Laplacian ideals, arrangements, and resolutions

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Abstract The Laplacian matrix of a graph G describes the combinatorial dynamics of the Abelian Sandpile Model and the more general Riemann–Roch theory of G. The lattice ideal associated to the lattice generated by the columns of the Laplacian provides an algebraic perspective on this recently (re)emerging field. This binomial ideal I_G has a distinguished monomial initial ideal M_G, characterized by the property that the standard monomials are in bijection with the G-parking functions of the graph G. The ideal M_G was also considered by Postnikov and Shapiro (Trans Am Math Soc 356:3109–3142, 2004) in the context of monotone monomial ideals. We study resolutions of M_G and show that a minimal-free cellular resolution is supported on the bounded subcomplex of a section of the graphical arrangement of G. This generalizes constructions from Postnikov and Shapiro (for the case of the complete graph) and connects to work of Manjunath and Sturmfels, and of Perkinson et al. on the commutative algebra of Sandpiles. As a corollary, we verify a conjecture of Perkinson et al. regarding the Betti numbers of M_G and in the process provide a combinatorial characterization in terms of acyclic orientations.

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1 Introduction

Let $G = (V, E)$ be an undirected and connected graph with vertex set $V = [n + 1] = \{1, 2, \ldots, n+1\}$. The *Laplacian* $\mathcal{L}(G)$ of G is the symmetric $(n + 1) \times (n + 1)$ matrix that encodes the dynamics of the *chip-firing game* on the graph G. More recently the Laplacian has been central to the study of a discrete version of the Riemann-Roch theorem for graphs, where chip-firing serves as the graph-theoretic notion of linear equivalence of divisors.

In this paper we are interested in a certain class of ideals arising from $\mathcal{L}(G)$. We fix a field $\mathbb{K}$ and consider the lattice ideal $I_G \subset \mathbb{K}[x_1, \ldots x_{n+1}]$ associated to $\mathcal{L}(G)$. By definition I_G is generated by binomials of the form $\mathbf{x^u} - \mathbf{x^v}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{N}^{n+1}$, and $\mathbf{u} - \mathbf{v}$ are in the lattice spanned by the rows of $\mathcal{L}(G)$. Following the lead of [12] we call this ideal the *toppling ideal* of the graph G; it was first introduced by Perkinson et al. in [16].

After fixing the vertex $n + 1$, the ideal I_G has a distinguished monomial initial ideal M_G with the property that the standard monomials of M_G are in bijection with the so-called *G-parking functions*. This monomial ideal was first studied by Postnikov and Shapiro [17] in the context of monotone monomial ideals and their deformations and can be defined by an explicit combinatorial rule (see below). As is illustrated in [12], the ideal M_G has interesting connections to the Riemann–Roch theory of G.

In each of the papers [12,16,17] various free resolutions of the ideals I_G and M_G are considered. In the case that $G = K_{n+1}$ is a complete graph on $n + 1$ vertices, it is shown in [17] that the monomial ideal M_G has a minimal cellular resolution supported on the first barycentric subdivision of a $(n-1)$-simplex. This fact is used in [17], where the authors describe resolutions of the lattice ideal I_G in the case that G is a *saturated* graph. Indeed in this case the monomial ideal in question is generic, and the resolution coincides with the *Scarf complex* of M_G. By results of [15], this resolution lifts to the Scarf complex of the lattice ideal I_G. In [17] the authors show that the barycentric subdivision of an $(n - 1)$-dimensional simplex supports a resolution of M_G for an arbitrary graph G on n vertices. However, these resolutions are typically far from minimal and thus directly accessible homological information (such as (graded) Betti numbers) is limited. In both [17] and [12] the issue of finding a minimal resolution for the case of a general graph G is left as an open question.

In this paper we describe a simple and explicit minimal cellular resolution of the monomial ideal M_G. The polyhedral complex supporting the resolution is obtained from the graphical hyperplane arrangement $\mathcal{A}_G$ associated to G. More precisely, the intersection of $\mathcal{A}_G$ with a certain affine subspace yields the essential affine hyperplane arrangement $\tilde{\mathcal{A}}_G$. Our main result (Theorem 3) is that $\mathcal{B}_G$, the bounded subcomplex of $\tilde{\mathcal{A}}_G$, supports a minimal-free resolution of the monomial ideal M_G. As a corollary of our result, we verify a conjecture of Perkinson et al. regarding the Betti numbers of M_G. In particular the Betti numbers of M_G enumerate acyclic orientations of certain contractions of the graph G (see Corollary 3).

In their work on Laplacian lattice ideals, Manjunath and Sturmfels [12] demonstrate how the duality involved in the discrete Riemann–Roch for certain graphs can be expressed in terms of M_G^*, the ideal Alexander dual to M_G with respect to the canonical

monomial of G. Our constructions also lead to an explicit description of a (co)cellular resolution of the ideal M_G^* for all graphs G; see Sect. 6.1.

This collaboration began in Berlin in the summer of 2012, and the results of this paper were first presented in the Combinatorics seminar at the University of Miami in September 2012. While this paper was being prepared, the two preprints [11] and [14] were posted on the arXiv announcing similar results. Both papers employ purely algebraic/combinatorial methods while our perspective is that of geometric combinatorics. In recent conversations with the authors of [14] we were made aware of their independent work-in-progress involving cellular resolutions.

2 Graphs, G-parking functions, and monomial ideals

Throughout the paper we let $G = (V, E)$ be a finite, undirected graph on the vertex set $V = [n + 1] = \{1, 2, \ldots, n + 1\}$ and edge set E. We assume that G is connected and without loops but with possibly parallel edges, i.e., multiple edges between vertices i and j.

We let $\mathcal{L}(G)$ denote the *Laplacian* of G. Recall that $\mathcal{L}(G)$ is the symmetric $(n + 1) \times (n + 1)$ matrix with $\mathcal{L}(G)_{ij} = -|\{\text{edges between } i \text{ and } j\}|$ if $i \neq j$ and equal to the degree of $i = j$, otherwise. We will denote by $\Lambda(G)$ the sublattice of $\mathbb{Z}^n$ generated by the rows of $\mathcal{L}(G)$. The Laplacian has been studied in various combinatorial settings including spectral graph theory [7]. Since G is assumed to be connected, $\mathcal{L}(G)$ has a one-dimensional kernel spanned by the vector $(1, 1, \ldots, 1)^t$. The celebrated Matrix-Tree theorem (see [7, Sect. 13.2]) asserts that $|\det \tilde{\mathcal{L}}(G)|$ is the number of spanning trees of the graph G. This is an application of the Binet-Cauchy theorem to the *truncated Laplace matrix* $\tilde{\mathcal{L}}(G)$, the matrix obtained by deleting the $(n + 1)$st (or any other) row and column from $\mathcal{L}(G)$. More recently, the Laplacian of G has appeared in the context of a discrete Riemann–Roch theory for graphs [1], where it encodes the dynamics of the so-called chip-firing moves (the discrete analog of linear equivalence of divisors).

We fix a field $\mathbb{K}$ and let $\mathbb{K}[x_1, x_2, \ldots, x_{n+1}]$ denote the polynomial ring in $n + 1$ generators. Associated to our graph G we let I_G denote the *lattice ideal* associated to $\Lambda(G)$. By definition I_G is generated by binomials of the form $\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}$, where $\mathbf{u} - \mathbf{v}$ is in the lattice $\Lambda(G)$, generated by the columns of $\mathcal{L}(G)$,

$$I_G = \langle \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} : \mathbf{u}, \mathbf{v} \in \mathbb{N}^{n+1}, \mathbf{u} - \mathbf{v} \in \Lambda(G) \rangle. \qquad (1)$$

Here we use the notation $\mathbf{x}^\mathbf{u} = x_1^{u_1} x_2^{u_2} \cdots x_{n+1}^{u_{n+1}}$. The ideal I_G is called the *toppling ideal* of the graph G in [12] and [16].

The binomial ideal I_G has a distinguished monomial initial ideal M_G, characterized by the property that the standard monomials of M_G are given by the G-parking functions. The ideals M_G have also been studied in the context of monotone monomial ideals in [17] and can be described explicitly as follows. For any nonempty subset $I \subseteq [n]$ we define the monomial

$$m_I := \prod \{x_i : ij \in E, i \in I, j \in [n+1] \setminus I\} = \prod_{i \in I} x_i^{d_I(i)}, \qquad (2)$$

where $d_I(i)$ denotes the number of edges from the vertex i to a vertex in $[n+1]\backslash I$. Now define M_G to be the ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$ generated by all m_I, as I ranges over all nonempty subsets of $[n]$:

$$M_G = \langle m_I : \emptyset \neq I \subseteq [n] \rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]. \tag{3}$$

Since G is connected and $n+1 \notin I$, M_G is a proper monomial ideal in R. For a subset $I \subseteq [n+1]$, let us denote by $G[I]$, the *vertex-induced subgraph*, i.e., the graph with vertex set I and edges $\{ij \in E : i, j \in I\}$.

Proposition 1 *The monomial m_I is a minimal generator of M_G if and only if $G[I]$ and $G[I^c]$ are connected.*

Proof Suppose $G[I] = G[I_1] \uplus G[I_2]$. Then $m_I = m_{I_1} \cdot m_{I_2}$. Similarly, if $G[I^c] = G[J_1] \uplus G[J_2]$ and $n+1 \in J_2$, then $m_{I \cup J_1}$ divides m_I. Thus, $G[I]$ and $G[I^c]$ are connected for every minimal generator.

Conversely, assume that $I \subseteq [n]$ satisfies the condition and assume that m_J divides m_I with $m_J \neq m_I$. Observe that J cannot be a subset of I. By the definition of m_I given in (2) above we see that

$$J \subseteq I \implies d_J(j) \geq d_I(j) \quad \text{for all } j \in J. \tag{4}$$

Thus, let $j \in J \backslash I$. Along a path from $j \in I^c$ to $n+1$ inside $G[I^c]$, there is an edge $st \in E$ such that $s \in J$ but $t \in J^c$. Since $s \in I^c$, we have $d_I(s) = 0$. But then $d_J(s) > 0 = d_I(s)$ which contradicts that $d_J(i) \leq d_I(i)$ for all $i \in [n]$.

According to [4], a *G-parking function* is a function $\phi : [n] \to \mathbb{Z}_{\geq 0}$ such that for every $\emptyset \neq I \subseteq [n]$, there is an $i \in I$ such that $0 \leq \phi(i) < d_I(i)$. It is easily seen that $\phi : [n] \to \mathbb{Z}_{\geq 0}$ is a G-parking function if and only if $\mathbf{x}^\phi = x_1^{\phi(1)} x_2^{\phi(2)} \cdots x_n^{\phi(n)} \notin M_G$. Thus, the G-parking functions of G are exactly the nonvanishing monomials in $\mathbb{K}[\mathbf{x}]/M_G$. It is known [6] that the number of G-parking functions (and hence the $\mathbb{K}$-vector space dimension of $\mathbb{K}[\mathbf{x}]/M_G$) equals the number of spanning trees of G which, as we have seen, is given by $|\det \tilde{\mathcal{L}}(G)|$.

To realize M_G as an initial ideal of I_G, we first fix a spanning tree T of G rooted at the vertex $n+1$ and choose an ordering of the variables that is a linear extension of T. More concretely, we choose a total order $\preceq$ of $\{x_1, \ldots, x_{n+1}\}$ that satisfies $x_i \succ x_j$ if in T the vertex i lies on the unique path from $n+1$ to j. We then take the reverse lexicographic term order for this ordering of variables. We call any such monomial ordering a *spanning tree order*, borrowing the term from [12].

Example 1 If $G = K_{n+1}$ is the complete graph on $n+1$ vertices, we can take the usual reverse lexicographic term order. The ideal M_G is the *tree ideal* on n variables, as described in [13, Sect. 4.3.4]. In this case the standard monomials of M_G are given by the classical parking functions studied in combinatorics, i.e., sequences of nonnegative integers $(a_1, a_2, \ldots, a_n)$ with the property that, when placed in weakly ascending order, sit below a staircase: $a_i \leq |\{j : a_j \leq a_i\}|$ for all $i \in [n]$. The *maximal* parking functions are given by the permutations of n, which also correspond to the generators of the ideal that is Alexander dual to M_G.

Example 2 A graph G is said to be *saturated* if G has $u_{ij} > 0$ edges between any two vertices $i \neq j$. In [12] it is shown that if G is a saturated graph on $n + 1$ vertices then the ideal I_G is a generic lattice ideal. For such graphs an explicit set of $2^n - 1$ binomial generators is given that form a Gröbner basis of I_G with respect to reverse lexicographic order.

We next introduce the example that will be used throughout the paper.

Example 3 (Running example) Let G be the 4-cycle with vertices $\{1, 2, 3, 4\}$ and edges $\{12, 23, 34, 14\}$. The Laplacian $\mathcal{L}(G)$ is the 4×4 matrix given below.

$$\mathcal{L}(G) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The ideal I_G is given by

$$I_G = \langle x_1^2 - x_2x_4, x_2^2 - x_1x_3, x_3^2 - x_2x_4, x_4^2 - x_1x_3, x_1x_2 - x_3x_4, x_2x_3 - x_1x_4 \rangle.$$

The monomial ideal M_G is given by

$$M_G = \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_1^2x_3^2, x_2x_3, x_1x_3 \rangle.$$

The generator $m_{\{1,3\}} = x_1^2x_3^2$ is redundant, so $M_G = \langle x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3 \rangle = \langle x_1, x_2, x_3 \rangle^2$.

3 Cellular resolutions

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ denote the polynomial ring on n variables with the standard $\mathbb{Z}^n$-grading given by $\deg(\mathbf{x}^a) = a \in \mathbb{Z}^n_{\geq 0}$ for any monomial $\mathbf{x}^a$. For any graded R-module M, a $\mathbb{Z}^n$-graded *free resolution* of M is an exact sequence

$$0 \leftarrow M \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_r} F_r \leftarrow 0,$$

where each F_i is a graded free R-module

$$F_i \cong \bigoplus_{\sigma \in \mathbb{Z}^n} R(-\sigma)^{\beta_{i,\sigma}}$$

and where each ϕ_i is a graded map. The resolution is called *minimal* if each of the $\beta_{i,\sigma}$ is minimal among all graded free resolutions of M. In this case the $\beta_{i,\sigma} = \beta_{i,\sigma}(M)$ are called the *finely* or $\mathbb{Z}^n$-*graded Betti numbers* of the module M. The $\mathbb{Z}$-*graded* Betti numbers of M are given by

$$\beta_{i,j}(M) = \sum_{|\sigma|=j} \beta_{i,\sigma}(M)$$

Fig. 1 The (minimal) free resolution of M_G for $G = K_4$ is supported on $\mathcal{B}(\Delta_2)$. Vertices are labeled by the subsets $I \subseteq [n]$ corresponding to generators m_I
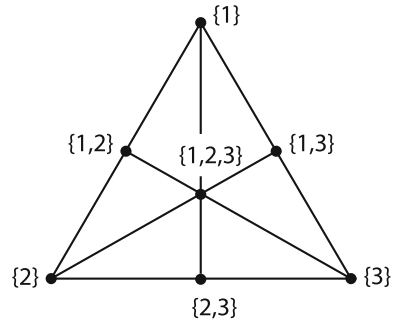
where $|\sigma| = \sigma_1 + \sigma_2 + \cdots + \sigma_n$. And finally, for each integer i, the ith (ungraded) Betti number of M is given by

$$\beta_i(M) = \sum_j \beta_{i,j}(M).$$

A *labeled polyhedral complex* is a polyhedral complex $\mathcal{X}$ together with an assignment $a_F \in \mathbb{N}^n$ to each face $F \in \mathcal{X}$ such that

$$(a_F)_i = \max\{(a_G)_i : G \subset F\}.$$

for all $i = 1, 2, \ldots, n$. Let $\mathcal{X}$ be a labeled polyhedral complex and let

$$M = M_{\mathcal{X}} = \langle \mathbf{x}^{a_F} : F \in \mathcal{X} \rangle \subseteq \mathbb{K}[x_1, \ldots, x_n]$$

be the monomial ideal generated by the labels. Clearly, $M_{\mathcal{X}}$ is generated by the 0-cells of $\mathcal{X}$. For $\sigma \in \mathbb{N}^n$, let $\mathcal{X}_{\leq \sigma} \subseteq \mathcal{X}$ be the subcomplex of faces F for which $a_F \leq \sigma$ componentwise. Bayer and Sturmfels [3] described how labeled complexes can encode resolutions of M, that is, when the chain complex $\mathcal{F}_{\mathcal{X}}$ of $\mathcal{X}$ gives rise to a $\mathbb{Z}^n$-graded resolution—a *cellular resolution*—of M. We refer to [13] for further details from which the following criterion is taken.

Proposition 2 ([13, Proposition 4.5]) *Let $\mathcal{X}$ be a labeled polyhedral complex and let $M = M_{\mathcal{X}} \subset \mathbb{K}[x_1, \ldots, x_n]$ be the associated monomial ideal. Then $\mathcal{X}$ supports a cellular resolution $\mathcal{F}_{\mathcal{X}}$ of M if and only if the subcomplex $\mathcal{X}_{\leq \sigma}$ is $\mathbb{K}$-acyclic for all $\sigma \in \mathbb{N}^n$. The resolution is minimal if $a_F \neq a_G$ for all faces $F \subset G \in \mathcal{X}$.*

Resolutions of I_G and M_G provide a new perspective on the duality expressed in the Riemann–Roch theory of the graph G via the notion of Alexander duality. The Betti numbers of M_G have combinatorial interpretations in terms of the graph G, and, as we will see, also relate to certain well-studied geometric complexes.

In [12] the authors consider resolutions of I_G and M_G. They show that in the case of saturated graphs G, the toppling ideal I_G has a minimal cellular resolution supported on what we will denote as $\mathcal{B}(\Delta_{n-1})$, the first barycentric subdivision of an $(n-1)$-dimensional simplex (Fig. 1).

In this case the complex $\mathcal{B}(\Delta_{n-1})$ can be lifted to the *Scarf complex* associated to the (in this case generic) lattice ideal I_G. This extends a result from [17], where it is shown that the monomial ideal M_G for $G = K_{n+1}$ has a resolution supported on the same complex $\mathcal{B}(\Delta_{n-1})$.

For an arbitrary (not necessarily saturated) graph G, it is shown in [12] that the complex $\mathcal{B}(\Delta_{n-1})$ supports a generally nonminimal cellular resolution of I_G. This leads to a formula for the Betti numbers of M_G in terms of the ranks of the reduced homology of certain induced subcomplexes of $\mathcal{B}(\Delta_{n-1})$, although it is not clear how one might obtain an explicit expression.

The description of a *minimal* resolution of M_G for an arbitrary graph G is stated as an open question in both [12] and [17]. In [16] the authors provide a conjecture for the Betti numbers of M_G in the context of combinatorial data arising from Riemann–Roch theory of G. In Sect. 6 we confirm this conjecture.

4 Chip-firing and superstable configurations

The Laplacian $\mathcal{L}(G)$ describes the dynamics of the so-called *chip-firing game* or *Abelian Sandpile Model* for G. We outline the basic construction here and refer to [16] for further details. In this context, let us choose $n + 1$ as a fixed sink. By a *configuration* c on G we mean a placement of a number c_i of "chips" or "grains of sand" on each nonsink vertex $i \in [n]$. A vertex i is said to be *unstable* if the number of chips c_i on i is greater or equal to the degree of i. In this case, one can then "fire" the vertex i, distributing chips to each of its neighbors (one chip along every edge). Chips that are sent to the sink vertex disappear. It is easily seen that "firing" the vertex i corresponds to subtracting the i-th column of the truncated Laplacian $\tilde{\mathcal{L}}(G)$ from c. If G is connected, repeated chip-firing of vertices eventually leads to a stable configuration $\bar{c}$. One of the fundamental results for chip-firing games is that this configuration $\bar{c}$ is independent of the sequence of firings; we call $\bar{c}$ the *stabilization* of c.

A stable configuration c on G is said to be *recurrent* if c has a nonnegative value on every nonsink vertex $[n]$, and if for any configuration a there exists a configuration b such that $a + b$ stabilizes to c. The stabilization process gives rise to a monoidal structure on the set of all configurations and forms a group when restricted to the recurrent configurations of G. This group is called the *sandpile group* of G, denoted $\mathcal{S}(G)$. The sandpile group can also be realized by considering the discrete subgroup $\mathbb{L}_G \subset \mathbb{Z}^n$ generated by the columns of the truncated Laplacian $\tilde{\mathcal{L}}(G)$. We then have an isomorphism

$$\mathcal{S}(G) \cong \mathbb{Z}^n / \mathbb{L}_G$$

given by $c \mapsto c + \mathbb{L}_G$. Thus, every element of $\mathbb{Z}^n$ is equivalent to a recurrent configuration modulo the reduced Laplacian. As a corollary to the Matrix-Tree theorem, we then see that the order of $\mathcal{S}(G)$ is given by the number of spanning trees of G.

The *canonical configuration* c_ω on G is the maximally stable configuration given by

$$(c_\omega)_i = |\{ij \in E : j \in [n]\}| - 1$$

for every nonsink $i \in [n]$. In the graph-theoretic Riemann–Roch theory developed in [1], configurations are naturally identified with "divisors" on the graph G. In this context the configuration $c_\omega - \mathbf{1}$ is closely related to the "canonical divisor" of G.

To fully describe our results we will need a few more notions from the chip-firing literature. As opposed to firing one vertex at a time, we consider a rule where one may fire sets of vertices simultaneously. This leads to a stronger version of stability and a resulting notion of *superstable* configurations. We will not detail the construction here as it will be enough for us to use the following characterization ([16]): a sequence $a = (a_1, a_2, \ldots, a_n)$ is a *superstable configuration* of G if and only if a is a G-parking function. For the next result, recall that an *acyclic orientation* of G is an orientation $\mathcal{O}$ of the edges of G with no directed cycles.

Theorem 1 ([4, Theorem 3.1]) *There is a bijection between the set of acyclic orientations of G with unique sink $n + 1$ and the set of maximal superstable configurations of G. Given an acyclic orientation $\mathcal{O}$, the corresponding configuration $c = c^{\mathcal{O}}$ is given by*

$$c_i = |\{i \to_{\mathcal{O}} j : j \in [n + 1]\}| - 1.$$

We note that the bijection described in [4] actually involves the set of acyclic orientations with unique *source* $n + 1$, but of course there is a bijection between these and orientations with unique sink $n + 1$ by reversing all arrows.

In [16] (Corollary 5.15) it is shown that a configuration c is superstable if and only if $c_\omega - c$ is recurrent. Hence for an undirected graph G on vertex set $[n + 1]$, there is a bijective correspondence between minimal recurrent configurations, maximal superstable configurations, maximal G-parking functions, and acyclic orientations with $n + 1$ as the unique sink vertex.

5 Graphical arrangements and Whitney numbers

In this section we introduce graphical hyperplane arrangements and other notions from geometric combinatorics that will serve to describe our resolutions. We describe the basic constructions and terminology here, and refer to [8] and [19] for more details.

Once again we fix our graph G on vertex set $V = [n + 1]$ with edge set E. For an edge $ij \in E$, we define a corresponding hyperplane

$$h_{ij} := \{x \in \mathbb{R}^{n+1} : x_i = x_j\}.$$

The arrangement $\mathcal{A}_G = \{h_{ij} : ij \in E\}$ of hyperplanes in $\mathbb{R}^{n+1}$ is called the *graphical arrangement* of G. The arrangement $\mathcal{A}_G$ dissects space into polyhedra, called *cells*, of various dimensions, and we use $f_k(\mathcal{A}_G)$ to denote the number of k-dimensional cells, or k-cells, for short. A *flat* of $\mathcal{A}_G$ is a nonempty intersection of elements of $\mathcal{A}_G$, and we let $L_G = L(\mathcal{A}_G)$ denote the collection of flats partially ordered by reverse inclusion. In fact L_G is a ranked (geometric) lattice, called the *lattice of flats*, with minimum $\mathbb{R}^{n+1}$ and maximum $\hat{1} = \bigcap_{ij \in E} h_{ij}$. The rank $\mathrm{rk}_G(x)$ of $x \in L_G$ is given

by $n + 1 - \dim x$ and, since G is connected, L_G has total rank $\mathrm{rk}(\hat{1}) = n$ (which we also take to be the rank of $\mathcal{A}_G$). The *lattice of partitions* of G is the collection of unordered partitions $V = V_1 \uplus V_2 \uplus \cdots \uplus V_m$ such that the vertex-induced graph $G[V_k]$ is connected for all $1 \le k \le m$. The partial order is by coarsening: a partition $\{V_k\}_k$ is smaller than $\{U_l\}_l$ if every U_s is contained in some V_t. The lattice of partitions is naturally isomorphic to L_G by associating to $\{V_k\}_k$, the flat

$$\bigcap \Big\{ h_{ij} : ij \in E, \{i, j\} \subseteq V_k \text{ for some } k \Big\}.$$

We will freely use both perspectives on the elements of L_G.

Central to the study of hyperplane arrangements (and more general matroids) is the notion of Whitney numbers. Here the *doubly-indexed Whitney numbers* of the first kind are given by

$$w_{ij}(L_G) = \sum \Big\{ \mu_{L_G}(x, y) \ : \ x, y \in L_G, \mathrm{rk}(x) = i, \mathrm{rk}(y) = j \Big\}$$

where μ_{L_G} is Möbius function of L_G. The *Whitney numbers* of L_G are the simply indexed versions

$$w_j(L_G) = w_{0j}(L_G).$$

There is a well-known connection (see [8]) between the Whitney numbers of L_G and the chromatic polynomial of G given by

$$\chi(t) = \sum_{j=0}^{n} w_j(L_G) t^{n-j}.$$

We say that a hyperplane H is in *general position* with respect to the arrangement $\mathcal{A}_G$ if $\dim(x \cap H) = \dim x - 1$ for all flats $x \in L_G$. If $\mathcal{A}$ is a hyperplane arrangement and U an affine subspace not parallel to any hyperplane, then the *restriction* of $\mathcal{A}$ to U is given by $\mathcal{A}|_U = \{ H \cap U : H \in \mathcal{A} \}$. We will need some further results from [8] that we collect here for future reference.

Theorem 2 ([8, Theorem 3.2]) *Let $\mathcal{A}$ be an arrangement of linear hyperplanes of rank r and let $k > 0$. Let H be a hyperplane general with respect to $\mathcal{A}$. Then the induced arrangement $\mathcal{A}|_H$ has $|\mu(0, 1)| = |w_r(L_G)|$ relatively bounded regions and $|w_{d-k,r}|$ relatively bounded $k - 1$ cells.*

Corollary 1 ([8, Corollary 7.3]) *Let G be a graph with vertex set $[n+1]$. The number of acyclic orientations of all contractions G/S in which $S \in L_G$ has k components, such that the vertex $n + 1$ is the only sink, equals $|w_{n+1-k,n}(L_G)|$.*

6 Cellular resolutions from graphical arrangements

In this section we describe our minimal cellular resolution of M_G and derive some consequences. As above we fix a connected graph G on vertex set $[n + 1]$ and let

Fig. 2 The complex $\mathcal{B}_G$ for our running example. The coordinates (as elements of $\mathbb{R}^4$) of the 0-cells are indicated
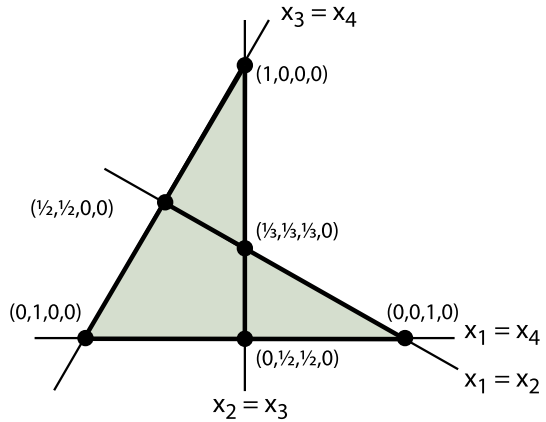
$\mathcal{A}_G \subset \mathbb{R}^{n+1}$ denote the associated graphical arrangement. Let $U \subset \mathbb{R}^{n+1}$ be the affine subspace

$$U := \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0, \ x_1 + x_2 + \cdots + x_n = 1 \right\} \cong \mathbb{R}^{n-1}$$

We let $\tilde{\mathcal{A}}_G = \{\tilde{h}_{ij} := h_{ij} \cap U : ij \in E\}$ be the restriction of $\mathcal{A}_G$ to U. Note that $\tilde{\mathcal{A}}_G$ is an essential arrangement of $|E|$ affine hyperplanes (two hyperplanes $\tilde{h}_{ij}$ and $\tilde{h}_{kl}$ coincide if and only if ij and kl are parallel).

For a point $p \in U$ such that $p_i \neq p_j$ for all $ij \in E$, we obtain an orientation on G by orienting $i \to j$ if $p_i > p_j$. It is easy to see that this orientation is in fact acyclic. If p takes the same value on some edges, we get an acyclic orientation on a certain contraction of G as follows. From p we get a partition of the vertex set $[n + 1] = V_1 \uplus V_2 \uplus \cdots \uplus V_s$ where for each k we place $i, j \in V_k$ if there is a path $i = i_0 i_1 \ldots i_t = j$ in G such that $p_{i_{h-1}} = p_{i_h}$ for all $1 \leq h \leq t$. In particular, the induced subgraphs $G[V_i]$ are connected, and we denote by G/p the result of contracting each $G[V_i]$ to a single vertex. The remaining edges satisfy $p_i \neq p_j$, and thus, we get an acyclic orientation on G/p.

The *bounded complex* $\mathcal{B}_G$ is the polyhedral complex of bounded cells of $\tilde{\mathcal{A}}_G$ in U. We let $|\mathcal{B}_G|$ denote the underlying pointset in U. The next result says that we can determine points p of $|\mathcal{B}_G|$ in terms of the associated graph G/p (Fig. 2).

Proposition 3 *For $p \in U$ let C be the inclusion-minimal cell having p in the relative interior. Then $\dim C + 2$ is the number of vertices of G/p, and $C \in \mathcal{B}_G$ if and only if the acyclic orientation on G/p has a unique sink given by the vertex class containing $n + 1$.*

Proof Let $L \subset U$ be the intersection of all $\tilde{h}_{ij}$ for which $p_i = p_j$. Then C is a full-dimensional cell in L, and we have to determine only $\dim L$. But the hyperplane arrangement induced in L by $\tilde{\mathcal{A}}_G$ is isomorphic to the arrangement $\tilde{\mathcal{A}}_{G/p}$ and thus $\dim L = |V(G/p)| - 2$. It thus suffices to assume that $p_i \neq p_j$ for all edges $ij \in E$.
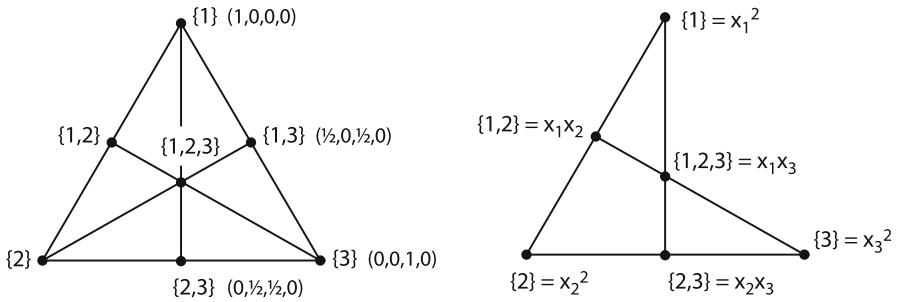
Fig. 3 The subset labeling of $\mathcal{B}_{K_{n+1}}$ corresponding to coordinates, and the induced monomial labeling of $\mathcal{B}_G$ for our running example G

If G has more than one sink, let $i \neq n+1$ be one of them and let j be an arbitrary source. We claim that p is not contained in a bounded cell. Consider the halfline $\{p(t) = p + t(e_j - e_i) : t \geq 0\}$. Since j is a source, we have $p_j > p_k$ for all $jk \in E$ and $p(t)_j \geq p_j > p_k \geq p(t)_k$ for all $t \geq 0$. An analogous argument applies for i and shows that the halfline is contained in the same inclusion-minimal cell of $\tilde{\mathcal{A}}_G$ as p.

Conversely, assume C is unbounded. Since $C \subset U$, there is a point $q \in \mathrm{relint}(C)$ with $q_i < 0$ for some i. All points in the relative interior of C induce the same orientation on G. However, since $q_{n+1} = 0 > q_i$, and G is connected, there is no directed path from i to $n+1$, and thus, $n+1$ is not a sink.

In order for $\mathcal{B}_G$ to support a cellular resolution, we have to label the zero-dimensional cells of $\mathcal{B}_G$. From Proposition 3, we infer that a 0-cell v of $\mathcal{B}_G$ is of the form $v = \frac{1}{|I|} e_I$ where $e_I \in \{0, 1\}^{n+1}$ is the characteristic vector of the nonempty subset $I \subseteq [n]$. Using the fact that subsets $I \subseteq [n]$ correspond to monomials m_I according to (2), this gives us a natural labeling of $\mathcal{B}_G$ (Fig. 3).

Corollary 2 *Under the labeling described above, the* 0*-cells of $\mathcal{B}_G$ are labeled by the minimal generators of M_G. The label $a_v \in \mathbb{N}^n$ of a 0-cell $v = \frac{1}{|I|} e_I$ of $\mathcal{B}_G$ is given by*

$$(a_v)_i := d_I(i)$$

for all $i \in [n]$.

Proof From (2) and (3), we know that $\mathbf{x}^{a_v} = m_I$ is in M_G. Now, by Proposition 1, m_I is a minimal generator if and only if $G[I]$ and $G[I^c]$ are connected and, by Proposition 3 this is the case if and only if $v = \frac{1}{|I|} e_I$ is a 0-cell of $\mathcal{B}_G$.

The labels on higher dimensional bounded cells are determined by the componentwise maximum of the labels of incident 0-cells. That is, for a cell $C \in \mathcal{B}_G$ and $i \in [n]$

$$(a_C)_i = \max\{(a_v)_i : v \in C \text{ is a 0-cell}\}.$$

However, we can also determine the labels of such cells directly.

Proposition 4 *Let $C \in \mathcal{B}_G$ be a bounded cell with label $a_C \in \mathbb{N}^n$. For $p \in \mathrm{relint}(C)$, we have*

$$(a_C)_i = \#\{ij \in E : p_i > p_j\}$$

Proof Fix $1 \le i \le n$ and let $D_i(p) := \#\{ij \in E : p_i > p_j\}$. Consider the set $I \subseteq [n]$ of vertices of G such that $k \in I$ if there is a path $k = i_0 i_1 \ldots i_s = i$ and $p_{i_j} \ge p_i$ for all $0 \le j \le s$. By construction $G[I]$ is connected. Assume that the complementary graph $G[I^c]$ is disconnected, and let $J \subset I^c$ be a connected component not containing $n + 1$. Since C is a cell in the hyperplane arrangement $\tilde{A}_G$, and p is a point in the relative interior, C is the set of points $x \in U$ with

$$x_i = x_j \text{ if } p_i = p_j \quad \text{and} \quad x_i \ge x_j \text{ if } p_i > p_j$$

for all $ij \in E$. Observe that for all $t \ge 0$ the point $p(t) = p + \frac{t}{|I|} e_I - \frac{t}{|J|} e_J$ is in C. Indeed, $p(t) \in U$ for all t. As for the defining equations and inequalities of C, the only relevant case to check is $kl \in E$ with $k \in I$ and $l \in J$. By definition of I, we have $p_k > p_l$ and $p(t)_k = p_k + \frac{t}{|I|} > p_l - \frac{t}{|J|}$ for all $t \ge 0$. This shows that C is not bounded.

It follows that $G[I^c]$ is connected and, by Proposition 3, the point $q = \frac{1}{|I|} e_I$ is a 0-cell of $\mathcal{B}_G$. To see that $q \in C$ let $kl \in E$ such that $p_k > p_l$ but $q_k < q_l$. This implies $l \in I$ and $k \in I^c$, but since $p_k > p_l$, and there is a path from l to i with values $\ge p_i$, this means $k \in I$. As for monomial labels, we have $(a_C)_i \ge (a_q)_i = \#\{ij \in E : j \notin I\} = D_i(p)$.

To see that $D(p)_i$ upper bounds $(a_C)_i$, observe that $\mathrm{relint}(C) \subseteq \{x \in U : x_i > x_j \text{ for all } ij \in E \text{ with } p_i > p_j\}$ for i fixed. In particular C is in the closure of this set and every point q in the closure satisfies $D(q)_i \le D(p)_i = (a_C)_i$.

Here is the main lemma that we need.

Lemma 1 *Let $G = ([n + 1], E)$ be a connected graph and $\mathcal{B}_G$ the bounded subcomplex labeled according to the monomial ideal M_G. For every $\sigma \in \mathbb{N}^n$, the set*

$$|B_G|_{\le \sigma} = \bigcup \{F \in \mathcal{B}_G : a_F \le \sigma\}$$

is star-convex. Hence $|B_G|_{\le \sigma}$ is contractible and in particular $\mathbb{K}$-acyclic for any field $\mathbb{K}$.

Proof Let $p^1, p^2, \ldots, p^m$ be the 0-dimensional cells of $(\mathcal{B}_G)_{\le \sigma}$ and for each i let $a^i = a_{p^i} \in \mathbb{N}^n$ be the corresponding exponent vector of the monomial label. Let us define

$$J = \bigcup_{i=1}^{m} \mathrm{supp}(a^i).$$

In the subgraph of G induced on the vertices $[n + 1] \setminus J$, let K be the set of vertices corresponding to the connected component containing the vertex $n + 1$. Finally, define

Fig. 4 The subspace $(\mathcal{B}_G)_{\leq\sigma}$ for our running example G with $\sigma = (1, 1, 2) = x_1 x_2 x_3^2$. The star point in this case is $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$
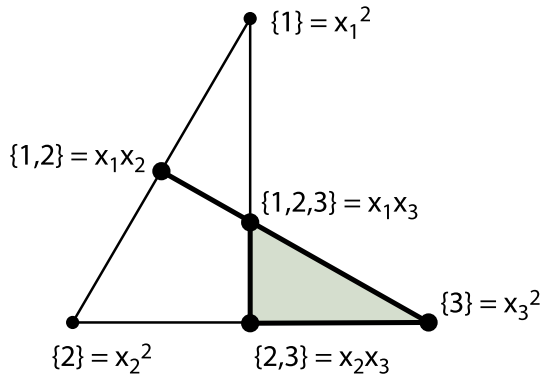
$$I = [n+1] \backslash K \quad \text{and} \quad q = \frac{1}{|I|} e_I.$$

We claim that q is a star point of $|\mathcal{B}_G|_{\leq\sigma}$. By construction, the contraction G/q has a unique sink, and thus, q is contained in the relative interior of some cell $\mathcal{B}_G$ with label a_q (Fig. 4).

Let us next verify that $a_q \leq \sigma$ so that indeed $q \in |\mathcal{B}_G|_{\leq\sigma}$. First we claim that $I_k := \text{supp}(p^k) \subseteq I$ for all k (note that $\text{supp}(a^k) \subseteq \text{supp}(p^k)$). For this suppose that $i \in I_k$. If $i \in \text{supp}(a^k)$ then $i \in J$, and hence $i \in I$. If $i \notin \text{supp}(a^k)$ then by definition $d_{I_k}(i) = 0$. Since p_k is a 0-cell of $\mathcal{B}_G$, both $G[I_k]$ and $G[I_k^c]$ are connected. Therefore, any path from $n+1$ to i has to include an edge st with $s \notin \text{supp}(a^k)$ and $t \in \text{supp}(a^k)$. Hence i is not in the connected component of $n+1$ in the subgraph of G induced on $[n+1]\backslash\text{supp}(a^k)$. Since $\text{supp}(a^k) \subseteq J$, this in turn implies that $i \notin K$ so that $i \in I$. Now let $i \in I$ with $(a_q)_i = d_I(i) > 0$ and let $j \in I^c = K$ with $ij \in E$. Then there is a path from $n+1$ to j that does not meet J. Hence, if $i \notin J$, there is a path in J^c from i to $n+1$ which would contradict $i \in I$. This implies $i \in J$ and thus $i \in I_k$ for some k. Now observe that $\sigma_i \geq d_{I_k}(i) \geq d_I(i)$.

Next, let $r \in |\mathcal{B}_G|_{\leq\sigma}$ be an arbitrary point. We need to show that the line segment connecting r and q is contained in $|\mathcal{B}_G|_{\leq\sigma}$. Recall that if $i \in I$, then either $d_I(i) = 0$ or else, by Proposition 4, there exists a point $p \in |\mathcal{B}_G|_{\leq\sigma}$ such that $p_i > p_j$ for $ij \in E$. Thus, if $r_i > r_j$ for some $ij \in E$, we have $i \in I$ and $q_i = \frac{1}{|I|}$. But $q_s \leq \frac{1}{|I|}$ for all $s \in [n]$ and therefore $q_i \geq q_j$. We conclude that no hyperplane $\tilde{h}_{ij}$ strictly separates r and q, and therefore, the open line segment (r, q) is contained in some (inclusion minimal) cell C of the arrangement $\tilde{\mathcal{A}}_G$.

We first confirm that $C \in \mathcal{B}_G$. Let $p \in (r, q) \subseteq \text{relint}(C)$. Appealing to Proposition 3, we need to show that the induced orientation on the contraction of G/p has a unique sink given by the class containing the vertex $n+1$. By contradiction, assume that $i \in [n]$ corresponds to a sink in G/p that is different from $n+1$. Let $i = i_0 i_1 \ldots i_m = n+1$ be a path such that $r_{i_{h-1}} \geq r_{i_h}$ for all $h = 1, \ldots, m$. As r is in the bounded subcomplex, and hence G/r has a unique sink, such a path exists. By assumption, the path is not weakly decreasing for p; that is, there is an index l with $p_{i_{l-1}} < p_{i_l}$. In particular we have $p_{i_l} > 0$. We have $\text{supp}(p) = \text{supp}(r) \cup \text{supp}(q)$ and, by construction,

$\mathrm{supp}(r) \subseteq \mathrm{supp}(q)$ and hence $i_l \in I = \mathrm{supp}(q)$ and $q_{i_l} = \frac{1}{|I|}$. Thus, the path is weakly decreasing for q which implies $p \in (r, q) \subseteq \{x \in U : x_{i_{l-1}} \geq x_{i_l}\}$. Recall that U is the affine slice $U = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0, \ x_1 + x_2 + \cdots + x_n = 1\}$.

It is left to show that $a_C \leq \sigma$. For this let $i \in [n]$ with $p_i > 0$, so that $i \in I$. If $r_i > 0$ then since $\mathrm{supp}(r) \subseteq \mathrm{supp}(q)$, we have $q_i > q_j \Rightarrow r_i > r_j$ and thus $\sigma_i \geq (a_r)_i \geq (a_C)_i$. If $r_i = 0$, then $p_i > p_j \Rightarrow q_i > q_j$ and thus $\sigma_i \geq (a_q)_i \geq (a_C)_i$, as desired.

Theorem 3 *With the monomial labeling described above, the complex $\mathcal{B}_G$ supports a minimal cellular resolution of $M_G \subset \mathbb{K}[x_1, \ldots, x_n]$ over every field $\mathbb{K}$.*

Proof We apply the criteria from Proposition 2. Corollary 2 and Lemma 1 imply that $\mathcal{B}_G$, with the monomial labeling described in Proposition 4, supports a cellular resolution of M_G. Proposition 4 asserts that the cellular resolution is indeed minimal.

Example 4 In the case of our running example, we obtain a free minimal resolution of M_G from the labeled complex in Fig. 3. Disregarding the grading, the resolution has the form

$$0 \leftarrow R \xleftarrow{\phi_1} R^6 \xleftarrow{\phi_2} R^8 \xleftarrow{\phi_3} R^3 \xleftarrow{\phi_4} 0.$$

In particular the Betti numbers β_i are given by the face numbers $f_{i-1}(\mathcal{B}_G)$, and the differentials are described by the incidence relations of $\mathcal{B}_G$.

In addition we obtain an explicit combinatorial formula for the Betti numbers of I_G, verifying a conjecture of Perkinson et al. [16, Conjecture 7.9].

Corollary 3 *For a graph G let P_k denote the elements of L_G that have k components. For $S \in P_k$ let G/S be the graph induced on S, with k vertices given by contracting the elements of S, while preserving the edges between elements of S. Then the (nongraded) Betti numbers of the ideal M_G are given by*

$$\beta_k(M_G) = \sum_{S \in P_{k+1}} \#\{c \ : \ c \ \text{a minimal recurrent configuration on } G/S\}.$$

Proof Let S be an element of P_{k+1}. As we have seen, the minimal recurrent configurations of G/S correspond to the maximal G/S-parking functions, which are in turn in bijection with acyclic orientations of G/S with unique sink $n + 1$. From Proposition 3 we know that this set is in bijection with the $k - 1$ cells of $\mathcal{B}_G$. Theorem 3 shows that $k - 1$ cells of $\mathcal{B}_G$ index $\beta_k(M_G)$ (Fig. 5).

Corollary 4 *If G is a tree on the vertices $[n + 1]$, then $M_G = \langle x_1, x_2, \ldots, x_n \rangle$, and $\mathcal{B}_G$ is an $(n - 1)$-dimensional simplex realizing the Koszul complex.*

Proof In this case $\tilde{\mathcal{A}}_G$ is an arrangement of n hyperplanes in $U \cong \mathbb{R}^{n-1}$. It is easy to see that the only acyclic orientation of G with unique sink $n + 1$ is obtained by orienting all edges toward $n + 1$. And, since any contraction of G is again a tree, we see that $\mathcal{B}_G$ has a unique bounded cell of dimension $n - 1$ with n facets, i.e., $\mathcal{B}_G$ is an $(n - 1)$-dimensional simplex.
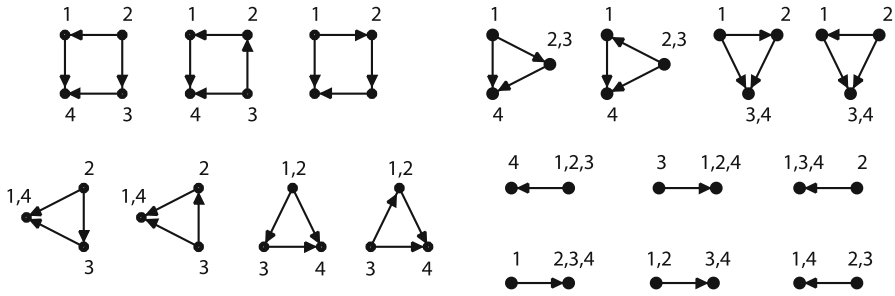
Fig. 5 The acyclic orientations of our example G with vertex 4 as the unique sink, enumerating the syzygies of M_G: $\beta_3 = 3$, $\beta_2 = 8$, $\beta_1 = 6$

Lastly we note that, in principle, Theorem 3 gives an *algebraic* approach to acyclic orientations of G and the study of the face poset of $\mathcal{B}_G$ by means of minimal $\mathbb{Z}^n$-graded resolutions. However, enumerating acyclic orientations of G is #P-hard [10], and we do not expect the algebraic method to be efficient.

6.1 Duality

As detailed in [12], the duality involved in the discrete Riemann–Roch theory of certain graphs G has a commutative algebraic analog in terms of the ideal $M_G^* = M_G^{[\mathbf{k+1}]}$. Here M_G^* is the Alexander dual ideal of M_G with respect to the monomial

$$\mathbf{x^{k+1}} = x_1^{\deg(1)} x_2^{\deg(2)} \cdots x_n^{\deg(n)},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

As a corollary to Theorem 3 we may apply the notion of duality of cellular resolutions [12] to obtain a (co)cellular minimal resolution of M_G^*. For this we will need the following result from the literature.

Proposition 5 ([13, Theorem 5.37]) *Fix a monomial ideal I generated in degrees preceding $\mathbf{a}$ and a cellular resolution $\mathcal{F}_X$ of $R/(I + \mathbf{x^{a+1}})$ such that all face labels on X precede $\mathbf{a} + \mathbf{1}$. If $Y = \mathbf{a} + \mathbf{1} - X$, then $\mathcal{F}^{Y_{\preceq \mathbf{a}}}$ is a weakly cocellular resolution of $I^{[\mathbf{a}]}$. The resolution supported by $Y_{\preceq \mathbf{a}}$ is minimal if $\mathcal{F}_X$ is minimal.*

In our context we take G to be our graph on vertex set $[n + 1]$, and let $\mathcal{B}_G$ denote the labeled polyhedral complex defined above. We define $\bar{\mathcal{B}}_G$ to be the *colabeled* polyhedral complex with underlying complex $\mathcal{B}_G$ but with monomial label on a face C given by

$$(\bar{a}_C)_i = \deg(i) + 1 - (a_C)_i.$$

Combining our Theorem 3 with Proposition 5 gives us the following.

Proposition 6 *Let G be a graph on vertex set $[n + 1]$ and set*
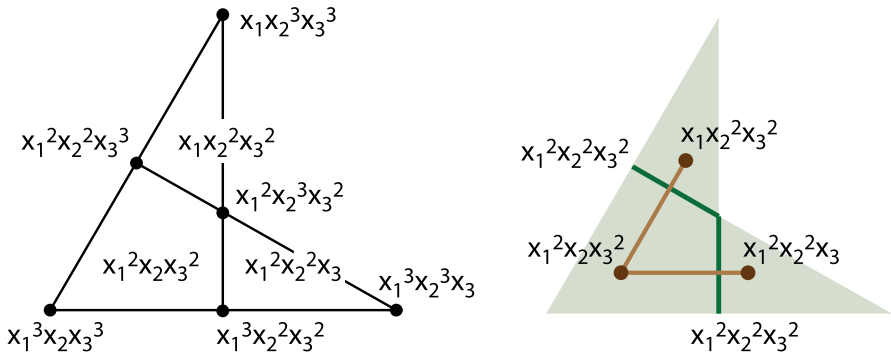
$$\mathbf{a} = (\deg(1), \deg(2), \ldots, \deg(n)).$$

Fig. 6 The colabeled complex $\bar{\mathcal{B}}_G$ (with 0-cells and 2-cells labeled) and the subcomplex $(\bar{\mathcal{B}}_G)_{\preceq \mathbf{a}}$, consisting of three 2-cells and two 1-cells. The dual complex is also depicted

Then with the notation established above, the labeled complex $(\bar{\mathcal{B}}_G)_{\preceq \mathbf{a}}$ supports a minimal cocellular resolution of the ideal M_G^*.

Example 5 In our running example, we have $\mathbf{a} = (2, 2, 2)$, and

$$M_G^* = M_G^{\mathbf{a}} = \langle x_1 x_2^2 x_3^2, x_1^2 x_2 x_3^2, x_1^2 x_2^2 x_3 \rangle.$$

The colabeled complex $\bar{\mathcal{B}}_G$ is depicted below, along with the subcomplex supporting the minimal resolution of M_G^* (Fig. 6).

Remark 1 For the case of $G = K_{n+1}$, the complete graph on $n + 1$ vertices, this is the duality between the (resolutions of the) *tree ideals* and the *permutohedron ideals* described in [13, Example 5.44].

7 Further questions

7.1 Toppling ideals

As mentioned in Sect. 2, there is a monomial term order $\preceq$ for $\mathbb{K}[x_1, \ldots, x_{n+1}]$ for which the ideal M_G is the initial ideal of I_G. A natural question to ask is whether one can describe a minimal cellular resolution of the lattice binomial ideal I_G. A construction presents itself in terms of a quotient of the unimodular *graphic lattice* associated to G, similar to the cellular resolutions of binomial Lawrence ideals described in [2]. Mohammadi and Shokrieh [14] informed us about progress along these lines that will be published in a sequel to their recent paper.

7.2 Monotone monomial ideals

In [17] the authors study *monotone monomial ideals*, a class of monomial ideals that are strictly more general than those arising as M_G for a graph G. We recall the definition

here. A *monotone monomial family* $\mathcal{M} = \{m_I : I \in \Sigma\}$ is a collection of monomials indexed by a set Σ of nonempty sets in $[n]$ that satisfy the conditions

(MM1) For $I \in \Sigma$, $\mathrm{supp}(m_I) \subseteq I$,
(MM2) For $I, J \in \Sigma$ such that $I \subset J$, we have m_I divides m_J.
(MM3) For $I, J \in \Sigma$, $\mathrm{lcm}(m_I, m_J)$ is divisible by m_k for some $K \supset I \cup J$ in Σ.

We then define the *monotone monomial ideal* $\langle \mathcal{M} \rangle$ associated to the family $\mathcal{M}$ to be the ideal generated by the monomials m_I in $\mathcal{M}$.

The monomial ideals M_G associated to a (directed) graph G described above are monotone. In this case Σ is the set of all nonempty subsets of $[n]$, and m_I is given by the formula 2. The natural question to ask is if a similar construction to $\mathcal{B}_G$ can be used to resolve the ideals $\langle \mathcal{M} \rangle$. Is there an arrangement of hyperplanes corresponding to a monotone family?

7.3 Shi arrangements and duality

In his study of the affine Weyl group of type A_{n-1}, Shi introduced the arrangement $\mathcal{S}_n$ of hyperplanes in $\mathbb{R}^n$ that now bears his name:

$$\mathcal{S}_n = \{x_i - x_j = 0, 1 : 1 \leq i < j \leq n\}.$$

Shi proved that the number of regions in the complement of $\mathcal{S}_n$ is given by $(n + 1)^{n-1}$ (the number of trees on $n + 1$ labeled vertices). Stanley [18] gave the first bijective proof of this fact by providing an explicit labeling of the regions with parking functions. Hopkins and Perkinson [9] generalize this picture, motivated by a conjecture of Duval, Klivans, and Martin. Associated to a graph G they define what they call a *bigraphical arrangement* and show that a Pak-Stanley type labeling of its regions is in bijection with the G-parking functions. Specifying certain parameters of the bigraphical arrangements recover the G-Shi and G-semiorder arrangements.

From Theorem 3, we obtain a minimal resolution of the ideal M_G from the graphical arrangement of G. We also know that when M_G is *Riemann–Roch* (in the sense of [12]), the generators of M_G^* (the Alexander dual of M_G) are given by the maximal G-parking functions. It would be interesting to find a connection between the bigraphical arrangements of [9] and the cellular resolutions that we have considered here.

7.4 Topology of the partition poset

Björner and Wachs [5] use the graphical hyperplane arrangement of the complete graph (the so-called *braid arrangement*) to give an explicit basis for the homology of the partition poset. It would be interesting to connect this study to the resolutions of the ideals studied here, and in particular to consider the case of a general graph G.

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