J Algebr Comb (2014) 40:785–804
DOI 10.1007/s10801-014-0507-8

Finite vertex-primitive edge-transitive metacirculants

Cai Heng Li · Zai Ping Lu · Jiangmin Pan

Received: 12 August 2011 / Accepted: 1 February 2014 / Published online: 21 March 2014

Abstract A classification is given of finite metacirculants which are vertex-primitive and edge-transitive. The classification forms a core part of a series of papers towards a classification of edge-transitive metacirculants.

Keywords Edge-transitive graph · Metacirculant · Primitive permutation group

Mathematics Subject Classification (2000) 20B15 · 20B30 · 05C25

C. H. Li
School of Mathematics and Statistics, Yunnan University,
Kunming, Yunnan 650091, People's Republic of China

C. H. Li
Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics,
The University of Western Australia, Crawley, WA 6009, Australia
e-mail: cai.heng.li@uwa.edu.au

Z. P. Lu
Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071,
People's Republic of China
e-mail: lu@nankai.edu.cn

J. Pan (✉)
School of Mathematics and Statistics, Yunnan University of Finance and Economics,
Kunming, Yunnan 650091, People's Republic of China
e-mail: jmpan@ynu.edu.cn

1 Introduction

Graphs considered in this paper are connected, undirected and simple. A graph $\Gamma = (V, E)$, with vertex set V and edge set E, is called a *metacirculant* if $\mathsf{Aut}\Gamma$ has a metacyclic subgroup R which is transitive on V (Recall that a group is called *metacyclic* if it is an extension of a cyclic group by a cyclic group). For convenience,we sometimes call Γ a *metacirculant of* R. Thus, Cayley graphs of metacyclic groups are metacirculants. We remark that metacirculants were first introduced by Alspach and Parsons [2] in 1982, with more restricted conditions, refer to [19,31]. The class of metacirculants provides a rich source of many interesting families of graphs, and has been extensively studied, see for example [25] and [3,8,23,30,33]. In particular, the following is a long-standing open problem in algebraic graph theory.

Problem A. Characterise edge-transitive metacirculants.

Some special classes of metacirculants have been well-characterised, see [1,12,14] for edge-transitive circulants (that is, Cayley graphs of cyclic groups); [9,20,21] for 2-arc transitive dihedrants (that is, Cayley graphs of dihedral groups); [18,34] for half-arc-transitive metacirculants of prime-power order; [24,35] for half-arc-transitive metacirculants of valency 4.

This paper is one of a series of papers to attack Problem A. A graph Γ is called *vertex-primitive* if $\mathsf{Aut}\Gamma$ is a primitive permutation group on its vertex set. Primitive permutation groups are divided into eight O'Nan-Scott types by O'Nan-Scott's theorem, refer to [27]. Five of the eight types can appear to contain a transitive metacyclic subgroup, see [17]. The purpose of this paper is to give a classification of the vertex-primitive edge-transitive metacirculants. As usual, $\mathbf{K}_n$ denotes a complete graph of order n, and by $\Delta \times \Sigma$, $\Delta \Box \Sigma$ we mean the direct product, cartesian product of two graphs Δ and Σ, respectively. Denote the line graph of a graph Σ by $\mathsf{line}(\Sigma)$. The complement of a graph Γ is denoted by $\overline{\Gamma}$. See Sect. 2 for the details and the definition of other notation.

Theorem 1.1 *Let $\Gamma = (V, E)$ be a G-edge-transitive metacirculant of R such that G is primitive on V, where $R \le G \le \mathsf{Aut}\Gamma$. Then, one of the following holds, where p is a prime.*

- *(i)* $\Gamma = \mathbf{K}_n$, $\mathbf{K}_n \times \mathbf{K}_n$ *or* $\mathbf{K}_n \Box \mathbf{K}_n$.
- *(ii)* $\Gamma = \mathsf{line}(\mathbf{K}_p)$ *or* $\overline{\mathsf{line}}(\mathbf{K}_p)$.
- *(iii)* $\Gamma = \mathrm{Cay}(T, S)$, *where* $T = \mathrm{PSL}(2, p)$ *and* $S = \{g^t \mid t \in \mathsf{Aut}(T)\}$ *for some non-identity element* $g \in T$, *and* G *is of diagonal type, and* Γ *is a Cayley graph of a metacyclic group* $\mathbb{Z}_{p(p+1)/2}{:}\mathbb{Z}_{p-1}$.
- *(iv)* $G = \mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$, *and* Γ *is a metacirculant of* $\mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ *or* $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$.
- *(v)* $G = \mathrm{P\Gamma L}(2, 16)$, $R = \mathbb{Z}_{17}{:}\mathbb{Z}_8$, *and* Γ *is of order* 68, *and valency* 12, 15, *or* 40.
- *(vi)* $G = \mathrm{PSL}(5, 2)$, *and* Γ *is the Grassmann graph* $\mathrm{G}_2(5, 2)$ *or its complement.*
- *(vii)* $G = \mathrm{PSU}(4, 2)$ *or* $\mathrm{PSU}(4, 2).2$, *and* Γ *is the Schläfli graph or its complement.*
- *(viii)* $G = \mathrm{M}_{23}$, Γ *is a Cayley graph of* $\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$ *of valency* 112 *or* 140.
- *(ix)* Γ *is a normal Cayley graph of* $\mathbb{Z}_p^d$, *where* $p^d = p$, p^2, 3^3, 2^3 *or* 2^4.

Most vertex-primitive edge-transitive metacirculants are Cayley graphs of metacyclic groups.

Corollary 1.2 *Let Γ be an edge-transitive and vertex-primitive metacirculant. Then, Γ is not a Cayley graph if and only if one of the following appears, where p is a prime.*

(i) $\Gamma = \mathsf{line}(\mathbf{K}_p)$ *or* $\overline{\mathsf{line}}(\mathbf{K}_p)$, *where* $p \equiv 1 \pmod 4$.
(ii) Γ *is a metacirculant of* $\mathbb{Z}_{17}{:}\mathbb{Z}_8$, *and* $\mathsf{Aut}\Gamma = \mathrm{P\Gamma L}(2, 16)$.
(iii) Γ *is a metacirculant of* $\mathbb{Z}_{19}{:}\mathbb{Z}_9$, *and* $\mathsf{Aut}\Gamma = \mathrm{PSL}(2, 19)$.
(iv) Γ *is a metacirculant of* $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, *and* $\mathsf{Aut}\Gamma = \mathrm{PGL}(2, p)$, *where* $p \equiv 1 \pmod 4$.

A graph $\Gamma = (V, E)$ is called *G-locally-primitive* where $G \le \mathsf{Aut}\Gamma$ if, for each vertex $v \in V$, G_v acts primitively on $\Gamma(v) := \{w \in V \mid w$ is adjacent to v in $\Gamma\}$. In particular, 2-arc-transitive graphs are locally-primitive. Some special classes of 2-arc-transitive metacirculants have been classified, see [1,9,20,21]. If a metacirculant is locally-primitive, then it is arc-transitive. In subsequent work, we will classify locally-primitive metacirculants, for which the following corollary plays an important role.

Corollary 1.3 *Let Γ be a G-locally-primitive metacirculant of R such that G is primitive on the vertex set, where $R \le G \le \mathsf{Aut}\Gamma$. Then, one of the following holds, where p is a prime.*

(i) Γ *is* $\mathbf{K}_n$, $\mathbf{K}_n \times \mathbf{K}_n$, $\overline{\mathsf{line}}(\mathbf{K}_p)$, $\mathrm{G}_2(5, 2)$, *or the Schläfli graph, or* $\mathrm{Cay}(\mathbb{Z}_2^4, S)$.
(ii) $\Gamma = \mathrm{Cay}(T, S)$, *where* $T = \mathrm{PSL}(2, p)$ *and* $S = g^T$ *with* $g \in T$ *an involution.*
(iii) $G = \mathrm{PSL}(2, p)$ *or* $\mathrm{PGL}(2, p)$, $\mathsf{val}(\Gamma) = 4, 6$, *or* $\frac{p+1}{2}$ *with* $\frac{p+1}{2}$ *a prime, and* Γ *is described in Examples 5.1–5.2 and Lemma 5.3.*

This paper is organised as follows. After this introduction section, in Sect. 2–5, we will construct and study examples of the edge-transitive metacirculants that appear in Theorem 1.1. Then, in Sect. 6, we present proofs of Theorem 1.1 and Corollaries 1.2–1.3.

2 Examples and constructions

We here construct and study some edge-transitive metacirculants that appear in the main theorem. Many of the graphs are Cayley graphs, defined as following.

2.1 Cayley graphs

A graph $\Gamma = (V, E)$ is a Cayley graph if there exists a group R and a subset $S \subset R\backslash\{1\}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$ such that the vertex set $V = R$ and x is adjacent to y if and only if $yx^{-1} \in S$. This Cayley graph is denoted by $\mathrm{Cay}(R, S)$. A well-known criterion for a graph to be a Cayley graph is as follows.

Lemma 2.1 ([4, Lemma 16.3]) *A graph $\Gamma = (V, E)$ is a Cayley graph of a group R if and only if $\mathsf{Aut}\Gamma$ contains a subgroup which is regular on V and isomorphic to R.*

For a Cayley graph $\Gamma = \mathrm{Cay}(R, S)$, if the regular subgroup R is normal in $\mathsf{Aut}\Gamma$, then Γ is called a *normal Cayley graph* of R.

We remark that a Cayley graph Γ may be expressed as a Cayley graph of different groups. It can be a normal Cayley graph for one of them, but is not for another; it can be a Cayley graph of a metacyclic group and of an insoluble group, see the graphs constructed in the next section.

The right multiplication of a group of order n on its elements gives rise to a regular permutation group of degree n. Hence each metacyclic group R of order n can be embedded into S_n as a regular subgroup, and so

$$\mathrm{S}_n = R\mathrm{S}_{n-1}.$$

For each positive integer n, there exists a metacyclic group R with order n, and the Cayley graph $\mathrm{Cay}(R, R\backslash\{1\}) \cong \mathbf{K}_n$ is a complete graph. Thus, all complete graphs are metacirculants. Moreover, a subgroup $G \leq \mathsf{Aut}\mathbf{K}_n$ acts on $\mathbf{K}_n$ edge-transitively if and only if G is 2-homogeneous on the vertex set.

Let $R = \mathbb{Z}_p^2$, where p is a prime. Let Γ be a Cayley graph of R. Then, Γ is a metacirculant. This gives rise to most examples of affine type, appeared in part (ix) of Theorem 1.1, see Lemma 6.2.

2.2 The line graphs of complete graphs

For a graph Σ with edge set F, the *line graph* $\mathsf{line}(\Sigma)$ is defined as the graph with vertex set F such that $e, f \in F$ are adjacent in $\mathsf{line}(\Sigma)$ if and only if e and f are incident in Σ.

Let $\Sigma = \mathbf{K}_n$ with vertex set Ω, a complete graph of order n. Assume that $\Gamma = \mathsf{line}(\Sigma)$ is a metacirculant of a metacyclic group R. Then, R is transitive on the edges of Γ, and so R is transitive on $\Omega^{\{2\}}$, the set of 2-subsets of Ω. Thus, R is 2-homogenous on the vertex set Ω. By the classification of 2-homogeneous groups, see [7, Corollary 3.5B], we conclude that R is an affine primitive permutation group on Ω. Since R is metacyclic, we have $n = p$ is a prime, and $\mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2} \leq R \leq \mathrm{AGL}(1, p)$. Moreover, if $R = \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$, then R is regular on $\Omega^{\{2\}}$, and so R has no involution, it follows that $p \equiv 3 \pmod 4$.

Conversely, for $\Gamma = \mathsf{line}(\mathbf{K}_p)$, the metacyclic subgroup $\mathrm{AGL}(1, p) = \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ of $\mathsf{Aut}\Gamma = \mathrm{S}_p$ is transitive on the vertex set of Γ, and thus Γ is a metacirculant. We therefore have the following statement.

Lemma 2.2 *The line graph $\mathsf{line}(\mathbf{K}_n)$ is a metacirculant if and only if $n = p$ is a prime. Moreover, if $\mathsf{line}(\mathbf{K}_p)$ is a metacirculant of R, then either $R = \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, or $R = \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ with $p \equiv 3 \pmod 4$.*

Next, we study the line graph $\mathsf{line}(\mathbf{K}_p)$.

Lemma 2.3 *Let $\Gamma = \mathsf{line}(\mathbf{K}_p)$ be a G-edge-transitive metacirculant, where $p \geq 5$ is a prime and $G \leq \mathsf{Aut}\Gamma$. Then, the following statements hold:*

(1) Γ and $\overline{\Gamma}$ are G-vertex-primitive arc-transitive metacirculants;

(2) Γ is a Cayley graph if and only if $p \equiv 3 \pmod 4$, so is $\overline{\Gamma}$;
(3) Γ is not G-locally-primitive;
(4) $\overline{\Gamma}$ is G-locally-primitive if and only if $G = \mathrm{A}_p$ or S_p;
(5) if $R \leq G$ is a metacyclic subgroup which is vertex-transitive on Γ, then (G, R, G_e) is listed in the following table, where e is a vertex of Γ.

G	R	G_e	conditions
A_p	$p{:}\frac{p-1}{2}$	S_{p-2}	$p \equiv 3 \pmod 4$
S_p	$p{:}\frac{p-1}{2}$	$\mathrm{S}_{p-2} \times \mathrm{S}_2$	$p \equiv 3 \pmod 4$
S_p	$p{:}(p-1)$	$\mathrm{S}_{p-2} \times \mathrm{S}_2$	
M_{11}	11:5	$\mathrm{M}_9.2$	
M_{23}	23:11	$\mathrm{M}_{21}.2$	

Proof Let Γ has vertex set V and edge set E. Then, $|V| = \frac{p(p-1)}{2}$, $|E| = \frac{p(p-1)(p-2)}{2}$, and Γ has valency $2(p-2)$. The complement $\overline{\Gamma}$ has valency $|V| - 1 - 2(p-2) = \frac{(p-2)(p-3)}{2} = \binom{p-2}{2}$. Let $\Omega = \{v_1, v_2, \ldots, v_p\}$ be the vertex set of $\mathbf{K}_p$. Then, $V = \Omega^{\{2\}}$ is the set of all unordered pairs of points of Ω.

Suppose $G \leq \mathsf{Aut}\Gamma$ acts transitively on E. Then, G is a 2-homogeneous permutation group on Ω. By the classification of 2-homogeneous permutation groups of prime degree (see [7, Corollary 3.5B]), we have that either G is affine, or G is almost simple and 2-transitive. If G is affine, then $G \leq \mathrm{AGL}(1, p)$ and $|G|$ divides $p(p-1)$, which is not possible because $|G|$ is not divisible by $|E| = \frac{p(p-1)(p-2)}{2}$. Thus, G is almost simple and 2-transitive of degree p.

If $G = \mathrm{PSL}(2, 11)$ and $p = 11$, then $|E| = \frac{11.10.9}{2}$ does not divide $|G|$, not possible. Suppose that $\mathsf{soc}(G) = \mathrm{PSL}(d, q)$ and $p = \frac{q^d-1}{q-1}$. Then, $|R|$ is divisible by $|V| = \frac{p(p-1)}{2} = \frac{q(q^d-1)(q^{d-1}-1)}{2(q-1)^2}$. However, $\mathrm{P\Gamma L}(d, q)$ does not contain such a metacyclic subgroup by [17], which is a contradiction . It then follows from [7, Corollary 3.5B] that either $G = \mathrm{A}_p$ or S_p, or (G, p) is $(\mathrm{M}_{11}, 11)$ or $(\mathrm{M}_{23}, 23)$. Thus, in particular, G is 4-transitive on Ω.

Let $e = \{v, w\} \in V$. Then, $\Gamma(e) = \{\{v, u\}, \{w, u\} \mid u \in \Omega \backslash \{v, w\}\}$, and $\overline{\Gamma}(e) = \{\{x, y\} \mid x, y \in \Omega \backslash \{v, w\}\}$. Since G is 4-transitive on Ω, we conclude that G_e is transitive on both $\Gamma(e)$ and $\overline{\Gamma}(e)$. So Γ and $\overline{\Gamma}$ are G-arc-transitive. Clearly, $\{\{v, u\} \mid u \in \Omega \backslash \{v, w\}\}$ and $\{\{w, u\} \mid u \in \Omega \backslash \{v, w\}\}$ are two blocks of G_e acting on $\Gamma(e)$. Hence, G_e is not primitive on $\Gamma(e)$, and Γ is not G-locally-primitive.

By Lemma 2.2, either $R = \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, or $R = \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ with $p \equiv 3 \pmod 4$. We next determine the vertex stabiliser G_e.

Suppose first that $G = \mathrm{A}_p$. Then, $G_e = \mathrm{S}_{p-2}$, and G has no subgroup isomorphic to $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$. Hence, $R = \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ with $p \equiv 3 \pmod 4$. So R is regular on V, and both Γ and $\overline{\Gamma}$ are Cayley graphs of R. Note that $G_e = \mathrm{S}_{p-2}$ is transitive on $\overline{\Gamma}(e)$ of degree $\binom{p-2}{2}$, and the only transitive permutation representation of S_{p-2} of this degree is primitive. So G is locally-primitive on $\overline{\Gamma}$.

Next, let $G = \mathrm{S}_p$. Then, $G_e = \mathrm{S}_{p-2} \times \mathrm{S}_2$. It is easily shown that any subgroup S of G of order $p(p-1)/2$ is isomorphic to $\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}} < \mathrm{AGL}(1, p)$. If $p \equiv 3 \pmod 4$, then $\frac{p-1}{2}$ is odd, and S is regular on V, so Γ is a Cayley graph. On the other hand, for $p \equiv 1 \pmod 4$, a subgroup S of order $p(p-1)/2$ is intransitive, and it follows that none of Γ and $\overline{\Gamma}$ is a Cayley graph. Similarly to the previous case for $G_e = \mathrm{S}_{p-2}$, the action of $G_e = \mathrm{S}_{p-2} \times \mathrm{S}_2$ in this case is also primitive on $\overline{\Gamma}(e)$. Hence, $\overline{\Gamma}$ is G-locally-primitive.

Now, let $G = \mathrm{M}_{11}$ and $p = 11$. By the Atlas [6], $G_e = \mathrm{M}_9.2$ and $R = \mathbb{Z}_{11}{:}\mathbb{Z}_5$. Then, R is regular on V, and $\Gamma, \overline{\Gamma}$ are Cayley graphs of R. The valency $|\overline{\Gamma}(e)| = \binom{9}{2} = 36$, and so $G_e = \mathrm{M}_9.2$ is not primitive on $\overline{\Gamma}(e)$, and $\overline{\Gamma}$ is not G-locally-primitive.

Finally, assume that $G = \mathrm{M}_{23}$ and $p = 23$. By the Atlas [6], we have $R = \mathbb{Z}_{11}{:}\mathbb{Z}_5$, and noticing that G_e has a subgroup G_{vw} of index 2, we conclude that $G_e = \mathrm{PSL}(3,4).2 = \mathrm{M}_{21}.2$. Then, R is regular on V, and $\Gamma, \overline{\Gamma}$ are Cayley graphs of R. Moreover, $|\overline{\Gamma}(e)| = \binom{21}{2} = 210$, and $G_e = \mathrm{M}_{21}.2$ has no primitive representation of degree 210 by the Atlas [6]. Thus $\overline{\Gamma}$ is not G-locally-primitive. □

2.3 Geometric graphs

We introduce here some geometric graphs associated with groups of Lie type which are metacirculants.

Let Ω be the set of 2-dimensional subspaces of the 5-dimensional space $\mathbb{F}_2^5$. Define Γ to be the graph with vertex set Ω such that two subspaces are adjacent if and only if they meet in a 1-subspace. This graph is called a *Grassmann graph* and denoted by $\mathrm{G}_2(5,2)$.

Lemma 2.4 *The Grassmann graph* $\mathrm{G}_2(5,2)$ *and its complement* $\overline{\mathrm{G}}_2(5,2)$ *are vertex-primitive edge-transitive Cayley graphs of* $\mathbb{Z}_{31}{:}\mathbb{Z}_5$*, of valency* 42 *and* 112*, respectively. None of them is locally-primitive.*

Proof There are exactly $(2^5-1)(2^5-2)$ ordered pairs of vectors which are linearly independent in $\mathbb{F}_2^5$, and each 2-subspace has exactly 6 ordered bases. Hence, the order $|\Omega| = (2^5-1)(2^5-2)/6 = 155$. Let $\omega = \langle x, y\rangle = \mathbb{F}_2^2$ be a vertex in Ω. Then, a neighbour of ω has the form $\langle x, z\rangle$, or $\langle y, z\rangle$, or $\langle x+y, z\rangle$, where $z \in \mathbb{F}_2^5 \backslash \langle x, y\rangle$. Thus the valency $|\Gamma(\omega)| = 3\frac{2^5-2^2}{2} = 42$, and the valency of the complement $\overline{\Gamma} = \overline{\mathrm{G}}_2(5,2)$ is equal to $155 - 1 - 42 = 112$.

Let $G = \mathrm{GL}(5,2)$. Then, $G \le \mathsf{Aut}\Gamma$ is vertex-primitive and edge-transitive on Γ. The stabiliser G_ω is isomorphic to $2^6{:}(\mathrm{S}_3 \times \mathrm{GL}(3,2))$. The neighbourhood $\Gamma(\omega)$ equals

$$\{\langle x, z\rangle \mid z \in \mathbb{F}_2^5\backslash\omega\} \cup \{\langle y, z\rangle \mid z \in \mathbb{F}_2^5\backslash\omega\} \cup \{\langle x+y, z\rangle \mid z \in \mathbb{F}_2^5\backslash\omega\},$$

and it forms a G_ω-invariant partition of $\Gamma(\omega)$. So Γ is not G-locally-primitive.

A vertex $\omega' = \langle x', y'\rangle \in \Omega$ is adjacent to ω if and only if x, y, x', y' are linearly independent. Since $G = \mathrm{GL}(5,2)$ is transitive on ordered bases of $\mathbb{F}_2^5$, we conclude that G_ω is transitive on $\overline{\Gamma}(\omega)$. Thus, the complement $\overline{\Gamma}$ is G-edge-transitive. The stabiliser $G_\omega = 2^6{:}(\mathrm{S}_3 \times \mathrm{GL}(3,2))$ does not have a primitive permutation representation of degree 112. So G_ω is not primitive on $\overline{\Gamma}(\omega)$, and $\overline{\Gamma}$ is not G-locally-primitive.

By the Atlas [6], the group $G = \mathrm{GL}(5,2) = \mathrm{PSL}(5,2)$ contains a subgroup $R = \mathrm{A\Gamma L}(1,2^5) \cong \mathbb{Z}_{31}{:}\mathbb{Z}_5$. Since $|G| = |G_\omega||R|$ and $(|G_\omega|, |R|) = 1$, we have $G = G_\omega R$, and R is regular on the vertex set Ω. In particular, Γ is a Cayley graph of $\mathbb{Z}_{31}{:}\mathbb{Z}_5$. □

The Schläfli graph is a graph arising from the $\mathrm{U}(4,2)$-geometry.

Let Ω be the set of isotropic lines in the unitary space of dimension 4 over $\mathbb{F}_4$. Define Γ to be the graph with vertex set Ω such that two lines in Ω are adjacent in Γ if and only if they are disjoint. This graph is called the *Schläfli graph*, refer to [5] or "http://www.win.tue.nl/~aeb/graphs/Schlaefli.html".

Lemma 2.5 *The Schläfli graph and its complement are vertex-primitive edge-transitive Cayley graph of* $\mathbb{Z}_9{:}\mathbb{Z}_3$*, of valency* 16 *and* 10*, respectively. Only the Schläfli graph is locally-primitive.*

Proof Let Γ be the Schläfli graph. Then, $\mathsf{Aut}\Gamma = \mathsf{Aut}\overline{\Gamma} = \mathrm{PSU}(4,2).2$ by [5]. Let $G = \mathrm{PSU}(4,2) \le \mathsf{Aut}\Gamma$ and let $\omega \in \Omega$ be a vertex. Then, the stabiliser $G_\omega = 2^4{:}\mathrm{A}_5$, refer to the Atlas [6], which is a maximal subgroup of G. Thus, G is primitive on the vertex set Ω.

The index $|G : G_\omega| = 27$, and hence a Sylow 3-subgroup G_3 of G is transitive on Ω. Moreover, G_3 has a subgroup which is isomorphic to $\mathbb{Z}_9{:}\mathbb{Z}_3$ and regular on the vertex set Ω. By [7, p. 317], G has rank 3, and so the graph Γ and its complement $\overline{\Gamma}$ are G-edge-transitive. The valency of Γ equals 16, and the valency of $\overline{\Gamma}$ equals $27-1-16 = 10$. Furthermore, Γ is G-locally-primitive but $\overline{\Gamma}$ is not, see [16, Lemma 2.6]. □

We remark that the Schläfli graph Γ is a strongly regular graph, and the complement $\overline{\Gamma}$ is the collinearity graph of the unique generalised quadrangle $\mathrm{GQ}(2,4)$, see [5].

2.4 Orbital graphs

For a transitive permutation group $G \le \mathrm{Sym}(\Omega)$, an *orbital graph* is a graph with vertex set Ω and arc set $(\alpha,\beta)^G$ with $\alpha, \beta \in \Omega$. The least interesting orbital graphs are in the case where $\alpha = \beta$. For convenience, by an orbital graph in the following, we always mean that $\alpha \ne \beta$. A *fused-orbital graph* is a graph with vertex set Ω and arc set $(\alpha,\beta)^G \cup (\beta,\alpha)^G$. We remark that if $(\alpha,\beta)^G = (\beta,\alpha)^G$, then the corresponding orbital graph is called *self-paired*, which is G-arc-transitive; on the other hand, if $(\alpha,\beta)^G \ne (\beta,\alpha)^G$, then the corresponding fused-orbital graph is the union of two orbital graphs and is G-half-transitive. Here are some examples of graphs appeared in the main theorem.

Lemma 2.6 *Let* $G = \mathrm{P\Gamma L}(2,16)$*, and let* $H < G$ *be isomorphic to* $(\mathrm{A}_5 \times 2).2$*. Then,* G *acting on* $[G : H]$ *is primitive of degree* 68 *and rank* 4*. Let* Γ *be a non-trivial fused-orbital graph. Then,* $\mathsf{Aut}\Gamma = G$*, and the following statements hold.*

(1) Γ *is self-paired, and has valency* 12*,* 15 *or* 40*;*
(2) Γ *is a metacirculant of* $\mathbb{Z}_{17}{:}\mathbb{Z}_8$*, but not a Cayley graph;*
(3) Γ *is not* G*-locally-primitive.*

Proof Let $\Omega = [G : H]$. Then, $|\Omega| = 68$. By [7, p. 310], G is primitive and of rank 4 on Ω with suborbits of length 1, 12, 15 or 40. So each orbital graph of G is self-paired and has valency 12, 15 or 40.

Let Γ be one of the orbital graphs. By the Atlas [6], G has a metacyclic subgroup $R = \mathbb{Z}_{17}{:}\mathbb{Z}_8$ such that $G = RH$. Thus, R is transitive on Ω, and Γ is a metacirculant. Since $\mathsf{Aut}\Gamma \geq \mathrm{P\Gamma L}(2, 16)$ is primitive on Ω of degree 68, by [7, Appendix B], we conclude that $\mathsf{Aut}\Gamma = G$.

By the Atlas [6], each subgroup A of G of order 68 is conjugate to a subgroup of $R \cong \mathbb{Z}_{17}{:}\mathbb{Z}_8$ of index 2. Hence $A_v = \mathbb{Z}_2$ where $v \in \Omega$, and A is intransitive on Ω. So G has no subgroup which is regular on the vertex set Ω, and Γ is not a Cayley graph.

Finally, for adjacent vertices v, w, we have $G_{vw} \cong (\mathbb{Z}_5{:}\mathbb{Z}_2).\mathbb{Z}_2$, $(\mathbb{Z}_2^2 \times 2).2$ or $\mathbb{Z}_3{:}\mathbb{Z}_2$, none of which is a maximal subgroup of $G_v \cong H$. Therefore, Γ is not G-locally-primitive. □

By Lemma 2.3, there are two Cayley graphs of the metacyclic group $\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$ which are M_{23}-vertex-primitive and M_{23}-edge-transitive. These two graphs are the line graph $\mathsf{line}(\mathbf{K}_{23})$ and the complement. The final example in this section shows that there are two more Cayley graphs of $\mathbb{Z}_{23}{:}\mathbb{Z}_{11}$, which are M_{23}-vertex-primitive and M_{23}-edge-transitive.

Example 2.7 Let $G = \mathrm{M}_{23}$. By the Atlas [6], G has a maximal subgroup $H \cong 2^4{:}\mathrm{A}_7$, so G is a primitive permutation group on $\Omega{:} = [G : H]$ with degree 253, induced by the coset action. Further, by [7, p. 322], G is of rank 3, and it is easy to show that the two non-trivial orbitals are of length 112 and 140. Thus, the two graphs are G-arc-transitive. Moreover, by the Atlas [6], G has subgroup $R \cong \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$. Since $|G| = |R||H|$ and $(|R|, |H|) = 1$, we have $G = RH$. So R is regular on Ω, and the graphs are metacirculants and Cayley graphs. Further, as $H = 2^4{:}\mathrm{A}_7$ has no primitive representation of degree 112 or 140, none of the graphs is G-locally-primitive. □

3 Examples of diagonal type

In this section, we study examples associated with primitive groups of diagonal type.

Let Γ be a Cayley graph of a group R. Then, the right multiplications of elements of R induce automorphisms of Γ, that is,

$$\hat{g} : \ x \mapsto xg, \ \text{for all } g, x \in R.$$

Further, $R \cong \hat{R} = \{\hat{g} \mid g \in R\}$, and $\hat{R} \leq \mathsf{Aut}\Gamma$. On the other hand, the left multiplication of an element g:

$$\check{g} : \ x \mapsto g^{-1}x, \ x \in R$$

is generally not an automorphism of Γ, and hence $\check{R} = \{\check{g} \mid g \in R\}$ is not necessarily a subgroup of $\mathsf{Aut}\Gamma$. As subgroups of $\mathrm{Sym}(R)$, $\check{R}$ centralises $\hat{R}$, namely, the central product $\hat{R}\circ\check{R} = \langle\hat{R},\check{R}\rangle < \mathrm{Sym}(R)$, see [7, Sect. 4.2]. We observe that, for an element $g \in R$,

$$\check{g}\hat{g} : x \mapsto g^{-1}xg,$$

is the inner automorphism of R induced by g, denoted by $\tilde{g}$. Let $\tilde{R} = \{\tilde{g} \mid g \in R\}$.

For a subgroup H of a group G, denote by $\mathbf{N}_G(H)$ and $\mathbf{C}_G(H)$ the normalizer and the centralizer of H in G, respectively. It is easily shown that $\mathbf{C}_{\mathrm{Sym}(R)}(\hat{R}) = \check{R}$, and $\hat{R}\mathbf{C}_{\mathrm{Sym}(R)}(\hat{R}) = \hat{R}\check{R} = \hat{R}{:}\mathsf{Inn}(R)$, where $\mathsf{Inn}(R) \cong \tilde{R}$ denotes the inner automorphism group of R. Moreover, for Cayley graphs, the following statements hold.

Lemma 3.1 ([10] and [15, Lemma 2.1]) *For a Cayley graph $\Gamma = \mathrm{Cay}(R,S)$, we have the following property:*

$$\mathbf{N}_{\mathsf{Aut}\Gamma}(\hat{R}) = \hat{R}{:}\mathsf{Aut}(R,S),\ \ \hat{R}\mathbf{C}_{\mathsf{Aut}\Gamma}(\hat{R}) = \hat{R}{:}\mathsf{Inn}(R,S),$$

where $\mathsf{Aut}(R,S) = \{\sigma \in \mathsf{Aut}(G) \mid s^\sigma \in S \text{ for each } s \in S\}$, *and* $\mathsf{Inn}(R,S) = \mathsf{Aut}(R,S) \cap \mathsf{Inn}(R)$.

For the case where S consists of full conjugate classes of elements of R, there are more properties of Cayley graph $\mathrm{Cay}(R,S)$.

Theorem 3.2 *Let $\Gamma = \mathrm{Cay}(R,S)$, where R is a group with centre $\mathbf{Z}(R) = 1$, and $S = \{g^x, (g^{-1})^x \mid x \in R\}$ or $\{g^x, (g^{-1})^x \mid x \in \mathsf{Aut}(R)\}$ for some non-identity element $g \in R$. Then, the following statements are true:*

(i) ([15, Lemma 2.4]) *The map $\pi : \ x \mapsto x^{-1}$, for all $x \in R$, is an automorphism of Γ, and $\pi^{-1}\hat{R}\pi = \check{R}$.*
(ii) ([15, Lemma 2.4]) $\mathsf{Aut}\Gamma \geq (\hat{R}\times\check{R}){:}\langle\pi\rangle \cong R \wr \mathbb{Z}_2 = R^2{:}\mathbb{Z}_2$.
(iii) If R is a nonabelian simple group, then $N := \hat{R}{:}\mathsf{Aut}(R,S) \geq \hat{R}\times\check{R}$ acting primitively on the vertex set V of Γ, and $\mathsf{Aut}\Gamma = N.\mathbb{Z}_2$.

Proof Since $\mathbf{Z}(R) = 1$, $\langle\hat{R},\check{R}\rangle = \hat{R}\times\check{R}$. For each $h \in R$, since $\tilde{h} \in \mathsf{Aut}(R,S)$ and $\check{h} = \tilde{h}(\hat{h})^{-1} \in \hat{R}{:}\mathsf{Aut}(R,S)$, we have $N = \hat{R}{:}\mathsf{Aut}(R,S) \geq \hat{R}\times\check{R}$. Let $G = \mathsf{Aut}\Gamma$. Then, G is an overgroup of N on the vertex set V of Γ as $\mathsf{Aut}(R,S) \leq \mathsf{Aut}\Gamma$. Noting that, as R is nonabelian simple, N is a primitive permutation group of holomorph simple type on V, and as $S \neq R\backslash\{1\}$, Γ is not a complete graph and so $\mathsf{Aut}\Gamma$ is not 2-transitive on V, then, by [26, Proposition 8.1], we have $\mathsf{soc}(G) = \mathsf{soc}(N)$, and either G is of holomorph simple or of simple diagonal type. It follows that $G \leq (\hat{R}\times\check{R}).(\mathsf{Out}(R)\times\langle\pi\rangle)$. Let $X = (\hat{R}\times\check{R}).\mathsf{Out}(R)$. Then, $\hat{R} \lhd X$, and by Lemma 3.1, $G\cap X = N$, and hence $G/N \cong GX/X = G/X \leq \langle\pi\rangle$. Now, as $\pi \in G\backslash N$ by part (i), we conclude that $\mathsf{Aut}\Gamma = N.\langle\pi\rangle \cong N.\mathbb{Z}_2$, as in part (iii). □

In the rest of this section, we always fix $T = \mathrm{PSL}(2,p)$ with $p \geq 5$ prime. We quote some properties of the group T below.

Proposition 3.3 (refer to [32, p. 419])

(1) All cyclic subgroups of T of the same order are conjugate in T.

(2) Elements of T of order p form two conjugate classes of T, and are conjugate in $\mathsf{Aut}(T)$.

(3) For an element $g \in T$, we have

(a) $o(g) = p$, or $o(g) \mid (p-1)$, or $o(g) \mid (p+1)$;

(b) if $o(g) \neq p$, then g, g^{-1} are conjugate in T, and further, g is not conjugate in $\mathsf{Aut}(T)$ *to g^i unless $g^i = g^{-1}$;*

(c) if $o(g) = p$, then g is conjugate to g^{-1} in T if and only if $4 \mid (p-1)$.

Now we construct a class of Cayley graphs of $T = \mathrm{PSL}(2, p)$, which will be shown to be Cayley graphs of a metacyclic group $\mathbb{Z}_{p(p+1)/2}{:}\mathbb{Z}_{p-1}$.

Construction 3.4 Let g be a non-identity element of T, and $S_g = \{g^t \mid t \in \mathsf{Aut}(T)\}$. Let

$$\Gamma_g = \mathrm{Cay}(T, S_g).$$

Lemma 3.5 *Using the notation defined above, we have the following:*

(i) $\mathsf{Aut}(T, S_g) = \mathsf{Aut}(T) = \mathrm{PGL}(2, p)$;

(ii) Γ is connected, undirected, and arc-transitive;

(iii) $\mathsf{Aut}\Gamma_g = (\hat{T} \times \check{T}).2^2$ *is primitive on the vertex set of simple diagonal type;*

(iv) Γ_g is a Cayley graph of a metacyclic group $\mathbb{Z}_{p(p+1)/2}{:}\mathbb{Z}_{p-1}$; in particular, Γ_g is a metacirculant.

Proof By definition, S_g is a full conjugacy class of g under $\mathsf{Aut}(T)$, and hence $\mathsf{Aut}(T, S_g) = \mathsf{Aut}(T) = \mathrm{PGL}(2, p)$, as in part (i).

Since T is simple, $\langle S_g \rangle = T$ and Γ_g is connected. By Proposition 3.3 (2) and (3)(b), g and g^{-1} are conjugate in $\mathsf{Aut}(T)$. Thus, Γ_g is undirected. By definition, $\mathsf{Aut}(T, S_g)$ is transitive on S_g. It follows that the Cayley graph Γ_g is arc-transitive. This proves part (ii).

Let $X = \mathsf{Aut}\Gamma_g$, and let α be the vertex of Γ_g corresponding to the identity of T. Then, the stabiliser $X_\alpha \geq \mathsf{Aut}(T, S_g) = \mathsf{Aut}(T)$, and so X contains the holomorph of T, namely, $X \geq \hat{T}{:}\mathsf{Aut}(T) = (\hat{T} \times \check{T}).2$. Furthermore, since every element of T is conjugate to its inverse in $\mathsf{Aut}(T)$, by Theorem 3.2 (ii), we have $\pi : x \mapsto x^{-1}$ is an automorphism of Γ_g. Then, by Theorem 3.2 (iii), we conclude that

$$X = (\hat{T} \times \check{T}).2^2,$$

as in part (iii).

Finally, by [17], the automorphism group $\mathsf{Aut}\Gamma_g \cong (\hat{T} \times \check{T}).2^2$ contains a metacyclic subgroup isomorphic to $\mathbb{Z}_{p(p+1)/2}{:}\mathbb{Z}_{p-1}$. Thus, Γ_g is a Cayley graph of this group, as in part (iv). □

The next lemma enumerates the graphs Γ_g where $g \in T$.

Lemma 3.6 *Given $T = \mathrm{PSL}(2, p)$, there are exactly $\frac{p+1}{2}$ graphs Γ_g which satisfy the following statements, where $\varepsilon = 1$ or -1 is such that $4 \mid (p - \varepsilon)$.*

(i) one has valency $p(p+\varepsilon)/2$;
(ii) one has valency p^2-1;
(iii) $\frac{p-3}{2}$ have valency $p(p+1)$ or $(p(p-1)$.

Among these graphs, the only locally-primitive one has valency $p(p+\varepsilon)/2$.

Proof To count the graphs Γ_g as in Construction 3.4, we need to compute the number of the full conjugacy classes of T under $\mathsf{Aut}(T)$.

Suppose that g is an involution. Then, S consists of all involutions of T, and the centraliser $\mathbf{C}_T(g) = \mathrm{D}_{p-1}$ or D_{p+1}, depending on $4 \mid (p-1)$ or $4 \mid (p+1)$, respectively. Since T is transitive on S, the valency of Γ is equal to $|S| = |T|/|\mathbf{C}_T(g)|$, which equals $\frac{p(p+1)}{2}$, or $\frac{p(p-1)}{2}$, respectively. Since S contains all involutions, Γ is unique, as in part (i).

Next, assume that g is of order p. Since all elements of T of order p are conjugate in $\mathsf{Aut}(T) = \mathrm{PGL}(2,p)$, we have that S consists of all elements of T of order p and hence Γ is unique. Further, $\mathsf{Aut}(T)$ acts on S transitively, and so $\mathsf{Aut}(T,S) = \mathsf{Aut}(T) = \mathrm{PGL}(2,p)$. The element g is self-centralising in $\mathsf{Aut}(T)$, namely $\mathbf{C}_{\mathsf{Aut}(T)}(g) = \langle g\rangle$, and so $|S| = |\mathsf{Aut}(T)|/p = p^2-1$, as in part (ii).

Now, assume that g is an element of T of order not equal to 2 or p. Then, g and g^{-1} are conjugate, and $\mathsf{Aut}(T,S) = \mathrm{PGL}(2,p)$. Further, the centralizer $\mathbf{C}_T(g) \cong \mathbb{Z}_{\frac{p-1}{2}}$ or $\mathbb{Z}_{\frac{p+1}{2}}$, for $o(g)$ dividing $p-1$ or $p+1$, respectively. Since T is transitive on S, the valency $|S| = |T|/|\mathbf{C}_T(g)|$, which equals $p(p+1)$ or $p(p-1)$, respectively.

We next compute the number of conjugacy classes of elements of order neither 2 nor p. It is known that $\mathbf{N}_{\mathsf{Aut}(T)}(\langle g\rangle) \cong \mathrm{D}_{2(p+\varepsilon)}$, and all cyclic subgroups of T of the same order are conjugate, see Proposition 3.3. For $p \equiv 1 \pmod 4$, cyclic groups $\mathbb{Z}_{\frac{p-1}{2}}$ and $\mathbb{Z}_{\frac{p+1}{2}}$ have exactly $\frac{p-1}{2}-2$ and $\frac{p+1}{2}-1$ elements of order greater than 2, respectively. Thus, the number of pairs $\{g,g^{-1}\}$ in T with $o(g)\neq 2$ or p is equal to $\frac{1}{2}\big(\frac{p-1}{2}-2\big)+\frac{1}{2}\big(\frac{p+1}{2}-1\big) = \frac{p-3}{2}$, as in part (iii). For $p \equiv 3 \pmod 4$, cyclic groups $\mathbb{Z}_{\frac{p-1}{2}}$ and $\mathbb{Z}_{\frac{p+1}{2}}$ have exactly $\frac{p-1}{2}-1$ and $\frac{p+1}{2}-2$ elements of order greater than 2, respectively. Thus, the number of pairs $\{g,g^{-1}\}$ in T with $o(g)\neq 2$ or p is also equal to $\frac{1}{2}\big(\frac{p-1}{2}-1\big)+\frac{1}{2}\big(\frac{p+1}{2}-2\big) = \frac{p-3}{2}$, as in part (iii). So there are exactly $\frac{p-3}{2}+2 = \frac{p+1}{2}$ graphs Γ_g for a given group T.

Finally, suppose $\Gamma = \Gamma_g$ is locally-primitive. By Lemma 3.5, $\mathsf{Aut}\Gamma_g = (\hat{T}\times\check{T}).2^2$, we have that $\mathsf{Aut}(T){:}\langle\pi\rangle$ is primitive on S_g, where $\pi : x \mapsto x^{-1}$ is an automorphism of Γ_g. If follows that elements in S_g are involutions, then the final statement of the lemma is true by part (i). □

Conversely, the next lemma shows that fused-orbital graphs of a primitive group of diagonal type with socle T^2 are the graphs in Construction 3.4.

Lemma 3.7 *Let G be a primitive group on V of diagonal type with $\mathsf{soc}(G) = T^2$. Let Γ be a G-edge-transitive metacirculant with vertex set V. Then, Γ is a graph Γ_g in Construction 3.4, and $\mathsf{Aut}\Gamma = (T{:}\mathsf{Aut}(T)).2$.*

Proof Let $M = \mathsf{soc}(G) = T_1\times T_2$, where $T_i \cong T \cong \mathrm{PSL}(2,p)$. Then, T_1 is regular on V, and $M_\alpha = \{(t,t) \mid t\in T\} \cong T$, where $\alpha\in V$ corresponds to the identity element of T_1. Thus, $\Gamma = \mathrm{Cay}(T, S)$, where S is a subset of $T \setminus \{1\}$ with $S = S^{-1}$. Let $X = \mathsf{Aut}\Gamma$. Since $S = S^{-1}$, by [15, Lemma 2.4], the map $\pi\colon\ x \mapsto x^{-1}$, for all $x \in R$, is an automorphism of Γ, so $X \geq M{:}\langle\pi\rangle = M.2$, and Γ is arc-transitive. If X is 2-transitive on V, then Γ is a complete graph, so G is 2-homogeneous on V and hence either affine or almost simple, which contradicts the assumption. Thus, X is not 2-transitive on V. Since T is nonabelian simple, M is a primitive group of holomorph simple type, by [26, Proposition 8.1], X is primitive of diagonal type. Therefore, $M.2 = M{:}\langle\pi\rangle \leq X \leq M.2^2$.

Assume $X = M{:}\langle\pi\rangle \cong M.2$. Then, $X_\alpha = T{:}\langle\pi\rangle$ and $S = \{g^t, (g^{-1})^t \mid t \in T\}$. If $o(g) \neq p$, by part (c) of Proposition 3.3, $S = \{g^t \mid t \in T\}$. Since $\mathbf{C}_{\mathsf{Aut}(T)}(g) = \mathbf{C}_T(g).2$, we have $\{g^t \mid t \in T\} = \{g^t \mid t \in \mathsf{Aut}(T)\}$ by comparing their sizes, that is, $S = \{g^t \mid t \in \mathsf{Aut}(T)\}$. It follows that $X_\alpha \geq \mathsf{Aut}(T){:}\langle\pi\rangle$, which is a contradiction. Assume that $o(g) = p$. We claim that $p \equiv 3 (\mathsf{mod}\ 4)$. Suppose that $p \equiv 1\ (\mathsf{mod}\ 4)$. Since Γ is a metacirculant, X has a metacyclic vertex-transitive subgroup R. Then, R has order divisible by $p(p^2-1)/2$. Let P be a Sylow p-subgroup of R. Then, $P \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. Since p is the largest prime divisor of $|G|$, it is easily shown that P is normal in R. If $P \cong \mathbb{Z}_p^2$, then $R \leq \mathbf{N}_G(P) \cong (\mathbb{Z}_p^2{:}(\mathbb{Z}_{\frac{p-1}{2}} \times \mathbb{Z}_{\frac{p-1}{2}})).2$, which contradicts that $|R|$ is divisible by $\frac{p+1}{2}$. Thus, $P \cong \mathbb{Z}_p$. Let H be a Hall π-subgroup of R, where π is the set of prime divisors of $\frac{p+1}{2}$. Then, $H \leq \hat{T} \times \check{T}$. Since T has no subgroups of order pq for any prime divisor q of $\frac{p+1}{2}$, it implies that either $P \leq \hat{T}$ and $H \leq \check{T}$, or $P \leq \check{T}$ and $H \leq \hat{T}$. Then, $H \cong \mathbb{Z}_{\frac{p+1}{2}}$. Without loss of generality, assume that $P \leq \hat{T}$. Then, $H \leq \mathbf{C}_R(P) \leq \mathbf{C}_G(P) = P \times \check{T}$, and it follows that $\mathbf{C}_R(P) = P \times H$ or $P \times (H{:}2)$. Thus, R has a normal subgroup PH, so $R \leq \mathbf{N}_G(PH) \cong (\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}) \times \mathrm{D}_{p+1}$. As $|R| \geq p(p^2-1)/2$, we further have $R \cong (\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}) \times \mathrm{D}_{p+1}$, which is not metacyclic as $4 \mid p-1$, yielding a contradiction. Hence, $p \equiv 3 (\mathsf{mod}\ 4)$. Now, by Proposition 3.3, we have $S = \{g^t, (g^{-1})^t \mid t \in T\} = \{g^t \mid t \in \mathsf{Aut}(T)\}$, implying $X_\alpha \geq \mathsf{Aut}(T){:}\langle\pi\rangle$, which is also a contradiction.

We therefore have $X = M.2^2$. Then, $X_\alpha = \mathsf{Aut}(T){:}\langle\pi\rangle$. Noting that Γ is arc-transitive and each element is conjugate to its inverse in $\mathsf{Aut}(T)$ by Proposition 3.3, we conclude that $S = g^{X_\alpha} = \{g^t \mid t \in \mathsf{Aut}(T)\}$, $\Gamma = \Gamma_g$, and $\mathsf{Aut}\Gamma = (T{:}\mathsf{Aut}(T)).2$, where g is a non-identity element of T. □

4 Products

For graphs $\Delta = (U, E)$ and $\Sigma = (W, F)$, we define *direct product* $\Delta \times \Sigma$ and *Cartesian product* $\Delta \square \Sigma$ as follows. They both have vertex set $V{:} = U \times W = \{(u, w) \mid u \in U, w \in W\}$, and given two vertices $v_1 = (u_1, w_1)$ and $v_2 = (u_2, w_2)$ in V,

(a) for $\Delta \times \Sigma$, (v_1, v_2) is an arc of $\Delta \times \Sigma$ if and only if $(u_1, u_2) \in E$ and $(w_1, w_2) \in F$;
(b) for $\Delta \square \Sigma$, (v_1, v_2) is an arc of $\Delta \square \Sigma$ if and only if $u_1 = u_2$ and $(w_1, w_2) \in F$, or $(u_1, u_2) \in E$ and $w_1 = w_2$.

Then, for $\Gamma = \mathbf{K}_n \times \mathbf{K}_n$ or $\mathbf{K}_n \Box \mathbf{K}_n$, we have $\mathsf{Aut}\Gamma = \mathrm{S}_n \wr \mathrm{S}_2$. It is easily shown that $\mathsf{Aut}\Gamma$ has metacyclic transitive subgroups, and Γ is an arc-transitive metacirculant. Generally, we have the following result, the proof of which is easy and omitted.

Lemma 4.1 *Let Σ be a circulant, and let $X \leq \mathsf{Aut}\Sigma$ contain a cyclic regular subgroup R. Then, both $\Sigma \Box \Sigma$ and $\Sigma \times \Sigma$ are metacirculants, and have an automorphism group $X \wr \mathrm{S}_2$ which contains a metacyclic regular subgroup $R \times R$.*

Moreover, if X is primitive but not regular on Δ, then G is primitive on $\Delta \times \Delta$, see [7, Lemma 2.7].

The next construction produces a metacyclic transitive subgroup which is not of the form $R \times R$ for certain degrees.

Construction 4.2 Let $n = 2m$ with m odd, and let P be a transitive permutation group on Δ of degree n. Assume that P contains a cyclic regular subgroup $C = \langle c \rangle$. Let $X = P \wr \langle \pi \rangle$, where $\pi : (x_1, x_2) \mapsto (x_2, x_1)$ for all elements $(x_1, x_2) \in P \times P$. Suppose that X acts on $\Omega := \Delta \times \Delta$ in product action. Then, the point stabiliser $X_\omega = P_\delta \wr \langle \pi \rangle$, where $\omega = (\delta, \delta) \in \Omega$.

Let $a = c^2$ and $t = c^m$. Suppose that P has a transitive subgroup Q such that $P = Q{:}\langle t \rangle$ and $P_\delta = Q_\delta{:}\langle t \rangle$. Let $\sigma = (t, 1)\pi \in X$. Then, $\sigma^2 = (t, t)$ and σ has order 4. Let

$$G = \langle Q \times Q, \sigma \rangle = (Q \times Q){:}\langle \sigma \rangle,$$
$$R = \langle (a, 1), (1, a), \sigma \rangle.$$

□

Some basic properties of the groups in Construction 4.2 are presented below.

Lemma 4.3 *Using the notation defined in Construction 4.2, we have that*

[i) $R = \mathbb{Z}_m{:}\mathbb{Z}_{4m}$ is a metacyclic subgroup of G and regular on Ω, and
(ii) G is primitive on Ω if and only if P is primitive on Δ.

Proof It is easy to check that $R = \langle (a, a^{-1}), (a, a)\sigma \rangle$, and $\langle (a, a^{-1}) \rangle$ is normal in R. Further, $\langle (a, a^{-1}) \rangle \cap \langle (a, a)\sigma \rangle = \{1\}$, and $(a, a)\sigma$ has order $4m$. Hence the order $|R|$ equals $n^2 = |\Omega|$, and $R = \langle (a, a^{-1}) \rangle{:}\langle (a, a)\sigma \rangle \cong \mathbb{Z}_m{:}\mathbb{Z}_{4m}$, that is, R is metacyclic.

We claim that R is regular on Ω. Obviously, $\langle c \rangle \times \langle c \rangle$ is regular on Ω and the subgroup $\langle (a, 1), (1, a), (t, t) \rangle$ is of index 2 in $\langle c \rangle \times \langle c \rangle$. If $x \in R \setminus \langle (a, 1), (1, a), (t, t) \rangle$, then x is of order 4, and conjugate to σ. Since $\sigma^2 = (t, t)$ fixes no point, so is σ. Hence R is semiregular on Ω, and as $|R| = |\Omega|$, R is regular on Ω, as in part (i).

By [7, Lemma 2.7], G is primitive on Ω if and only if P is primitive on Δ, as in part (ii). □

The following are a few examples.

Example 4.4 Let $\Delta = \{1, 2, \ldots, n\}$ where $n = 2m$ with m odd. Let $P = \mathrm{Sym}(\Delta) = \mathrm{S}_n$, and let $Q = \mathrm{A}_n$. Applying Construction 4.2, we have a primitive permutation group $G = (\mathrm{A}_n \times \mathrm{A}_n){:}\mathbb{Z}_4$, of product action type on $\Omega = \Delta \times \Delta$, and G has a regular metacyclic subgroup $R = \mathbb{Z}_m^2{:}\mathbb{Z}_4 = \mathbb{Z}_m{:}\mathbb{Z}_{4m}$.

It is easily shown that G has rank 3, and each orbital graph of G is self-paired. This gives rise to arc-transitive metacirculants: $\mathbf{K}_n \times \mathbf{K}_n$, and $\mathbf{K}_n \square \mathbf{K}_n$. Since G contains a regular metacyclic subgroup $R = (\mathbb{Z}_m \times \mathbb{Z}_m){:}\mathbb{Z}_4$, the two graphs are metacirculant of R. □

Example 4.5 Let $q = p^f$, where $p \equiv 1 \pmod 4$ is a prime and f is odd. Let $P = \mathrm{PGL}(2, q)$, and $Q = \mathrm{PSL}(2, q)$. Let $H = [q]{:}(q-1)$ be a subgroup of P, and $\Delta = [P : H]$, which is of size $n = q + 1 = 2 \cdot (q+1)/2$ with $(q+1)/2$ odd. By Construction 4.2, we have a primitive permutation group $G = (\mathrm{PSL}(2, q) \times \mathrm{PSL}(2, q)){:}\mathbb{Z}_4$ on $\Omega = \Delta \times \Delta$ of product action type, which contains a regular metacyclic subgroup $R = (\mathbb{Z}_{\frac{q+1}{2}} \times \mathbb{Z}_{\frac{q+1}{2}}){:}\mathbb{Z}_4$. □

Almost simple primitive permutation groups with socle T of degree n which contain a regular cyclic subgroup are 2-transitive, as listed below, refer to [11].

T	A_n	$\mathrm{PSL}(d, q)$	$\mathrm{PSL}(2, 11)$	M_{11}	M_{23}
n	n	$\frac{q^d-1}{q-1}$	11	11	23

Lemma 4.6 *Let P be an almost simple primitive permutation group on Δ of degree $n = |\Delta|$, and let $T = \mathsf{soc}(P)$. Assume that P contains a regular cyclic subgroup. Let G be primitive of product action type with socle $N = T \times T$, as constructed in Construction 4.2. Let Γ be a fused-orbital graph of G acting on $\Delta \times \Delta$. Then, $\Gamma \cong \mathbf{K}_n \square \mathbf{K}_n$ or $\mathbf{K}_n \times \mathbf{K}_n$, where $n \geq 5$. Moreover, if Γ is G-locally-primitive, then $\Gamma = \mathbf{K}_n \times \mathbf{K}_n$, and $T \neq \mathrm{PSL}(d, q)$ with $d \geq 3$.*

Proof By the assumption, T and n are as in the above table. Since G is primitive on $\Omega = \Delta \times \Delta$, the socle $N = \mathsf{soc}(G) = T \times T$ is transitive on Ω. Let $\omega = (\delta, \delta) \in \Omega$. Then, $N_\omega = T_\delta \times T_\delta$.

Since T is 2-transitive on Δ, T_δ is transitive on $\Delta \backslash \{\delta\}$, we conclude that N_ω is transitive on

$$\{(\delta_1, \delta_2) \mid \delta_1, \delta_2 \in \Delta \backslash \{\delta\}\},$$

which is of size $(n-1) \times (n-1) = (n-1)^2$. Thus, the orbital graph of G containing the edge $\{(\delta, \delta), (\delta_1, \delta_2)\}$ is isomorphic to $\mathbf{K}_n \times \mathbf{K}_n$, where $\delta_1, \delta_2 \in \Delta \backslash \{\delta\}$.

Similarly, N_ω is transitive on $\{(\delta, \delta') \mid \delta' \in \Delta \backslash \{\delta\}\}$ and $\{(\delta', \delta) \mid \delta' \in \Delta \backslash \{\delta\}\}$. Since G acting on Ω is of product action type, there exists an element $x \in G \backslash N$ which interchanges (t_1, t_2) and (t_2, t_1) for all elements $t_1, t_2 \in T$, and so interchanges points (δ_1, δ_2) and (δ_2, δ_1) for all $\delta_1, \delta_2 \in \Delta$. The element x fixes $\omega = (\delta, \delta)$, and fuses $\{(\delta, \delta') \mid \delta' \in \Delta \backslash \{\delta\}\}$ and $\{(\delta', \delta) \mid \delta' \in \Delta \backslash \{\delta\}\}$. Therefore, the orbital graph of G containing $\{(\delta, \delta), (\delta, \delta')\}$ is isomorphic to $\mathbf{K}_n \square \mathbf{K}_n$, where $\delta' \in \Delta \backslash \{\delta\}$.

Let $\Gamma = \mathbf{K}_n \square \mathbf{K}_n$. Then, $\Gamma(\omega) = \{(\delta, \delta') \mid \delta' \in \Delta \backslash \{\delta\}\} \cup \{(\delta', \delta) \mid \delta' \in \Delta \backslash \{\delta\}\}$, and $\{(\delta, \delta') \mid \delta' \in \Delta \backslash \{\delta\}\}$ and $\{(\delta', \delta) \mid \delta' \in \Delta \backslash \{\delta\}\}$ are two blocks of G_ω acting of $\Gamma(\omega)$. Thus, Γ is not G-locally-primitive.

On the other hand, assume that $\Gamma = \mathbf{K}_n \times \mathbf{K}_n$. Then, $\Gamma(\omega) = \{(\delta_1, \delta_2) \mid \delta_1, \delta_2 \in \Delta\backslash\{\delta\}\}$. If $T = \mathrm{PSL}(d, q)$ with $d \geq 3$, then by [16, Lemma 2.5], Γ is not G-locally-primitive. Suppose that $T \neq \mathrm{PSL}(d, q)$ with $d \geq 3$. Then, T acting on Δ is 2-primitive. It follows that the arc-stabiliser $G_{(\omega,(\delta_1,\delta_2))}$ is a maximal subgroup of G_ω. So Γ is G-locally-primitive. □

5 Graphs associated with PSL(2, p)

We study now examples associated with PSL(2, p) with p a prime.

Consider the case where $p \in \{11, 19, 29, 59\}$ and $G = \mathrm{PSL}(2, p)$ first. Note that G has a factorization $G = RH$, where $R = \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$, and $H = \mathrm{A}_5$. Let $\Omega = [G : H]$. Then G is a primitive permutation group on Ω of degree 11, or of degree pq, where q is a prime divisor of $\frac{p-1}{2}$. For the latter, each fused-orbital graph Γ of G on Ω has order equal to a product of two primes. Such graphs Γ of G were classified in [28,29] (with two graphs associated with M_{23} missed and pointed out on [22]), stated as follows.

Example 5.1 Let $G = \mathrm{PSL}(2, p)$ with $p = 11$, 19, 29 or 59, and let $H < G$ be isomorphic to A_5. If $p \neq 19$, then each fused-orbital graph of G on $\Omega = [G : H]$ is a Cayley graph of a metacyclic group R, and moreover, we have the following statements:

- (i) For $p = 11$, then $R = \mathbb{Z}_{11}$ and G is 2-transitive on Ω, so $\Gamma = \mathbf{K}_{11}$ and $\mathsf{Aut}\Gamma = \mathrm{S}_{11}$;
- (ii) For $p = 19$, then there are three fused-orbital graphs, all of which are arc-transitive of valency 6, 20 or 30, and have automorphism group G. The three graphs are metacirculants of $\mathbb{Z}_{19}{:}\mathbb{Z}_9$ but not Cayley graphs.
- (iii) For $p = 29$, then $R = \mathbb{Z}_{29}{:}\mathbb{Z}_7$ and there are seven fused-orbital graphs, all of which are arc-transitive and have automorphism group equal to G, one of valency 12, two of valency 20, three of valency 30, and one of valency 60.
- (iv) For $p = 59$, then $R = \mathbb{Z}_{59}{:}\mathbb{Z}_{29}$ and there are 33 fused-orbital graphs, which have automorphism group equal to G. Four of them are half-transitive of valency 120, and twenty-nine of them are arc-transitive: one of valency 6 or 10, two of valency 12, four of valency 20, five of valency 30, and sixteen of valency 60.

Example 5.2 Let $G = \mathrm{PSL}(2, 23)$, and $\mathrm{S}_4 \cong H < G$. Let $\Omega = [G : H]$, of size 253. Then, G is a primitive permutation group on Ω, and contains a metacyclic subgroup $R = \mathbb{Z}_{23}{:}\mathbb{Z}_{11}$ which is regular on Ω. By [29, Lemma 4.3], there are 13 fused-orbital graphs, and all of which have automorphism group equal to G. Two of them are half-transitive of valency 24 or 48, and the other eleven are arc-transitive graphs, one of valency 4 or 8 or 12, two of valency 6, and six of valency 24. Moreover, among the graphs, the graph of valency 4 is the unique G-locally-primitive graph.

Praeger and Xu in [29, Lemma 4.4] also determined edge-transitive graphs admitting PSL(2, p) and PGL(2, p) with stabiliser D_{p+1} and $\mathrm{D}_{2(p+1)}$, respectively.

Table 1 The almost simple primitive groups with a transitive metacyclic subgroup

row	G	A	B	conditions
1	S_p	$p{:}(p-1)$	$\mathrm{S}_{p-2}, \mathrm{S}_{p-2}\times \mathrm{S}_2$	
	$\mathrm{A}_p.o$	$p{:}\frac{p-1}{2}$	$\mathrm{S}_{p-2}\times o$	$o\le 2$
2	$\mathrm{PSL}(d,q).o$	$G(q^d).o_1$	$P_1.o$, parabolic	where$q=p^f$, and
	$\mathrm{PSL}(d,q).o.2$	$G(q^d).o_1.2$	$P_1.o$	$o_1\le o\le f.(d,q-1)$
3	$\mathrm{PGL}(2,p)$	$p{:}(p-1)$	$\mathrm{D}_{2(p+1)}$	
	$\mathrm{PSL}(2,p).o$	$p{:}\frac{p-1}{2}.o_1$	$\mathrm{D}_{(p+1)o}$	$o_1\le o\le 2$, $p\equiv 3 \pmod 4$
4	$\mathrm{PSL}(2,11)$	11, 11:5	A_5	
5	$\mathrm{PSL}(2,29)$	29:7	A_5	
6	$\mathrm{PSL}(2,p)$	$p{:}\frac{p-1}{2}$	A_5	$p=11,19,29,59$
	$\mathrm{PGL}(2,p)$	$p{:}(p-1)$	A_5	
7	$\mathrm{PSL}(2,23)$	23:11	S_4	
	$\mathrm{PGL}(2,23)$	23:22	S_4	
8	$\mathrm{P\Gamma L}(2,16)$	17:8	$\mathrm{PSL}(2,4).4$	
9	$\mathrm{PSL}(5,2).o$	$31{:}(5\times o)$	$2^6{:}(\mathrm{S}_3\times \mathrm{PSL}(3,2))$	$o\le 2$
10	$\mathrm{PSU}(3,8).3^2.o$	$(3\times 19{:}9).o_1$	$(2^{3+6}{:}63{:}3).o$	$o_1\le o\le 2$
11	$\mathrm{PSU}(4,2).o$	$9{:}3.o_1$	$2^4{:}\mathrm{A}_5.o$	$o_1\le 2, o\le 4$
12	M_{11}	11, 11:5	$\mathrm{M}_{10}, \mathrm{M}_9.2$	
13	M_{12}	6×2	M_{11}	
	$\mathrm{M}_{12}.2$	D_{24}	M_{11}	
14	$\mathrm{M}_{22}.2$	D_{22}	$\mathrm{PSL}(3,4).2$	
15	M_{23}	23, 23:11	$\mathrm{M}_{22}, \mathrm{M}_{21}.2, 2^4.\mathrm{A}_7$	
16	M_{24}	D_{24}	M_{23}	

Lemma 5.3 *(1) For $p\equiv 3 \pmod 4$, the simple group $T=\mathrm{PSL}(2,p)$ is a product of subgroups $R\cong \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$ and $H\cong \mathrm{D}_{p+1}$, each fused-orbital graph of T acting on $[T:H]$ is a vertex-primitive metacirculant Cayley graph of R.*

(2) The group $G=\mathrm{PGL}(2,p)$ is a product of subgroups $R\cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ and $H\cong \mathrm{D}_{2(p+1)}$, each fused-orbital graph of G acting on $[G:H]$ is a vertex-primitive metacirculant graph of R.

Moreover, such a graph is a Cayley graph if and only if $p\equiv 3 \pmod 4$, and is G-locally-primitive if and only if its valency equals to $\frac{p+1}{2}$ with $\frac{p+1}{2}$ prime.

6 Proofs of Theorem 1.1 and the Corollaries

Let $\Gamma=(V,E)$ be a connected metacirculant, and assume further that $G\le \mathrm{Aut}\Gamma$ is primitive on V and transitive on E, and contains a transitive metacyclic subgroup R. In particular, G is a primitive permutation group on V.

The proof of Theorem 1.1 depends on the classification of primitive permutation groups which contain a transitive metacyclic subgroup, obtained in [17], as stated in the following theorem.

Theorem 6.1 *Let G be a finite primitive permutation group on Ω, and let R be a transitive metacyclic subgroup of G. Then, one of the following holds:*

(1) G is an almost simple group, and either $(G, G_\omega) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ or $(\mathrm{S}_n, \mathrm{S}_{n-1})$, or $(G, R, G_\omega) = (G, A, B)$ such that $R = A$ and $G_\omega = B$ as in Table 1;
(2) G is of diagonal type with socle $T^2 = \mathrm{PSL}(2, p)^2$, R is regular, and either
  (i) $p \equiv 3 \pmod 4$, $R \cong \mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong (\mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}) \times \mathrm{D}_{p+1}$, or
  (ii) $p \equiv 1 \pmod 4$, $G \geq T{:}\mathsf{Aut}(T)$, and $R \cong \mathbb{Z}_{\frac{p(p+1)}{2}}{:}\mathbb{Z}_{p-1} \cong (\mathbb{Z}_p{:}\mathbb{Z}_{p-1}) \times \mathbb{Z}_{\frac{p+1}{2}}$;
(3) G is of product action type of degree n^2 with socle T^2, and $R = \mathbb{Z}_n^2$ or $\mathbb{Z}_m^2{:}\mathbb{Z}_4$ with $m = \frac{n}{2}$ odd, and $T = \mathrm{A}_n$, or $\mathrm{PSL}(d, q)$ with $n = \frac{q^d-1}{q-1}$, or $(T, n) = (\mathrm{PSL}(2, 11), 11)$, $(\mathrm{M}_{11}, 11)$ or $(\mathrm{M}_{23}, 23)$.
(4) G is an affine group, and either G is 2-transitive, or $p^d = p$, p^2, 3^3, 2^3 or 2^4.

We first treat the affine groups.

Lemma 6.2 *Let G be an affine primitive permutation group with socle $\mathbb{Z}_p^d$, where $p^d = p$, p^2, 3^3, 2^3 or 2^4. Then, one of the following holds:*

(1) G is 2-homogeneous, and $\Gamma = \mathbf{K}_{p^d}$;
(2) $\Gamma = \mathbf{K}_p \square \mathbf{K}_p$, or $\mathbf{K}_p \times \mathbf{K}_p$;
(3) Γ is a normal Cayley graph of $\mathbb{Z}_p^d$.

Proof Let $X = \mathsf{Aut}\Gamma$. If X is affine, then, the socle of X is $\mathbb{Z}_p^d$ and regular on V, and hence Γ is a normal Cayley graph, as in part (3). Suppose that X is not affine. Since the degree is p^d, either X is almost simple or of product action type. If X is almost simple, then by [13], X is 2-transitive, so $\Gamma = \mathbf{K}_{p^d}$ is a complete graph and G is 2-homogeneous, as in part (1). If X is of product action, then by Theorem 6.1, X satisfies part (3) of Theorem 6.1, and in particular $\mathsf{soc}(X) = T^2$ and d is even. It follows that $d = 2$, and $\Gamma = \mathbf{K}_p \square \mathbf{K}_p$ or $\mathbf{K}_p \times \mathbf{K}_p$ by Lemma 4.6, as in part (2). □

It would be interesting to give a classification of edge-transitive metacirculants associated with a primitive affine automorphism group. Here, we only mention a special case. Let $G = \mathbb{Z}_p^2{:}\mathrm{Q}_8 < \mathrm{AGL}(2, p)$ with p an odd prime, and let $H = \mathrm{Q}_8$. Then, H acts semiregularly on $\mathbb{Z}_p^2 \setminus \{0\}$, and hence there are $\frac{p^2-1}{8}$ different G-edge-transitive graphs, which are of valency 8.

We observe that the edge-transitive group G is 2-homogeneous on the vertex set V if and only if Γ is a complete graph. Many of the almost simple primitive groups G listed in TABLE 1 are 2-homogeneous, which correspond to complete graphs.

Lemma 6.3 *Assume that $G \leq \mathsf{Aut}\Gamma$ is 2-homogeneous on V and contains a metacyclic subgroup R which is transitive on V. Then, $\Gamma = (V, E)$ is a complete graph of order n, and one of the following holds.*

(1) $G = \mathrm{A}_n$ *or* S_n, *and* $G_\omega = \mathrm{A}_{n-1}$ *or* S_{n-1}, *respectively;*
(2) $G \rhd \mathrm{PSL}(d, q)$, $n = \frac{q^d-1}{q-1}$, *and* $R \leq \Gamma\mathrm{L}(1, q^d)$*;*
(3) $G = \mathrm{PSU}(3, 8).3^2.o$, $R = (57{:}9).o_1$, *and* $G_\omega = (2^{3+6}{:}63{:}3).o$, *where* $o_1 \leq o \leq 2$*;*
(4) $(G, G_\omega, n) = (\mathrm{PSL}(2, 11), \mathrm{A}_5, 11)$, *or* $(\mathrm{M}_{22}.2, \mathrm{PSL}(3, 4).2, 22)$*;*
(5) $G = \mathrm{M}_n$, *where* $n = 11, 12, 23,$ *or* 24*;*
(6) *G is an affine 2-homogeneous group of degree n, where* $n = p$, p^2, 3^3, 2^3 *or* 2^4.

Moreover, if $\Gamma = \mathbf{K}_n$ *is G-locally-primitive, then G is* 2*-primitive, and* $\mathsf{soc}(G) = \mathrm{A}_n$, $\mathrm{PSL}(2, q)$ *with* $n = q + 1$, *or* M_n *with* $n \in \{11, 12, 22, 23, 24\}$.

Proof Since G is 2-homogeneous on V, then graph $\Gamma = \mathbf{K}_n$, and G is almost simple or affine. By Theorem 6.1, if G is almost simple, then G satisfies part (1) of Theorem 6.1; if G is affine, then G satisfies part (4) of Theorem 6.1. Analyzing these candidates, we obtain that G satisfies one of parts (1)–(6).

It is easily shown that $\Gamma = \mathbf{K}_n$ is G-locally-primitive if and only if G is 2-primitive, so $\mathsf{soc}(G) = \mathrm{A}_n$, $\mathrm{PSL}(2, q)$ with $n = q + 1$, or M_n with $n \in \{11, 12, 23, 24\}$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By assumption, for a vertex $v \in V$, the triple (G, R, G_v) satisfies Theorem 6.1.

If G is affine, then G satisfies Theorem 6.1 (4), and then by Lemma 6.2, Γ satisfies part (i) or part (ix) of Theorem 1.1.

If G is of product action type, as in Theorem 6.1 (3), then by Lemma 4.6, the graph Γ satisfies part (i) of Theorem 1.1.

If G is of diagonal type, then G satisfies Theorem 6.1 (2). By Lemma 3.7, Γ satisfies part (iii) of Theorem 1.1.

Finally, we consider the almost simple case. In this case, G satisfies Theorem 6.1 (1). If G is 2-homogeneous on V, then Γ is a complete graph, as in part (i) of Theorem 1.1. We thus assume that G is not 2-homogeneous.

Assume that $G = \mathrm{A}_p$ or S_p as in row 1 of Table 1. Then, the vertex set V is the set of 2-subsets of a set $\Omega = \{1, 2, \ldots, p\}$, namely, $V = \Omega^{(2)}$, and G is 4-transitive on Ω. Let $\alpha = \{1, 2\}$ be a vertex of Γ. Then, G_α has exactly two orbits on $V \backslash \{\alpha\} = \Omega^{(2)} \backslash \{\{1, 2\}\}$, with representatives $\{1, 3\}$ and $\{3, 4\}$. If $\{1, 3\}$ is adjacent to $\alpha = \{1, 2\}$ in Γ, then Γ is the line graph of $\mathbf{K}_p$, while if $\{3, 4\}$ lies in $\Gamma(\alpha)$, then Γ is the complement of $\mathsf{line}(\mathbf{K}_p)$, as stated in part (ii) of Theorem 1.1. Similarly, if $G = \mathrm{M}_{11}$ with $G_v \cong \mathrm{M}_9.2$ as in row 14 of Table 1, or $G = \mathrm{M}_{23}$ with $G_v = \mathrm{M}_{21}.2$ as in row 17 of Table 1, then $\Gamma = \mathsf{line}(\mathbf{K}_{11})$, $\overline{\mathsf{line}}(\mathbf{K}_{11})$, $\mathsf{line}(\mathbf{K}_{23})$ or $\overline{\mathsf{line}}(\mathbf{K}_{23})$, as in part (ii) of Theorem 1.1.

If $\mathsf{soc}(G) = \mathrm{PSL}(2, p)$, as in rows 3-7 of Table 1, then Γ is described in Examples 5.1–5.2 and Lemma 5.3. This is as claimed in Theorem 1.1 (iv).

For $G = \mathrm{P\Gamma L}(2, 16)$, the graph Γ is described in Lemma 2.6, as in part (v).

For $\mathsf{soc}(G) = \mathrm{PSL}(5, 2)$, Lemma 2.4 shows that the graph $\Gamma = \mathrm{G}_2(5, 2)$ is the Grassmann graph or the complement $\overline{\Gamma} = \overline{\mathrm{G}}_2(5, 2)$, as in part (vi).

For $\mathsf{soc}(G) = \mathrm{PSU}(4, 2)$, by Lemma 2.5, the graph Γ is the Schläfli graph or its complement, as in part (vii).

Finally, for $G = \mathrm{M}_{23}$ and $G_v = 2^4{:}\mathrm{A}_7$, by Example 2.7, the graph Γ is of valency 112 or 140, as in part (viii). □

Proof of Corollary 1.2 The graphs in parts (i), (iii), and (vi)-(ix) of Theorem 1.1 are all Cayley graphs, by the corresponding lemmas or examples in Sect. 2–5 which define or describe these graphs.

For the graphs in part (ii) of Theorem 1.1, by Lemma 2.3, a line graph $\mathsf{line}(\mathbf{K}_p)$ and its complement are Cayley graphs if and only if $p \equiv 3 \pmod 4$.

For graphs in part (iv) of Theorem 1.1, if G acts on V with exceptional action, Γ is a Cayley graph with the only exception that $\mathsf{Aut}\Gamma = \mathrm{PSL}(2, 19)$, see Examples 5.1–5.2; for the other actions, Γ is not a Cayley graph if and only if $G = \mathrm{PGL}(2, p)$ and $p \equiv 1 \pmod 4$, see Lemma 5.3.

Finally, the three graphs associated with $\mathrm{P\Gamma L}(2, 16)$, stated in Theorem 1.1 (v), are not Cayley graphs, see Lemma 2.6. □

Proof of Corollary 1.3 The local-primitivity of each graph listed in Theorem 1.1 is determined in the corresponding lemmas and examples in Sect. 2–5, from which the proof of Corollary 1.3 follows. □

Acknowledgments This paper was partially supported by the National Natural Science Foundation of China and an ARC Discovery Grant Project.

References

1. Alspach, B., Conder, M., Marusic, D., Xu, M.Y.: A classification of 2-arc-transitive circulants. J. Algebraic Combin. **5**(2), 83–86 (1996)
2. Alspach, B., Parsons, T.D.: A construction for vertex-transitive graphs. Canad. J. Math. **34**(2), 307–318 (1982)
3. Alspach, B., Marušič, D., Nowitz, L.: Constructing graphs which are 1/2-transitive. J. Austral. Math. Soc. Ser. A **56**(3), 391–402 (1994)
4. Biggs, N.: Algebraic Graph Theory, 2nd edn. Cambridge University Press, New York (1992)
5. Brouwer, A.E., Wilbrink, H.A.: Ovoids and fans in the generalized quadrangle $GQ(4, 2)$. Geom. Dedicata **36**, 121–124 (1990)
6. Conway, J.H., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of Finite Groups. Oxford University Press, New York (1985)
7. Dixon, J.D., Mortimer, B.: Permutation Groups. Springer, New York (1996)
8. Dobson, E.: Isomorphism problem for metacirculant graphs of order a product of distinct primes. Canad. J. Math. **50**(6), 1176–1188 (1998)
9. Du, S.F., Malnič, A., Marušič, D.: Classification of 2-arc-transitive dihedrants. J. Combin. Theory Ser. B **98**, 1349–1372 (2008)
10. Godsil, C.D.: On the full automorphism group of a graph. Combinatorica **1**, 243–256 (1981)
11. Jones, G.: Cyclic regular subgroups of primitive permutation groups. J. Group Theory **5**(4), 403–407 (2002)
12. Kovacs, I.: Classifying arc-transitive circulants. J. Algebraic Combin. **20**, 353–358 (2004)
13. Li, C.H.: The finite primitive permutation groups containing an abelian regular subgroup. Proc. London Math. Soc. **87**, 725–747 (2003)
14. Li, C.H.: Permutation groups with a cyclic regular subgroup and arc transitive circulants. J. Algebraic Combin. **21**, 131–136 (2005)
15. Li, C.H.: Finite edge-transitive Cayley graphs and rotary Cayley maps. Trans. Amer. Math. Soc. **358**, 4605–4635 (2006)
16. Li, C.H., Pan, J.M., Ma, L.: Locally primitive graphs of prime-power order. J. Austral. Math. Soc. **86**, 111–122 (2009)

17. Li, C.H., Pan, J.M.: Finite primitive permutation groups which contain a transitive metacyclic subgroup, Proc. London Math. Soc. (to appear)
18. Li, C.H., Sims, H.: On half-transitive metacirculant graphs of prime-power order. J. Combin. Theory Ser. B **81**, 45–57 (2001)
19. Li, C.H., Song, S.J., Wang, D.J.: A characterization of metacirculants. J. Combin. Theory Ser. A **120**, 39–48 (2013)
20. Marušič, D.: On 2-arc-transitivity of Cayley graphs. J. Combin. Theory Ser. B **87**, 162–196 (2003)
21. Marušič, D.: On 2-arc-transitivity of Cayley graphs. J. Combin. Theory Ser. B **96**, 761–764 (2006)
22. Marušič, D., Scapellato, R.: Characteriing vertex-transitive pq-graphs with an imprimitive automorphism subgroups. J. Graph Theory **16**(4), 375–387 (1992)
23. Marušič, D., Scapellato, R.: Classifying vertex-transitive graphs whose order is a product of two primes. Combinatorica **14**(2), 187–201 (1994)
24. Marušič, D., Šparl, P.: On quartic half-arc-transitive metacirculants. J. Algebraic Combin. **28**, 365–395 (2008)
25. Morris, J., Šajna, M.: Brian Alspach and his work. Discrete Math. **299**(1–3), 269–287 (2005)
26. Praeger, C.E.: The inclusion problem for finite primitive permutation groups. Proc. London Math. Soc. (3) **60**, 68–88 (1990)
27. Praeger, C.E.: An O'Nan-Scott theorem for finite primitive permutation groups and an application to 2-arctransitive graphs. J. London Math. Soc. **47**, 227–239 (1992)
28. Praeger, C.E., Xu, M.: Symmetric graphs of order a product of two distinct primes. J. Combin. Theory Ser. B **58**, 299–318 (1993)
29. Praeger, C.E., Xu, M.: Vertex-primitive graphs of order a product of two distinct primes. J. Combin. Theory Ser. B **59**, 245–266 (1993)
30. Šajna, M.: Half-transitivity of some metacirculants. Discrete Math. **185**(1–3), 117–136 (1998)
31. Šparl, P.: On the classification of quartic half-arc-transitive metacirculants. Discrete Math. **309**, 2271–2283 (2009)
32. Suzuki, M.: Group Theory I. New York, II, Springer, Berlin (1982)
33. Tan, N.D.: Non-Cayley tetravalent metacirculant graphs and their hamiltonicity. J. Graph Theory **23**(3), 273–287 (1996)
34. Xu, M.Y.: Half-transitive graphs of prime-cube order. J. Algebraic Combin. **1**, 275–282 (1992)
35. Zhou, Chuixiang, Feng, Yan-Quan: An infinite family of tetravalent half-arc-transitive graphs. Discrete Math. **306**, 2205–2211 (2006)