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A construction for infinite families of semisymmetric graphs revealing their full automorphism group

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Abstract We give a general construction leading to different non-isomorphic families $\Gamma_{n,q}(\mathcal{K})$ of connected q-regular semisymmetric graphs of order $2q^{n+1}$ embedded in $\mathrm{PG}(n+1,q)$, for a prime power $q=p^h$, using the *linear representation* of a particular point set $\mathcal{K}$ of size q contained in a hyperplane of $\mathrm{PG}(n+1,q)$. We show that, when $\mathcal{K}$ is a *normal rational curve* with one point removed, the graphs $\Gamma_{n,q}(\mathcal{K})$ are isomorphic to the graphs constructed for $q=p^h$ in Lazebnik and Viglione (J. Graph Theory 41, 249–258, 2002) and to the graphs constructed for q prime in Du et al. (Eur. J. Comb. 24, 897–902, 2003). These graphs were known to be semisymmetric but their full automorphism group was up to now unknown. For $q \geq n+3$ or $q=p=n+2$, $n \geq 2$, we obtain their full automorphism group from our construction by showing that, for an *arc* $\mathcal{K}$, every automorphism of $\Gamma_{n,q}(\mathcal{K})$ is induced by a collineation of the ambient space $\mathrm{PG}(n+1,q)$. We also give some other examples of semisymmetric graphs $\Gamma_{n,q}(\mathcal{K})$ for which not every automorphism is induced by a collineation of their ambient space.

Keywords Semisymmetric graph · Linear representation · Automorphism group · Arc · Normal rational curve

1 Introduction

In the following, all graphs are assumed to be finite and simple, i.e. they are undirected graphs which contain no loops or multiple edges.

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Definition We say that a graph is *vertex-transitive* if its automorphism group acts transitively on the vertices. Similarly, a graph is *edge-transitive* if its automorphism group acts transitively on the edges. A graph is *semisymmetric* if it is regular and edge-transitive but not vertex-transitive (see [10]).

One can easily prove that a semisymmetric graph must be bipartite with equal partition sizes. Moreover, the automorphism group must be transitive on both partition sets. General constructions of semisymmetric graphs are quite rare.

We construct several infinite families $\Gamma_{n,q}(\mathcal{K})$ of semisymmetric graphs using affine points and some selected lines of a projective space $\mathrm{PG}(n+1,q)$. In Sect. 5 we show that the infinite series of semisymmetric graphs given in [19] is exactly one of the families that we construct in this paper; the graphs in [9] are shown to be part of the same series. Using our construction, in many cases, the structure of the full automorphism group of the graphs $\Gamma_{n,q}(\mathcal{K})$ can be clarified (at least for $q \geq n+3$ and $q = p = n+2$). This structure was not given in [9, 19] where only part of the automorphism group is constructed, enough to show edge-transitivity.

2 Construction and properties of the graph $\Gamma_{n,q}(\mathcal{K})$

Let $\mathrm{PG}(n,q)$ denote the n-dimensional projective space over the finite field $\mathbb{F}_q$, $q = p^h$. Throughout this paper we assume $n \geq 2$.

We wish to emphasise the distinction we will make between a *subspace* and a *subgeometry*. A subspace of $\mathrm{PG}(n,q)$ is a projective space $\mathrm{PG}(m,q)$ contained in $\mathrm{PG}(n,q)$, $m \leq n$, over the same finite field $\mathbb{F}_q$. An n-dimensional subgeometry of $\mathrm{PG}(n,q)$ is a projective space $\mathrm{PG}(n,q_0)$ contained in $\mathrm{PG}(n,q)$ over the finite field $\mathbb{F}_{q_0}$ where $\mathbb{F}_{q_0} \subseteq \mathbb{F}_q$.

Definition Let $\mathcal{K}$ be a point set in $H_\infty = \mathrm{PG}(n,q)$ and embed H_∞ in $\mathrm{PG}(n+1,q)$. The *linear representation* $T_n^*(\mathcal{K})$ of $\mathcal{K}$ is a point-line incidence structure with natural incidence, point set $\mathcal{P}$ and line set $\mathcal{L}$ as follows:

$\mathcal{P}$: *affine* points of $\mathrm{PG}(n+1,q)$ (i.e. the points of $\mathrm{PG}(n+1,q) \setminus H_\infty$),
$\mathcal{L}$: lines of $\mathrm{PG}(n+1,q)$ through a point of $\mathcal{K}$, but not lying in H_∞.

For more information on linear representations of geometries, we refer to [8].

Definition We denote the point-line incidence graph of $T_n^*(\mathcal{K})$ by $\Gamma_{n,q}(\mathcal{K})$, i.e. the bipartite graph with classes $\mathcal{P}$ and $\mathcal{L}$ and adjacency corresponding to the natural incidence of the geometry.

Whenever we consider the incidence graph $\Gamma_{n,q}(\mathcal{K})$ of some linear representation $T_n^*(\mathcal{K})$ of $\mathcal{K}$, we still regard the set of vertices as a set of points and lines in $\mathrm{PG}(n+1,q)$. In this way we can use the inherited properties of this space and borrow expressions such as the span of points, a subspace, incidence, and others.

We define the closure of a set of points in $\mathrm{PG}(n,q)$ as follows:

Definition If a point set S contains a frame of $\mathrm{PG}(n,q)$, then its *closure* $\overline{S}$ consists of the points of the smallest n-dimensional subgeometry of $\mathrm{PG}(n,q)$ containing all points of S.

The closure $\overline{S}$ of a point set S can be constructed recursively as follows:

(i) determine the set $\mathcal{A}$ of all subspaces of $\mathrm{PG}(n,q)$ spanned by an arbitrary number of points of S;
(ii) determine the set $\overline{S}$ of points P for which there exist two subspaces in $\mathcal{A}$ that intersect only at P, if $\overline{S} \neq S$ replace S by $\overline{S}$ and go to (i), otherwise stop.

This construction corresponds to the definition of a closure of a set of points in a projective plane in [16, Chap. XI]. Here the authors show that if S is contained in a projective plane and contains a quadrangle, the points of $\overline{S}$ form the smallest subplane containing all points of S.

Result 2.1 [7, Corollary 4.3] The graph $\Gamma_{n,q}(\mathcal{K})$ is connected if and only if the span $\langle\mathcal{K}\rangle$ has dimension n.

Remark Suppose the set $\mathcal{K}$ spans a t-dimensional subspace $\mathrm{PG}(t,q)$ of $H_\infty = \mathrm{PG}(n,q)$, $t < n$. One can check that in this case the graph $\Gamma_{n,q}(\mathcal{K})$ is a non-connected graph with q^{n-t} connected components, where each component is isomorphic to the graph $\Gamma_{t,q}(\mathcal{K})$. This explains why we will only consider graphs $\Gamma_{n,q}(\mathcal{K})$ with set $\mathcal{K}$ such that $\langle\mathcal{K}\rangle = H_\infty$.

Throughout this paper, we use the following theorems of [4].

Result 2.2 [4] Let $|\mathcal{K}| \neq q$ or let $\mathcal{K}$ be a set of q points of H_∞ such that every point of $H_\infty \setminus \mathcal{K}$ lies on at least one tangent line to $\mathcal{K}$. Suppose α is an isomorphism between $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{n,q}(\mathcal{K}')$, for some set $\mathcal{K}'$ in H_∞, then α stabilises $\mathcal{P}$.

Result 2.3 [4] Let $q > 2$. Let $\mathcal{K}$ and $\mathcal{K}'$ be sets of q points such that $\overline{\mathcal{K}}$ is equal to H_∞ and such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$. Consider an isomorphism α between $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{n,q}(\mathcal{K}')$. Then α is induced by an element of the stabiliser $\mathrm{P\Gamma L}(n+2,q)_{H_\infty}$ of H_∞ mapping $\mathcal{K}$ onto $\mathcal{K}'$.

Result 2.4 [4] Let $q > 2$ and let $\mathcal{K}$ be a set of q points such that $\overline{\mathcal{K}}$ is equal to H_∞ and such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$. Then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$.

From Result 2.2 we now easily deduce the following corollary.

Corollary 2.5 *If $\mathcal{K}$ is a set of q points of H_∞ such that every point of H_∞ lies on at least one tangent line to $\mathcal{K}$, then $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive.*

Recall that if a group G has a normal subgroup N and the quotient G/N is isomorphic to some group H, we say that G is an *extension* of N by H. This is written as $G = N.H$.

An extension $G = N.H$ which is a semidirect product is also called a *split extension*. This means that one can find a subgroup $\overline{H} \cong H$ in G such that $G = N\overline{H}$ and $N \cap \overline{H} = \{e_G\}$ and is denoted by $G = N \rtimes H$.

A *perspectivity* is an element of $\mathrm{P\Gamma L}(n+2, q)$ which fixes a hyperplane of $\mathrm{PG}(n+1, q)$ pointwise, this hyperplane is called the axis. Every perspectivity also has a centre, i.e. a point such that every line through it is stabilised. If this centre belongs to the axis, such a perspectivity is called an *elation*.

The subgroup of $\mathrm{P\Gamma L}(n+2, q)$ consisting of all perspectivities with axis H_∞ is written as $\mathrm{Persp}(H_\infty)$. A perspectivity ϕ is uniquely determined by its axis and centre and one ordered pair $(P, \phi(P))$ for a point P different from the centre and not on the axis. Hence, one can easily count $|\mathrm{Persp}(H_\infty)| = q^{n+1}(q-1)$.

Result 2.6 [4] Let $\mathcal{K}$ be a point set spanning $H_\infty = \mathrm{PG}(n, q)$. If the setwise stabilisers $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ and $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$, respectively, fix a point of H_∞, then $\mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ and $\mathrm{PGL}(n+2, q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{PGL}(n+1, q)_{\mathcal{K}}$, respectively.

Result 2.7 [4] Let $\mathcal{K}$ be a point set spanning $H_\infty = \mathrm{PG}(n, q)$, $q = p^h$. If the setwise stabiliser $\mathrm{PGL}(n+1, q)_{\mathcal{K}}$ fixes a point of H_∞, and $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1, q)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_{q_0})$, for $q_0 = p^{h_0}$, $h_0 | h$ or $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1, q)_{\mathcal{K}}$, then $\mathrm{P\Gamma L}(n+2, q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$.

The following theorem is easy to prove. We will use it to show the edge-transitivity of the constructed graphs.

Theorem 2.8 *If the stabiliser* $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ *of* $\mathcal{K}$ *in the full collineation group of* H_∞ *acts transitively on the points of* $\mathcal{K}$, *then* $\Gamma_{n,q}(\mathcal{K})$ *is an edge-transitive graph.*

Proof Consider two edges (R_i, L_i), $i = 1, 2$, where $R_i \in \mathcal{P}$, $L_i \in \mathcal{L}$, $R_i \in L_i$, we will construct a mapping from one edge to the other. Let P_i be $L_i \cap H_\infty$. Since $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ acts transitively on $\mathcal{K}$, we may take an element β of $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ such that $\beta(P_1) = P_2$. This element extends to an element β' of $(\mathrm{P\Gamma L}(n+2, q)_{H_\infty})_{\mathcal{K}}$ mapping P_1 onto P_2.

If $\beta'(R_1) = R_2$, then $\beta'(L_1) = L_2$, hence the statement follows. If $\beta'(R_1) \neq R_2$, then let S be the point at infinity of the line $\beta'(R_1)R_2$. There is a (unique) elation γ with centre S and axis H_∞ mapping $\beta'(R_1)$ to R_2. This elation maps $\beta'(L_1)$ onto L_2. Since $\gamma \circ \beta'$ is an element of $(\mathrm{P\Gamma L}(n+2, q)_{H_\infty})_{\mathcal{K}}$ mapping (R_1, L_1) onto (R_2, L_2), the statement follows. □

The main goal of this paper is the construction of infinite families of semisymmetric graphs. The results of [4] introduced in this section will enable us in some cases to explicitly describe the automorphism group of the constructed graphs. Note that, since a semisymmetric graph is regular, any graph $\Gamma_{n,q}(\mathcal{K})$ that is semisymmetric, necessarily has $|\mathcal{K}| = q$. For this reason, we will investigate point sets of size q in $\mathrm{PG}(n, q)$. Considering Theorem 2.8 we need point sets $\mathcal{K}$ of H_∞ such that $\mathrm{P\Gamma L}(n+1, q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$.

In Sect. 3, considering Result 2.4, we will look for point sets $\mathcal{K}$ such that the closure $\overline{\mathcal{K}}$ is equal to H_∞. In Sect. 4 we will look for point sets $\mathcal{K}$ spanning H_∞ such that the closure $\overline{\mathcal{K}}$ is equal to a subgeometry of H_∞.

We give a brief overview of all constructions to come. We use the abbreviation NRC for a normal rational curve, for its definition see Sect. 3. When the size of the automorphism group is given, all automorphisms are geometric, i.e. induced by a collineation of the ambient space. If the size is larger than a given bound, this means there exist automorphisms that are not geometric.

$\mathcal{K}$	Condition	$\lvert\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))\rvert$	Reference
Basis	$q=n+1$	$hq^{n+1}(q-1)q!$	Section 3.1
Frame	$q=n+2$	$hq^{n+1}(q-1)^n q!$	Section 3.1, [19]
$\subset$NRC	$q\geq n+3$	$hq^{n+2}(q-1)^2$	Section 3.2, [9, 19] ($q=p$)
$\subset$non-classical arc	$q>4$ even	$hq^5(q-1)^2$	Section 3.3
$\subset$Glynn-arc	$q=9$	$9^6 8^2$	Section 3.4
$\subset Q^-(3,q)$	$q>4$ square	$>2hq^5(q-1)^2$	Section 4.1
$\subset$Tits-ovoid	$q=2^{2(2e+1)}$	$>hq^5(q-1)(\sqrt{q}-1)$	Section 4.2
$\subset Q^+(3,q)$	$q>4$ square	$>2hq^5(q-1)(\sqrt{q}-1)^2$	Section 4.3
$\subset$cone $V\mathcal{O}$	$q={q_0}^h$	$>hq^{2n+1}(q-1)^2\lvert\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}\rvert$	Section 4.4

3 Families of semisymmetric graphs arising from arcs

We are in search of point sets $\mathcal{K}$ such that the closure $\overline{\mathcal{K}}$ is equal to H_∞ and such that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$. An arc of size q turns out to be an excellent choice.

Definition A *k-arc* in $\mathrm{PG}(n,q)$ is a set of k points, $k\geq n+1$, such that no $n+1$ points lie on a hyperplane.

If $\mathcal{A}$ is a k-arc in $\mathrm{PG}(n,q)$, then $k\geq n+1$, hence, we will only consider the case where $q\geq n+1$. If $q=n+1$, then it is easy to see that an arc of size q in $\mathrm{PG}(n,q)$ is a basis, if $q=n+2$, then every arc of size q is a frame. Hence, when $q=n+1$ or $q=n+2$, all arcs of size q in $\mathrm{PG}(n,q)$ are $\mathrm{P\Gamma L}$-equivalent. Because of the isomorphism of the graph $\Gamma_{n,q}(\mathcal{K})$ with other graphs (see Sect. 5), we will explicitly investigate these cases, but the more interesting examples occur when $q\geq n+3$.

It is conjectured that an arc in $\mathrm{PG}(n,q)$, $3\leq n\leq q-3$, has at most $q+1$ points (this is the well-known MDS-conjecture, in view of its coding-theoretical description). An example of an arc of size $q+1$ is given by the *normal rational curve*.

Definition [15, Sect. 27.5] A *normal rational curve* in $\mathrm{PG}(n,q)$, $2\leq n\leq q$, is a $(q+1)$-arc PGL-equivalent to the $(q+1)$-arc $\{(0,\ldots,0,1)\}\cup\{(1,t,t^2,t^3,\ldots,t^n)\mid t\in\mathbb{F}_q\}$

Remark There are results showing that, if n is sufficiently large w.r.t. q, an arc of size q in $\mathrm{PG}(n,q)$ can be extended to an arc of size $q+1$. Moreover, other results show that for many values of q and n, all $(q+1)$-arcs in $\mathrm{PG}(n,q)$ are normal rational curves. The combination of these results leads to the understanding why there are not many known examples of q-arcs in $\mathrm{PG}(n,q)$ that are not contained in a normal rational curve. For an overview, we refer to [14].

We will construct different families of graphs, arising from non-PΓL-equivalent arcs of size q. Since these arcs satisfy the conditions of Result 2.3, we see that the obtained graphs are non-isomorphic.

In view of Result 2.4, our first goal is to show that the closure of a set of q points of an arc in $\mathrm{PG}(n,q)$, $q \geq n+3$ or $q=p=n+2$ prime, is H_∞. When $n=2$, this follows immediately. In the following lemmas, we deal with the case $n \geq 3$.

Lemma 3.1 *Let $\mathcal{K}$ be an arc of size q in* $\mathrm{PG}(n,q)$, $n \geq 3$. *Let P_1 and P_2 be any two points of $\mathcal{K}$;*

- *if $q=n+2$, there is at least one additional point in $\overline{\mathcal{K}}$ (the closure of $\mathcal{K}$) on the line P_1P_2,*
- *if $q \geq n+3$, there are at least $q/2$ additional points in $\overline{\mathcal{K}}$ on the line P_1P_2.*

Proof Note that a k-space π, $k \leq n-2$, with $k+1$ points of $\mathcal{K}$, different from P_1 and P_2, does not intersect P_1P_2, since otherwise $\langle \pi, P_1P_2 \rangle$ would be a $(k+1)$-space containing $k+3$ points of $\mathcal{K}$, contradicting the arc condition.

Let $P_3, \ldots, P_{n+2}$ be n points of $\mathcal{K}$, different from P_1 and P_2. The space $\langle P_3, \ldots, P_{n+2} \rangle$ is a hyperplane of H_∞, hence, it meets the line P_1P_2 in a point Q. This point Q is contained in $\overline{\mathcal{K}}$ but not contained in $\mathcal{K}$ since $\mathcal{K}$ is an arc. If $q=n+2$, there is exactly one set $\{P_3, \ldots, P_{n+2}\}$ of n points of $\mathcal{K}$, different from P_1 and P_2, yielding an extra point in $\overline{\mathcal{K}}$ on P_1P_2.

If $n+3 \leq q \leq 2n+2$, then let $\{P_3, \ldots, P_{n+3}\}$ be a set of $n+1$ points of $\mathcal{K}$, different from P_1 and P_2. Any subset with n points of $\{P_3, \ldots, P_{n+3}\}$ defines a hyperplane intersecting P_1P_2 in a point $Q \neq P_1, P_2$ contained in $\overline{\mathcal{K}}$. These points Q are all different since any two considered hyperplanes intersect in a $(n-2)$-space with $n-1$ points of $\mathcal{K}$, and hence this space does not intersect P_1P_2. There are $n+1$ such subsets, so the line P_1P_2 contains $q/2 \leq n+1 \leq q-2$ additional points in $\overline{\mathcal{K}}$ different from P_1 and P_2.

If $q \geq 2n+2$, then let $P_3, \ldots, P_{n+1}$ be $n-1$ points of $\mathcal{K}$, different from P_1 and P_2. Clearly, $\langle P_3, \ldots, P_{n+1} \rangle$ is disjoint from P_1P_2. There are $q-n-1$ points of $\mathcal{K}$ different from all P_i, $i=1, \ldots, n+1$. For every such point R, the hyperplane $\langle P_3, \ldots, P_{n+1}, R \rangle$ intersects P_1P_2 in a point of $\overline{\mathcal{K}}$ different from P_1 and P_2. Again, all these points are different since two such hyperplanes intersect in $\langle P_3, \ldots, P_{n+1} \rangle$. The line P_1P_2 contains $q-n-1 \geq q/2$ points of $\overline{\mathcal{K}}$ different from P_1 and P_2. □

Lemma 3.2 *Let $\mathcal{K}$ be an arc of size q in* $\mathrm{PG}(n,q)$, $n \geq 2$. *Let $q \geq n+3$ or $q=p=n+2$ and let μ_∞ be a plane containing 3 points of $\mathcal{K}$. Then every point of μ_∞ is contained in $\overline{\mathcal{K}}$.*

Proof Let P_1, P_2, P_3 be 3 points of $\mathcal{K}$ and let μ_∞ be the plane $\langle P_1, P_2, P_3\rangle = \mathrm{PG}(2,q)$. Consider $q \geq n+3$. By Lemma 3.1, we know that there exist at least $q/2$ points in $\overline{\mathcal{K}}$ on each of the lines P_2P_3, P_1P_3 and P_1P_2, different from P_1, P_2 and P_3. Consider the set S containing all these points and the points P_1, P_2 and P_3. Its closure $\overline{S}$ forms a subplane π of μ_∞ consisting of only points of $\overline{\mathcal{K}}$. Since a proper subplane of $\mathrm{PG}(2,q)$ contains at most $\sqrt{q}+1 < q/2+2$ points of the line P_1P_2, we see that π must be μ_∞.

If $q = n+2$ is prime, by Lemma 3.1, we find an extra point $Q_i \in \overline{\mathcal{K}}$, $i = 2, 3$, on the line P_1P_i. The closure of $\{P_1, P_2, P_3, Q_2, Q_3\}$ forms a subplane with all points in $\overline{\mathcal{K}}$. By the fact that q is prime, this subplane equals $\mu_\infty = \mathrm{PG}(2,q)$. □

Lemma 3.3 *Let L be a line such that every point is in $\overline{\mathcal{K}}$, let π_∞ be a plane of H_∞ through L, containing at least two points R_1 and R_2 of $\overline{\mathcal{K}}$ outside L. Then every point in the plane π_∞ is in $\overline{\mathcal{K}}$.*

Proof The closure of the set of points of $\overline{\mathcal{K}}$ on the line L, together with the points R_1 and R_2 is clearly the plane π_∞ itself. □

Lemma 3.4 *For $n \geq 2$, let $q \geq n+3$ or $q = p = n+2$ and let $\mathcal{K}$ be an arc of size q in* $\mathrm{PG}(n,q)$, *then* $\overline{\mathcal{K}} = \mathrm{PG}(n,q)$.

Proof For $n = 2$, this easily follows. Let $P_1, \ldots, P_q$ be the points of $\mathcal{K}$. By Lemma 3.2, we know that every point of $\langle P_1, P_2, P_3\rangle$ is in $\overline{\mathcal{K}}$. Suppose, by induction, that every point in $\langle P_1, \ldots, P_k\rangle$, $k \leq n$ is in $\overline{\mathcal{K}}$. The point P_{k+1} is not contained in $\langle P_1, \ldots, P_k\rangle$. There exists an additional point Q in $\overline{\mathcal{K}}$ on the line P_1P_{k+1} by Lemma 3.1. Let S be a point of $\langle P_1, \ldots, P_{k+1}\rangle$, not on the line P_1P_{k+1}, and let R be the intersection of the line SP_{k+1} with $\langle P_1, \ldots, P_k\rangle$. Since every point on the line RP_1 is in $\overline{\mathcal{K}}$, and $\langle RP_1, P_{k+1}\rangle$ contains the points Q and P_{k+1} of $\overline{\mathcal{K}}$, Lemma 3.3 implies that the point S is in $\overline{\mathcal{K}}$, as are the points of P_1P_{k+1}. This shows that every point in $\langle P_1, \ldots, P_{k+1}\rangle$ is in $\overline{\mathcal{K}}$. The lemma follows by induction and the fact that $H_\infty = \langle P_1, \ldots, P_{n+1}\rangle$. □

Theorem 3.5 *Let $n = 2$ and q odd or $n \geq 3$, $q \geq n+3$ or $q = p = n+2$, and let $\mathcal{K}$ be an arc in* $\mathrm{PG}(n,q)$, *then* $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$.

Proof It is clear that if $n = 2$, q odd or $n \geq 3$, then every point of H_∞ lies on a tangent line to the arc. By Lemma 3.4, $\overline{\mathcal{K}}$ equals $\mathrm{PG}(n,q)$. The theorem follows from Result 2.4. □

3.1 $\mathcal{K}$ is a q-arc in $\mathrm{PG}(n,q)$ with $q = n+1$ or $q = n+2$

As noted before, a q-arc in $\mathrm{PG}(n,q)$ with $q = n+1$ is a basis, a q-arc in $\mathrm{PG}(n,q)$ with $q = n+2$ is a frame. In these cases, the linear representation of a q-arc gives rise to a semisymmetric graph, however, the description of the automorphism group is different from the case $q \geq n+3$. In the following proof, we cannot use the same techniques as in [4] to show that $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ splits over $\mathrm{Persp}(H_\infty)$.

We introduce some definitions.

Definition A *permutation matrix* is a square binary matrix that has exactly one entry 1 in each row and each column and 0's elsewhere. A *monomial matrix* or generalised permutation matrix has exactly one non-zero entry in each row and each column. The monomial matrices form a group.

Let $\mathrm{PMon}(q)$ denote the quotient group of the monomial matrices over $\mathbb{F}_q$ by the scalar matrices. Let S_k denote the *symmetric group* of degree k, meaning the group of all permutations of $\{1, 2, \ldots, k\}$.

Theorem 3.6 *If $\mathcal{K}$ is a q-arc in $\mathrm{PG}(n,q)$, $n \geq 2$, $q = n+1$ or $q = n+2$, with $(n,q) \neq (2,4)$ then $\Gamma_{n,q}(\mathcal{K})$ is a semisymmetric graph. The group $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ is a subgroup of $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ and is isomorphic to $\mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, where $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ is isomorphic to*

(i) $\mathrm{S}_q \rtimes \mathrm{Aut}(\mathbb{F}_q)$ *if $q = n+2$, having size $hq^{n+1}(q-1)q!$;*
(ii) $\mathrm{PMon}(q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ *if $q = n+1$, having size $hq^{n+1}(q-1)^n q!$.*

Moreover, if $q = n+2$ and q is prime, then $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ is isomorphic to $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$.

Proof (i) If $q = n+2$, then $\mathcal{K}$ is PGL-equivalent to the frame $\mathcal{K}'$ of $\mathrm{PG}(n,q)$ with points $P_1, \ldots, P_{n+2}$, where P_i has coordinates v_i, and $v_1 = (1,0,\ldots,0)$, $v_2 = (0,1,0,\ldots,0), \ldots, v_{n+1} = (0,\ldots,0,1)$, $v_{n+2} = (-1,-1,\ldots,-1)$. Let $B_k = (b_{ij})_k$, $1 \leq k \leq n+1$, be the matrix with $b_{ii} = 1$, $i \neq k$, $1 \leq i \leq n+1$, $b_{ik} = -1$, $1 \leq i \leq n+1$, and $b_{ij} = 0$ for all other i, j. The considered action of B_k on the points of $\mathrm{PG}(n,q)$ is by left-multiplication on the column vector of their coordinates. Let G_{per} denote the subgroup of permutation matrices of $\mathrm{GL}(n+1,q)$, and consider the subgroup G of $\mathrm{GL}(n+1,q)$, generated by the elements of G_{per} and the matrices B_k, $1 \leq k \leq n+1$.

For every matrix $B = (b_{ij})$, $1 \leq i,j \leq n+1$, in G, we can define a matrix $A = (a_{ij})$, $0 \leq i,j \leq n+1$, as the $(n+2) \times (n+2)$ matrix with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i,j \geq 1$ and $a_{ij} = b_{ij}$ for $1 \leq i,j \leq n+1$. Let $\widetilde{G}$ be the group obtained by extending all matrices of G in this way. It is clear that the elements of G are exactly the permutations of the elements of $\{v_1, \ldots, v_{n+2}\}$ and hence that $\widetilde{G}$ is isomorphic to $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{S}_q$.

It follows that the only element of $\widetilde{G}$ fixing $\mathcal{K}$ pointwise corresponds to the identity matrix, which implies that any element of $\mathrm{Persp}(H_\infty)$ contained in $\widetilde{G}$ is trivial. Hence, $\mathrm{PGL}(n+2,q)_{\mathcal{K}}$ is isomorphic to $\mathrm{Persp}(H_\infty) \rtimes \mathrm{PGL}(n+1,q)_{\mathcal{K}}$. Clearly, $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$, hence by Theorem 2.8 the graph $\Gamma_{n,q}(\mathcal{K})$ is edge-transitive.

(ii) Now suppose $q = n+1$. The group $\mathrm{PSL}(n+1,q)$ is a subgroup of $\mathrm{PGL}(n+1,q)$ and a quotient of $\mathrm{SL}(n+1,q)$. When $q = n+1$, all three groups have the same order and thus are all isomorphic. Hence, $\mathrm{PGL}(n+1,q)$ can be embedded in $\mathrm{PGL}(n+2,q)_{H_\infty}$ by taking all matrices $B = (b_{ij})$, $1 \leq i,j \leq n+1$, of $\mathrm{SL}(n+1,q)$ and, as before, defining $A = (a_{ij})$, $0 \leq i,j \leq n+1$, with $a_{00} = 1$, $a_{i0} = a_{0j} = 0$ for $i,j \geq 1$ and $a_{ij} = b_{ij}$ for $1 \leq i,j \leq n+1$. An element of $\mathrm{Persp}(H_\infty)$ corresponds to a matrix of the form $D = (d_{ij})$, $0 \leq i,j \leq n+1$, with $d_{0j} = \lambda_j$, $0 \leq j \leq n+1$, $d_{ii} = \mu$, $1 \leq i \leq n+1$, for some $\lambda_j, \mu \in \mathbb{F}_q$, and $d_{ij} = 0$ otherwise. This implies that the group $\widetilde{G}$ of matrices A defined in this way meets $\mathrm{Persp}(H_\infty)$ trivially. Hence, $\mathrm{PGL}(n+2,q)_\mathcal{K}$ is isomorphic to $\mathrm{Persp}(H_\infty) \rtimes \mathrm{PGL}(n+1,q)_\mathcal{K}$.

Since $q = n+1$, the curve $\mathcal{K}$ is PGL-equivalent to the set $\mathcal{K}'$ of points $P_1, \ldots, P_{n+1}$ in $\mathrm{PG}(n,q)$, where P_i has coordinates v_i, and $v_1 = (1,0,\ldots,0)$, $v_2 = (0,1,0,\ldots,0)$, $\ldots$, $v_{n+1} = (0,\ldots,0,1)$. Using this, it is clear that $\mathrm{PGL}(n+1,q)_\mathcal{K}$ is isomorphic to the quotient group of monomial matrices by scalar matrices and that $\mathrm{PGL}(n+1,q)_\mathcal{K}$ acts transitively on $\mathcal{K}$. Hence, $\Gamma_{n,q}(\mathcal{K})$ is an edge-transitive graph.

In both cases, it is clear that $\mathcal{K}'$ is stabilised by the Frobenius automorphism, hence, using Result 2.6, it also follows that $\mathrm{P\Gamma L}(n+2,q)_\mathcal{K} \cong \mathrm{Persp}(H_\infty) \rtimes (\mathrm{PGL}(n+1,q)_\mathcal{K} \rtimes \mathrm{Aut}(\mathbb{F}_q))$. The observation on the sizes follows from $|\mathrm{S}_q| = q!$ and $|\mathrm{PMon}| = |\mathrm{S}_q|.|(\mathbb{F}_q^*)^n|/(q-1) = q!(q-1)^{n-1}$.

Since through every point of H_∞ there is a tangent line to $\mathcal{K}$, Corollary 2.5 shows that $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive. Since $\mathcal{K}$ spans H_∞ and $|\mathcal{K}| = q$, we get that $\Gamma_{n,q}(\mathcal{K})$ is semisymmetric.

The last part of the statement follows from Theorem 3.5. □

Remark For $n = 2, q = 3$, and $\mathcal{K}$ a basis is $\mathrm{PG}(2,3)$, we have shown, by using the computer program GAP [12], that all automorphisms are induced by a collineation of $\mathrm{PG}(3,3)$ so we have that the automorphism group of $\Gamma_{2,3}(\mathcal{K})$ is isomorphic to $\mathrm{P\Gamma L}(4,3)_\mathcal{K}$. For $n = 3, q = 4$, however, again using the computer, we find that $[\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) : \mathrm{P\Gamma L}(n+2,q)_\mathcal{K}] = 8$. This implies that there exist automorphisms of the graph $\Gamma_{3,4}(\mathcal{K})$ that are not collineations of $\mathrm{PG}(4,4)$. For $n = 4, q = 5$, this index is already 7776. This might indicate that the general description of the full automorphism group of $\Gamma_{n,q}(\mathcal{K})$, with $n+1 = q$ and $\mathcal{K}$ a basis, is a hard problem.

3.2 $\mathcal{K}$ is contained in a normal rational curve and $q \geq n+3$

We will use the following theorem by Segre.

Result 3.7 [22] If $q \geq n+2$ and S is a set of $n+3$ points in $\mathrm{PG}(n,q)$, no $n+1$ of which lie in a hyperplane, then there is a unique normal rational curve in $\mathrm{PG}(n,q)$ containing the points of S.

Corollary 3.8 *If $\mathcal{K}$ is a set of q points of a normal rational curve $\mathcal{N}$ in* $\mathrm{PG}(n,q)$, $q \geq n+3$, *then $\mathcal{N}$ is the unique normal rational curve through the points of $\mathcal{K}$.*

The following theorem is well known, a proof can be found in e.g. [15, Theorem 27.5.3].

Result 3.9 If $q \geq n+2$ and $\mathcal{N}$ is a normal rational curve in $\mathrm{PG}(n,q)$, then the stabiliser of $\mathcal{N}$ in $\mathrm{P\Gamma L}(n+1,q)$ is isomorphic to $\mathrm{P\Gamma L}(2,q)$ (in its faithful action on $q+1$ points).

These results enable us to give a construction for the following infinite two-parameter family of semisymmetric graphs.

The subgroup of $\mathrm{P\Gamma L}(2,q)$ fixing one point in its natural action is isomorphic to the affine semilinear group $\mathrm{A\Gamma L}(1,q)$ in dimension 1.

Theorem 3.10 *If $\mathcal{K}$ is a set of q points, contained in a normal rational curve of* $\mathrm{PG}(n,q)$, $q=p^h$, $n\geq 3$, $q\geq n+3$, *or* $n=2$, *q odd, then $\Gamma_{n,q}(\mathcal{K})$ is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}(H_\infty)\rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^{n+2}(q-1)^2$.

Proof Since $|\mathcal{K}|=q$, the graph $\Gamma_{n,q}(\mathcal{K})$ is q-regular. The set $\mathcal{K}$ is an arc spanning the space $\mathrm{PG}(n,q)$. It is clear that if $n\geq 3$, or if q is odd, every point of $\mathrm{PG}(n,q)$ lies on at least one tangent line to $\mathcal{K}$. Hence, by Result 2.1, Corollary 2.5 and Theorem 3.5, $\Gamma_{n,q}(\mathcal{K})$ is a connected non-vertex-transitive graph for which $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K}))\cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$. By Corollary 3.8, $\mathcal{K}$ extends by a point P to a unique normal rational curve $\mathcal{N}$. Since P must be fixed by the stabiliser of $\mathcal{K}$ and $\mathrm{P\Gamma L}(2,q)_P\cong \mathrm{A\Gamma L}(1,q)$, we get $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}\cong \mathrm{Persp}(H_\infty)\rtimes \mathrm{A\Gamma L}(1,q)$, by Result 2.6. The size of this group follows when considering that $|\mathrm{Persp}(H_\infty)|=q^{n+1}(q-1)$ and $|\mathrm{A\Gamma L}(1,q)|=hq(q-1)$. By Theorem 2.8 the graph $\Gamma_{n,q}(\mathcal{K})$ is edge-transitive and thus semisymmetric. □

3.3 $\mathcal{K}$ is contained in a non-classical arc in $\mathrm{PG}(3,q)$, q even

The $(q+1)$-arcs in $\mathrm{PG}(3,q)$, q even, have been classified, each of them has the same stabiliser group as the normal rational curve.

Result 3.11 [6] In $\mathrm{PG}(3,q)$, $q=2^h$, $h>2$, every $(q+1)$-arc is PGL-equivalent to some $\mathcal{C}(\sigma)=\{(1,x,x^\sigma,x^{\sigma+1})\mid x\in\mathbb{F}_q\}\cup\{(0,0,0,1)\}$ where σ is a generator of $\mathrm{Aut}(\mathbb{F}_q)$.

Result 3.12 [20] In $\mathrm{PG}(3,q)$, $q=2^h$, $h>2$, the stabiliser of $\mathcal{C}(\sigma)$ in $\mathrm{P\Gamma L}(4,q)$ is isomorphic to $\mathrm{P\Gamma L}(2,q)$ (in its faithful action on $q+1$ points).

The case $q=4$ is already discussed in Sect. 3.1.

Result 3.13 [5] For any k-arc of $\mathrm{PG}(3,q)$, $q=2^h$, $h>1$, we have $k\leq q+1$.

Result 3.14 [3] Let $\mathcal{K}$ be any k-arc in $\mathrm{PG}(3,q)$, $q=2^h$. If $k>(q+4)/2$, then $\mathcal{K}$ is contained in a unique complete arc.

Corollary 3.15 *Consider a $(q+1)$-arc $\mathcal{C}(\sigma)$ of* $\mathrm{PG}(3,q)$, $q=2^h$, $h>2$. *If $\mathcal{K}$ is a set of q points contained in $\mathcal{C}(\sigma)$, then there is a unique $(q+1)$-arc through the points of $\mathcal{K}$, namely $\mathcal{C}(\sigma)$.*

Proof Using Result 3.14, since $q>(q+4)/2$, $q>4$, we find a unique complete arc through $\mathcal{K}$. This arc has size at most $q+1$ by Result 3.13 and thus is equal to $\mathcal{C}(\sigma)$. □

Theorem 3.16 *If $\mathcal{K}$ is a set of q points contained in any $(q+1)$-arc of* $\mathrm{PG}(3,q)$, $q\geq 8$ *even, then $\Gamma_{3,q}(\mathcal{K})$ is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{3,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}(H_\infty)\rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^5(q-1)^2$.

Proof The proof goes in exactly the same way as the proof of Theorem 3.10, by making use of Corollary 3.15 and Results 3.11 and 3.12. □

3.4 $\mathcal{K}$ is contained in the Glynn arc in PG(4, 9)

In [11] Glynn constructs an example of an arc of size 10 in PG(4, 9), which is not a normal rational curve. We call this 10-arc the *Glynn arc (of size 10)*. He also shows that an arc in PG(4, 9) of size 10 is a normal rational curve or a Glynn arc.

Result 3.17 [11] The stabiliser in $\mathrm{P\Gamma L}(5,9)$ of the Glynn arc of size 10 in PG(4, 9) is isomorphic to PGL(2, 9).

Result 3.18 [2] A k-arc in $\mathrm{PG}(n,q)$, $n \geq 3$, q odd and $k \geq \frac{2}{3}(q-1)+n$ is contained in a unique complete arc of $\mathrm{PG}(n,q)$.

Corollary 3.19 *If* $\mathcal{K}$ *is a set of* 9 *points contained in a Glynn* 10*-arc* $\mathcal{C}$ *of* PG(4, 9), *then* $\mathcal{K}$ *is contained in a unique* 10*-arc, namely* $\mathcal{C}$.

Theorem 3.20 *If* $\mathcal{K}$ *is a* 9*-arc contained in a Glynn* 10*-arc of* PG(4, 9), *then* $\Gamma_{4,9}(\mathcal{K})$ *is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{4,9}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes \mathrm{AGL}(1,9)$ *and has size* $9^6 8^2$.

Proof Since $|\mathcal{K}| = 9$, $\Gamma_{4,9}(\mathcal{K})$ is a 9-regular graph. The set $\mathcal{K}$ is an arc spanning the space PG(4, 9). It is clear that every point of PG(4, 9) lies on at least one tangent line to $\mathcal{K}$. Hence, by Result 2.1, Corollary 2.5 and Theorem 3.5, $\Gamma_{4,9}(\mathcal{K})$ is a connected non-vertex-transitive graph for which $\mathrm{Aut}(\Gamma_{4,9}(\mathcal{K})) \cong \mathrm{P\Gamma L}(6,9)_{\mathcal{K}}$. By Corollary 3.19, $\mathcal{K}$ extends by a point P to a unique Glynn 10-arc $\mathcal{C}$. By Result 2.6 we have $\mathrm{P\Gamma L}(6,9)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(5,9)_{\mathcal{K}}$. Since $\mathrm{PGL}(2,9)_P \cong \mathrm{AGL}(1,9)$, we find $\mathrm{P\Gamma L}(6,9)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{AGL}(1,9)$. As before, the size easily follows. By Theorem 2.8 the graph $\Gamma_{4,9}(\mathcal{K})$ is edge-transitive and thus semisymmetric. □

3.5 Using the dual arc construction

Let $\mathcal{K} = \{P_1, \ldots, P_k\}$ be a k-arc in $\mathrm{PG}(n,q)$, $k \geq n+4$. Consider the respective coordinates $(a_{0j}, \ldots, a_{nj})$ of P_j, $1 \leq j \leq k$, then the $(n+1) \times k$-matrix $A = (a_{ij})$ determines a vector space (an MDS code) $V_1 = V(n+1,q)$, which is a subspace of $V(k,q)$. The space V_1 has a unique orthogonal complement $V_2 = V(k-n-1,q)$ in $V(k,q)$. Then V_2 is also an MDS code [21, p. 319]. A k-arc $\hat{\mathcal{K}} = \{Q_l, \ldots, Q_k\}$ of $\mathrm{PG}(k-n-2,q)$ with respective coordinates $(b_{0j}, \ldots, b_{k-n-2,j})$ of Q_j, $1 \leq j \leq k$, such that the $(k-n-1) \times k$-matrix $B = (b_{ij})$ generates V_2, is called a *dual* k*-arc* $\hat{\mathcal{K}}$ of the k-arc $\mathcal{K}$ [27].

It should be noted that duality for arcs is a $1-1$-correspondence between *equivalence classes* of arcs, rather than a correspondence between arcs: with another ordering of $\mathcal{K}$ and choosing other coordinates for the points of $\mathcal{K}$, we obtain the same set of dual k-arcs.

Result 3.21 [26, Theorem 2.1] A k-arc $\mathcal{K}$ in $\mathrm{PG}(n,q)$, $k \geq n+4$, and a dual k-arc $\hat{\mathcal{K}}$ of $\mathcal{K}$ in $\mathrm{PG}(k-n-2,q)$ have isomorphic collineation groups and isomorphic projective groups.

The duality transformation maps normal rational curves to normal rational curves and non-classical arcs to non-classical arcs. This implies that the arcs in Sects. 3.3 and 3.4 give rise to a different family of semisymmetric graphs. This follows from the following theorem.

Theorem 3.22 *Let $\mathcal{K}$ be a q-arc in $H_\infty = \mathrm{PG}(n,q)$, $q \geq n+4$, and let $\hat{\mathcal{K}}$ be a dual arc of $\mathcal{K}$ in $\hat{H}_\infty = \mathrm{PG}(q-n-2,q)$. Suppose that one of the groups $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ or $\mathrm{P\Gamma L}(q-n-1,q)_{\hat{\mathcal{K}}}$ fixes a point outside $\mathcal{K}$, $\hat{\mathcal{K}}$, respectively, and acts transitively on the points of $\mathcal{K}$, $\hat{\mathcal{K}}$, respectively, then $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\hat{\mathcal{K}})$ are semisymmetric, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{Aut}(\Gamma_{q-n-2,q}(\hat{\mathcal{K}})) \cong \mathrm{Persp}(\hat{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.*

Proof In the same way as before, using Result 2.1, Corollary 2.5 and Theorem 3.5, we see that $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\hat{\mathcal{K}})$ are connected non-vertex-transitive graphs for which $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ and $\mathrm{Aut}(\Gamma_{q-n-2,q}(\hat{\mathcal{K}})) \cong \mathrm{P\Gamma L}(q-n,q)_{\hat{\mathcal{K}}}$.

Suppose w.l.o.g. that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ fixes a point outside $\mathcal{K}$, then by Result 2.6, $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$. The embedding of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ in $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ used to show this result was constructed by adding a 1 at the lower right corner of every matrix B corresponding to an element (B,θ) of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, for some $\theta \in \mathrm{Aut}(\mathbb{F}_q)$ to obtain a matrix B' corresponding to an element (B',θ) of $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$. This subgroup meets $\mathrm{Persp}(H_\infty)$ trivially, which implies that in the group of matrices defining elements of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$, no proper scalar multiple of the identity matrix occurs. Now, from the isomorphism of Result 3.21, it follows that the group $\mathrm{PGL}(q-n-1,q)_{\hat{\mathcal{K}}}$, which is isomorphic to $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$, also contains no proper scalar multiple of the identity matrix. Hence, by embedding $\mathrm{P\Gamma L}(q-n-1,q)_{\hat{\mathcal{K}}}$ in $\mathrm{P\Gamma L}(q-n,q)$ in the same way (by adding a 1 at the lower right corner), we see that it meets $\mathrm{Persp}(\hat{H}_\infty)$ trivially. This implies that $\mathrm{P\Gamma L}(q-n,q)_{\hat{\mathcal{K}}} \cong \mathrm{Persp}(\hat{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.

We know that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{P\Gamma L}(q-n-1,q)_{\hat{\mathcal{K}}}$ are permutation isomorphic, hence, if one of them acts transitively on the points of $\mathcal{K}$ or $\hat{\mathcal{K}}$, so does the other. By Theorem 2.8, the graphs $\Gamma_{n,q}(\mathcal{K})$ and $\Gamma_{q-n-2,q}(\hat{\mathcal{K}})$ are edge-transitive and hence semisymmetric. □

If we restrict ourselves in the previous theorem to elements of the projective groups, using Result 2.7 we get the following corollary.

Corollary 3.23 *Let $\mathcal{K}$ be a q-arc in $H_\infty = \mathrm{PG}(n,q)$, $q \geq n+4$, and let $\hat{\mathcal{K}}$ be a dual arc of $\mathcal{K}$ in $\hat{H}_\infty = \mathrm{PG}(q-n-2,q)$. Suppose that one of the groups $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ or $\mathrm{PGL}(q-n-1,q)_{\hat{\mathcal{K}}}$ fixes a point outside $\mathcal{K}$, $\hat{\mathcal{K}}$, respectively, and acts transitively on the points of $\mathcal{K}$, $\hat{\mathcal{K}}$, respectively. Suppose $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_{q_0})$ or $\mathrm{P\Gamma L}(q-n-1,q)_{\hat{\mathcal{K}}} \cong \mathrm{PGL}(q-n-1,q)_{\hat{\mathcal{K}}} \rtimes \mathrm{Aut}(\mathbb{F}_{q_0})$, respectively,* *for* $q_0 = p^{h_0}$, $h_0|h$ *or* $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q)_{\mathcal{K}}$, $\mathrm{P\Gamma L}(q-n-1,q)_{\hat{\mathcal{K}}} \cong \mathrm{PGL}(q-n-1,q)_{\hat{\mathcal{K}}}$, *respectively. Then* $\Gamma_{n,q}(\mathcal{K})$ *and* $\Gamma_{q-n-2,q}(\hat{\mathcal{K}})$ *are semisymmetric*, $\mathrm{Aut}(\Gamma_{n,q}(\mathcal{K})) \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ *and* $\mathrm{Aut}(\Gamma_{q-n-2,q}(\hat{\mathcal{K}})) \cong \mathrm{Persp}(\hat{H}_\infty) \rtimes \mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$.

Consider the Glynn 10-arc contained in $\mathrm{PG}(4,9)$ and take any point P of this 10-arc; if we project the arc from P onto a $\mathrm{PG}(3,9)$ skew to P, then we obtain a complete 9-arc of $\mathrm{PG}(3,9)$. In [11] the author also shows that all complete 9-arcs in $\mathrm{PG}(3,9)$ can be obtained in this way, i.e. all complete 9-arc of $\mathrm{PG}(3,9)$ are $\mathrm{P\Gamma L}$-equivalent. It follows from [25] that the complete 9-arc in $\mathrm{PG}(3,9)$ is the dual of a 9-arc that is contained in the Glynn arc in $\mathrm{PG}(4,9)$. If we apply Theorem 3.22 to the Glynn 10-arc, we obtain the following corollary. The size of the automorphism group follows as before.

Corollary 3.24 *If* $\mathcal{K}$ *is a complete* 9*-arc of* $\mathrm{PG}(3,9)$, *then* $\Gamma_{3,9}(\mathcal{K})$ *is a semisymmetric graph. Moreover,* $\mathrm{Aut}(\Gamma_{3,9}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes \mathrm{AGL}(1,9)$ *and has size* $9^5 8^2$.

We can also apply Theorem 3.22 to the arcs of Sect. 3.3.

Corollary 3.25 *Let* $\mathcal{K}$ *be an arc of size* q *contained in any* $(q+1)$*-arc of* $\mathrm{PG}(q-4,q)$, $q = 2^h > 8$, *then* $\Gamma_{q-4,q}(\mathcal{K})$ *is a semisymmetric graph.*

Moreover, $\mathrm{Aut}(\Gamma_{q-4,q}(\mathcal{K}))$ *is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes \mathrm{A\Gamma L}(1,q)$ *and has size* $hq^{q-2}(q-1)^2$.

4 Families of semisymmetric graphs arising from other sets

By Result 2.4, if $\mathcal{K}$ is a set of points such that its closure $\overline{\mathcal{K}}$ is the whole space H_∞, then every automorphism of the graph $\Gamma_{n,q}(\mathcal{K})$ is induced by a collineation of its ambient space $\mathrm{PG}(n+1,q)$. However, we do not need this property for the construction of semisymmetric graphs. From the results and theorems of Sect. 2, the following theorem clearly follows.

Theorem 4.1 *Let* $\mathcal{K}$ *be a point set of* $H_\infty = \mathrm{PG}(n,q)$ *of size* q *spanning* H_∞ *such that every point of* $H_\infty \setminus \mathcal{K}$ *lies on at least one tangent line to* $\mathcal{K}$, *and such that* $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ *acts transitively on the points of* $\mathcal{K}$. *Then the graph* $\Gamma_{n,q}(\mathcal{K})$ *is a connected semisymmetric graph.*

The subgroup of the automorphism group of the graph $\Gamma_{n,q}(\mathcal{K})$ for which the elements are induced by collineations of the space $\mathrm{PG}(n+1,q)$ will be called the *geometric automorphism group* of $\Gamma_{n,q}(\mathcal{K})$.

We now give some examples of semisymmetric graphs for which $\overline{\mathcal{K}}$ is a subgeometry of H_∞. In the first three examples $\overline{\mathcal{K}}$ is a Baer subgeometry, obviously this only works if we look at a projective space over a field of square order. We will also construct their geometric automorphism group.

4.1 $\mathcal{K}$ is contained in an elliptic quadric

Let π be a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ embedded in $H_\infty = \mathrm{PG}(3,q)$, q a square. Let $\mathcal{K}$ denote the set of points of an elliptic quadric $Q^-(3,\sqrt{q})$ in π with one point removed. This set $\mathcal{K}$ has q points and clearly every point not in $\mathcal{K}$ lies on at least one tangent line to $\mathcal{K}$.

We introduce the definition of a *cap* and some results.

Definition A *k-cap* in $\mathrm{PG}(n,q)$ is a set of k points such that no 3 points lie on a line.

Result 4.2 [1] A q-cap in $\mathrm{PG}(3,\sqrt{q})$, q an odd square, is uniquely extendable to an elliptic quadric $Q^-(3,\sqrt{q})$.

Result 4.3 [23, Chap. IV] In $\mathrm{PG}(3,\sqrt{q})$, $q > 4$ an even square, a k-cap with $q - \sqrt[4]{q}/2 + 1 < k < q + 1$ lies on a unique complete $(q+1)$-cap.

Result 4.4 [13, Sect. 15.3] The stabiliser in $\mathrm{P\Gamma L}(4,\sqrt{q})$ of an elliptic quadric in $\mathrm{PG}(3,\sqrt{q})$ is $\mathrm{P\Gamma O}^-(4,\sqrt{q})$, which is isomorphic to $\mathrm{P\Gamma L}(2,q)$ (in its faithful action on $q+1$ points).

Theorem 4.5 *The graph $\Gamma_{3,q}(\mathcal{K})$, $q > 4$ square, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes (\mathrm{A\Gamma L}(1,q) \rtimes 2)$ *and has size* $2hq^5(q-1)^2$.

Proof Since $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(3,q)$, the graph $\Gamma_{3,q}(\mathcal{K})$ is q-regular and it is connected by Result 2.1. The graph $\Gamma_{3,q}(\mathcal{K})$ is not vertex-transitive by Corollary 2.5. The geometric automorphism group of $\Gamma_{3,q}(\mathcal{K})$ is $\mathrm{P\Gamma L}(5,q)_{\mathcal{K}}$. By Results 4.2 ($q$ odd) and 4.3 (q even), the cap $\mathcal{K}$ extends uniquely to an elliptic quadric in $\mathrm{PG}(3,\sqrt{q})$ by a point P. This point is obviously fixed by the stabiliser of $\mathcal{K}$ and hence, by Result 2.6, we find $\mathrm{P\Gamma L}(5,q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(4,q)_{\mathcal{K}}$. The group stabilising $\mathcal{K}$ stabilises the subgeometry $\overline{\mathcal{K}}$, hence $\mathrm{P\Gamma L}(4,q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(4,\sqrt{q})_{\mathcal{K}} \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{\sqrt{q}})) \cong \mathrm{P\Gamma L}(4,\sqrt{q})_{\mathcal{K}} \rtimes 2$. The stabiliser of $\mathcal{K}$ stabilises the elliptic quadric and fixes its point P, hence we find $\mathrm{P\Gamma L}(4,\sqrt{q})_{\mathcal{K}} \cong \mathrm{P\Gamma O}^-(4,\sqrt{q})_P \cong \mathrm{P\Gamma L}(2,q)_P \cong \mathrm{A\Gamma L}(1,q)$. Since $\mathrm{A\Gamma L}(1,q)$ acts transitively on the points of $\mathcal{K}$, the graph is semisymmetric. The size of this group follows from $|\mathrm{Persp}(H_\infty)| = q^4(q-1)$ and $|\mathrm{A\Gamma L}(1,q)| = hq(q-1)$. □

4.2 $\mathcal{K}$ is contained in a Tits-ovoid

Let π be a Baer subgeometry $\mathrm{PG}(3,\sqrt{q})$ embedded in $H_\infty = \mathrm{PG}(3,q)$, $q = 2^{2(2e+1)}$, $e > 0$. Let $\mathcal{K}$ denote the set of points of a Tits-ovoid in π with one point removed. This set $\mathcal{K}$ has q points and forms a cap in $\mathrm{PG}(3,q)$.

The canonical form of a Tits-ovoid in $\mathrm{PG}(3,\sqrt{q})$, $\sqrt{q} = 2^{2e+1}$ is

$$\{(1,s,t,st + s^{\sigma+2} + t^\sigma) \mid s,t \in \mathbb{F}_{\sqrt{q}}\} \cup \{(0,0,0,1)\},$$

where $\sigma : \mathbb{F}_{\sqrt{q}} \to \mathbb{F}_{\sqrt{q}} : x \mapsto x^{2^{e+1}}$. Let the set $\mathcal{K}$ correspond to the points of this ovoid minus the point $(0,0,0,1)$, then $\mathcal{K}$ is clearly stabilised by $\mathrm{Aut}(\mathbb{F}_q)$.

Result 4.6 [28] The stabiliser of $\mathcal{K}$ in $\mathrm{PGL}(4, \sqrt{q})$ is the 2-transitive Suzuki simple group $\mathrm{Sz}(\sqrt{q})$.

Following the notation of [17, Chap. 11], the point stabiliser of $\mathrm{Sz}(\sqrt{q})$ will be denoted by $\mathfrak{F}\mathfrak{H}$. Since $\mathrm{Sz}(\sqrt{q})$ is 2-transitive, the group $\mathfrak{F}\mathfrak{H}$ is transitive.

Theorem 4.7 *The graph $\Gamma_{3,q}(\mathcal{K})$, $q = 2^{2(2e+1)}$, $e > 0$, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes (\mathfrak{F}\mathfrak{H} \rtimes \mathrm{Aut}(\mathbb{F}_q))$ *and has size $hq^5(q-1)(\sqrt{q}-1)$.*

Proof The proof works in almost the same way as for the elliptic quadric. The size of the group follows when considering that $|\mathrm{Persp}(H_\infty)| = q^4(q-1)$ and $|\mathfrak{F}\mathfrak{H}| = q(\sqrt{q}-1)$. □

4.3 $\mathcal{K}$ is contained in a hyperbolic quadric $Q^+(3, q)$

Let π be a Baer subgeometry $\mathrm{PG}(3, \sqrt{q})$ embedded in $H_\infty = \mathrm{PG}(3, q)$, $q > 4$ square. Let $\mathcal{K}$ denote the set of points of a hyperbolic quadric $Q^+(3, \sqrt{q})$ in π with two lines of different reguli removed. This set $\mathcal{K}$ has q points.

Result 4.8 [13, Sect. 15.3] The stabiliser in $\mathrm{P\Gamma L}(4, \sqrt{q})$ of a hyperbolic quadric in $\mathrm{PG}(3, \sqrt{q})$ is $\mathrm{P\Gamma O}^+(4, \sqrt{q})$, which is isomorphic to $((\mathrm{PGL}(2, \sqrt{q}) \times \mathrm{PGL}(2, \sqrt{q})) \rtimes 2) \rtimes \mathrm{Aut}(\mathbb{F}_{\sqrt{q}})$ for $\sqrt{q} > 2$.

Corollary 4.9 *For $\sqrt{q} > 2$, the stabiliser in* $\mathrm{P\Gamma L}(4, \sqrt{q})$ *of a hyperbolic quadric in* $\mathrm{PG}(3, \sqrt{q})$ *fixing two lines of different reguli is isomorphic to* $((\mathrm{AGL}(1, \sqrt{q}) \times \mathrm{AGL}(1, \sqrt{q})) \rtimes 2) \rtimes \mathrm{Aut}(\mathbb{F}_{\sqrt{q}})$.

Theorem 4.10 *The graph $\Gamma_{3,q}(\mathcal{K})$, $q = p^h > 4$ square, is semisymmetric. Moreover, the geometric automorphism group is isomorphic to* $\mathrm{Persp}(H_\infty) \rtimes ((\mathrm{AGL}(2, \sqrt{q}) \times \mathrm{AGL}(2, \sqrt{q})) \rtimes 2) \rtimes \mathrm{Aut}(\mathbb{F}_q)$ *and has size $2hq^5(q-1)(\sqrt{q}-1)^2$.*

Proof Since $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(3, q)$, $\Gamma_{3,q}(\mathcal{K})$ is q-regular and is connected by Result 2.1. Clearly, every point of $\mathrm{PG}(3, q)$ not in $\mathcal{K}$ lies on at least one tangent to $\mathcal{K}$, hence $\Gamma_{3,q}(\mathcal{K})$ is not vertex-transitive by Corollary 2.5. The geometric automorphism group is $\mathrm{P\Gamma L}(5, q)_{\mathcal{K}}$. Clearly, $\mathcal{K}$ extends uniquely to a hyperbolic quadric in $\mathrm{PG}(3, \sqrt{q})$ by adding the missing line of each regulus. Since the intersection point of these lines will be fixed by the stabiliser of $\mathcal{K}$, we find by Result 2.6 that $\mathrm{P\Gamma L}(5, q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{P\Gamma L}(4, q)_{\mathcal{K}}$. Since the group stabilising the hyperbolic quadric also stabilises the subgeometry $\overline{\mathcal{K}} = \mathrm{PG}(3, \sqrt{q})$ and the canonical form of $Q^+(3, \sqrt{q})$ is fixed by $\mathrm{Aut}(\mathbb{F}_q)$, we find $\mathrm{P\Gamma L}(4, q)_{\mathcal{K}} \cong \mathrm{PGL}(4, \sqrt{q})_{\mathcal{K}} \rtimes \mathrm{Aut}(\mathbb{F}_q) \cong ((\mathrm{AGL}(1, \sqrt{q}) \times \mathrm{AGL}(1, \sqrt{q})) \rtimes 2) \rtimes \mathrm{Aut}(\mathbb{F}_q)$, by Result 2.7. Since $(\mathrm{AGL}(1, \sqrt{q}) \times \mathrm{AGL}(1, \sqrt{q})) \rtimes 2$ acts transitively on the points of $\mathcal{K}$, the graph is semisymmetric. □

4.4 $\mathcal{K}$ is contained in a cone

Let Π be a subgeometry $\mathrm{PG}(n, q_0)$ embedded in $H_\infty = \mathrm{PG}(n, q)$, $q = q_0^h$. Let π be a hyperplane of Π. Consider a set $\mathcal{O}$ of q_0^{h-1} points of π. Let V be a point of $\Pi \backslash \pi$ and let $V\mathcal{O}$ denote the set of points of the cone in Π with vertex V and base $\mathcal{O}$. This set minus its vertex V has q points.

For a vertex v in a graph Γ and a positive integer i we write $\Gamma_i(v)$ for the set of vertices at distance i from v.

Lemma 4.11 *Let $\mathcal{K}$ be the cone $V\mathcal{O}$ of Π minus its vertex V, such that every point of $\pi \backslash \mathcal{O}$ lies on at least one tangent line to $\mathcal{O}$, then $\forall P \in \mathcal{P}, \forall L \in \mathcal{L} : \Gamma_{n,q}(\mathcal{K})_4(P) \ncong \Gamma_{n,q}(\mathcal{K})_4(L)$.*

Proof Let $\Gamma = \Gamma_{n,q}(\mathcal{K})$. We will prove that, for every line $L \in \mathcal{L}$, the set of vertices $\Gamma_4(L)$ contains more than $q-1$ vertices that have all their neighbours in $\Gamma_3(L)$, while for every point $P \in \mathcal{P}$, there are exactly $q-1$ vertices in the set $\Gamma_4(P)$ that have all their neighbours in $\Gamma_3(P)$.

To prove the first claim, consider a line $L \in \mathcal{L}$ with $L \cap H_\infty = P_1 \in \mathcal{K}$. Choose an affine point Q on L and a point $P_2 \in \mathcal{K}$ different from P_1. Take a point R on QP_2, not equal to Q or P_2, then clearly the line $RP_1 \in \Gamma_4(L)$. We will show that RP_1 has all its neighbours in $\Gamma_3(L)$. Consider a neighbour S of RP_1, i.e $S \in RP_1 \setminus \{P_1\}$. The line SP_2 meets L in a point T. Since $T \in \Gamma_1(L)$ and $TP_2 \in \Gamma_2(L)$, it follows that $S \in \Gamma_3(L)$. Clearly, any line $M \in \mathcal{L}$ through P_1, such that $\langle M, L\rangle \cap H_\infty$ contains at least two points in $\mathcal{K}$, belongs to $\Gamma_4(L)$ and has all its neighbours in $\Gamma_3(L)$. Since the points of $\mathcal{K}$ do not lie on one line, there are more than $q-1$ such lines M.

Consider now a point $P \in \mathcal{P}$ and a point $T \in \Gamma_4(P)$. Look at the following minimal path of length 4 from T to P: the point T, a line $Q_1P_1 \in \Gamma_3(P)$ containing T for some $P_1 \in \mathcal{K}$, an affine point $Q_1 \in \Gamma_2(P)$, the line $PP_2 \in \Gamma_1(P)$ containing Q_1, for some $P_2 \in \mathcal{K}$ different from P_1, and finally the point P. Consider the point $R = PT \cap H_\infty$, then it follows from our construction that R lies on the line P_1P_2. Since $PR \notin \Gamma_1(P)$, we have R not in $\mathcal{K}$. First, suppose there is a tangent line of $\mathcal{K}$ through R, say RP_3, with $P_3 \in \mathcal{K}$. The line TP_3 is a neighbour of T. If TP_3 belongs to $\Gamma_3(P)$, then there exists a line PT' through a point $P_4 \in \mathcal{K}$, with T' on TP_3, which implies that RP_3 contains the point $P_4 \in \mathcal{K}$, a contradiction. Hence in this case there are neighbours of T that do not belong to $\Gamma_3(P)$. Now suppose there is no tangent line of $\mathcal{K}$ through R, then by construction, R is the vertex V of the cone. A line through V either contains 0 or q_0 points of $\mathcal{K}$, so in this case, any neighbour of T belongs to $\Gamma_3(P)$. There are exactly $q-1$ points on the line VP different from P and V. □

Corollary 4.12 *The graph $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive.*

Proof Since any graph automorphism preserves distance and hence neighbourhoods, no automorphism of $\Gamma_{n,q}(\mathcal{K})$ can map a vertex in $\mathcal{P}$ to a vertex in $\mathcal{L}$. □

Denote the subgroup of $\mathrm{P\Gamma L}(n+1, q)$ consisting of the perspectivities with centre V by $\mathrm{Persp}(V)$.

Lemma 4.13 *Consider $\mathcal{K}$, the point set of the cone $V\mathcal{O}$ in $\mathrm{PG}(n,q_0)$, minus its vertex V, where $\mathcal{O}$ spans π. If $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ and $\mathrm{PGL}(n,q_0)_{\mathcal{O}}$, respectively, fix a point of π, then $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{Persp}(V) \rtimes \mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}} \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0}))$ and $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{Persp}(V) \rtimes \mathrm{PGL}(n,q_0)_{\mathcal{O}}$, respectively.*

Proof First, it should be noted that the $\mathbb{F}_{q_0}$-span of $\mathcal{O}$ is π, the $\mathbb{F}_{q_0}$-span of $\mathcal{K}$ is Π, so $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ and $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ stabilise the subgeometry Π of H_∞. This implies that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong (\mathrm{P\Gamma L}(n+1,q)_{\Pi})_{\mathcal{K}}$, and $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong (\mathrm{PGL}(n+1,q)_{\Pi})_{\mathcal{K}}$, respectively. Since $\mathrm{P\Gamma L}(n+1,q)_{\Pi}$ is clearly isomorphic to $\mathrm{PGL}(n+1,q_0) \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0}))$, we have that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}} \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0}))$. Also, since $\mathrm{PGL}(n+1,q)_{\Pi}$ is isomorphic to $\mathrm{PGL}(n+1,q_0)$, we have that $\mathrm{PGL}(n+1,q)_{\mathcal{K}} \cong \mathrm{PGL}(n+1,q_0)_{\mathcal{K}}$.

Let ϕ be an element of $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}}$, then ϕ preserves the lines through V. Define the action of ϕ on π to be the mapping taking $L\cap\pi$ to $\phi(L)\cap\pi$.

The kernel of this action of $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}}$ on π is clearly isomorphic to $\mathrm{Persp}(V)$, as it consists of all collineations fixing the lines through V. The image of the action is isomorphic to $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$, showing that $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}}$ is an extension of $\mathrm{Persp}(V)$ by $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$. To show that this extension splits, we embed $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ in $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}}$ in such a way that it intersects trivially with $\mathrm{Persp}(V)$. By assumption, $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ fixes a point $P\in\pi$. W.l.o.g. let π be the hyperplane with equation $X_0=0$ and let V be the point $(1,0,\dots,0)$. Suppose that P has coordinates $(0,c_1,c_2,\dots,c_n)$, where the first non-zero coordinate equals one. This implies that for each $\beta\in\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$, there exists a unique $n\times n$ matrix $B=(b_{ij})$, $1\le i,j\le n$, and $\theta\in\mathrm{Aut}(\mathbb{F}_{q_0})$ corresponding to β, such that $(c_1,c_2,\dots,c_n)^\theta.B=(c_1,c_2,\dots,c_n)$. Moreover, the obtained matrices B form a subgroup of $\Gamma\mathrm{L}(n,q_0)$. Let $A_\beta=(a_{ij})$, $0\le i,j\le n$, be the $(n+1)\times(n+1)$ matrix with $a_{00}=1$, $a_{i0}=a_{0j}=0$ for $i,j\ge 1$ and $a_{ij}=b_{ij}$ for $1\le i,j\le n$. It is clear that the semi-linear map (A_β,θ) defines an element of $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}}$, corresponding to a collineation α acting in the same way as β on H_∞. If θ is not the identity $\mathbb{1}$, then α is not a perspectivity. If $\theta=\mathbb{1}$, then α fixes every point on the line through P and V, thus fixes at least two affine points and hence is not a perspectivity. This implies that the elements α form a subgroup of $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ isomorphic to $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ and intersecting $\mathrm{Persp}(V)$ trivially. This implies that $\mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}} \cong \mathrm{Persp}(V)\rtimes\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$, and we have seen before that $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}} \cong \mathrm{P\Gamma L}(n+1,q_0)_{\mathcal{K}} \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0}))$. Since $\mathrm{Persp}(V)$ intersects trivially with the standard embedding of $\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0})$, the claim follows.

The claim for $\mathrm{PGL}(n+1,q)_{\mathcal{K}}$ can be proved in the same way. □

The following corollary follows easily when we take into account that $\mathrm{Persp}(V)$ acts transitively on the points of each line through V.

Corollary 4.14 *If $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ acts transitively on $\mathcal{O}$, then $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on $\mathcal{K}$.*

Theorem 4.15 *Suppose that $\mathcal{O}$ spans π, that every point of $\pi\setminus\mathcal{O}$ lies on a tangent line to $\mathcal{O}$ and that $\mathrm{P\Gamma L}(n,q_0)_{\mathcal{O}}$ acts transitively on $\mathcal{O}$. Then the graph $\Gamma_{n,q}(\mathcal{K})$*

is semisymmetric. Moreover, the geometric automorphism group is isomorphic to $\mathrm{Persp}(H_\infty) \rtimes \mathrm{Persp}(V) \rtimes \mathrm{P\Gamma L}(n, q_0)_{\mathcal{O}} \rtimes (\mathrm{Aut}(\mathbb{F}_q)/\mathrm{Aut}(\mathbb{F}_{q_0}))$.

Proof Since $\mathcal{K}$ consists of q points spanning $\mathrm{PG}(n,q)$, $\Gamma_{n,q}(\mathcal{K})$ is q-regular and is connected by Result 2.1. The graph $\Gamma_{n,q}(\mathcal{K})$ is not vertex-transitive by Lemma 4.11. Clearly, $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ stabilises the point V, so we find by Result 2.6 that $\mathrm{PGL}(n+2,q)_{\mathcal{K}} \cong \mathrm{Persp}(H_\infty) \rtimes \mathrm{PGL}(n+1,q)_{\mathcal{K}}$. The expression for the geometric automorphism group follows from Lemma 4.13. Since $\mathrm{P\Gamma L}(n+1,q)_{\mathcal{K}}$ acts transitively on the points of $\mathcal{K}$, by Theorem 2.8, the graph is edge-transitive, and hence semisymmetric. □

5 Isomorphisms of $\Gamma_{n,q}(\mathcal{K})$ with other graphs

In this section, we will show that the graphs constructed by Du, Wang and Zhang [9], and the graphs of Lazebnik and Viglione [19] belong to the family $\Gamma_{n,q}(\mathcal{K})$, where $\mathcal{K}$ is a q-arc contained in a normal rational curve (see Sect. 3.2).

5.1 The graph of Du, Wang and Zhang

If $q = p$ prime, then the point of $\mathrm{PG}(n,p)$ with coordinates $(0,\dots,0,1)$ and the orbit of the point P with coordinates $(1,0,\dots,0)$ under the element $\phi \in \mathrm{PGL}(n+1,p)$ of order p, defined by the matrix A_ϕ, form a normal rational curve $\mathcal{N}$ in $\mathrm{PG}(n,p)$ (see [24]):

$$A_\phi = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

When we use the orbit of P for the point set $\mathcal{K}$ at infinity, we obtain a reformulation of the construction of the semisymmetric graphs found by Du, Wang and Zhang in [9]. This shows that our construction of the graph $\Gamma_{n,p}(\mathcal{K})$, with $\mathcal{K}$ a set of p points, contained in a normal rational curve, contains their family (and extends their construction to the case where q is not a prime). Moreover, the edge-transitive group of automorphisms described by the authors is not the full automorphism group of the graph: they only consider automorphisms induced by the group $\langle\phi\rangle$ of order p acting on the points of $\mathcal{K}$, together with $\mathrm{Persp}(H_\infty)$.

5.2 The graph of Lazebnik and Viglione

In [19], the authors define the graph $\Lambda_{n,q}$ as follows. Let $\mathcal{P}_n$ and $\mathcal{L}_n$ be two $(n+1)$-dimensional vector spaces over $\mathbb{F}_q$, $q = p^h$. The vertex set of $\Lambda_{n,q}$ is $\mathcal{P}_n \cup \mathcal{L}_n$, and we declare a point $(p) = (p_1, p_2, \dots, p_{n+1})$ adjacent to a line $[l] = [l_1, l_2, \dots, l_{n+1}]$ if and only if the following n relations on their coordinates hold.

$$l_2 + p_2 = p_1 l_1,$$

$$l_3 + p_3 = p_1 l_2,$$
$$\vdots$$
$$l_{n+1} + p_{n+1} = p_1 l_n.$$

In the following theorem, we will show that the graph $\Lambda_{n,q}$ is isomorphic to the graph $\Gamma_{n,q}(\mathcal{K})$, where $\mathcal{K}$ is contained in a normal rational curve; hence, $\Gamma_{n,q}(\mathcal{K})$ provides an embedding of the Lazebnik–Viglione graph in $\mathrm{PG}(n+1,q)$. It should be noted that in [19], the authors provide some automorphisms, acting on the graph $\Lambda_{n,q}$, to show that this graph is semisymmetric. From the isomorphism with $\Gamma_{n,q}(\mathcal{K})$ it follows that $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$ is also the full automorphism group of the Lazebnik–Viglione graph when $q \geq n+3$ or $q = p = n+2$.

Theorem 5.1 $\Lambda_{n,q} \cong \Gamma_{n,q}(\mathcal{K})$, *where* $\mathcal{K}$ *is a* q*-arc contained in a normal rational curve.*

Proof The graph $\Lambda_{n,q}$ is isomorphic to the graph $\Lambda'_{n,q}$ obtained by reversing the role of the points and the lines in the definition of $\Lambda_{n,q}$. So, $\Lambda'_{n,q}$ is the bipartite graph with parts $\mathcal{P}_n$ and $\mathcal{L}_n$, where $(p_1, \ldots, p_{n+1}) \in \mathcal{P}_n$ is incident with $(l_1, \ldots, l_{n+1}) \in \mathcal{L}_n$ if and only if $p_{i+1} + l_{i+1} = l_1 p_i$ for all $1 \leq i \leq n$. Let $\ell = (l_1, \ldots, l_{n+1})$ be a vertex of $\Lambda'_{n,q}$, then the points, incident with ℓ form a line of $\mathrm{AG}(n+1,q)$: suppose $(p_1, \ldots, p_{n+1})$ and $(p'_1, \ldots, p'_{n+1})$ are vertices, adjacent with ℓ, then so is the vertex $(p_1 + \lambda(p'_1 - p_1), \ldots, p_{n+1} + \lambda(p'_{n+1} - p_{n+1}))$, for any $\lambda \in \mathbb{F}_q$.

Now let $(p_1, \ldots, p_{n+1})$ and $(p'_1, \ldots, p'_{n+1})$ be vertices of $\Lambda'_{n,q}$ and embed these points of $\mathrm{AG}(n+1,q)$ in $\mathrm{PG}(n+1,q)$, by identifying $(p_1, \ldots, p_{n+1})$ with $(1, p_1, \ldots, p_{n+1})$. The line L determined by these points meets the hyperplane at infinity with equation $X_0 = 0$ of $\mathrm{AG}(n+1,q)$ in the point $P_\infty = (0, p_1 - p'_1, \ldots, p_{n+1} - p'_{n+1})$. Now the affine point set of L is a vertex of $\Lambda'_{n,q}$ if and only if there is an element $(l_1, \ldots, l_{n+1}) \in \mathcal{L}_n$ such that for all $1 \leq i \leq n$:

$$p_{i+1} + l_{i+1} = l_1 p_i,$$
$$p'_{i+1} + l_{i+1} = l_1 p'_i.$$

This implies that there exists some $l_1 \in \mathbb{F}_q$ such that $p_{i+1} - p'_{i+1} = l_1(p_i - p'_i)$ for all $1 \leq i \leq n$. Hence, the point P_∞ has coordinates $(0, 1, l_1, l_1^2, \ldots, l_1^n)$, which implies that all the vertices $(l_1, \ldots, l_{n+1})$ of $\Lambda'_{n,q}$ define a line in $\mathrm{PG}(n+1,q)$ through a point of the standard normal rational curve $\mathcal{K}$, minus the point $(0, \ldots, 0, 1)$. This is exactly the description of the graph $\Gamma_{n,q}(\mathcal{K})$. □

Corollary 5.2 *The automorphism group* $\mathrm{Aut}(\Lambda_{n,q})$ *of the graph* $\Lambda_{n,q}$ *is isomorphic to the edge-transitive group* $\mathrm{P\Gamma L}(n+2,q)_{\mathcal{K}}$. *Moreover*:

- *If* $q \geq n+3$, $q = p^h$, p *prime*, $n \geq 3$ *or* $n = 2$ *and* q *odd, then* $\mathrm{Aut}(\Lambda_{n,q})$ *has size* $hq^{n+2}(q-1)^2$;
- *If* $q = p = n+2$, *then* $\mathrm{Aut}(\Lambda_{n,q})$ *has size* $q^{n+1}(q-1)q!$.

5.3 The graph of Wenger and cycles in $\Gamma_{n,q}(\mathcal{K})$

We use the symbol C^k for a cycle of length k. The infinite family of graphs $H_n(q)$ introduced in [18] and [29] are clearly isomorphic to the graphs $\Lambda_{n-1,q}$ of Sect. 5.2, and thus isomorphic to the graphs $\Gamma_{n-1,q}(\mathcal{K})$, where $\mathcal{K}$ is a q-arc contained in a normal rational curve. Wenger [29] proved that the graphs $H_2(p)$, $H_3(p)$, $H_5(p)$ do not contain a C^4, C^6, C^{10}, respectively, for any prime p. In [18] the authors notice that, for a prime power q (implicitly assuming $n \geq 5$), the graph $H_n(q)$ contains no C^{10} and prove it has girth 8 for $n \geq 3$.

We now prove a similar theorem for the graph $\Gamma_{n,q}(\mathcal{K})$ using its geometric properties.

Theorem 5.3 *Let $\mathcal{K}$ be any arc in* $\mathrm{PG}(n,q)$, $q \geq n+1$, *then the graph $\Gamma_{n,q}(\mathcal{K})$ does not contain a C^4, C^6 and has girth* 8. *For $n=2$ and $|\mathcal{K}| \geq 4$ or $n=3$, $q>4$ and $|\mathcal{K}| \geq 5$, the graph $\Gamma_{n,q}(\mathcal{K})$ contains cycles of length* 10. *If $n \geq 4$, the graph $\Gamma_{n,q}(\mathcal{K})$ is C^{10}-free.*

Proof Since $\Gamma_{n,q}(\mathcal{K})$ is bipartite, every cycle has even length. Note that a cycle C^{2k} of $\Gamma_{n,q}(\mathcal{K})$ contains k points of $\mathcal{P}$ and k lines of $\mathcal{L}$. Since there is at most one line of $\mathcal{L}$ through any two affine points, the graph does not contain a C^4. Suppose $\Gamma_{n,q}(\mathcal{K})$ contains a C^6, $R_1 \sim R_1R_2 \sim R_2 \sim R_2R_3 \sim R_3 \sim R_3R_1$, $R_i \in \mathcal{P}$, $R_iR_j \in \mathcal{L}$. Clearly, the affine points R_1, R_2, R_3 are not collinear. The plane $\langle R_1, R_2, R_3\rangle$ intersects H_∞ in a line. The lines R_1R_2, R_2R_3 and R_3R_1 define three different points of $\mathcal{K}$, all lying on this line, a contradiction since $\mathcal{K}$ is an arc.

Consider two points $P_1, P_2 \in \mathcal{K}$ and a plane π through P_1P_2 not contained in H_∞. For $i=1,2$ consider distinct lines L_i through P_1 and distinct lines M_i through P_2, different from P_1P_2. Define the intersection points $R_{ij} = L_i \cap M_j$. The path $R_{11} \sim L_1 \sim R_{12} \sim M_2 \sim R_{22} \sim L_2 \sim R_{21} \sim M_1$ is a cycle C^8. Since $\Gamma_{n,q}(\mathcal{K})$ does not contain a C^4 or C^6, it has girth 8.

Let $\mathcal{K}$ be an arc in $\mathrm{PG}(2,q)$ and let P_1, P_2, P_3, P_4 be four points of $\mathcal{K}$. Let R_1 be an affine point. Let π be a plane through P_3P_4, not through R_1. Let R_2 be $\pi \cap R_1P_2$ and R_5 be $\pi \cap R_1P_1$. For $q>2$, we can choose an affine point R_3 on R_2P_3, different from R_2, but not lying on the line P_4R_5. Let R_4 be the point $R_3P_4 \cap R_5P_3$. Then $R_1 \sim R_1R_2 \sim \cdots \sim R_5 \sim R_5R_1$, $R_i \in \mathcal{P}$, $R_iR_j \in \mathcal{L}$, is a cycle of length 10.

Suppose $n=3$ and $|\mathcal{K}| \geq 5$. For $q>4$, one can consider five points $P_1, \ldots, P_5$ in $\mathrm{PG}(4,q)$ disjoint from H_∞ forming a basis of $\mathrm{PG}(4,q)$. The five points $P_1P_2 \cap H_\infty, \ldots, P_4P_5 \cap H_\infty, P_5P_1 \cap H_\infty$ form a frame of H_∞. Any five points of an arc in $\mathrm{PG}(3,q)$ form a frame and all frames are PGL-equivalent. Hence, we can assume w.l.o.g. that these five points of H_∞ belong to $\mathcal{K}$. It is clear that $P_1 \sim P_1P_2 \sim \cdots \sim P_5 \sim P_5P_1$, $P_i \in \mathcal{P}$, $P_iP_j \in \mathcal{L}$, is a cycle of length 10.

Now let $n \geq 4$, let $\mathcal{K}$ be an arc and assume $\Gamma_{n,q}(\mathcal{K})$ contains a C^{10}, $R_1 \sim R_1R_2 \sim \cdots \sim R_5 \sim R_5R_1$, $R_i \in \mathcal{P}$, $R_iR_j \in \mathcal{L}$. Note that two lines at distance 2 intersect H_∞ in different points of $\mathcal{K}$; hence the five lines intersect H_∞ in at least three different points of $\mathcal{K}$. The space $\pi = \langle R_1, R_2, R_3, R_4, R_5\rangle$ has dimension at most 4 and at least 3, so intersects H_∞ in at most a 3-space, containing at most 4 points of $\mathcal{K}$. Hence there are at least two lines of our set intersecting in a point of $\mathcal{K}$, these lines are not at distance two of each other, so without loss of generality, assume these are the lines R_1R_2 and R_3R_4. It follows that π is a 3-space, intersecting H_∞ in a plane containing 3 points of $\mathcal{K}$. However, the points R_1, R_2, R_3 and R_4 lie in a plane containing two points P_1 and P_2 of $\mathcal{K}$. The point R_5 does not lie in this plane, so the lines R_4R_5 and R_5R_1 intersect H_∞ in two new points P_3 and P_4. The points P_1, P_2, P_3 and P_4 lie in a plane of H_∞, a contradiction since $\mathcal{K}$ is an arc. □

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