

# The Brill–Noether rank of a tropical curve

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**Abstract** We construct a space classifying divisor classes of a fixed degree on all tropical curves of a fixed combinatorial type and show that the function taking a divisor class to its rank is upper semicontinuous. We extend the definition of the Brill–Noether rank of a metric graph to tropical curves and use the upper semicontinuity of the rank function on divisors to show that the Brill–Noether rank varies upper semicontinuously in families of tropical curves. Furthermore, we present a specialization lemma relating the Brill–Noether rank of a tropical curve with the dimension of the Brill–Noether locus of an algebraic curve.

**Keywords** Tropical geometry · Tropical curves · Brill–Noether theory · Specialization Lemma

## 1 Introduction

Let  $\Gamma$  be a tropical curve in the sense of [2], so that  $\Gamma = (G, w, \ell)$  is a connected compact metric graph with a non-negative integer weight function on its vertices. For integers  $d$  and  $r$ , the Brill–Noether locus of  $\Gamma$ , denoted  $W_d^r(\Gamma)$ , is the set of equivalence classes of divisors of degree  $d$  and rank at least  $r$ . The Brill–Noether rank  $w_d^r(\Gamma)$  of the curve  $\Gamma$  is the largest number  $\rho$  such that every effective divisor of degree  $r + \rho$  is contained in an effective divisor of degree  $d$  and rank at least  $r$ . The main result in this paper is the following:

**Theorem (5.1)** *The Brill–Noether rank is upper semicontinuous on  $M_g^{tr}$ , the moduli space of tropical curves of genus  $g$ .*

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Moreover, we extend the Brill–Noether rank specialization Lemma ([10, Theorem 1.7]) to the setting of tropical curves:

**Theorem (5.3)** *Let  $X$  be a smooth projective curve over a discretely valued field with a regular semistable model whose special fiber has weighted dual graph  $\Gamma$ . Then for every  $r, d \in \mathbb{N}$ ,  $\dim W_d^r(X) \leq w_d^r(\Gamma)$ .*

The notion of Brill–Noether rank was introduced in [10] in the context of metric graphs and was motivated by the fact that the function taking a metric graph to the dimension of its Brill–Noether locus is not upper semicontinuous on the moduli space of metric graphs of a fixed genus. This is in contrast with the analogous statement for algebraic curves. However, as shown in Theorem 1.6 of the same paper, the function taking a metric graph to its Brill–Noether rank is indeed upper semicontinuous which implies that this notion serves as a good tropical analog for the dimension of the Brill–Noether locus of an algebraic curve.

The generalization of this upper semicontinuity to the setting of tropical curves is not just a simple extension of the metric graph case. The difficulty lies in the fact that, for a converging sequence of curves in  $M_g^{\text{tr}}$ , the limit may be a curve with an entirely different topology. To overcome this difficulty, we study the divisor theory of tropical curves as the diameters of certain subcurves become sufficiently small. See Proposition 4.3 for a precise statement. Proposition 4.4 then shows that every divisor class contains an effective representative whose restriction to these subcurves has a sufficiently high degree. It follows that the rank of the divisor does not drop in the limit, despite the change in topology.

In order to manage sequences of pairs of tropical curves and divisors, we construct in Sect. 3 a universal Jacobian classifying divisors on all tropical curves of a fixed combinatorial type, with the property that the fiber over each point is the Jacobian of the corresponding curve. Within this space we will identify the universal Brill–Noether locus and show in Sect. 4 that it is a closed set, or equivalently, that the function taking a divisor class to its rank is upper semicontinuous. We will then use this result in Sect. 5 to conclude that the Brill–Noether rank is upper semicontinuous on the moduli space of tropical curves of genus  $g$ .

## 2 Preliminaries

In this section we briefly review some of the basics of divisor theory on tropical curves. We refer the reader to [2], [5], [11], and [10, Sect. 2] for further details. We also give a mild generalization of Luo’s theory of rank determining sets from metric graphs to tropical curves.

### 2.1 Tropical curves

**Definition 2.1** By a metric graph we mean a pair  $(G, \ell)$  such that  $G$  is a connected graph and

$$\ell : E(G) \rightarrow \mathbb{R}_{>0}$$

is a function known as the length function. A **tropical curve**  $\Gamma$  is a triple  $(G, w, \ell)$  such that  $(G, \ell)$  is a metric graph and

$$w : V(G) \rightarrow \mathbb{Z}_{\geq 0}$$

is a function known as the weight function. A **pure tropical curve** is a tropical curve whose weight function is identically zero. To emphasize that a curve is not pure we will sometimes call it a weighted tropical curve.

*Remark 2.1* We implicitly identify a tropical graph  $\Gamma$  with the underlying metric space of the metric graph  $(G, \ell)$ .

### 2.2 Divisor theory of a tropical curve

Let  $\Gamma = (G, 0, \ell)$  be a pure tropical curve. Recall that a **rational function** on  $\Gamma$  is a piecewise linear function with integer slopes. To a rational function  $f$ , we associate a divisor  $\text{div}(f)$  whose value at each point  $p$  of  $\Gamma$  is the sum of the incoming slopes of  $f$  at  $p$ . The group of divisors of the form  $\text{div}(f)$  is denoted  $\text{Prin}(\Gamma)$ . The **rank** of a divisor  $D$  is the smallest number  $r$  such that for every effective divisor  $E$  of degree  $r$ ,  $D$  is linearly equivalent to a divisor containing  $E$  (where two divisors are linearly equivalent whenever their difference is  $\text{div}(f)$  for some rational function  $f$ ).

Now let  $\Gamma = (G, w, \ell)$  be any tropical curve. The pure tropical curve associated to  $\Gamma$  is the curve  $\Gamma^0 = (G, 0, \ell)$ . For  $\epsilon > 0$  we denote  $\Gamma_\epsilon^w = (G^w, 0, \ell_\epsilon^w)$  the pure tropical curve obtained from  $\Gamma^0$  by attaching  $w(v)$  loops of length  $\epsilon > 0$  at every vertex  $v$  of  $\Gamma$ . As shown in [2, Theorem 5.4], for a divisor  $D$  supported on  $\Gamma^0$ , the rank of  $D$  as a divisor on the curve  $\Gamma_\epsilon^w$  is independent of the choice of  $\epsilon$ .

**Definition 2.2** ([2]) Let  $\Gamma$  be a tropical curve.

1. The divisor group of  $\Gamma$  is defined to be the divisor group of the underlying pure tropical curve, namely  $\text{Div}(\Gamma) := \text{Div}(\Gamma^0)$ .
2. Let  $D$  be a divisor on  $\Gamma$ . The **rank** of  $D$ , denoted  $r_\Gamma(D)$ , is its rank as a divisor on  $\Gamma_\epsilon^w$  for any  $\epsilon > 0$ . Since  $r_{\Gamma_\epsilon^w}$  is independent of  $\epsilon$ , this is well defined. When the curve  $\Gamma$  is known from context we shall simply write  $r(D)$ .

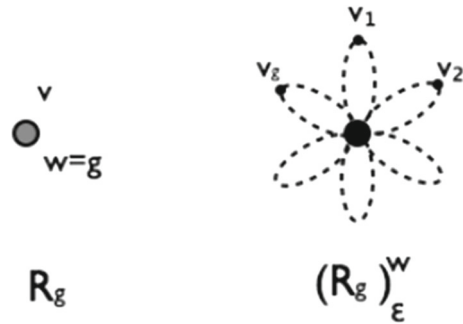
The **genus** of a graph  $G$  is its first Betti number  $g(G) = |E| - |V| + 1$ , and the genus of the tropical curve  $\Gamma = (G, w, \ell)$  is

$$g(\Gamma) = g(G) + \sum_{v \in V(G)} w(v).$$

### 2.3 Rank-determining sets for tropical curves

The theory of rank-determining sets, introduced by Luo, shows that in order to determine the rank of a divisor  $D$  on a (non-weighted) metric graph, one only needs to determine whether or not  $D - E$  is linearly equivalent to an effective divisor for a finite set of divisors  $E$ . We recall the basic notions of the theory.

**Fig. 1** The rose with  $g$  petals



**Definition 2.3** ([11]) Let  $A$  be a subset of a metric graph  $\Gamma = (G, \ell)$  and let  $D$  be a divisor on  $\Gamma$ . The  $A$ -rank of  $D$ , denoted  $r_{\Gamma,A}(D)$ , is the largest number  $r$  such that whenever  $E$  is an effective divisor of degree  $r$  which is supported on the set  $A$ , the divisor  $D - E$  is linearly equivalent to an effective divisor. The set  $A$  is said to be a rank determining set if the rank of every divisor  $D$  on  $\Gamma$  coincides with its  $A$ -rank.

For instance, if  $G$  is a loopless graph then  $V(G)$  is a rank- determining set for  $\Gamma$  [11, Theorem 1.6]. In Corollary 2.1, we extend the notion of rank-determining sets to tropical curves and show the existence of finite rank- determining sets supported on the underlying pure tropical curve.

Next, we wish to give an equivalent definition of the rank in terms of the underlying pure tropical curve.

Let us first consider an example:

*Example 2.1* Let  $R_g$  be the tropical curve consisting of a single vertex  $v$  of weight  $g$  and no edges. The pure tropical curve  $(R_g)_\epsilon^w$ , obtained by attaching  $g$  loops to the vertex  $v$ , is known as the “rose with  $g$  petals” (see Fig. 1).

Let  $D$  be the divisor  $d \cdot v$  on  $R_g$  for some  $d \geq 0$ . Then

$$r(D) = \begin{cases} d - g & d > 2g \\ \lfloor \frac{d}{2} \rfloor & d \leq 2g \end{cases}.$$

In particular,  $r(D) \geq r$  if and only if  $d \geq r + \min\{r, g\}$ . If  $d > 2g$  then this follows immediately from the Riemann–Roch Theorem for tropical curves ([2, Theorem 5.4]). The case where  $d < 2g$  appears in [2, Lemma 3.7] for weighted non-metric graphs and the proof in general is similar: let  $v_1, \dots, v_g$  be the vertices in the middle of the attached loops of  $(R_g)_1^w$ . Then  $v, v_1, \dots, v_g$  is a rank- determining set for  $(R_g)_1^w$ . Let  $E = b \cdot v + b_1 \cdot v_1 + \dots + b_g \cdot v_g$  be an effective divisor of degree  $\lfloor \frac{d}{2} \rfloor$  supported on  $v, v_1, \dots, v_g$ . For each  $i$ , identify the interior of the loop containing  $v_i$  with the interval  $(0, 1)$ . Now define a rational function  $f_i$  as follows

$$f_i = \begin{cases} x & 0 < x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} < x < 1 \end{cases}$$

and extend  $f_i$  to the rest of  $(R_g)_1^w$  by zero. Then  $D + \sum_i b_i \cdot \text{div}(f_i)$  is effective and contains  $E$ , Hence  $r(D) \geq \lfloor \frac{d}{2} \rfloor$ . For the converse,  $D$  could either be a special divisor (i.e.,  $r > d - g$ ) or a non-special divisor. If  $D$  is a non-special divisor then  $2r = 2d - 2g < 2d - d = d$ . Otherwise,  $D$  is special and  $r \leq 2d$  by Clifford’s Theorem ([8, Theorem 1]).

Roughly speaking, Example 2.1 shows that a single vertex of weight  $w(v)$  places  $r + \min\{r, w(v)\}$  conditions on a divisor of rank  $r$ . In general, for a divisor  $E = b_1v_1 + \dots + b_s v_s$  on a tropical curve, we write  $E^*$  for the divisor

$$E^* = \sum_{i=1}^s (b_i + \min\{b_i, w(v_i)\})v_i.$$

*Remark 2.2* A similar idea already appeared in the proof of the Specialization Lemma for weighted graphs ([2, Theorem 4.9]). To show that the divisor obtained by specialization has the desired rank it was first proved that for a fixed vertex  $v$ , its equivalence class contains a divisor whose value at  $v$  is at least  $r + \min\{r, w(v)\}$ .

**Proposition 2.1** *Let  $D$  be a divisor on a tropical curve  $\Gamma = (G, w, \ell)$ . Then  $D$  has rank  $r$  if and only if for every effective divisor  $E$  of degree  $r$ , the divisor  $D - E^*$  is linearly equivalent to an effective divisor.*

*Moreover, if  $A$  is a rank-determining set for  $\Gamma_0$  which contains the vertices with non-trivial weights, then it suffices to only consider divisors  $E$  that are supported on  $A$ .*

*Proof* To prove this we will use the following characterization of the rank of a divisor on a tropical curve:

$$r(D) = \min_{0 \leq F \leq \mathcal{W}} (\text{deg}(F) + r_{\Gamma_0}(D - 2F)), \tag{2.1}$$

where  $\mathcal{W} = \sum_{v \in V(G)} w(v) \cdot v$  ([1, Corollary 5.5]).

Suppose first that  $D - E^*$  is linearly equivalent to an effective divisor whenever  $E$  is effective of degree  $r$  supported on  $A$ , and let us show that the rank of  $D$  is at least  $r$ . By (2.1), we need to show that whenever  $F$  is an effective divisor which is contained in  $\mathcal{W}$ , then  $\text{deg}(F) + r_{\Gamma_0}(D - 2F) \geq r$ . Let  $F$  be such a divisor. If  $\text{deg}(F) > r$ , then  $\text{deg}(F) + r_{\Gamma_0}(D - 2F) \geq r + 1 - 1 = r$ . Therefore, we may assume that  $\text{deg}(F) \leq r$ . Since  $0 \leq F \leq \mathcal{W}$ ,  $F^*$  is just  $2F$ . Let  $F'$  be any effective divisor of degree  $r - \text{deg}(F)$  supported on  $A$ . Since  $F' + 2F \leq (F' + F)^*$  and  $\text{deg}(F' + F) = r$ , together with the fact that  $F, F'$  are supported on  $A$ , we conclude that  $D$  is linearly equivalent to an effective divisor that contains  $F' + 2F$ . Hence  $D - 2F$  is equivalent to a divisor containing  $F'$ . Since this is true for every effective divisor  $F'$  of degree  $r - \text{deg}(F)$  which is supported on a rank-determining set for  $\Gamma_0$ , we see that  $r_{\Gamma_0}(D - 2F) \geq r - \text{deg} F$ , hence  $\text{deg}(F) + r_{\Gamma_0}(D - 2F) \geq r$ .

Conversely, suppose that the rank of  $D$  is  $r$ , and let  $E$  be an effective divisor of degree  $r$  on  $\Gamma_0$ . Write  $E = E_w + E_0$ , where  $E_w(v) = \min\{E(v), w(v)\}$  at every vertex  $v$ , zero everywhere else, and  $E_0 = E - E_w$ . Then  $E^* = 2E_w + E_0$ . To prove that

$D - 2E_w - E_0$  is equivalent to an effective divisor it suffices to show that its rank on  $\Gamma_0$  is non-negative. From the definition of the rank, subtracting an effective divisor may only decrease the rank by the degree of the subtracted divisor, hence  $r_{\Gamma_0}(D - 2E_w - E_0) \geq r_{\Gamma_0}(D - 2E_w) - \deg(E_0)$ . The fact that  $E_w \leq \mathcal{W}$  implies  $r \leq \deg(E_w) + r_{\Gamma_0}(D - 2E_w)$ , hence  $r_{\Gamma_0}(D - 2E_w) - \deg(E_0) \geq r - \deg(E_w) - \deg(E_0) = 0$ . This is true for every such  $E$ , therefore  $r_{\Gamma, V(G)}^*(D) \geq r$ .

In light of Proposition 2.1, we introduce the following definition, which is a slight generalization of the notion of rank-determining sets for tropical curves.

**Definition 2.4** Let  $A$  be a subset of a tropical curve  $\Gamma$  and let  $D$  be a divisor on  $\Gamma$ . The weighted  $A$ -rank of  $D$ , denoted  $r_{\Gamma, A}^*(D)$  is the largest number  $r$  such that whenever  $E$  is an effective divisor of degree  $r$  which is supported on  $A$ , the divisor  $D - E^*$  is linearly equivalent (as a divisor on  $\Gamma^0$ ) to an effective divisor. We say that  $A$  is a weighted rank-determining set if the rank of every divisor  $D$  on  $\Gamma$  coincides with its weighted  $A$ -rank.

An immediate corollary of Proposition 2.1 is

**Corollary 2.1** Let  $\Gamma = (G, w, \ell)$  be a tropical curve. Then every rank-determining set for  $\Gamma_0$  which contains the vertices of positive weight is a weighted rank-determining set for  $\Gamma$ .

In particular, the set of vertices of a tropical curve is a weighted rank-determining set. □

### 3 The universal Jacobian

Let  $\Gamma$  be a tropical curve of genus  $g$ . The **Jacobian** of  $\Gamma$  is the group of divisor classes of degree zero on  $\Gamma$ :

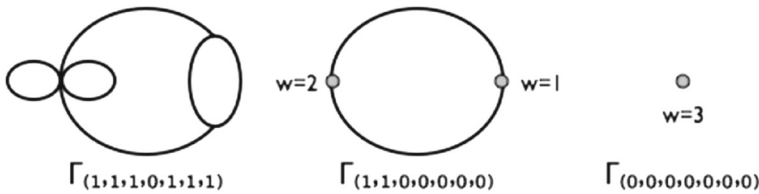
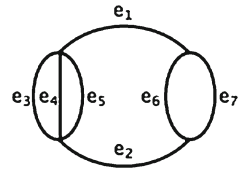
$$J(\Gamma) = \text{Div}^0(\Gamma) / \text{Prin}(\Gamma).$$

The Jacobian is identified with the torus  $H_1(\Gamma^0, \mathbb{R}) / H_1(\Gamma^0, \mathbb{Z}) \simeq \mathbb{R}^g(\Gamma^0) / \mathbb{Z}^g(\Gamma^0)$ , where  $g(\Gamma^0)$  is the genus of  $\Gamma^0$ . See [4], [12] for details. In this section, we introduce the universal space of divisor classes over the space of all tropical curves of a fixed combinatorial type.

**Definition 3.1** Let  $\Gamma = (G, w, \ell)$  be a tropical curve. The combinatorial type of  $\Gamma$  is  $(G, w)$ .

Fix a combinatorial type  $(G, w)$ . Let  $E(G) = \{e_1, \dots, e_n\}$  and denote  $\sigma = \mathbb{R}_{\geq 0}^n$ . We can identify any point  $(s_1, \dots, s_n)$  in the interior of  $\sigma$  with the tropical curve of type  $(G, w)$  with  $\ell(e_i) = s_i$  which we denote by  $\Gamma_{(s_1, \dots, s_n)}$ . Next, we extend this definition to the boundary of  $\mathbb{R}_{\geq 0}^n$ . For a point  $s$  on the boundary of  $\sigma$  we construct a tropical curve  $\Gamma_s = (G_s, \ell_s, w_s)$  as follows. Let  $G_s$  be the graph obtained from  $G$  by contracting an edge  $e_k$  whenever  $s_k = 0$ . Define the length function by  $\ell(e_i) = s_i$  whenever  $s_i > 0$ , and the weight  $w_s$  at a vertex  $v$  of  $G_s$  by  $w_s(v) = g(H_v)$ , where  $H_v$  is the subgraph of

**Fig. 2** The finite graph  $G$



**Fig. 3** Different tropical curves obtained from the graph  $G$

$G$  being collapsed to the point  $v$  in  $G_s$ . For instance, by contracting a loop edge, the weight at the base is increased by one, and contracting an edge between two vertices identifies both of them to a single vertex whose weight is the sum of the weights at both ends.

*Remark 3.1* Points on the boundary of  $\sigma$  correspond to curves of different combinatorial types than points in the interior. All points in the interior of a face of  $\sigma$  correspond to tropical curves of the same type.

*Example 3.1* If  $G$  is the graph shown in Fig. 2 and  $w$  is the zero function, then the tropical curves shown in Fig. 3 are all on different faces of  $\sigma$ .

For every  $s = (s_1, \dots, s_n) \in \sigma$  there is a natural rescaling map  $\alpha_s : \Gamma_{(1, \dots, 1)} \rightarrow \Gamma_s$  that induces a map  $H_1(\Gamma_{(1, \dots, 1)}, \mathbb{Z}) \rightarrow H_1(\Gamma_s, \mathbb{Z})$  and a surjection of homology groups  $H_1(\Gamma_{(1, \dots, 1)}, \mathbb{R}) \rightarrow H_1(\Gamma_s, \mathbb{R})$ , hence a surjection  $\alpha_{s*} : J(\Gamma_{(1, \dots, 1)}) \rightarrow J(\Gamma_s)$ . Note that this map is surjective even when edges are contracted. Therefore, we obtain a map

$$\alpha : \sigma \times J(\Gamma_{(1, \dots, 1)}) \rightarrow \prod_{s \in \sigma} \{s\} \times J(\Gamma_s)$$

given by

$$\alpha(s, D) = (s, \alpha_{s*}(D)).$$

**Definition 3.2** Let  $(G, w)$  be a combinatorial type of tropical curves.

1. The universal Jacobian of the space of tropical curves of type  $(G, w)$  is

$$J(G, w) = \prod_{s \in \sigma} \{s\} \times J(\Gamma_s)$$

endowed with the quotient topology, that is to say the largest topology making  $\alpha$  continuous.

- Fix a point  $p$  in  $\Gamma_{(1,\dots,1)}$ . For every  $s$  in  $\sigma$ , let  $\alpha_s(p)$  be its image in  $\Gamma_s$  under the natural rescaling map. By identifying pairs of the form  $(s, [D])$  in  $\{s\} \times J(\Gamma_s)$  with pairs of the form  $(s, [D - d \cdot \alpha_s(p)])$  in  $\{s\} \times \text{Pic}_d(\Gamma_s)$  we obtain the universal Picard space

$$\text{Pic}_d(G, w) = \coprod_{s \in \sigma} \{s\} \times \text{Pic}_d(\Gamma_s),$$

with the property that the map  $\sigma \times (\Gamma_{(1,\dots,1)})^d \rightarrow \text{Pic}_d(G, w)$ , sending  $(s, E)$  to  $(s, \alpha_{s*}[E - d \cdot p])$ , is continuous.

- The universal Brill–Noether locus, denoted  $W'_d(G, w)$ , is the subset of  $\text{Pic}_d(G, w)$  consisting of pairs  $(s, [D])$  such that the rank of  $D$  is at least  $r$ .

We can also exhibit the universal Jacobian explicitly as a quotient of a polyhedral set inside  $\mathbb{R}^{2n}$ . Let  $c_1, \dots, c_g$  be a basis for the  $\mathbb{Z}$ -module  $H_1(\Gamma_{1,\dots,1}, \mathbb{Z})$ , and suppose that  $c_i = \sum_j a_{ij}e_j$ , where  $e_j$  are the edges of  $G$  and each  $a_{ij}$  is an integer. Notice that for any choice of  $s = (s_1, \dots, s_n)$  in  $\mathbb{R}_{\geq 0}^n$ , the corresponding sums of edges in  $\Gamma_s$  obtained by the rescaling map also generate  $H_1(\Gamma_s, \mathbb{Z})$ . For each  $i$ , let  $C_i$  be the vector of coefficients of  $c_i$ , namely  $C_i = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{R}^n$ , and for  $(s_1, \dots, s_n)$  in  $\sigma$  denote  $C_i^{s_1, \dots, s_n} = (s_1 a_{i1}, \dots, s_n a_{in})$ . Define

$$\mathcal{H} = \coprod_{(s_1, \dots, s_n) \in \sigma} \{(s_1, \dots, s_n, \sum_{i=1}^g t_i C_i^{s_1, \dots, s_n}) \mid 0 \leq t_i \leq 1\} \subseteq \sigma \times \mathbb{R}^n$$

(where  $t_i$  runs through all the values between 0 and 1) and let

$$\mathcal{J} = \mathcal{H} / \sim,$$

where  $\sim$  is the equivalence relation identifying opposite faces in the parallelogram above each point in  $\sigma$ . In other words, for each  $k = 1, \dots, g$ , the relation  $\sim$  identifies points of the form

$$(s_1, \dots, s_n, \sum_{i \neq k} t_i C_i^{s_1, \dots, s_n})$$

with those of the form

$$(s_1, \dots, s_n, \sum_{i \neq k} t_i C_i^{s_1, \dots, s_n} + C_k^{s_1, \dots, s_n}).$$

The restriction of  $\mathcal{J}$  to the fiber above any point of  $\sigma$  is naturally isomorphic to the Jacobian of the corresponding tropical curve and we have a natural map

$$\psi : \sigma \times J(\Gamma_{(1,\dots,1)}) \rightarrow \mathcal{J}$$



sending

$$(s_1, \dots, s_n) \times \sum_{i=1}^g t_i C_i^{1, \dots, 1}$$

to

$$(s_1, \dots, s_n) \times \sum_{i=1}^g t_i C_i^{s_1, \dots, s_n}.$$

To show that this definition agrees with the one above, consider the following commuting diagram:

$$\begin{array}{ccc}
 & \sigma \times J(\Gamma_{(1, \dots, 1)}) & \\
 \alpha \swarrow & & \searrow \psi \\
 J(G, w) & \xrightarrow{\theta} & \mathcal{J}
 \end{array}$$

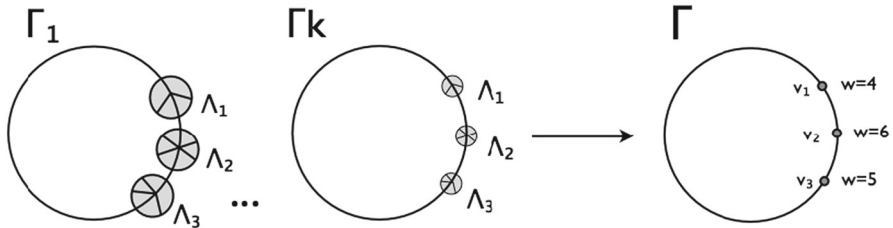
with  $\theta$  defined on each fiber  $\{s\} \times J(\Gamma_s)$  of  $J(G, w)$  as the one to one and onto map identifying  $J(\Gamma_s)$  with a torus. Let us show that  $\theta$  is continuous: suppose  $U \subseteq \mathcal{J}$  is an open set. By definition, a set in  $J(G, w)$  is open if and only if its preimage in  $\sigma \times J(\Gamma_{(1, \dots, 1)})$  is, hence  $\theta^{-1}(U)$  is open if and only if  $\alpha^{-1}(\theta^{-1}(U))$  is. But by the commutativity of the diagram  $\alpha^{-1}(\theta^{-1}(U)) = \psi^{-1}(U)$  which is open since  $\psi$  is continuous. Hence  $\theta$  is continuous. The Jacobian  $J(G, w)$  is compact because it is the continuous image of a compact space and  $\mathcal{J}$  is seen from definition to be Hausdorff. The map  $\theta$  is a bijection between a compact and a Hausdorff space, hence a homeomorphism.

#### 4 The universal Brill–Noether locus

This section is devoted to proving the following proposition:

**Theorem 4.1** *Let  $(G, w)$  be a combinatorial type of tropical curves. Then the universal Brill–Noether locus  $W_d^r(G, w)$  is closed in  $\text{Pic}_d(G, w)$ .*

The proposition is equivalent to the fact that, whenever a sequence  $(s_i, [D_i])$  converges to a pair  $(s, [D])$  in  $\text{Pic}_d(G, w)$ , and for every  $i$  the rank of  $D_i$  is at least  $r$ , then the limit divisor class  $[D]$  has rank at least  $r$  as well. Suppose that  $\Gamma_s = (G_s, w_s, \ell_s)$  and the vertices of  $G_s$  are  $v_1, \dots, v_m$ . By the discussion above on rank-determining sets, this is the same as showing that for every effective divisor  $E = b_1 v_1 + \dots + b_m v_m$  of degree  $r$ , the divisor  $D$  is equivalent to a divisor containing  $b_j + \min\{b_j, w(v_j)\}$  points at every vertex  $v_j$ . Each vertex  $v_j$  was obtained, roughly speaking, by collapsing a subgraph  $H_j$  of genus  $w(v_j)$  of  $G$  and that subgraph is realized as a subcurve  $\Lambda_i$ . Hence we expect that, when  $i$  is large enough, the divisor  $D_i$  will be linearly equivalent to a divisor whose restriction to a neighborhood of each subcurve  $\Lambda_i$  is of degree of



**Fig. 4** A sequence of tropical curves converging in  $\text{Pic}_d(G, w)$

at least  $b_j + \min\{b_j, w(v_j)\}$ . This leads us to explore the behavior of divisors when the diameter of certain subcurves is very small with respect to the rest of the curve. Figure 4 illustrates a sequence  $(\Gamma_k)$  of pure tropical curves converging to a weighted tropical curve  $\Gamma$ . The subcurves  $\Lambda_1, \Lambda_2, \Lambda_3$  and collapse in the limit to the weighted vertices  $v_1, v_2,$  and  $v_3$ .

See also [9] for a different example, where the limiting behavior of families of metric graphs was studied in order to extend the Riemann–Roch Theorem to the metric graph case.

**Lemma 4.1** *Let  $D$  and  $E$  be two effective linearly equivalent divisors on a tropical curve  $\Gamma$ , and let  $f$  be a piecewise linear function such that  $D - E = \text{div}(f)$ . Then at every point  $x$  where  $f$  is differentiable we have  $|f'(x)| \leq \text{deg}(D)$ .*

A similar claim for the case where  $\text{deg}(D) = 2$  appeared in the second part of [7, Lemma 3.1] and the proof of the general case is almost identical. We include the details for completeness.

*Proof* Write  $F = \text{div}(f) = D - E = x_1 + \dots + x_d - y_1 - \dots - y_d$ . Pick a point  $z$  in  $\Gamma$ . Let  $s^+(z)$  and  $s^-(z)$  denote the sum of the positive and negative incoming slopes of  $f$  at  $z$ , respectively. We will show that  $s^+(z) \leq d$ . An analogous argument shows that  $s^-(z) \geq -d$ , so the lemma will follow. Let  $U$  be the union of all the paths in  $\Gamma$  emanating from  $z$  along which  $f$  is non-decreasing. Let  $z_1, \dots, z_n$  be the set of points in  $U$  that are either vertices of  $\Gamma$  or are points at which  $f$  is non-differentiable. Let  $W = \{z, z_1, \dots, z_n\}$ . Then  $U \setminus W$  consists of finitely many open segments. Let  $S = \{s_1, \dots, s_k\}$  be the set of closures of these segments. So  $s_1, \dots, s_k$  (which could be either closed intervals or loops) cover  $U$  and intersect at points in  $W$ , and  $f$  is linear along each of them. Orient each segment in  $S$  for reference, and for each point  $y$  in  $W$  let

$$\delta^+(y) = \{j | s_j \text{ is incoming at } y\}$$

$$\delta^-(y) = \{j | s_j \text{ is outgoing at } y\}.$$

Finally, for each  $i = 1, \dots, k$ , let  $m_i$  be the slope of  $f$  along the oriented segment  $s_i$ . Note that the slope of  $f$  along any edge  $e$  leaving  $U$  must be negative, because otherwise  $e$  would lie in  $U$ . Therefore, we have

$$\begin{aligned}
 F(z_i) &\geq \sum_{j \in \delta^+(z_i)} m_j - \sum_{j \in \delta^-(z_i)} m_j \\
 F(z) &\geq \sum_{j \in \delta^+(z)} m_j - \sum_{j \in \delta^-(z)} m_j + s^+(z)
 \end{aligned}$$

for  $i = 1, \dots, n$ . Summing over all  $i$ , we have  $F(z) + F(z_1) + \dots + F(z_n) \geq s^+(z)$ . Since  $F = x_1 + \dots + x_d - y_1 - \dots - y_d$  it must be that  $F(z) + F(z_1) + \dots + F(z_n)$  and hence  $s^+(z)$  is at most  $d$ .

**Definition 4.1** Let  $\Gamma = (G, w, \ell)$  be a tropical curve. A tropical subcurve  $\Lambda$  is a triple  $(H, w', \ell')$  such that  $H$  is a connected subgraph of  $G$ ,  $w'$  is the restriction of  $w$  to the vertices of  $H$  and  $\ell'$  is the restriction of  $\ell$  to the edges of  $H$ . For a tropical subcurve  $\Lambda$ , we write  $N_\delta(\Lambda)$  for the set of points in  $\Gamma$  whose distance from  $\Lambda$  is at most  $\delta$ .

We now fix notation for the remainder of the section. Let  $\Gamma$  be a tropical curve such that  $\Gamma = (G, w, \ell)$  and let  $\Lambda$  be a tropical subcurve of diameter  $\epsilon > 0$  with  $\Lambda = (H, w', \ell')$ .  $H$  is assumed to be a loopless graph with  $m$  vertices.  $D$  will always denote an effective divisor of degree  $d$  and rank at least  $r$  on  $\Gamma$ .

**Proposition 4.1** *Let  $E$  be an effective divisor of degree  $r$  supported on  $\Lambda$ . Then there is a closed neighborhood  $\Lambda'$  of  $\Lambda$  which is contained in  $N_{d\epsilon}(\Lambda)$  and an effective divisor  $D'$  such that  $D$  and  $D'$  coincide on  $\Lambda$ ,  $D'$  is linearly equivalent to  $D$  (as a divisor on  $\Gamma$ ), and  $D'|_{\Lambda'} - E^*$  is equivalent (as a divisor on  $\Lambda'$ ) to an effective divisor.*

*Proof* Since  $r(D) \geq r$ , Proposition 2.1 implies that there is a piecewise linear function  $f$  on  $\Gamma$  satisfying  $E^* \leq D + \text{div}(f)$ . Denote  $\mu = \inf\{f(x) \mid x \in \partial N_\epsilon(\Lambda)\}$ , and let

$$\bar{f} = \begin{cases} \mu & x \in N_\epsilon(\Lambda) \\ \min\{f, \mu\} & x \notin N_\epsilon(\Lambda) \end{cases}$$

Define  $D' = D + \text{div}(\bar{f})$ , (see Sect. 2.2 for the definition of  $\text{div}(\bar{f})$ ).  $D'$  identifies with  $D$  on  $\Lambda$  since  $\bar{f}$  is constant on a neighborhood of  $\Lambda$ , and one easily checks that  $D'$  is effective. Let  $\Lambda'$  be the closure of the union of  $\Lambda$  with the paths emanating from it along which  $f$  is decreasing and strictly larger than  $\mu$ . By Lemma 4.1, the absolute value of the slope of  $f$  is at most  $d$ , hence

$$\sup\{f(x) - f(y) \mid x, y \in \Lambda\} \leq d \cdot \epsilon,$$

implying that these paths cannot be longer than  $d\epsilon$  and  $\Lambda'$  is contained in  $N_{d\epsilon}(\Lambda)$ . It remains to show that the restriction of  $D'$  to  $\Lambda'$  is linearly equivalent (as a divisor on  $\Lambda'$ ) to an effective divisor containing  $E^*$ . Denote  $g = (f - \bar{f})|_{\Lambda'}$  and let  $D''$  be the divisor  $D'' = D'|_{\Lambda'} + \text{div}(g)$  on  $\Lambda'$ .  $D''$  is equivalent on  $\Lambda'$  to  $D'|_{\Lambda'}$  and the claim is that it is effective and contains  $E^*$ . Since this is true for  $(D + \text{div}(f))|_{\Lambda'}$ , it suffices to check that  $D'' \geq (D + \text{div}(f))|_{\Lambda'}$ .

Let  $y$  be any point of  $\Lambda'$ . If  $y$  is in the interior of  $\Lambda'$  then  $g$  identifies with  $(f - \bar{f})$  on a neighborhood of  $y$ , hence  $\text{div}(g)(y) = \text{div}(f - \bar{f})(y)$  and  $D''(y) = (D + \text{div}(f))(y)$ .

If  $y \in \partial\Lambda'$  and  $f(y) > \mu$  then on each path emanating from  $\Lambda'$  at  $y$ ,  $f$  is non-decreasing (otherwise, by the construction of  $\Lambda'$  that path would have been included in  $\Lambda'$ ). Hence  $\text{div}(g)(y) \geq \text{div}(f - \tilde{f})(y)$  and  $D''(y) \geq (D + \text{div}(f))(y)$ . Finally, suppose that  $y \in \partial\Lambda'$  and  $f(y) = \mu$ . Let  $e$  be a segment leading to  $y$  from outside  $\Lambda'$  along which  $f$  has constant slope. If the incoming slope of  $f$  is non-negative then  $\tilde{f}$  identifies with  $f$  on  $e$  and  $(f - \tilde{f}) = 0$  on  $e$ . If the incoming slope of  $f$  is negative then the incoming slope of  $f - \tilde{f}$  is negative as well. In either case,  $\text{div}(g)(y) \geq \text{div}(f - \tilde{f})(y)$  and  $D''(y) \geq D + (\text{div}(f))(y)$ .

**Proposition 4.2** *Suppose that the closed neighborhood  $N_{\epsilon \cdot (3d)^{d-r+1}}(\Lambda)$  deformation retracts onto  $\Lambda$ . Then there is a neighborhood  $\tilde{\Lambda}$  of  $\Lambda$  which is contained in  $N_{\epsilon \cdot (3d)^{d-r+1}}(\Lambda)$  and an effective divisor  $\tilde{D}$  such that*

1.  $D$  is equivalent to  $\tilde{D}$  as a divisor on  $\Gamma$ .
2.  $D$  and  $\tilde{D}$  coincide on  $\Lambda$ .
3. The rank of  $\tilde{D}$  as a divisor on  $\tilde{\Lambda}$  is at least  $r$ .

*Proof* Recall that by Corollary 2.1, the vertices of  $\Lambda$  are a rank-determining set for  $\Lambda$ . Since  $N_{\epsilon \cdot (3d)^{d-r+1}}(\Lambda)$  deformation retracts onto  $\Lambda$  and does not contain any weighted vertices outside  $\Lambda$ , they are a rank-determining set for  $N_{\epsilon \cdot (3d)^{d-r+1}}(\Lambda)$  as well. Let  $S_1, \dots, S_M$  be the different effective divisors of degree  $r$  supported on the vertices of  $\Lambda$ . By Proposition (4.1) with  $\Lambda$  and  $S_1$  we obtain a curve  $\Lambda_1$  of diameter  $\epsilon_1 \leq 3d \cdot \epsilon$  and a divisor  $D_1$ , such that  $D_1$  is linearly equivalent to  $D$ , identifies with it on  $\Lambda$ , and  $D_1|_{\Lambda_1} - S_1^*$  is linearly equivalent, as a divisor on  $\Lambda_1$ , to an effective divisor. Note that the degree of  $D_1|_{\Lambda_1}$  is at least  $r$ .

We now repeat the process successively for  $S_2, \dots, S_M$  as follows. For  $2 \leq i \leq M$ , we use Proposition (4.1) again to obtain a subcurve  $\Lambda_i$  and a divisor  $D_i$ , such that the restriction of  $D_i$  to  $\Lambda_{i-1}$  identifies with  $D_{i-1}$ , and for every  $j = 1, \dots, i$ ,  $D_i|_{\Lambda_i} - S_j^*$  is equivalent, as a divisor on  $\Lambda_i$ , to an effective divisor.

If we denote  $\epsilon_i$  the diameter of  $\Lambda_i$ , then by construction,  $\epsilon_i < 3d \cdot \epsilon_{i-1}$ . Now, if  $\text{deg}(D_i|_{\Lambda_i})$  is the same as  $\text{deg}(D_{i-1}|_{\Lambda_i})$ , then  $D_{i-1}$  is already equivalent (as a divisor on  $\Lambda_{i-1}$ ) to a divisor that contains  $S_i$ . Therefore, we may choose  $\Lambda_i$  to be the same subcurve as  $\Lambda_{i-1}$ , and in this case,  $\epsilon_i = \epsilon_{i-1}$ . In particular, since the total degree of  $D$  is  $d$ , there can be at most  $d - r + 1$  stages where the degree of the restriction increases, and so  $\epsilon_M < \epsilon \cdot (3d)^{d-r+1}$ .

Now choose  $\tilde{D} = D_M$  and  $\tilde{\Lambda} = \Lambda_M$ . By construction,  $\tilde{D}$  is linearly equivalent to  $D$ , and for every subset  $S$  of degree  $r$  of the rank-determining set,  $\tilde{D}|_{\tilde{\Lambda}}$  is equivalent on  $\tilde{\Lambda}$  to a divisor containing  $S^*$ . It follows that  $r_{\tilde{\Lambda}}(\tilde{D}|_{\tilde{\Lambda}}) \geq r$ .

**Definition 4.2** Let  $\Lambda$  be a subcurve of a tropical curve  $\Gamma$  and let  $r \in \mathbb{Z}_{\geq 0}$ . Then we write  $r^\Lambda = r + \min\{r, g(\Lambda)\}$ .

**Corollary 4.1** *In the same conditions and notations as Proposition 4.2,  $D$  is linearly equivalent to an effective divisor whose restriction to  $\tilde{\Lambda}$  has degree at least  $r^\Lambda$ . Moreover, this divisor coincides with  $D$  on  $\Lambda$ .*

*Proof* Let  $\tilde{D}$  be the divisor linearly equivalent to  $D$  obtained in Proposition (4.2). Then  $\tilde{D}$  identifies with  $D$  on  $\Lambda$  and  $r_{\tilde{\Lambda}}(\tilde{D}|_{\tilde{\Lambda}}) \geq r$ . Let  $E$  be the restriction of  $\tilde{D}$  to  $\tilde{\Lambda}$ , denote  $e = \text{deg } E$  and let  $h$  be the genus of  $\tilde{\Lambda}$ . We must show that  $e \geq r^\Lambda$ .

If  $e \geq 2h$  then by the Riemann–Roch theorem for tropical curves ([2, Theorem 5.4]),  $r_{\tilde{\Lambda}}(E) = e - h$ , hence  $e = r_{\tilde{\Lambda}}(E) + h \geq r + h \geq r + \min\{r, h\}$ .

Now suppose  $0 \leq e < 2h$ .  $E$  could either be a special (i.e.,  $r_{\tilde{\Lambda}}(E) > e - h$ ) or a non-special divisor, and we claim that in any case  $2 \cdot r_{\tilde{\Lambda}}(E) \leq e$ : if  $E$  is special this is exactly Clifford’s theorem ([8, Theorem 1]). Otherwise,  $E$  is non-special, namely  $r_{\tilde{\Lambda}}(E) = e - h$ . Assume by contradiction that  $e < 2 \cdot r_{\tilde{\Lambda}}(E) = 2e - 2h$ . But then  $2h < e$ , a contradiction. In any case,  $2r \leq 2 \cdot r_{\tilde{\Lambda}}(E) \leq e$ . Together with the fact that  $e < 2h$ , it follows that  $r < h$ , hence  $e \geq 2r = r + \min\{r, h\} = r^\Lambda$ .

Suppose that instead of a single subcurve we are given two subcurves  $\Lambda_1$  and  $\Lambda_2$  and two integers  $r_1$  and  $r_2$  such that  $r_1 + r_2 = r$ . One could be tempted to use Corollary 4.1 twice to show that  $D$  is equivalent to an effective divisor whose restriction to both  $\tilde{\Lambda}_1$  and  $\tilde{\Lambda}_2$  has degree at least  $r_1^{\Lambda_1}$  and  $r_1^{\Lambda_2}$ , respectively. However, when using the corollary a second time in an attempt to move points of the divisor into  $\tilde{\Lambda}_2$ , there is no guarantee, a priori, that enough of its points will remain inside  $\tilde{\Lambda}_1$ . The following lemma shows that every subcurve contains special configurations of points which are “stuck” inside the subcurve. That is to say, whenever the restriction of an effective divisor to the subcurve is exactly that configuration, it cannot be moved to contain fewer points in the subcurve.

**Lemma 4.2** *Let  $h$  be the genus of the underlying metric space  $\Lambda^0$  and let  $0 \leq k \leq h$ . Then there exists a divisor  $v_1 + \dots + v_k$  of degree  $k$  supported on  $\Lambda$  such that whenever  $E$  and  $E'$  are two linearly equivalent effective divisors, and the restriction of  $E$  to  $\Lambda$  is exactly  $v_1 + \dots + v_k$ , then the degree of the restriction of  $E'$  to  $\Lambda$  is at least  $k$ .*

*Proof* First assume that  $k = h$ . We will show that the set of classes of effective divisors on  $\Lambda$  that do not satisfy the Lemma cannot be the entire  $h$ -dimensional torus  $\text{Pic}_h(\Lambda)$ . Suppose that  $E_\Lambda$  is such a divisor. Namely, there is an effective divisor  $E$  that restricts to  $E_\Lambda$  on  $\Lambda$ , and  $E$  is linearly equivalent to an effective divisor  $E'$  whose restriction to  $\Lambda$  has degree less than  $h$ . Write

$$E = E' + \text{div}(f)$$

for some piecewise linear function  $f$ . The divisors  $\text{div}(f)|_\Lambda$  and  $\text{div}(f|_\Lambda)$  coincide on the interior of  $\Lambda$  hence differ by a combination of its boundary points. Therefore, if we denote the boundary points of  $\Lambda$  by  $y_1, \dots, y_\ell$ , then

$$E|_\Lambda = E'|_\Lambda + \text{div}(f|_\Lambda) + a_1 \cdot y_1 + \dots + a_\ell \cdot y_\ell$$

for some integers  $a_1, \dots, a_\ell$ . Hence the divisor  $E|_\Lambda$  is linearly equivalent, as a divisor on  $\Lambda$ , to  $E'|_\Lambda + a_1 \cdot y_1 + \dots + a_\ell \cdot y_\ell$ . For a fixed  $a_1 y_1 + \dots + a_\ell y_\ell$ , every choice of  $E|_\Lambda$  corresponds to a choice of  $E'|_\Lambda$  which is a divisor of degree strictly smaller than  $h$ . It follows that the set of all such  $[E|_\Lambda]$  is the translation by  $a_1 y_1 + \dots + a_\ell y_\ell$  of the set of effective divisors of a fixed degree smaller than  $h$ . Recall that this set has positive codimension inside  $\text{Pic}_h(\Lambda)$  ([10, Proposition 3.6]). As we let  $a_1 y_1 + \dots + a_\ell y_\ell$  vary over all the divisors supported on the boundary we see that the set of classes  $[E|_\Lambda]$

that do not satisfy the lemma is a countable union of sets of positive codimension, and hence is a proper subset of  $\text{Pic}_h(\Lambda)$ .

Now, let  $E$  be an effective divisor whose restriction to  $\Lambda$  is  $v_1 + \dots + v_k$  for some  $k < h$  and let  $E'$  be an effective divisor such that  $E' = E + \text{div}(f)$  for some piecewise linear function  $f$ . Then  $E' = (E + v_{k+1} + \dots + v_h + \text{div}(f)) - v_{k+1} - \dots - v_h$ . By the first part,  $\text{deg}((E + v_{k+1} + \dots + v_h + \text{div}(f))|_\Lambda) \geq h$ . Therefore,  $\text{deg}(E'|_\Lambda) \geq h - (h - k) = k$ .

Since the lemma above requires the restriction of the divisor to  $\Lambda$  to be exactly  $v_1 + \dots + v_k$ , we shall require a tool to remove superfluous points away from the subcurve. This is given by the following lemma:

**Lemma 4.3** *Let  $E$  be an effective divisor on  $\Gamma$ . Suppose that the restriction of  $E$  to  $\Lambda$  has degree greater than  $k$  but  $E$  is equivalent to an effective divisor  $F$  whose restriction to  $\Lambda$  has degree smaller than  $k$ . Then there exists an arbitrarily small neighborhood  $\Lambda'$  of  $\Lambda$  and an effective divisor  $E'$  linearly equivalent to  $E$  such that its restriction to  $\Lambda'$  has degree exactly  $k$ .*

*Proof* Let  $f$  be the piecewise linear function satisfying  $E + \text{div}(f) = F$ . Let  $\text{outdeg}_\Lambda(f)$  denote the sum of all the slopes of  $f$  emanating from  $\Lambda$ . Then  $\text{outdeg}_\Lambda(f) = \text{deg}(E|_\Lambda) - \text{deg}(F|_\Lambda) > 0$ . For each path emanating from  $\Lambda$ , choose a segment homeomorphic to an interval along which  $f$  has a constant slope and contains no points of  $E$  except possibly at their intersection with  $\Lambda$ . Let  $\{e_1, \dots, e_l\}$  be the sets of those paths. Note that each  $e_i$  can be chosen arbitrarily small. For each  $e_i$ , let  $p_i$  be its endpoint touching  $\Lambda$  and  $q_i$  the other endpoint, and denote  $c_i$  the outgoing slope of  $f$  on  $e_i$ . Rearrange the segments so that  $i_2 > i_1$  whenever  $f(p_{i_2}) > f(p_{i_1})$  or  $f(p_{i_2}) = f(p_{i_1})$  and  $f(q_{i_2}) > f(q_{i_1})$ . Let  $\alpha$  be the first index satisfying

$$\text{deg}(E|_\Lambda) - \sum_{i=1}^{\alpha} c_i < k.$$

Let  $\Lambda'$  be the union of  $\Lambda$  with the segments  $e_{\alpha+1}, \dots, e_l$ . Denote  $c = \text{deg}(E|_\Lambda) - \sum_{i=1}^{\alpha-1} c_i + k > 0$  and let  $M = f(p_\alpha) + \epsilon \cdot c$  for some positive number  $\epsilon$  smaller than the length of  $e_\alpha$ . Now define

$$\bar{f}(x) = \begin{cases} f(p_\alpha) + c \cdot x & x \in e_\alpha, 0 \leq x \leq \epsilon \\ \max(f(x), M) & \text{otherwise} \end{cases},$$

and let  $E' := E + \text{div}(\bar{f})$ . Then

$$\text{outdeg}_{\Lambda'}(\bar{f}) = \sum_{i=1}^{\alpha-1} c_i + c = \sum_{i=1}^{\alpha-1} c_i + \text{deg}(E|_\Lambda) - (k + \sum_{i=1}^{\alpha-1} c_i) = \text{deg}(E|_\Lambda) - k,$$

hence  $\text{deg}(E'|_{\Lambda'}) = \text{deg}(E|_\Lambda) - \text{outdeg}_{\Lambda'}(\bar{f}) = k$  as required.

The next proposition is a stronger version of Corollary 4.1. We use it in Proposition 4.4 to generalize the result of the corollary to multiple tropical subcurves.

**Proposition 4.3** *Suppose that the neighborhood  $N_{\epsilon_i \cdot (3d)^{d-r+1}}(\Lambda)$  deformation retracts onto  $\Lambda$  and does not contain any weighted vertices outside  $\Lambda$ . Then there is an effective divisor  $\tilde{D}$  which is linearly equivalent to  $D$ , a neighborhood  $\tilde{\Lambda}$  of  $\Lambda$  which is contained in  $N_{\epsilon_i \cdot (3d)^{d-r+1}}(\Lambda)$ , and an effective divisor  $U$  of degree  $r$  supported on  $\tilde{\Lambda}$ , such that*

1.  $\tilde{D}$  contains  $U$ .
2. Any divisor that contains  $U$  and is linearly equivalent to  $\tilde{D}$  restricts to a divisor of degree at least  $r^\Lambda$  on  $\tilde{\Lambda}$ .

*Proof* Let  $s = \min\{r, h\}$  where  $h$  is the genus of  $\Lambda^0$ . Let  $V$  be the effective divisor of degree  $s$  supported on  $\Lambda$  that was obtained in Lemma 4.2. Since  $r(D) \geq r$ , we can replace  $D$  with a linearly equivalent effective divisor that contains  $V$ . By Corollary 4.1, there is a divisor  $D'$  which is equivalent to  $D$ , identifies with it on  $\Lambda$ , and satisfies  $\deg(D'|_{\Lambda'}) \geq r^\Lambda$  for some neighborhood  $\Lambda'$  of  $\Lambda$  in  $N_{\epsilon_i \cdot (3d)^{d-r+1}}(\Lambda)$ . In particular,  $D'$  contains  $V$ , and  $D' - V$  has degree at least  $r$  on  $\Lambda'$ .

Suppose that  $D' - V$  is equivalent to some divisor whose restriction to  $\Lambda'$  has degree less than  $r$ . Then by Lemma 4.3,  $D' - V$  is equivalent to an effective divisor  $D''$  whose restriction to an arbitrarily small neighborhood  $\tilde{\Lambda}$  of  $\Lambda'$  has degree exactly  $r$ . Choose  $\tilde{\Lambda}$  small enough so that it is still contained in  $N_{\epsilon_i \cdot (3d)^{d-r+1}}(\Lambda)$ . Let  $U$  be the restriction of  $D''$  to  $\tilde{\Lambda}$ . Take  $\tilde{D} = D'' + V$ . Then  $\tilde{D}$  is equivalent to  $D'$  which is equivalent to  $D$ . Notice that the restriction of  $\tilde{D}$  to  $\tilde{\Lambda}$  is exactly  $V + U$ , hence the restriction of  $\tilde{D} - U$  is exactly  $V$ . Now, let  $E$  be any other effective divisor that contains  $U$  and is linearly equivalent to  $\tilde{D}$ . Then  $E - U$  is effective and linearly equivalent to  $\tilde{D} - U$ , so by the choice of  $V$ , the restriction of  $E - U$  to  $\tilde{\Lambda}$  has degree at least  $s = r^\Lambda - r$ . Therefore,  $\deg(E|_{\tilde{\Lambda}}) \geq s + r = r^\Lambda$ .

Otherwise, the restriction to  $\Lambda'$  of every effective divisor that is linearly equivalent to  $D' - V$  has degree of at least  $r$ . In this case, the restriction to  $\Lambda'$  of every divisor that is linearly equivalent to  $D'$  and contains  $V$  has degree of at least  $r + s = r^\Lambda$ . Hence the proposition is satisfied by choosing  $\tilde{D} = D'$ ,  $\tilde{\Lambda} = \Lambda'$ , and  $U$  to be any effective divisor of degree  $r$ , which contains  $V$ , contained in  $\tilde{D}$ , and is supported on  $\Lambda$ .

The following is the generalization of Corollary 4.1 to multiple subcurves. Suppose that  $r_1, \dots, r_k$  are non-negative numbers such that  $r_1 + \dots + r_k \leq r$ , and  $\Lambda_1, \dots, \Lambda_k$  are tropical subcurves of  $\Gamma$  such that  $\Lambda_i = (H_i, w_i, l_i)$ , each  $H_i$  is loopless, and the diameter of each  $\Lambda_i$  is  $\epsilon_i$ .

**Proposition 4.4** *Suppose that for every  $i$ , the curve  $N_{\epsilon_i \cdot (3d)^{d-r_i+1}}(\Lambda_i)$  deformation retracts onto  $\Lambda_i$  and contains no weighted vertices outside  $\Lambda_i$ . Then there is a neighborhood  $\tilde{\Lambda}_i$  of each  $\Lambda_i$  which is contained in  $N_{\epsilon_i \cdot (3d)^{d-r_i+1}}(\Lambda_i)$ , and an effective divisor linearly equivalent to  $D$ , whose restriction to each  $\tilde{\Lambda}_i$  is of degree at least  $r_i^{\Lambda_i}$ .*

*Proof* We will construct, using induction on  $k$ , an effective divisor  $D_k$ , a sequence of effective divisors  $U_1, \dots, U_k$ , and neighborhoods  $\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_k$  with the following

properties: for every  $i$ ,  $\tilde{\Lambda}_i$  is contained in  $N_{\epsilon_i, (3d)^{d-r_i+1}}(\Lambda_i)$  and contains  $\Lambda_i$ ;  $U_i$  is of degree  $r_i$  and is supported on  $\tilde{\Lambda}_i$ ; the restriction to  $\tilde{\Lambda}_i$  of any divisor that is linearly equivalent to  $D$  and contains  $U_i$  is of degree of at least  $r_i^{A_i}$ ;  $D_k$  is linearly equivalent to  $D$  and contains  $U_1, \dots, U_k$ . Then  $D_k$  satisfies the statement of the proposition.

For  $k = 1$ , this is just Proposition 4.3. Now suppose that  $k$  is at least 2 and that the claim is true for  $i = k - 1$ . Then  $D_{k-1}$  contains  $U_1, \dots, U_{k-1}$ , hence  $D_{k-1} - U_1 - \dots - U_{k-1}$  is effective and of rank at least  $r - r_1 - \dots - r_{k-1} \geq r_k$ . Moreover,  $D_{k-1}$  is linearly equivalent to  $D$ , so we can use Proposition 4.3 again for the the divisor  $D_{k-1} - U_1 - \dots - U_{k-1}$  and the subcurve  $\Lambda_k$ . We obtain an effective divisor  $U_k$  of degree  $r_k$ , an effective divisor  $D'_k$  that contains  $U_k$ , and a neighborhood  $\tilde{\Lambda}_k$  of  $\Lambda_k$  such that  $D'_k$  is linearly equivalent to  $D_{k-1} - U_1 - \dots - U_{k-1}$  and every effective divisor which is linearly equivalent to  $D'_k$  and contains  $U_k$  has degree of at least  $r_k^{A_k}$  on  $\tilde{\Lambda}_k$ . Now take  $D_k = D'_k + U_1 + \dots + U_{k-1}$  and the statement of the induction is satisfied.

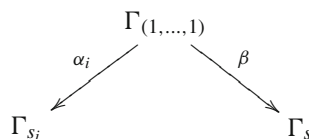
We are now in a position to prove that the Brill–Noether locus is closed in the universal Jacobian:

*Proof* (of Theorem 4.1) Let  $(s_i, [D_i])$  be a sequence in  $\text{Pic}_h(G, w)$  converging to  $(s, [D])$ , and suppose that for every  $i$ ,  $D_i$  is a divisor of degree  $d$  and rank at least  $r$  on  $\Gamma_{s_i}$ . By passing to a subsequence, we may assume that the points  $s_i$  are all in the interior of the same face  $\sigma$  of  $\mathbb{R}_{\geq 0}^{|E(G)|}$ . By replacing  $(G, w)$  with the combinatorial type corresponding to  $\sigma$ , we can assume that every curve  $\Gamma_{s_i}$  is of type  $(G, w)$  and that  $s$  is a point of  $\sigma$ , possibly on the boundary.

Recall the map

$$\Psi^d : \sigma \times (\Gamma_{(1, \dots, 1)})^d \rightarrow \text{Pic}_h(G, w)$$

defined in Sect. 3, and let  $\beta, \alpha_i$  be the natural rescaling maps



(note that every  $\alpha_i$  is a homeomorphism while  $\beta$  is a retract). Let  $v_1, \dots, v_k$  be the vertices of  $\Gamma_s$  which are either weighted or whose preimage under  $\beta$  is more than a single point. Let  $\Lambda_1, \dots, \Lambda_k$  be the disjoint preimages of  $v_1, \dots, v_k$  in  $\Gamma_{(1, \dots, 1)}$ . The weight at every point  $x$  of  $\Gamma_s$  equals the genus of  $\beta^{-1}(x)$ , hence the preimage of any point other than  $v_1, \dots, v_k$  is a single non-weighted point. For every  $1 \leq j \leq k$ , and every  $i$ , let  $\epsilon_i^j$  be the diameter of  $\alpha_i(\Lambda_j)$ . Fix a rank-determining set for  $\Gamma_s$  that contains  $v_1, \dots, v_k$ , namely  $A = \{v_1, \dots, v_k, v_{k+1}, \dots, v_K\}$  for some points  $v_{k+1}, \dots, v_K$ .

Now, let  $E = r_1 v_1 + \dots + r_k v_k + r_{k+1} v_{k+1} + \dots + r_K v_K$  be an effective divisor of degree  $r$  on  $\Gamma_s$ . We need to show that  $D$  is linearly equivalent to an effective divisor containing  $E^* = r_1^{A_1} v_1 + \dots + r_k^{A_k} v_k + r_{k+1} v_{k+1} + \dots + r_K v_K$ . Since for every  $j$ , the subcurve  $\alpha_i(\Lambda_j)$  collapses to a point in the limit  $\beta(\Lambda_j)$ , we can find  $i$  large



enough so that each of the neighborhoods  $N_{\epsilon_i^j(3d)^{d-r_{j+1}}(\alpha_i(\Lambda_j))}$  deformation retracts onto  $\alpha_i(\Lambda_j)$ . Since  $D_i - (r_{k+1}v_{k+1} + \dots + r_K v_K)$  has rank at least  $r_1 + \dots + r_k$ , Proposition 4.4 implies that there are closed neighborhoods  $\Delta_i^j$  of each curve  $\alpha_i(\Lambda_j)$  such that, by replacing  $D_i$  with a linearly equivalent divisor, we can assume that it contains  $r_{k+1}v_{k+1} + \dots + r_K v_K$  and its restriction to  $\Delta_i^j$  has degree at least  $r_j^{A_j}$ .

Let  $\hat{D}_i$  be the preimage of  $D_i$  in  $(\Gamma_{(1,\dots,1)})^d$ . The fact that  $\alpha_i$  is a homeomorphism implies that the restriction of  $\hat{D}_i$  to  $\alpha_i^{-1}(\Delta_i^j)$  has degree  $r_j^{A_j}$  as well. Hence, for every  $j$  we may choose an effective divisor  $C_i^j$  of degree  $r_j^{A_j}$  on  $\Gamma_{(1,\dots,1)}$  that is contained in the restriction of  $\hat{D}_i$  to  $\alpha_i^{-1}(\Delta_i^j)$ . By compactness we may assume that the sequence  $\hat{D}_i$  converges in  $(\Gamma_{(1,\dots,1)})^d$  to some  $\hat{D}$ , and similarly that each sequence  $C_i^j$  converges to some  $C^j$  such that  $C^j \leq \hat{D}$ . Each  $C_i^j$  is supported on  $\alpha_i^{-1}(\Delta_i^j)$ , so  $C^j$  is supported on  $\bigcap_{i=1}^\infty \alpha_i^{-1}(\Delta_i^j)$ . Since the diameters of  $\Delta_i^j$  tend to zero as  $i$  tends to infinity, the set  $\bigcap_{i=1}^\infty \alpha_i^{-1}(\Delta_i^j)$  in  $\Gamma_{(1,\dots,1)}$  is exactly  $\Lambda_i^j$ . Therefore,  $C^j$  is supported on  $\Lambda_i^j$  and the degree of the restriction of  $\hat{D}$  to  $\Lambda_i^j$  is at least  $r_j^{A_j}$  for every  $j$ . By continuity of  $\Psi^d$ , the pair  $(s, \hat{D})$  is mapped to  $(s, [D])$  in  $\text{Pic}_h(G, w)$ , hence  $D$  is equivalent to a divisor that contains  $E^*$ .

This is true for every effective divisor  $E$  of degree  $r$  supported on the rank-determining set  $A$ , and it follows that  $r(D) \geq r$ .

### 5 The Brill–Noether rank of a tropical curve

In the classical case, the dimension of the Brill–Noether locus is an important invariant of an algebraic curve. Among its different properties, the function that takes a curve  $C$  to  $\dim(W_d^r(C))$  is upper semicontinuous on the moduli space of algebraic curves of a fixed genus. However, this is no longer true for metric graphs ([10, Example 2.1]). As a substitute for the dimension of the locus, Lim, Payne, and Potashnik introduced the notion of Brill–Noether rank:

**Definition 5.1** ([10]) Let  $\Gamma = (G, \ell)$  be a metric graph and let  $r, d$  be non-negative integers. The Brill–Noether rank  $w_d^r(\Gamma)$  is the largest number  $\rho$  such that every effective divisor of degree  $r + \rho$  is contained in an effective divisor  $D$  of degree  $d$  and rank at least  $r$ .

The authors showed that the function  $w_d^r$  is in fact upper semicontinuous on the space classifying metric graphs of a given genus ([10, Theorem 1.6]). For the case of a tropical curve we propose the following definition:

**Definition 5.2** Let  $\Gamma = (G, w, \ell)$  be a tropical curve and let  $r, d$  be non-negative integers. The Brill–Noether rank  $w_d^r(\Gamma)$  is the largest number  $\rho$  such that every effective divisor  $E$  of degree  $r + \rho$  is contained in an effective divisor  $D$  of degree  $d$  and rank at least  $r$ .

Recall that, by definition, this means that  $E$  and  $D$  are divisors on  $\Gamma^0$ , but the rank of  $D$  is computed as a divisor on the curve  $\Gamma_\epsilon^w$  for any  $\epsilon > 0$ . We will show in 5.1 that with this definition, the function assigning a tropical curve its Brill–Noether rank is upper semicontinuous on  $M_g^{\text{tr}}$ , the moduli space of tropical curve. We briefly review the construction of  $M_g^{\text{tr}}$ , following [6].

Fix a combinatorial type  $(G, w)$ . Recall that we may identify points in  $\mathbb{R}_{>0}^{E(G)}$  with curves of type  $(G, w)$ , by assigning lengths to the edges of  $G$  according to the coordinates of the points. We may then identify points on the boundary of  $\mathbb{R}_{\geq 0}^{E(G)}$  with curves in which the appropriate edges are contracted. To get a unique representation we should identify curves obtained from weight-preserving symmetries. Let  $\text{Aut}(G, w)$  be the group of weight-preserving automorphisms of  $G$ . Then  $\text{Aut}(G, w)$  acts on  $\mathbb{R}_{\geq 0}^{E(G)}$  by simply permutating coordinates. Denote

$$\overline{C(G, w)} = \mathbb{R}_{\geq 0}^{E(G)} / \text{Aut}(G, w)$$

the closure of the cell classifying all curves of type  $(G, w)$ . Consider  $\coprod \overline{C(G, w)}$ , the disjoint union of all the cells, and let  $\sim$  be the equivalence relation identifying two points if they correspond to isomorphic curves.

**Definition 5.3** The moduli space of a tropical curve of genus  $g$  is

$$M_g^{\text{tr}} = \coprod \overline{C(G, w)} / \sim .$$

**Theorem 5.1** *The Brill–Noether rank is upper semicontinuous on the moduli space of tropical curves of genus  $g$ .*

*Proof* It suffices to show that the Brill–Noether rank is upper semicontinuous on the closure of each of the cells comprising  $M_g^{\text{tr}}$ . Fix a combinatorial type  $(G, w)$ , and let  $\sigma = \mathbb{R}_{\geq 0}^{|E(G)|}$ . Let  $[s_i]$  be a sequence in  $\overline{C(G, w)}$  such that the Brill–Noether rank of each  $\Gamma_{s_i}$  is at least  $\rho$ , and suppose that  $[s_i]$  converges to  $[s]$ . We may choose representatives  $s_i$  so that they converge to  $s$  in  $\sigma$ . Now, we need to show that  $w_d^r(\Gamma_s) \geq \rho$ .

Let  $E$  be an effective divisor of degree  $r + \rho$  on  $\Gamma_s$ . For every  $i$ , choose an effective divisor  $E_i$  of degree  $r + \rho$  on  $\Gamma_{s_i}$ , so that the sequence  $(s_i, [E_i])$  converges to  $(s, [E])$  in  $\text{Pic}_{r+\rho}(G, w)$ . Since each of the curves  $\Gamma_{s_i}$  has Brill–Noether rank at least  $\rho$ , there exists a divisor  $D_i$  containing  $E_i$ , such that each  $[D_i]$  is in  $W_d^r(G, w)$ . By passing to a subsequence we may assume that the sequence  $(s_i, [D_i])$  converges in  $\text{Pic}_d(G, w)$  to some  $(s, [D])$  such that  $E$  is contained in  $D$ . By Theorem 4.1,  $W_d^r(G, w)$  is closed in  $\text{Pic}_d(G, w)$ , so  $(s, [D])$  is in  $W_d^r(G, w)$  as well, and the rank of  $D$  is at least  $r$ .

To conclude the paper we wish to examine the relation between the Brill–Noether rank of a tropical curve and the dimension of the Brill–Noether locus of an algebraic curve.

Let  $X_0$  be a nodal curve. The dual graph of  $X_0$  is the metric graph  $(G, \ell)$ , where the vertices of  $G$  are identified with the irreducible components of  $X_0$ , and an edge of length 1 connects two vertices whenever the two corresponding components meet. In particular, loop edges correspond to self-intersection of components of the special fiber.

As defined in [2], the dual weighted graph of  $X_0$  is the weighted metric graph  $(G, w, \ell)$ , where  $(G, \ell)$  is the dual graph of  $X_0$ , and the weight at a vertex  $v$  is the geometric genus of the corresponding component. Note that the genus of the dual weighted graph is the same as the genus of  $X_0$ .

In [3] the author presented a specialization lemma relating the rank of divisors on algebraic curves with the rank of divisors on their dual graphs. This lemma was generalized in [2, Theorem 4.9] and we will need the following special case:

**Theorem 5.2** ([1, Theorem 1.2]) *Let  $X$  be a smooth projective curve over a discretely valued field with a regular semistable model  $\mathcal{X}$  whose special fiber  $X_0$  has dual weighted graph  $\Gamma = (G, w, \ell)$ . Then there is a degree-preserving specialization map  $\tau : \text{Div}(X) \rightarrow \text{Div}(G)$  such that for every  $D \in \text{Pic}(X)$ ,  $r_X(D) \leq r_\Gamma(\tau(D))$ .*

In [10, Theorem 1.7] the authors proved a similar inequality relating the dimension of the Brill–Noether locus of an algebraic curve with the Brill–Noether rank of a tropical curve. The proof used Baker’s specialization lemma together with the following fact:

**Proposition 5.1** ([10, Proposition 5.1]) *Let  $X$  be a smooth projective curve. Suppose  $W_d^r(X)$  is not empty, and let  $E$  be an effective divisor of degree  $r + \dim(W_d^r(X))$  on  $X$ . Then there is a divisor  $D$  whose class is in  $W_d^r(X)$  such that  $D - E$  is effective.*

In what follows we shall use the above proposition together with the specialization lemma for weighted graphs to do the same for the dual weighted graph of the special fiber of a curve:

**Theorem 5.3** *Let  $X$  be a smooth projective curve over a discretely valued field with a regular semistable model whose special fiber has weighted dual graph  $\Gamma$ . Then for every  $r, d \in \mathbb{N}$ ,  $\dim W_d^r(X) \leq w_d^r(\Gamma)$ .*

*Proof* Denote the dimension of  $W_d^r(X)$  by  $\rho$ . To show that the above inequality is true we must show that every effective divisor  $E$  of degree  $r + \rho$  is contained in an effective divisor  $D$  of degree  $d$  and rank at least  $r$ . Note that by definition, both  $D$  and  $E$  are divisors supported on  $\Gamma^0$ .

We start by showing that the statement is true whenever  $E$  is a rational divisor, namely that its distance from any vertex is rational. Let  $E$  be such a divisor. Then  $E$  is the specialization of some effective divisor  $E^X$  of degree  $r + \rho$  on  $X$ . Since  $w_d^r(X) = \rho$ , it follows by [10, Proposition 5.1] that  $E^X$  is contained in an effective divisor  $D^X$  of degree  $d$  and rank at least  $r$ , which specializes to a divisor  $D$  that contains  $E$ . By the Specialization Lemma, the rank of  $D$  on  $\Gamma$  is at least  $r$ .

Now, let  $E$  be any effective divisor on  $\Gamma^0$  and let  $(E_i)$  be a sequence of rational effective divisors converging to  $E$ . For each  $i$ , choose a divisor  $D_i$  of rank  $r$  that contains  $E_i$ . By compactness of  $\text{Pic}_d(\Gamma)$ , there is a subsequence of  $\{D_i\}$  which converges to a divisor  $D$  that contains  $E$ . Since each  $D_i$  is in  $W_d^r(\Gamma)$ , and  $W_d^r(\Gamma)$  is closed,  $D$  is of rank  $r$  as well.

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