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Completely regular clique graphs

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Abstract Let $\Gamma = (X, R)$ be a connected graph. Then Γ is said to be a completely regular clique graph of parameters (s, c) with $s \geq 1$ and $c \geq 1$, if there is a collection $\mathcal{C}$ of completely regular cliques of size $s + 1$ such that every edge is contained in exactly c members of $\mathcal{C}$. In this paper, we show that the parameters of $C \in \mathcal{C}$ as a completely regular code do not depend on $C \in \mathcal{C}$. As a by-product we have that all completely regular clique graphs are distance-regular whenever $\mathcal{C}$ consists of edges. We investigate the case when Γ is distance-regular, and show that Γ is a completely regular clique graph if and only if it is a bipartite half of a distance-semiregular graph.

Keywords Distance-regular graph · Association scheme · Subconstituent algebra · Terwilliger algebra · Completely regular code · Distance-semiregular graph

1 Introduction

In this paper, we only consider finite graphs. Let $\Gamma = (X, R)$ be a connected graph with vertex set X and edge set R consisting of 2-element subsets of X. When $\{x, y\} \in R$, i.e., x and y are adjacent, we write $x \sim y$. For $x, y \in X$, $\partial_\Gamma(x, y) = \partial(x, y)$ denotes the distance between x and y, i.e., the length of a shortest path between x and y in Γ. The diameter $d(\Gamma)$ is the maximal distance between two vertices. A nonempty subset C of X is said to be a clique if every distinct vertices in C are adjacent.

A subset C of X is often called a *code* in $\Gamma = (X, R)$, and $\Gamma_i(C) = \{x \in X \mid \partial(x, C) = i\}$ is called the *ith subconstituent* with respect to C, where $\partial(x, C) = \min\{\partial(x, y) \mid y \in C\}$. We write $\Gamma(C)$ for $\Gamma_1(C)$. The number $t = t(C) = \max\{i \mid \Gamma_i(C) \neq \emptyset\}$ is called the *covering radius* of C. If C and C' are subsets of X, $\partial(C, C') = \min\{\partial(x, y) \mid x \in C, y \in C'\}$. When $C = \{x\}$, we write $\Gamma_i(x)$ for $\Gamma_i(\{x\})$,

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and set $\Gamma(x)$ for $\Gamma_1(x)$. The number $k(x) = |\Gamma(x)|$ is called the valency of x. For $x, y \in X$ with $\partial(x, y) = i$ with $i = 0, 1, \ldots, d(\Gamma)$, let

$$B_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma(y), \qquad A_i(x, y) = \Gamma_i(x) \cap \Gamma(y),$$
$$C_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma(y),$$

and $b_i(x, y) = |B_i(x, y)|$, $a_i(x, y) = |A_i(x, y)|$ and $c_i(x, y) = |C_i(x, y)|$.

A connected graph $\Gamma = (X, R)$ of diameter $D = d(\Gamma)$ is said to be *distance-regular* if for each $i \in \{0, 1, 2, \ldots, D\}$, the numbers $c_i = c_i(x, y)$, $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ depend only on $i = \partial(x, y)$. In this case, the numbers b_i, a_i, c_i with $i = 0, 1, \ldots, D$ are called the parameters of Γ. For distance-regular graphs we refer the reader [4]. We mainly follow the notation and the terminologies in the monograph.

Let $\Gamma = (X \cup Y, R)$ be a connected bipartite graph with the bipartition $X \cup Y$, i.e., there is no edge within X and Y. Let $d^X = d^X(\Gamma) = \max\{\partial(x, y) \mid x \in X, y \in X \cup Y\}$. Then Γ is said to be *distance-semiregular* on X, if for each $i \in \{0, 1, 2, \ldots, d^X\}$, the numbers $c_i^X = c_i(x, y)$, and $b_i^X = b_i(x, y)$ depend only on $i = \partial(x, y)$ whenever $x \in X$ and $y \in X \cup Y$. In this case the numbers b_i^X, c_i^X with $i = 0, 1, \ldots, d^X$ are called the parameters of Γ. Note that each vertex $y \in Y$ is of valency $b_0^Y = b_1^X + c_1^X$ and distance-semiregular graphs are biregular, i.e., the valency of a vertex depends only on the part the vertex belongs to. If $\Gamma = (X \cup Y, R)$ is distance-semiregular on both X and Y, Γ is called *distance-biregular*. For more information on distance-biregular graphs and distance-semiregular graphs, see [9, 11, 14].

Let $\Gamma = (X, R)$ be a connected graph, and C a nonempty subset of X with covering radius $t = t(C)$. Then C is said to be a *completely regular code* if $\gamma_i = \gamma_i(C) = |\Gamma_{i-1}(C) \cap \Gamma(x)|$, $\alpha_i = \alpha_i(C) = |\Gamma_i(C) \cap \Gamma(x)|$, $\beta_i = \beta_i(C) = |\Gamma_{i+1}(C) \cap \Gamma(x)|$ do not depend on $x \in \Gamma_i(C)$ for $i \in \{0, 1, \ldots, t\}$. In this case the numbers $\gamma_i, \alpha_i, \beta_i$ with $i = 0, 1, \ldots, t$ are called the parameters of C. We also write $\gamma_i(C)$, $\alpha_i(C)$ or $\beta_i(C)$ exists when the corresponding number does not depend on the choice of $x \in \Gamma_i(C)$. For completely regular codes of distance-regular graphs, see [4, Sect. 11.1] and [13]. For a special type of completely regular codes in a regular graph, see [7].

Let $\Gamma = (X, R)$ be a connected graph. In [11], C.D. Godsil and J. Shawe-Taylor showed that $\{x\}$ is completely regular for each $x \in X$, if and only if Γ is either distance-regular or distance-biregular. As a corollary, we have that distance-regular graphs can be characterized as regular connected graphs such that $\{x\}$ is completely regular for each $x \in X$. It is not difficult to show that a connected bipartite graph $\Gamma = (X \cup Y, R)$ with the bipartition $X \cup Y$ is distance-semiregular on X, if and only if it is biregular and $\{x\}$ is completely regular for each $x \in X$. Recently it was shown in [6] that each edge of Γ is completely regular with the same parameters, if and only if Γ is either bipartite or almost bipartite distance-regular, i.e., distance-regular graphs of diameter D with $a_0 = a_1 = \cdots = a_{D-1} = 0$. These results can be viewed as characterizations of distance-regularity by complete regularity of its substructures, i.e., cliques.

On the other hand, many distance-regular graphs contain many completely regular codes of various sizes. These completely regular codes correspond to substructures of the geometry associated with them. If Γ is a distance-regular graph of diameter D isomorphic to one of the Johnson graphs, the Hamming graphs, the

Grassmann graphs, the dual polar graphs, the bilinear forms graphs, then for each $i \in \{0, 1, \ldots, D\}$ and vertices x, y at distance i, there is a geodetically closed completely regular code of diameter i containing x and y. In each case, there is a set $\mathcal{C}$ of completely regular maximal cliques such that the incidence graph on $X \cup \mathcal{C}$ associated with it is distance-semiregular.

Definition 1 Let $\Gamma = (X, R)$ be a connected graph, and let $\mathcal{C}$ be a collection of cliques of Γ. Then Γ is said to be a *completely regular clique graph with parameters* (s, c) *with respect to* $\mathcal{C}$, if the following are satisfied.

(i) Each member $C \in \mathcal{C}$ is a completely regular code of size $s + 1 \geq 2$.
(ii) Each edge is contained in exactly c members of $\mathcal{C}$ and $c \geq 1$.

When $\mathcal{C}$ consists of Delsarte cliques, it is called a Delsarte clique graph with parameters (s, c) in [2, 3], and Delsarte clique graphs with parameters $(s, 1)$ are called geometric in [1]. Many examples are listed in [2].

Our first result in this paper is concerning the parameters of $C \in \mathcal{C}$.

Theorem 1 *Let Γ be a completely regular clique graph with parameters (s, c) with respect to $\mathcal{C}$. Then the parameters of a completely regular code $C \in \mathcal{C}$ do not depend on C.*

We prove Theorem 1 in Sect. 2 using modules of Terwilliger algebra $\mathcal{T}(C)$ with respect to $C \in \mathcal{C}$, and applying the fact that C is completely regular if and only if the primary $\mathcal{T}(C)$-module is thin. For Terwilliger algebras $\mathcal{T}(C)$ and their modules, see [15].

A connected graph is called *edge distance-regular* if every edge is a completely regular code with the same parameters. See [5, 10]. Because of Theorem 1, the condition on parameters is not necessary. Combining with the result in [6] mentioned above, we have the following.

Corollary 2 *Let $\Gamma = (X, R)$ be a connected graph of diameter D. Then the following are equivalent.*

(i) *For every $\{x, y\} \in R$, $\{x, y\}$ is a completely regular code.*
(ii) *Γ is a distance-regular graph with $a_1 = \cdots = a_{D-1} = 0$.*

Next result is a characterization of distance-regular completely regular clique graphs. The proof will be given in Sect. 3. It can be viewed as a characterization of the collinearity graphs of distance-regular geometries in [8, 12].

Theorem 3 *Let $\Gamma = (X, R)$ be a distance-regular graph. Then Γ is a completely regular clique graph if and only if Γ is the bipartite half of a distance-semiregular graph $\tilde{\Gamma} = (X \cup Y, \tilde{R})$ on X. Here the bipartite half of $\tilde{\Gamma} = (X \cup Y, \tilde{R})$ on X is the graph with vertex set X such that two vertices are adjacent whenever they are at distance 2 in $\tilde{\Gamma}$.*

2 Parameter set of completely regular clique codes

The main objective of this section is to prove Theorem 1.

Lemma 4 *Let $\Gamma = (X, R)$ be a connected graph of diameter $D > 1$. Let $\mathcal{C}$ be a collection of cliques of Γ. Then the following hold.*

- (i) *Let $C, C' \in \mathcal{C}$ with $C \cap C' \neq \emptyset$. Suppose $\alpha_0(C)$, $\beta_0(C)$, $\alpha_0(C')$ and $\beta_0(C')$ exist. Then the valencies of the vertices in $C \cup C'$ are the same.*
- (ii) *If every edge is contained in at least one $C \in \mathcal{C}$, and both $\alpha_0(C)$ and $\beta_0(C)$ exist for all $C \in \mathcal{C}$, then Γ is regular with valency $k = \alpha_0(C) + \beta_0(C)$ for any $C \in \mathcal{C}$.*

Proof (i) For all $x, y \in C$, $k(x) = \beta_0(C) + \alpha_0(C) = k(y)$, and for all $x', y' \in C'$, $k(x') = \beta_0(C') + \alpha_0(C') = k(y')$. Since $C \cap C' \neq \emptyset$, the valency $k(x)$ is constant on $C \cup C'$.

(ii) Since Γ is connected, the assertion follows from (i). □

Let $\Gamma = (X, R)$ be a connected graph of diameter D and C a nonempty subset of X with covering radius $t(C)$. Let $V = \boldsymbol{R}^X$ denote the real vector space consisting of column vectors whose entries are indexed by X. For $\boldsymbol{u}, \boldsymbol{v} \in V$, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^T \boldsymbol{v}$ and $\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u}\rangle}$. Let $A \in \mathrm{Mat}_X(\boldsymbol{R})$ be the adjacency matrix of Γ. Let $\theta_0 > \theta_1 > \cdots > \theta_r$ be all the distinct eigenvalues of A, and $E_0, E_1, \ldots, E_r \in \boldsymbol{R}[A]$ the corresponding primitive idempotents, where $\boldsymbol{R}[A]$ is the polynomial algebra in A over the real number field. Thus,

$$E_0 + E_1 + \cdots + E_r = I, \qquad E_i E_j = \delta_{i,j} E_i, \quad \text{and}$$
$$A E_i = \theta_i E_i \quad \text{for } i, j \in \{0, 1, \ldots, r\}.$$

For each $i \in \{0, 1, \ldots, t(C)\}$, let $E_i^*(C)$ denote the projection onto the subspace of V spanned by unit vectors corresponding to vertices in $\Gamma_i(C)$. We let $\mathcal{T}(C)$ denote the subalgebra of $\mathrm{Mat}_X(\boldsymbol{R})$ generated by A and $E_0^*(C), E_1^*(C), \ldots, E_{t(C)}^*(C)$. Let $\mathbf{1} \in V$ be the all one vector. A $\mathcal{T}(C)$-module W, i.e., a vector subspace of V invariant under the action of $\mathcal{T}(C)$, is said to be *thin* if $\dim E_i^*(C)W \leq 1$ for all $i \in \{0, 1, \ldots, t(C)\}$. $\mathcal{T}(C)\mathbf{1}$ is called the primary module of $\mathcal{T}(C)$. Note that $\boldsymbol{w}_i = E_i^*(C)\mathbf{1}$ is the characteristic vector of $\Gamma_i(C)$ for $i \in \{0, 1, \ldots, t(C)\}$ and $W_C = \mathrm{Span}(\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{t(C)}) \subset \mathcal{T}(C)\mathbf{1}$. It is easy to see that if C is a completely regular code with parameters $\gamma_i, \alpha_i, \beta_i$ $(i = 0, 1, \ldots, t(C))$, then

$$A\boldsymbol{w}_i = \beta_{i-1}\boldsymbol{w}_{i-1} + \alpha_i \boldsymbol{w}_i + \gamma_{i+1}\boldsymbol{w}_{i+1} \quad \text{for } i = 0, 1, \ldots, t(C).$$

Here $\boldsymbol{w}_{-1} = \boldsymbol{w}_{t(C)+1} = \mathbf{0}$, and β_{-1} and $\gamma_{t(C)+1}$ are indeterminate. Hence in this case $W_C = \mathcal{T}(C)\mathbf{1}$ and W_C is a thin irreducible $\mathcal{T}(C)$-module. See [15, Proposition 7.2].

The ideas and techniques of proofs of the following results are taken from the lecture note by P. Terwilliger [16].

Proposition 5 *Let $\Gamma = (X, R)$ be a connected graph of diameter $D > 1$. Suppose C, C' are completely regular codes that are cliques of Γ with $|C \cap C'| = e > 0$. Let*

$$W = W_C = \mathrm{Span}(\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_t), \quad \textit{where } \boldsymbol{w}_i = E_i^*(C)\mathbf{1},$$
$$A\boldsymbol{w}_i = \beta_{i-1}\boldsymbol{w}_{i-1} + \alpha_i \boldsymbol{w}_i + \gamma_{i+1}\boldsymbol{w}_{i+1},$$
$$W' = W_{C'} = \mathrm{Span}\big(\boldsymbol{w}_0', \boldsymbol{w}_1', \ldots, \boldsymbol{w}_{t'}'\big), \quad \textit{where } \boldsymbol{w}_i' = E_i^*\big(C'\big)\mathbf{1},$$
$$A\boldsymbol{w}_i' = \beta_{i-1}'\boldsymbol{w}_{i-1}' + \alpha_i' \boldsymbol{w}_i' + \gamma_{i+1}'\boldsymbol{w}_{i+1}',$$

with $t = t(C)$ and $t' = t(C')$. Then the following hold.

(i) *There are polynomials $p(\lambda), p'(\lambda) \in \boldsymbol{R}[\lambda]$ such that*

$$\mathrm{proj}_{W'}\, \boldsymbol{w}_0 = p(A)\frac{\|\boldsymbol{w}_0\|}{\|\boldsymbol{w}_0'\|}\boldsymbol{w}_0', \qquad \mathrm{proj}_W\, \boldsymbol{w}_0' = p'(A)\frac{\|\boldsymbol{w}_0'\|}{\|\boldsymbol{w}_0\|}\boldsymbol{w}_0,$$

where with $k = \alpha_0 + \beta_0$,

$$p(\lambda) = \frac{\|\boldsymbol{w}_0\|^2 - e}{(k - \|\boldsymbol{w}_0'\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|}\lambda + \frac{ke - \|\boldsymbol{w}_0\|^2(\|\boldsymbol{w}_0'\|^2 - 1)}{(k - \|\boldsymbol{w}_0'\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|},$$
$$p'(\lambda) = \frac{\|\boldsymbol{w}_0'\|^2 - e}{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|}\lambda + \frac{ke - \|\boldsymbol{w}_0'\|^2(\|\boldsymbol{w}_0\|^2 - 1)}{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|}.$$

(ii) *$p'(\theta_i)m_W(\theta_i) = p(\theta_i)m_{W'}(\theta_i)$ for all i, where*

$$m_W(\theta_i) = \frac{\|E_i\boldsymbol{w}_0\|^2}{\|\boldsymbol{w}_0\|^2} \quad \textit{and} \quad m_{W'}(\theta_i) = \frac{\|E_i\boldsymbol{w}_0'\|^2}{\|\boldsymbol{w}_0'\|^2}.$$

(iii) *If $\|\boldsymbol{w}_0\| = \|\boldsymbol{w}_0'\|$, i.e., $|C| = |C'|$, then $m_W(\theta_i) = m_{W'}(\theta_i)$ for all i.*

Proof Since $C \cap C' \neq \emptyset$, the valencies of the vertices in $C \cup C'$ are the same by Lemma 4. Let $k = \alpha_0 + \beta_0 = \alpha_0' + \beta_0'$ be the valency of vertices in $C \cup C'$.

(i) Since $A\boldsymbol{w}_0 = \alpha_0\boldsymbol{w}_0 + \gamma_1\boldsymbol{w}_1$ with $\gamma_1 \neq 0$ and $\alpha_0 = |C| - 1 = \|\boldsymbol{w}_0\|^2 - 1$,

$$\boldsymbol{w}_1 = \frac{1}{\gamma_1}\big(A - \big(\|\boldsymbol{w}_0\|^2 - 1\big)I\big)\boldsymbol{w}_0.$$

Since $C' \subset C \cup \Gamma(C)$, $\langle \boldsymbol{w}_0', \boldsymbol{w}_i\rangle = 0$ for $i = 2, 3, \ldots, t$. Let $\mathrm{proj}_W\, \boldsymbol{w}_0' = \xi_0\boldsymbol{w}_0 + \xi_1\boldsymbol{w}_1$. By counting the edges between C and $\Gamma(C)$ in two ways, we have

$$\|\boldsymbol{w}_1\|^2 = \langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle = \big|\Gamma(C)\big| = \frac{\beta_0|C|}{\gamma_1} = \frac{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|^2}{\gamma_1},$$

$e = \langle \boldsymbol{w}_0', \boldsymbol{w}_0\rangle = \langle \mathrm{proj}_W\, \boldsymbol{w}_0', \boldsymbol{w}_0\rangle = \xi_0\|\boldsymbol{w}_0\|^2$, and

$$\big\|\boldsymbol{w}_0'\big\|^2 - e = \big\langle \boldsymbol{w}_0', \boldsymbol{w}_1\big\rangle = \big\langle \mathrm{proj}_W\, \boldsymbol{w}_0', \boldsymbol{w}_1\big\rangle = \xi_1\|\boldsymbol{w}_1\|^2 = \xi_1\frac{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|^2}{\gamma_1}.$$

Therefore,

$$\begin{aligned}
&\operatorname{proj}_W \boldsymbol{w}_0' \\
&\quad = \xi_0 \boldsymbol{w}_0 + \xi_1 \boldsymbol{w}_1 \\
&\quad = \frac{e}{\|\boldsymbol{w}_0\|^2}\boldsymbol{w}_0 + \frac{\gamma_1(\|\boldsymbol{w}_0'\|^2 - e)}{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|^2} \cdot \frac{1}{\gamma_1}\big(A - \big(\|\boldsymbol{w}_0\|^2 - 1\big)I\big)\boldsymbol{w}_0 \\
&\quad = \frac{\|\boldsymbol{w}_0'\|}{\|\boldsymbol{w}_0\|}\left(\frac{\|\boldsymbol{w}_0'\|^2 - e}{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|}A + \frac{ke - \|\boldsymbol{w}_0'\|^2(\|\boldsymbol{w}_0\|^2 - 1)}{(k - \|\boldsymbol{w}_0\|^2 + 1)\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|}I\right)\boldsymbol{w}_0 \\
&\quad = p'(A)\frac{\|\boldsymbol{w}_0'\|}{\|\boldsymbol{w}_0\|}\boldsymbol{w}_0.
\end{aligned}$$

By symmetry we obtain the formula of $\operatorname{proj}_{W'} \boldsymbol{w}_0$ as well.

(ii) Since $E_i \boldsymbol{w}_0 \in W$ and A is a real symmetric matrix,

$$\begin{aligned}
\frac{\langle E_i \boldsymbol{w}_0, E_i \boldsymbol{w}_0'\rangle}{\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|} &= \frac{\langle E_i \boldsymbol{w}_0, \boldsymbol{w}_0'\rangle}{\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|} = \frac{\langle E_i \boldsymbol{w}_0, \operatorname{proj}_W \boldsymbol{w}_0'\rangle}{\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|} = \frac{\|\boldsymbol{w}_0'\|}{\|\boldsymbol{w}_0\|}\frac{\langle E_i \boldsymbol{w}_0, p'(A)\boldsymbol{w}_0\rangle}{\|\boldsymbol{w}_0\|\|\boldsymbol{w}_0'\|} \\
&= \frac{\langle p'(A)E_i \boldsymbol{w}_0, \boldsymbol{w}_0\rangle}{\|\boldsymbol{w}_0\|^2} = p'(\theta_i)\frac{\langle E_i \boldsymbol{w}_0, E_i \boldsymbol{w}_0\rangle}{\|\boldsymbol{w}_0\|^2} = p'(\theta_i)\frac{\|E_i \boldsymbol{w}_0\|^2}{\|\boldsymbol{w}_0\|^2} \\
&= p'(\theta_i) m_W(\theta_i).
\end{aligned}$$

By symmetry we have (ii).

(iii) Suppose $\|\boldsymbol{w}_0\| = \|\boldsymbol{w}_0'\|$. Then $p(\lambda) = p'(\lambda)$ by (i). Therefore $m_W(\theta_i) = m_{W'}(\theta_i)$ for all i except possibly one i for which $p(\theta_i) = 0$. Since

$$1 = \frac{\langle \boldsymbol{w}_0, \boldsymbol{w}_0\rangle}{\|\boldsymbol{w}_0\|^2} = \sum_{i=0}^{r} \frac{\|E_i \boldsymbol{w}_0\|^2}{\|\boldsymbol{w}_0\|^2} = \sum_{i=0}^{r} m_W(\theta_i),$$

$m_W(\theta_i) = m_{W'}(\theta_i)$ for all i without an exception. □

Proposition 6 *Let $\Gamma = (X, R)$ be a connected graph of diameter $D > 1$. Suppose Γ is regular of valency k. Let C, C' be completely regular codes that are cliques in Γ with $C \cap C' \neq \emptyset$. If $|C| = |C'|$, then the parameters of C and C' coincide.*

Proof We use the notation in the proof of Proposition 5. By Proposition 5(iii), $m_W(\theta_i) = m_{W'}(\theta_i)$ for all i. Since $E_i E_j = \delta_{i,j} E_i$ for $i, j \in \{0, 1, \dots, r\}$, nonzero vectors in the set $\{E_0 \boldsymbol{w}_0, E_1 \boldsymbol{w}_0, \dots, E_r \boldsymbol{w}_0\}$ are perpendicular to each other, and hence they form a linearly independent set of vectors. Since $\boldsymbol{R}[A]\boldsymbol{w}_0 = \operatorname{Span}\{E_0 \boldsymbol{w}_0, E_1 \boldsymbol{w}_0, \dots, E_r \boldsymbol{w}_0\}$, $\dim \boldsymbol{R}[A]\boldsymbol{w}_0 = \dim(W) = t(C) + 1$ is equal to the number of i such that $E_i \boldsymbol{w}_0 \neq \boldsymbol{0}$, we have $t(C) = t(C')$. Let t be this number. Let $m(\theta_i) = m_W(\theta_i) = \|E_i \boldsymbol{w}_0\|^2/\|\boldsymbol{w}_0\|^2$, and let $\boldsymbol{R}_t[\lambda]$ be the set of all polynomials of degree at most t. Then, for $f, g \in \boldsymbol{R}_t[\lambda]$,

$$\langle f, g\rangle_m = \sum_{i=0}^{r} f(\theta_i) g(\theta_i) m(\theta_i)$$

defines an inner product on $\boldsymbol{R}_t[\lambda]$. Let $q_0, q_1, \dots, q_t$ be uniquely determined monic orthogonal polynomials with respect to this inner product. Let $p_0, p_1, \dots, p_t, p_{t+1}$ and $p'_0, p'_1, \dots, p'_t, p'_{t+1}$ be polynomials defined by the following recursive relations:

$$p_0 = 1, \quad \lambda p_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \quad \text{for } i = 0, 1, \dots, t \text{ with } p_{-1} = 0,$$

$$p'_0 = 1, \quad \lambda p'_i = \beta'_{i-1} p'_{i-1} + \alpha'_i p'_i + \gamma'_{i+1} p'_{i+1}, \quad \text{for } i = 0, 1, \dots, t \text{ with } p'_{-1} = 0.$$

Here, $\beta_i = \beta_i(C)$, $\alpha_i = \alpha_i(C)$, $\gamma_i = \gamma_i(C)$, $\beta'_i = \beta_i(C')$, $\alpha'_i = \alpha_i(C')$, $\gamma'_i = \gamma_i(C')$ $(i = 0, 1, \dots, t)$, and assume $\gamma_{t+1} = \gamma'_{t+1} = 1$, $\beta_{-1} = \beta'_{-1} = 0$. Then we have $p_i(A)\boldsymbol{w}_0 = \boldsymbol{w}_i$, $p'_i(A)\boldsymbol{w}'_0 = \boldsymbol{w}'_i$ and $p_{t+1}(A)\boldsymbol{w}_0 = p'_{t+1}(A)\boldsymbol{w}'_0 = \boldsymbol{0}$. Moreover, for $i, j \in \{0, 1, \dots, t+1\}$ with $\boldsymbol{w}_{t+1} = \boldsymbol{w}'_{t+1} = \boldsymbol{0}$,

$$\begin{aligned} \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle &= \langle p_i(A)\boldsymbol{w}_0, p_j(A)\boldsymbol{w}_0 \rangle \\ &= \left\langle p_i(A) \sum_{h=0}^{r} E_h \boldsymbol{w}_0, p_j(A) \sum_{h'=0}^{r} E_{h'} \boldsymbol{w}_0 \right\rangle \\ &= \sum_{h=0}^{r} \sum_{h'=0}^{r} p_i(\theta_h) p_j(\theta_{h'}) \langle E_h \boldsymbol{w}_0, E_{h'} \boldsymbol{w}_0 \rangle \\ &= \sum_{h=0}^{r} p_i(\theta_h) p_j(\theta_h) m(\theta_h) \|\boldsymbol{w}_0\|^2 \\ &= \langle p_i, p_j \rangle_m \|\boldsymbol{w}_0\|^2. \end{aligned}$$

Therefore we have for $i, j \in \{0, 1, \dots, t\}$,

$$\langle p_i, p_j \rangle_m = \delta_{i,j} = \langle p'_i, p'_j \rangle_m,$$

and $\langle p_{t+1}, p_{t+1} \rangle_m = \langle p'_{t+1}, p'_{t+1} \rangle_m = 0$. Considering the leading coefficients of p_i and p_j, we have

$$\gamma_1 \gamma_2 \cdots \gamma_i p_i = q_i = \gamma'_1 \gamma'_2 \cdots \gamma'_i p'_i \quad \text{for all } i \in \{0, 1, \dots, t\}.$$

In particular, $q_0, q_2, \dots q_t$ with q_{t+1}, monic characteristic polynomial of A on W, satisfy

$$\lambda q_i = \beta_{i-1} \gamma_i q_{i-1} + \alpha_i q_i + q_{i+1} = \beta'_{i-1} \gamma'_i q_{i-1} + \alpha'_i q_i + q_{i+1} \quad \text{for } i = 0, 1, 2, \dots, t.$$

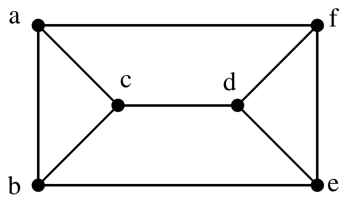
Therefore we have

$$\beta_{i-1} \gamma_i = \beta'_{i-1} \gamma'_i \quad \text{and} \quad \alpha_i = \alpha'_i \quad \text{for } i = 0, 1, 2, \dots, t.$$

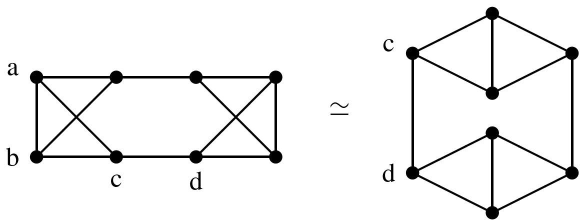
Since Γ is regular of valency k, $\alpha_0 = \alpha'_0$ implies $\beta_0 = k - \alpha_0 = k - \alpha'_0 = \beta'_0$. Hence we have $\gamma_1 = \gamma'_1$ by above. Since $\alpha_i = \alpha'_i$ for all i, by induction, we can conclude that all parameters are equal. □

Proof of Theorem 1 Let Γ be a completely regular clique graph with parameters (s, c) with respect to $\mathcal{C}$. Since every edge is contained in a member of $\mathcal{C}$, Γ is regular by Lemma 4. Now the fact that the parameters of completely regular codes $C \in \mathcal{C}$ do not depend on C follows from Proposition 6. □

Example 1 The prism graph below has two types of completely regular cliques, i.e., triangles $\{a, b, c\}$ and $\{d, e, f\}$, and edges $\{a, f\}, \{c, d\}$ and $\{b, e\}$. Parameters are different. If $C = \{a, b, c\}$ and $C' = \{a, f\}$, then $p(\lambda) = \frac{1}{6}\lambda$ and $p'(\lambda) = \frac{1}{6}(\lambda - 1)$.



Example 2 The following graph is 3-regular with 8 vertices. Both edges $\{a, b\}$ and $\{c, d\}$ are completely regular but parameters are different. Note that these two edges do not have a common vertex.



3 Completely regular clique graphs

In this section, we prove Theorem 3. We need the following result.

Proposition 7 (A. Neumaier [13, Theorem 4.1]) *Let $\Gamma = (X, R)$ be a distance-regular graph and let C be a nonempty subset of X with covering radius $t = t(C)$. For $i = 0, 1, \dots, t$, let $\mu_i = |\Gamma_i(x) \cap C|$ and $\lambda_i = |\Gamma_{i+1}(x) \cap C|$ when $x \in \Gamma_i(C)$.*

(i) *C is completely regular if and only if μ_i and λ_i are independent of the choice of $x \in \Gamma_i(C)$.*
(ii) *Suppose C is completely regular with the parameters $\gamma_i, \alpha_i, \beta_i$ with $i \in \{1, 2, \dots, t\}$. Then*

$$\gamma_i = \frac{\mu_i c_i}{\mu_{i-1}}, \qquad \alpha_i = a_i + \frac{\lambda_i c_{i+1}}{\mu_i} - \frac{\lambda_{i-1} c_i}{\mu_{i-1}}, \quad \textit{and}$$
$$\beta_i = b_i - \frac{(\mu_i - \mu_{i-1} - \lambda_{i-1}) c_i}{\mu_{i-1}} - \frac{\lambda_i c_{i+1}}{\mu_i}.$$

Proposition 8 *Let $\widetilde{\Gamma} = (X \cup Y, \widetilde{R})$ be a distance-semiregular graph on X with parameters b_i^X, c_i^X with $i = 0, 1, \dots, d^X$. Let Γ be the bipartite half of $\widetilde{\Gamma}$ on X. For $y \in Y$, write $C_y = \widetilde{\Gamma}(y) \subset X$, and set $\mathcal{C} = \{C_y \mid y \in Y\}$. Then the following hold.*

(i) *Γ is distance-regular.*
(ii) *Each element $C \in \mathcal{C}$ is a clique and a completely regular code in Γ.*
(iii) *Each edge in Γ is contained in c_2^X members of $\mathcal{C}$. In particular, $\Gamma = (X, R)$ is a completely regular clique graph of parameters $(b_0^Y - 1, c_2^X)$.*

Proof (i) This is clear. See [14].

(ii) Since Γ is a distance-2-graph of $\widetilde{\Gamma}$, each C_y is a clique in Γ. Let $C = C_y \in \mathcal{C}$. We apply Proposition 7 to show that C is completely regular. Let $x \in X$ with $\partial_\Gamma(x, C) = i$. Then $\partial_{\widetilde{\Gamma}}(x, y) = 2i + 1$. Hence

$$\mu_i(x) = \left|\Gamma_i(x) \cap C\right| = \left|\widetilde{\Gamma}_{2i}(x) \cap \widetilde{\Gamma}(y)\right| = c_{2i+1}^X.$$

Therefore $\mu_i(x)$ does not depend on the choice of $x \in \Gamma_i(C)$. Since C is a clique, $\lambda_i(x) = |C| - \mu_i(x)$ and $\lambda_i(x)$ does not depend on the choice of $x \in \Gamma_i(C)$ either.

(iii) For each edge $\{x_1, x_2\}$ of Γ there exist c_2^X vertices $y \in Y$ such that $\{x_1, x_2\} \subset C_y$. Thus we have the assertions. □

Let $\Gamma = (X, R)$ be a completely regular clique graph with parameters (s, c) with respect to $\mathcal{C}$. The *incidence graph* of Γ is a bipartite graph $\widetilde{\Gamma} = (X \cup Y, \widetilde{R})$ with vertex set $X \cup Y$, where $Y = \mathcal{C}$, and edge set $\widetilde{R} = \{(x, y) \mid x \in X, y \in Y \text{ such that } x \in y\}$. Let c_i and b_i be parameters of Γ if they exist. Let c_i^X and b_i^X denote the parameters of $\widetilde{\Gamma}$ when the base vertex is in X and they exist. We define c_i^Y and b_i^Y similarly.

Lemma 9 *Let $\widetilde{\Gamma} = (X \cup Y, \widetilde{R})$ be the incidence graph of a completely regular clique graph $\Gamma = (X, R)$ with parameters (s, c) with respect to $\mathcal{C}$. Then the following hold.*

(i) *$\widetilde{\Gamma}$ is biregular of valencies (b_0^X, b_0^Y), where $b_0^X = (\beta_0 + s)c/s$ and $b_0^Y = s + 1$. Moreover, $c_1^X = 1, c_2^X = c, c_3^X = \gamma_1, b_1^X = s, b_2^X = \beta_0 c/s, b_3^X = s + 1 - \gamma_1$, and $b_1^Y = b_0^X - 1$.*
(ii) *Γ is edge regular with $a_1 = (s - 1) + \beta_0(\gamma_1 - 1)/s$.*
(iii) *If $d(\Gamma) > 1$, then Γ is $K_{2,1,1}$-free if and only if $\gamma_1 = 1$ if and only if $a_1 = s - 1$. In this case $c = 1$, and each member of $\mathcal{C}$ is a maximal clique.*

Proof (i) By Lemma 4(ii), Γ is regular of valency $k = b_0 = \alpha_0 + \beta_0$. By definition, we have $b_0^Y = s + 1$, $c_1^X = 1$, $b_1^X = b_0^Y - c_1^X = s \geq 1$, $c_2^X = c$ and $c_3^X = \gamma_1$. We show that b_0^X exists.

Let $x \in X$ and let

$$S = \left\{\left(x', y\right) \in X \times Y \mid \partial_{\tilde{\Gamma}}\left(x, x'\right) = 2,\ \partial_{\tilde{\Gamma}}(x, y) = \partial_{\tilde{\Gamma}}\left(y, x'\right) = 1\right\}.$$

By counting the cardinality of S, we have $|S| = |\widetilde{\Gamma}(x)| b_1^X = b_0 c_2^X$. Hence b_0^X exists and $b_0^X = b_0 c_2^X / b_1^X = kc/s$.

Hence $\widetilde{\Gamma}$ is biregular. Therefore b_i^X exists if and only if c_i^X exists, $b_{2i}^X + c_{2i}^X = b_0^X$, $b_{2i-1}^X + c_{2i-1}^X = b_0^Y$. Since $\alpha_0 = s$,

$$b_2^X = b_0^X - c_2^X = kc/s - c = c(k-s)/s = \beta_0 c/s.$$

The rest follow immediately.

(ii) Since $b_0 = b_0^X b_1^X / c_2^X$ and $b_1 = b_2^X b_3^X / c_2^X = \beta_0(s+1-\gamma_1)/s$, we have

$$a_1 = b_0 - c_1 - b_1 = \beta_0 + s - 1 - \beta_0(s+1-\gamma_1)/s = (s-1) + \beta_0(\gamma_1 - 1)/s.$$

(iii) We first prove that Γ is $K_{2,1,1}$-free if and only if $\gamma_1 = 1$. Suppose Γ is $K_{2,1,1}$-free. Let $y \in X$, $x, z \in \Gamma(y)$ with $\partial(x,z) = 2$ and $\{x, y\} \subset C \in \mathcal{C}$. Since Γ is $K_{2,1,1}$-free, $\gamma_1 = |\Gamma(z) \cap C| = 1$. Conversely, suppose $\gamma_1 = 1$. If x, y, z, w form a $K_{2,1,1}$ with $\partial(z,w) = 2$ and $\{x, y\} \subset C \in \mathcal{C}$, then either z or w is not in C. This contradicts $\gamma_1 = 1$.

By (ii) it is clear that $\gamma_1 = 1$ is equivalent to $a_1 = s - 1$.

Assume these three equivalent conditions. Then clearly $c = 1$ and each member of $\mathcal{C}$ is a maximal clique. This proves (iii). □

Next we show that the incidence graph of a distance-regular completely regular clique graph is distance-semiregular.

Proposition 10 *Let $\Gamma = (X, R)$ be a distance-regular graph of valency k and diameter D. Suppose Γ is a completely regular clique graph with parameters (s, c) with respect to $\mathcal{C}$. Let t be the uniquely determined covering radius and $\gamma_i, \alpha_i, \beta_i$ with $i = 0, 1, \ldots, t$ the parameters of completely regular codes in $\mathcal{C}$. Let $\widetilde{\Gamma} = (X \cup Y, \widetilde{R})$ be its incidence graph with $Y = \mathcal{C}$. Then the following hold.*

(i) *$\widetilde{\Gamma}$ is distance-semiregular on X and Γ is a bipartite half of $\widetilde{\Gamma}$ on X.*
(ii) *The diameter $d(\widetilde{\Gamma}) = 2D$ if $t = t(C) = D - 1$ and $d(\widetilde{\Gamma}) = 2D + 1$ if $t = t(C) = D$ and the parameters of $\widetilde{\Gamma}$ are as follows.*

$$b_0^Y = s+1, \qquad b_0^X = kc/s, \qquad c_1^X = 1, \qquad c_2^X = c,$$
$$c_{2i+1}^X = \frac{\gamma_1\gamma_2\cdots\gamma_i}{c_1 c_2 \cdots c_i}, \qquad c_{2j}^X = \frac{c_1 c_1 \cdots c_{j-1} c_j c}{\gamma_1 \gamma_2 \cdots \gamma_{j-1}}, \quad \text{and}$$
$$b_{2i+1}^X = s + 1 - c_{2i+1}^X, \qquad b_{2j}^X = kc/s - c_{2j}^X$$

for i, j with $0 \le 2i+1, 2j \le d(\widetilde{\Gamma})$.

Proof Let μ_i with $i = 0, 1, \ldots, t$ be the numbers defined in Proposition 7.

(i) For $y \in Y$, a subset $\widetilde{\Gamma}(y)$ of X forms a clique of Γ in $\mathcal{C}$ by definition. We write $C_y = \widetilde{\Gamma}(y)$. Since $\widetilde{\Gamma}$ is biregular by Lemma 9 (i), b_i^X exists if and only if c_i^X exists, and $b_{2i}^X + c_{2i}^X = b_0^X$ and $b_{2i-1}^X + c_{2i-1}^X = b_0^Y$. Moreover, if $b_0^X, b_1^X, \ldots, b_{i-1}^X, c_1^X, c_2^X, \ldots, c_i^X$ exist, $k_i^X = |\widetilde{\Gamma}_i(x)| = (b_0^X b_1^X \cdots b_{i-1}^X)/(c_1^X c_2^X \cdots c_i^X)$ does not depend on the choice of $x \in X$.

Let $y \in Y$ and $x \in \Gamma_i(C_y)$. Then $\partial_{\widetilde{\Gamma}}(x, y) = 2i + 1$ and

$$c_{2i+1}(x, y) = \left|\widetilde{\Gamma}_{2i}(x) \cap \widetilde{\Gamma}(y)\right| = \left|\Gamma_i(x) \cap C_y\right| = \mu_i.$$

Therefore c^X_{2i+1} and hence b^X_{2i+1} exists for all i.

Now suppose $b^X_0, b^X_1, \dots, b^X_{2i+1}, c^X_0, c^X_1, \dots, c^X_{2i+1}$ exist for $i \geq 1$. For $x' \in \Gamma_{i+1}(x)$, by counting the cardinality of the set

$$\begin{aligned} S &= \left\{\left(x'', y\right) \mid y \in \widetilde{\Gamma}_{2i+1}(x) \cap \widetilde{\Gamma}\left(x'\right),\ x'' \in \widetilde{\Gamma}_{2i}(x) \cap \widetilde{\Gamma}(y)\right\} \\ &= \left\{\left(x'', y\right) \mid x'' \in \Gamma_i(x) \cap \Gamma\left(x'\right),\ y \in \widetilde{\Gamma}\left(x'\right) \cap \widetilde{\Gamma}\left(x''\right)\right\} \end{aligned}$$

we have $c_{i+1}c^X_2 = c^X_{2i+1}|\widetilde{\Gamma}_{2i+1}(x) \cap \widetilde{\Gamma}(x')| = c^X_{2i+1}c_{2i+2}(x, x')$. Thus c^X_{2i+2} exists and

$$c^X_{2i+2} = c_{i+1}c^X_2 / c^X_{2i+1}.$$

By induction, we have that $\widetilde{\Gamma}$ is distance-semiregular on X.

(ii) Note that $\mu_0 = 1$. Now by Proposition 7,

$$c^X_{2i+1} = \mu_i = \frac{\gamma_1\gamma_2\cdots\gamma_i}{c_1c_2\cdots c_i}, \quad \text{and} \quad c^X_{2i+2} = \frac{c_{i+1}c^X_2}{c^X_{2i+1}} = \frac{c_1c_1\cdots c_ic_{i+1}c}{\gamma_1\gamma_2\cdots\gamma_i}.$$

Thus we have the assertion. □

Theorem 3 now follows directly from Propositions 8 and 10.

4 Notes

It is useful to define the notion of completely regular clique graphs in terms of an incidence structure.

Definition 2 Let $\mathcal{I} = (X, Y, I)$ be an incidence structure, where X and Y are finite set and I a relation on $X \times Y$. Let $\Gamma = (X, R)$ be the collinearity graph of $\mathcal{I}$ with vertex set X and edge set $R = \{\{x, x'\} \subset X \mid x \neq x'$ and there exists $y \in Y$ such that xIy and $x'Iy\}$. Then $\mathcal{I}$ is said to be a *CRC geometry with parameters* (s, c), if the following are satisfied.

(i) For each $y \in Y$, let $C_y = \{x \in X \mid xIy\}$. Then C_y is a completely regular code of Γ of size $s + 1 \geq 2$.

(ii) For each distinct $x, x' \in X$, $|\{y \in Y \mid xIy$ and $x'Iy\}| \in \{0, c\}$ and $c \geq 1$.

Let $\Gamma = (X, R)$ be one of the Johnson graphs or the Grassmann graphs of diameter D. Then for each $i \in \{0, 1, \dots, D-1\}$ and $x, y \in X$ with $\partial(x, y) = i$ we can choose a geodetically closed completely regular code $C_{x,y}$ of diameter i containing x and y. Define $\mathcal{C}_i = \{C_{x,y} \mid x, y \in X$ with $\partial(x, y) = i\}$ and $\mathcal{C}_{i+1}$ similarly. Then the pair $(\mathcal{C}_i, \mathcal{C}_{i+1})$ defines a CRC geometry when incidence is defined by inclusion.

We conclude this paper by proposing three problems related to completely regular clique graphs.

Problem 1 Prove or disprove that completely regular clique graphs are distance-regular.

Problem 2 Classify completely regular clique graphs with respect to $\mathcal{C}$ such that there is a nonempty collection $\mathcal{C}'$ of completely regular codes of width two such that $(\mathcal{C}, \mathcal{C}', \subset)$ is a CRC geometry when incidence is defined by inclusion.

Problem 3 Characterize connected graphs $\Gamma = (X, R)$ with the following properties.

(i) There is a nonempty collection $\mathcal{C}$ of cliques of size $s+1$ and that for each $C \in \mathcal{C}$ and $x \in C$, $\mathcal{T}(C)\boldsymbol{c}_x$ is a thin irreducible $\mathcal{T}(C)$-module, where $\boldsymbol{c}_x = \hat{x} - \frac{1}{|C|}\sum_{y \in C} \hat{y} \in \boldsymbol{C}^X$. Here $\hat{z}$ is a unit vector in $\boldsymbol{C}^X$ corresponding to a vertex z.
(ii) Each edge is contained in exactly c members of $\mathcal{C}$ and $c \geq 1$.

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