in \mathcal{C}_{\Phi\, fund}$ *the automorphism group of the Dynkin diagram of* Φ *is not isomorphic to the group* G_C *then we have* $Aut(P_{\mathcal{G}}(\Phi)) = Aut(\Phi)$. *In particular, if the automorphism group of the Dynkin diagram of* Φ *is trivial then* $Aut(P_{\mathcal{G}}(\Phi)) = Aut(\Phi) = W$.

Proof This is an immediate application of Theorem 7.1 since the subgroup of $Aut(P_{\mathcal{G}}(\Phi))$ which leaves $P_{\mathcal{G}_{fund}}$ invariant is, as we showed in the proof of Theorem 7.1, the automorphism group Γ of the Dynkin diagram. Let $\varphi \in Aut(P_{\mathcal{G}}(\Phi))$; then the subgroup of $Aut(P_{\mathcal{G}}(\Phi))$ which leaves the facet $\varphi(P_{\mathcal{G}_{fund}})$ invariant is a conjugate of Γ, and, under our hypothesis, this implies that $\varphi(P_{\mathcal{G}_{fund}})$ is one of the facets in $\{wP_{\mathcal{G}_{fund}} \mid w \in W\}$. The claim of Theorem 7.1 then implies that $Aut(P_{\mathcal{G}}(\Phi)) = Aut(\Phi)$. □

Remark 7.3 The corollary above is easy to apply when Φ is irreducible, since the automorphism group of the Dynkin diagram is small (it is either trivial or $\mathbb{Z}_2$, or S_3 in the case D_4) and it is easy to compare it with the groups G_C.

8 A remark on the S_{n+1} action on the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$

As we have seen in the preceding section, the face poset of a permutonestohedron $P_{\mathcal{G}}(\Phi)$ provides nice geometrical realizations of representations of W or even of $Aut(\Phi)$. In this section we focus on a special case, where $W = S_n$ and some representations of S_{n+1} also come into play.

First we recall that there is a well-known 'extended' S_{n+1} action on the De Concini–Procesi model $Y_{\mathcal{F}_{A_{n-1}}}$ that is a quotient of $CY_{\mathcal{F}_{A_{n-1}}}$: it comes from the isomorphism with the moduli space $M_{0,n+1}$ (see [7,18]), and its character has been computed in [13].

This extended action can be lifted to the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$, which is a subposet of $P_{\mathcal{F}_{A_{n-1}}}(A_{n-1})$. We illustrate this lifting using our description of this subposet.

Let $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be a basis for the root system of type A_n, where we added to a basis of A_{n-1} the extra root α_0, and let $\tilde{\Delta} = \{\tilde{\alpha}, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the set of roots that appear in the affine diagram. We identify in the standard way S_{n+1} with the group which permutes $\{0, 1, \ldots, n\}$ and s_{α_0} with the transposition $(0, 1)$. Therefore, S_n, the subgroup generated by $\{s_{\alpha_1}, \ldots, s_{\alpha_{n-1}}\}$, is identified with the subgroup which permutes $\{1, \ldots, n\}$.

Let $\mathcal{S} = \{V, A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_s\}$ be a nested set in $\mathcal{F}_{A_{n-1}fund}$ and let $\sigma \in S_{n+1}$. Let us then denote by C the cyclic subgroup generated by $(0, 1, 2, 3, 4, 5, \ldots, n)$ and by $w = \sigma(0, 1, 2, 3, 4, 5, \ldots, n)^r$ the only element in the coset σC which fixes 0; we notice that w belongs to S_n.

Moreover, let us suppose that, for every subspace A_j, some of the roots contained in σA_j have α_0 in their support (when they are written with respect to the basis Δ), while this does not happen for the subspaces σB_t. Then for every j we denote by $\overline{A}_j$ the subspace generated by all the roots of $\tilde{\Delta}$ which are orthogonal to A_j.
As a first step in the description of the S_{n+1} action, we put

$$\sigma \cdot (\{e\}, \mathcal{S}) = (w\{e\}, \{V, .., w^{-1}\sigma\overline{A_j}, .., w^{-1}\sigma B_s, ..\}).$$

As one can quickly check, this can be extended to a S_{n+1} action on the full face poset of $CY_{\mathcal{F}_{A_{n-1}}}$ by imposing that σ sends the face $(\gamma\{e\}, \mathcal{S})$, where $\gamma \in S_n$, to the face $\sigma\gamma \cdot (\{e\}, \mathcal{S})$.

Example 8.1 Let $\mathcal{S}$ be the nested set of $\mathcal{F}_{A_4 fund}$ made by V, $A =< \alpha_1, \alpha_2 >$ and $B =< \alpha_4 >$. The group S_6 is generated by the reflections $s_{\alpha_0}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$ and we identify S_5 with the subgroup generated by $s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}, s_{\alpha_4}$. Now we compute $s_{\alpha_0}(\{e\}, \{V, A, B\})$.

We notice that the root $s_{\alpha_0}\alpha_1$ contains α_0 in its support (when it is written with respect to the basis Δ). We then denote by $\overline{A}$ the subspace generated by all the roots of $\tilde{\Delta}$ which are orthogonal to A: $\overline{A} =< \tilde{\alpha}, \alpha_4 >$.

Let $w = (0, 1)(0, 1, 2, 3, 4, 5)$ i.e. the representative of the coset $(0, 1)C$ in S_6 which leaves 0 fixed. Then $s_{\alpha_0} = (0, 1)$ sends the face $(\{e\}, \{V, A, B\})$ to the face

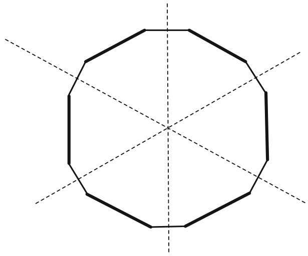
$$(w\{e\}, \{V, w^{-1}s_{\alpha_0}\overline{A}, w^{-1}s_{\alpha_0}B\}) = (w\{e\}, \{V, < \alpha_3, \alpha_4 >, < \alpha_3 >\}).$$

Therefore, the group S_{n+1} acts on the face poset of $CY_{\mathcal{F}_{A_{n-1}}}$, which is the disjoint union of $n!$ associahedra: the action of $\sigma \in S_{n+1}$ sends the face poset of the associahedron that lies in the fundamental chamber onto the face poset of an associahedron that lies in a chamber which may be different from the fundamental one. But it is easy to provide examples where two associahedra that lie in two adjacent chambers are sent to two associahedra whose chambers are not adjacent.

This shows that this lifted action is not induced by an isometry and it cannot be extended to the full permutoassociahedron. Anyway it provides geometrical realizations of all the representations $Ind_G^{S_n+1} Id$, where G is any subgroup of the cyclic group C and Id is its trivial representation (the case $G = \{e\}$, i.e. the regular representation of S_{n+1}, occurs only if $n \geq 4$).

In fact the stabilizer of an element of the face poset is by construction a subgroup of the cyclic group C: in the notation above, $w = e$ only if $\sigma \in C$. Now we want to show that all the above mentioned representations appear. As for the regular representation of

Fig. 4 The permutonestohedron of type A_2 (i.e. the two-dimensional Kapranov's permutoassociahedron). The *thick edges* provide a linear realization of $CY_{\mathcal{F}_{A_2}}$: once ϵ_2 is fixed, there is only one admissible value for ϵ_1 such that the dodecagon is regular



S_{n+1}, we notice that, for instance, if $n \geq 4$ the stabilizer of $(\{e\}, \{V, < \alpha_1, \ldots, \alpha_{n-2} > \})$ is the trivial subgroup.

Let then $d < n + 1$ be a divisor of $n + 1$. We will exhibit an element of the face poset whose stabilizer is generated by $(0, 1, 2, 3, 4, 5, \ldots, n)^d$.

If $d > 2$ and $dk = n + 1$ we consider for instance the face $(\{e\}, \{V, < \alpha_1, \ldots, \alpha_{d-2} >, < \alpha_{d+1}, \ldots, \alpha_{2d-2} >, \ldots, < \alpha_{(k-1)d+1}, \ldots, \alpha_{kd-2} >\})$: its stabilizer is generated by $(0, 1, 2, 3, 4, 5, \ldots, n)^d$.

If $d = 2$ and $2k = n + 1$ we consider $(\{e\}, \{V, < \alpha_1, \alpha_2, \ldots, \alpha_{n-2} >, < \alpha_1 >, < \alpha_3 >, \cdots, < \alpha_{n-2} >\})$. Its stabilizer is generated by $(0, 1, 2, 3, 4, 5, \ldots, n)^2$.

If $d = 1$ we consider $(\{e\}, \{V\})$. Its stabilizer is, by definition of the S_{n+1} action, the full cyclic group C.

9 Examples

9.1 Some low-dimensional cases

In this section we will show some examples and pictures of permutonestohedra.

Let us start from A_2. There is only one building set associated to this root system, since the maximal and the minimal building set coincide. There are six chambers and in every chamber the nestohedron is a segment. Therefore, in this case the permutonestohedron is a dodecagon, and it is a Kapranov's permutoassociahedron (see Fig. 4).

It is not necessarily regular; this depends on the choice of ϵ_1, ϵ_2: once ϵ_2 is fixed, there is only one admissible value for ϵ_1 such that $P_{\mathcal{F}_{A_2}}(A_2)$ is regular. If it is not regular its edges have two different lengths and its automorphism group coincides with $Aut\ A_2 \cong S_3 \rtimes \mathbb{Z}_2$; if it is regular its automorphism group, which is the full dihedral group with 24 elements, strictly contains $Aut\ A_2$.

In the A_3 case, there are two distinct S_4 invariant building sets: the building set of the irreducibles $\mathcal{F}_{A_3}$ and the maximal building set. A picture of the corresponding permutonestohedra, which are a Kapranov's permutoassociahedron ($P_{\mathcal{F}_{A_3}}(A_3)$) and a 'permutopermutohedron', is in Fig. 5. It is easy to show that for every choice of a suitable list $\epsilon_1 < \epsilon_2 < \epsilon_3 = a$, their automorphism group coincides with $Aut\ A_3 \cong S_4 \rtimes \mathbb{Z}_2$. In the minimal case, the nestohedra that lie inside the chambers are pentagons (i.e. the two-dimensional Stasheff's associahedra), while in the maximal case they are hexagons (i.e. the two-dimensional permutohedra).

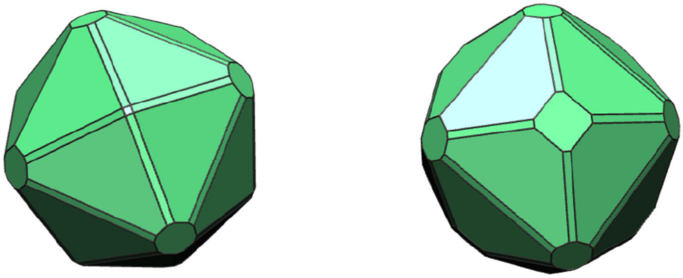


Fig. 5 The minimal (on the *left*) and the maximal permutonestohedron of type A_3. In accordance with Corollary 6.2, the maximal permutonestohedron is a simple polytope, while the minimal one is not simple

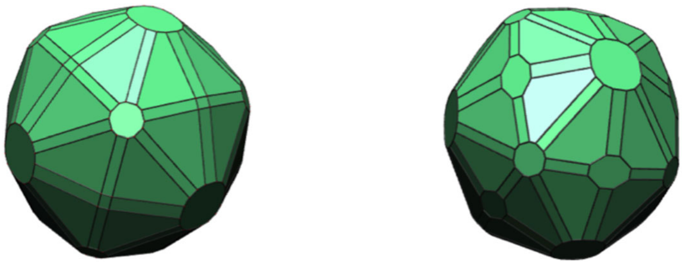


Fig. 6 The minimal (on the *left*) and the maximal permutonestohedron of type B_3

In the B_2 case there is only one building set, and the corresponding permutonestohedron is a polygon with 16 edges. It is not regular and, depending on the choice of ϵ_1, its edges can have two or three different lengths; its automorphism group is $W_{B_2} \cong \mathbb{Z}_2^2 \rtimes S_2$, i.e. the dihedral group with eight elements.

In the B_3 case we have two W_{B_3}-invariant building sets. The corresponding permutonestohedra appear in Fig. 6. As in the A_3 case, in the minimal case the nestohedra inside the chambers are pentagons, while in the maximal case they are hexagons.

The automorphism group of these permutonestohedra is $W_{B_3} \cong \mathbb{Z}_2^3 \rtimes S_3$ as it follows for instance from Corollary 7.1.

Let us now consider the boolean arrangements $Bo(n)$, i.e. the arrangements associated with the root systems A_1^n. The nestohedra $P_{\mathcal{G}_{fund}}$, as $\mathcal{G}$ varies among all the $W = \mathbb{Z}_2^n$ - invariant building sets containing V, are all the nestohedra in the 'interval simplex-permutohedron' (see [27]).

In the $A_1 \times A_1$ case, there is only one possible building set which contains V and the corresponding permutonestohedron is an octagon. It may be regular, depending on the choice of ϵ_1. If it is not regular its edges have two different lengths and its automorphism group coincides with $Aut\ A_1^2 \cong \mathbb{Z}_2^2 \rtimes S_2$ ($\cong W_{B_2}$).

In Fig. 7, there are two pictures of the maximal permutonestohedron of type $A_1 \times A_1 \times A_1$. Its automorphism group coincides with $Aut\ A_1^3 \cong \mathbb{Z}_2^3 \rtimes S_3$ ($\cong W_{B_3}$). We recall that the real De Concini–Procesi model associated with the maximal building set of the boolean arrangement $Bo(n)$ is isomorphic to the toric variety of type A_{n-1} (see Procesi [30], Henderson [23]).

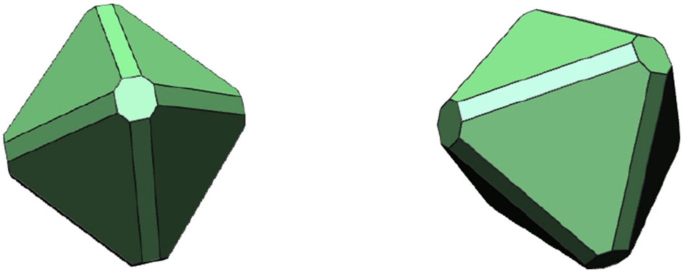


Fig. 7 Two pictures of the maximal permutonestohedron in the three-dimensional boolean case ($A_1 \times A_1 \times A_1$)

In the case of the root system A_4, there is one S_5-invariant building set which strictly contains the minimal one and is different from the maximal one. Therefore, there is a permutonestohedron which is intermediate between the minimal and the maximal one. For any irreducible root system of dimension $n \geq 4$ intermediate building sets (i.e. not minimal or maximal) appear: in [21] all the W-invariant building sets $\mathcal{G}$ of type A_n, B_n, C_n, D_n have been classified.

We observe that, if $\Phi = A_n, B_n, C_n$ and we consider the minimal building set, the corresponding permutonestohedron is the convex hull of $|W|$ (resp. $n!$, $2^n n!$, $2^n n!$) $(n-1)$-dimensional Stasheff 's associahedra. In the A_n case this minimal permutonestohedron is a Kapranov's permutoassociahedron, and therefore, it is combinatorially equivalent to Reiner and Ziegler's Coxeter associahedron of type A_n (see [32]). In the B_n and C_n case it is easy to check that the minimal permutoassociahedron is different from the corresponding Coxeter associahedron, since the latter is the convex hull of $|W|$ Stasheff's associahedra whose dimension is $n-2$, not $n-1$. For instance, the Coxeter associahedron of type B_2 is an octagon, while the unique permutonestohedron of type B_2 is a polygon with 16 edges, as we observed before.

Furthermore, we remark that if $\Phi = D_n$ the minimal permutonestohedon is not the convex hull of some Stasheff's associahedra: it is the convex hull of $2^{n-1}n!$ graph associahedra of type D_n (for a description of these polytopes see for instance [2] and Sect. 8 of [28]).

For any root system Φ, the maximal permutonestohedron is the convex hull of $|W|$ permutohedra.

9.2 Counting faces

As an example of face counting on permutonestohedra, in this section we will compute the f-vectors of the minimal and maximal permutonestohedra associated to the root system A_n (resp. $P_{\mathcal{F}_{A_n}}(A_n)$ and $P_{\mathcal{C}_{A_n}}(A_n)$). These computations are variations of the well-known computations of the f-vectors of the Stasheff's associahedron and of the permutohedron. We first need to introduce some notation.

For any positive integer n let us denote by Λ_n the set of the partitions of n and, if $\lambda \in \Lambda_n$ we denote by $l(\lambda)$ the number of parts of λ. Therefore, we can write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)})$. A partition $\lambda \in \Lambda_n$ can also be written as $\lambda = \prod_{1 \leq i \leq n} i^{m_i}$:

this means that in λ the number i appears m_i times. We then put

$$w(\lambda) = \frac{l(\lambda)!}{\prod_{1\le i\le n} m_i!}$$

There is a bijective correspondence between the elements of $\mathcal{F}_{A_{n-1}}$ and the subsets of $\{1,\cdots,n\}$ of cardinality at least two: if $A^{\perp}$ is the subspace described by the equation $x_{i_1}=x_{i_2}=\cdots=x_{i_k}$ then we represent $A\in\mathcal{F}_{A_{n-1}}$ by the set $\{i_1,i_2,\ldots,i_k\}$.

In an analogous way we can establish a bijective correspondence between the elements of the maximal building set $\mathcal{C}_{A_{n-1}}$ and the unordered partitions of the set $\{1,\cdots,n\}$ in which at least one part has more than one element: for instance, $\{1,5,6\}\{2,3\}\{4\}\{7,8\}$ represents the subspace B in $\mathcal{C}_{A_7}$ of dimension 4 such that $B^{\perp}$ is described by the system of equations $x_1=x_5=x_6$, $x_2=x_3$ and $x_7=x_8$.

Now, to each unordered partition of $\{1,\ldots,n\}$ we can associate, considering the cardinalities of its parts, a partition in Λ_n. Therefore, we can associate a partition in Λ_n to every subspace in $\mathcal{C}_{A_{n-1}}$. We will say that a subspace in $\mathcal{C}_{A_{n-1}}$ has the form $\lambda\in\Lambda_n$ if its associated partition is λ. For instance, the subspace $\{1,5,6\}\{2,3\}\{4\}\{7,8\}$ in $\mathcal{C}_{A_7}$ has the form $(3,2,2,1)$.

Theorem 9.1 *For every* $0\le k\le n-2$ *the number of faces of codimension* $k+1$ *of the minimal permutonestohedron* $P_{\mathcal{F}_{A_{n-1}}}(A_{n-1})$ *is*

$$\sum_{\lambda\in\Lambda_n,\ l(\lambda)\ge 2+k} w(\lambda)\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_{l(\lambda)}!}\left[\frac{1}{k+1}\binom{l(\lambda)-2}{k}\binom{l(\lambda)+k}{k}\right]$$

Proof Let us denote by $\mathcal{G}$, for brevity, the minimal building set $\mathcal{F}_{A_{n-1}}$. As a first step we consider the faces represented in the poset by the pais $(\sigma H,\mathcal{S})$ such that the sum of the minimal subspaces of $\mathcal{S}$ which are nontrivially labelled has the form λ for a fixed $\lambda\in\Lambda_n$. It is immediate to check that there are $w(\lambda)$ choices of a set of minimal subspaces with this property in $\mathcal{G}_{fund}$.

We then notice that, if $\mathcal{S}'$ is the complement in $\mathcal{S}$ to the subset given by V and by the nontrivially labelled minimal subspaces, the codimension of the face represented by $(\sigma H,\mathcal{S})$ is equal to $|\mathcal{S}'|+1$. So, once the set of minimal nontrivially labelled subspaces is fixed (with the form λ), we have to count in how many ways it is possible to complete this set of subspaces to a nested set $\mathcal{S}$ adding V and k other subspaces in $\mathcal{G}_{fund}$.

This is equivalent to finding the number of distinct parenthesizations with k couples of parentheses of an ordered list of $l(\lambda)$ distinct elements. This number was computed by Cayley in [5] and is equal to

$$\frac{1}{k+1}\binom{l(\lambda)-2}{k}\binom{l(\lambda)+k}{k}$$

To finish our proof it if sufficient to observe that the number of faces represented by the pairs $(\sigma H,\mathcal{S})$, once $\mathcal{S}$ and H are fixed, is equal to the index of H in W, which in our case is $\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_{l(\lambda)}!}$. □

Remark 9.1 In the case of the faces of codimension $n-1$, i.e. the vertices, the formula of Theorem 9.1 specializes, as expected, to $C_{n-1}n!$ where C_{n-1} is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$.

Theorem 9.2 *For every $0 \le k \le n-2$ the number of faces of codimension $k+1$ of the maximal permutonestohedron $P_{\mathcal{C}_{A_{n-1}}}(A_{n-1})$ is*

$$\sum_{\lambda\in\Lambda_n,\ l(\lambda)\geq 2+k} w(\lambda)\frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_{l(\lambda)}!}\left[\sum_{1<j_1<\cdots<j_k<l(\lambda)=j_{k+1}}\prod_{t=1}^{k}\binom{j_{i+1}-1}{j_i-1}\right].$$

Proof Let us denote by $\mathcal{G}$, for brevity, the maximal building set $\mathcal{C}_{A_{n-1}}$. As in the first part of the proof of the preceding theorem, we observe that, given $\lambda \in \Lambda_n$ there are $w(\lambda)$ subspaces in $\mathcal{G}_{fund}$ whose form is λ. So, once λ if fixed, we have $w(\lambda)$ ways to choose a minimal nontrivially labelled subspace whose form is λ. Now we have to count in how many ways it is possible to complete this subspace to a nested set $\mathcal{S}$ adding V and k other subspaces in $\mathcal{G}_{fund}$. Since the nested sets in this maximal case are linearly ordered by inclusion, we can do this in k steps. At the first step we have to add a subspace which contains the minimal one, which is equivalent to split into j_k parts (with $1 < j_k < l(\lambda)$) an ordered list of $l(\lambda)$ distinct elements. Then at the second step we have to add a subspace which contains the one added at the first step, which is equivalent to split into j_{k-1} (with $1 < j_{k-1} < j_k$) parts an ordered list of j_k distinct elements. This process is successful if we manage to complete k steps of this kind; this happens in

$$\sum_{1<j_1<\cdots<j_k<l(\lambda)=j_{k+1}}\prod_{t=1}^{k}\binom{j_{i+1}-1}{j_i-1}$$

different ways, where the sum ranges over all possible lists of k numbers j_i that satisfy the required inequalities. The claim then follows, as in the preceding theorem, by counting the index of the subgroup given by the nontrivial label. □

Remark 9.2 The formula of Theorem 9.2 in particular claims that the vertices of $P_{\mathcal{C}_{A_{n-1}}}(A_{n-1})$ are $(n-1)!n!$, as expected from a 'permutopermutohedron'.

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