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Betti tables of p-Borel-fixed ideals

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Abstract In this note we provide a counterexample to a conjecture of Pardue (Thesis (Ph.D.), Brandeis University, 1994), which asserts that if a monomial ideal is p-Borel-fixed, then its $\mathbb{N}$-graded Betti table, after passing to any field, does not depend on the field. More precisely, we show that, for any monomial ideal I in a polynomial ring S over the ring $\mathbb{Z}$ of integers and for any prime number p, there is a p-Borel-fixed monomial S-ideal J such that a region of the multigraded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ is in one-to-one correspondence with the multigraded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$ for all fields ℓ of arbitrary characteristic. There is no analogous statement for Borel-fixed ideals in characteristic zero.

Additionally, the construction also shows that there are p-Borel-fixed ideals with noncellular minimal resolutions.

Keywords Graded free resolutions · Positive characteristic · Borel-fixed ideals · Cellular resolutions

1 Introduction

Let $x_1, \ldots, x_n$ be indeterminates over the ring $\mathbb{Z}$ of integers, and $S = \mathbb{Z}[x_1, \ldots, x_n]$. Let p be zero or a prime number. For any field $\mathbb{k}$, the general linear group $\mathrm{GL}_n(\mathbb{k})$ acts on $S \otimes_{\mathbb{Z}} \mathbb{k}$. We say that a monomial S-ideal I is *p-Borel-fixed* if $I(S \otimes_{\mathbb{Z}} \mathbb{k})$ is fixed under the action of the Borel subgroup of $\mathrm{GL}_n(\mathbb{k})$ consisting of all the upper

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triangular invertible matrices over $\Bbbk$ for any infinite field $\Bbbk$ of characteristic p. (This definition does not depend on the choice of $\Bbbk$; see Proposition 2.5.)

Let I be any monomial S-ideal. In Theorem 3.2 we will show that, for any prime number p, there exists a (monomial) S-ideal J that is p-Borel-fixed and that, for any field ℓ, there is a region (independent of ℓ) in the multigraded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ (as a module over $S \otimes_{\mathbb{Z}} \ell$) that is determined by the multigraded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$. This shows that, homologically, the class of Borel-fixed ideals in positive characteristic is as bad as the class of all monomial ideals.

There is a combinatorial characterization of p-Borel-fixed S-ideals; see Proposition 2.5. It follows from this characterization that if I is 0-Borel-fixed, then $I(S \otimes_{\mathbb{Z}} \ell)$ if Borel-fixed for all fields ℓ, irrespective of $\operatorname{char} \ell$; the converse is not true. The Eliahou–Kervaire complex [4, Theorem 2.1] gives S-free resolutions of 0-Borel-fixed ideals in S, which specialize to minimal resolutions over any field ℓ. In particular, the $\mathbb{N}^n$-graded Betti table (and, hence, the $\mathbb{N}$-graded Betti table) of a 0-Borel-fixed S-ideal remains unchanged after passing to any field. On the other hand, if we only assume that I is p-Borel-fixed, with $p > 0$, then little is known about minimal resolutions of $I(S \otimes_{\mathbb{Z}} \ell)$ for some field ℓ, including when $\operatorname{char} \ell = p$.

A systematic study of Borel-fixed ideals in positive characteristic was begun by Pardue [11]. In positive characteristic, Proposition 2.5 was proved by him. He gave a conjectural formula for the (Castelnuovo–Mumford) regularity of principal p-Borel-fixed ideals. Aramova and Herzog [1, Theorem 3.2] showed that the conjectured formula is a lower bound for regularity; Herzog and Popescu [6, Theorem 2.2] finished the proof of the conjecture by showing that it is also an upper bound. Ene, Pfister, and Popescu [5] determined Betti numbers and Koszul homology of a class of Borel-fixed ideals in $\Bbbk[x_1, \dots, x_n]$, where $\operatorname{char} \Bbbk = p > 0$, which they called "$p$-stable."

Our main result (Theorem 3.2) arose in the following way. It is known that the Eliahou–Kervaire resolution is cellular [9]. Using algebraic discrete Morse theory, Jöllenbeck and Welker [7, Chap. 6] constructed minimal cellular free resolutions of principal Borel-fixed ideals in positive characteristic; also, see [14]. We were trying to see whether this extends to more general p-Borel-fixed ideals when we realized the possibility of the existence of p-Borel-fixed ideals whose Betti tables might depend on the characteristic. As a corollary of our construction and the result of Velasco [15] that there are monomial ideals with a noncellular minimal resolution, we conclude that there are p-Borel-fixed ideals that admit a noncellular minimal resolution.

We remarked earlier that the $\mathbb{N}$-graded Betti table of a 0-Borel-fixed S ideal remains identical over any field. Pardue [11, Conjecture V.4, p. 43] conjectured that this is true also for p-Borel-fixed ideals; see Conjecture 2.6 for the statement. (This conjecture also appears in [13, 4.3].) There has been some evidence that the conjecture is true. If J is a p-Borel-fixed S-ideal, then the projective dimension of $J(S \otimes_{\mathbb{Z}} \ell)$ is determined by the largest i such that x_i divides some minimal monomial generator of J. The regularity of $J(S \otimes_{\mathbb{Z}} \ell)$ does not depend on ℓ [11, Corollary VI.9]; this is part of the motivation for Pardue to make this conjecture. Later, Popescu [12] showed that the extremal Betti numbers of $J(S \otimes_{\mathbb{Z}} \ell)$ do not depend on ℓ. However, Example 3.5 shows that the conjecture is not true.

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2 Preliminaries

We begin with some preliminaries on estimating the graded Betti numbers of monomial ideals and on p-Borel-fixed ideals. By $\mathbb{N}$ we denote the set of nonnegative integers. When we say that p is a prime number, we will mean that $p > 0$. By $\mathbf{e}_1, \dots, \mathbf{e}_n$, we mean the standard vectors in $\mathbb{N}^n$.

Let A be an $\mathbb{N}^d$-graded polynomial ring (for some integer $d \geq 1$) over a field $\Bbbk$, with $A_{\mathbf{0}} = \Bbbk$. Let M be an $\mathbb{N}^d$-graded A-module. (All the modules that we deal with in this paper are ideals or quotients of ideals.) The $\mathbb{N}^d$*-graded Betti numbers* of M are $\beta^A_{i,\mathbf{a}}(M) := \dim_{\Bbbk} \operatorname{Tor}^A_i(M, \Bbbk)_{\mathbf{a}}$. The $\mathbb{N}^d$*-graded Betti table* of M is the element $(\beta^A_{i,\mathbf{a}}(M))_{i,\mathbf{a}} \in \mathbb{Z}^{\mathbb{N}\times\mathbb{N}^d}$. For $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$, we write $|\mathbf{a}| = a_1 + \cdots + a_d$.

Notation 2.1 Let A be a Noetherian ring, and z an indeterminate over A. Let $B = A[z]$; it is a graded A-algebra with $\deg z = 1$. For a graded B-ideal I, define A-ideals $I_{\langle i\rangle} = ((I : z^i) \cap A)$ for all $i \in \mathbb{N}$.

Note that for all $i \in \mathbb{N}$, $I_{\langle i\rangle} \subseteq I_{\langle i+1\rangle}$. Moreover, since A is Noetherian, $I_{\langle i\rangle} = I_{\langle i+1\rangle}$ for all $i \gg 0$.

Lemma 2.2 *Adopt Notation 2.1. Suppose that A is an $\mathbb{N}^d$-graded polynomial ring (for some integer $d \geq 1$) over a field $\Bbbk$ of arbitrary characteristic, with $A_{\mathbf{0}} = \Bbbk$. Let I be a graded B-ideal (in the natural $\mathbb{N}^{d+1}$-grading of B). Then, for all $\mathbf{a} \in \mathbb{N}^d$,*

$$\beta^B_{i,(\mathbf{a},j)}(I) = \begin{cases} 0 & \text{if } j < 0, \\ \beta^A_{i,\mathbf{a}}(I_{\langle 0\rangle}) & \text{if } j = 0, \quad \text{and} \\ \beta^A_{i-1,\mathbf{a}}(I_{\langle j\rangle}/I_{\langle j-1\rangle}) & \text{otherwise.} \end{cases}$$

Proof Fix $\mathbf{a} \in \mathbb{N}^d$. Let $M := I_{\langle 0\rangle}B \oplus \bigoplus_{l\geq 1}(I_{\langle l\rangle}/I_{\langle l-1\rangle}) \otimes_A B(-(\mathbf{0}, l))$. We need to prove that $\beta^B_{i,(\mathbf{a},j)}(I) = \beta^B_{i,(\mathbf{a},j)}(M)$ for all i, j. Note that z is a non-zero-divisor on M. Moreover, $M/zM \simeq I_{\langle 0\rangle} \otimes_A (B/zB) \oplus \bigoplus_{l\geq 1}(I_{\langle l\rangle}/I_{\langle l-1\rangle}) \otimes_A (B/zB)(-(\mathbf{0}, l)) \simeq I/zI$. Therefore, there are two exact sequences

$$0 \longrightarrow I\bigl(-(\mathbf{0},1)\bigr) \xrightarrow{z} I \longrightarrow I/zI \longrightarrow 0,$$

$$0 \longrightarrow M\bigl(-(\mathbf{0},1)\bigr) \xrightarrow{z} M \longrightarrow I/zI \longrightarrow 0.$$

The maps $\operatorname{Tor}^B_i(I(-(\mathbf{0},1)), \Bbbk) \xrightarrow{z} \operatorname{Tor}^B_i(I, \Bbbk)$ and $\operatorname{Tor}^B_i(M(-(\mathbf{0},1)), \Bbbk) \xrightarrow{z} \operatorname{Tor}^B_i(M, \Bbbk)$ are zero. Therefore, for all i and for all $j > 0$,

$$\beta^B_{i,(\mathbf{a},j)}(I) + \beta^B_{i-1,(\mathbf{a},j-1)}(I) = \beta^B_{i,(\mathbf{a},j)}(I/zI) = \beta^B_{i,(\mathbf{a},j)}(M) + \beta^B_{i-1,(\mathbf{a},j-1)}(M). \quad (2.1)$$

Note that outside a bounded rectangle inside $\mathbb{Z}^2$, the functions $(i, j) \mapsto \beta^B_{i,(\mathbf{a},j)}(I)$ and $(i, j) \mapsto \beta^B_{i,(\mathbf{a},j)}(M)$ take the value zero. Therefore, it follows from (2.1) that $\beta^B_{i,(\mathbf{a},j)}(I) = \beta^B_{i,(\mathbf{a},j)}(M)$ for all i, j. □

Definition 2.3 Adopt Notation 2.1. Let $d = (d_0 < d_1 < \cdots)$ be an increasing sequence of natural numbers. Define the operation Φ_d on graded B-ideals by setting $\Phi_d(I)$ to be the B-ideal generated by $\bigoplus_{i \in \mathbb{N}} I_{\langle i \rangle} z^{d_i}$.

Proposition 2.4 *Adopt the hypothesis of Lemma 2.2. Then*

$$\beta_{i,(\mathbf{a},j)}\big(\Phi_d(I)\big) = \begin{cases} \beta_{i,(\mathbf{a},l)}(I) & \text{if } j = d_l, \\ 0 & \text{otherwise}. \end{cases}$$

Proof This follows immediately by noting that, for all $j \in \mathbb{N}$, $(\Phi_d(I))_{\langle j \rangle} = I_{\langle l \rangle}$ where l is such that $d_l \leq j < d_{l+1}$. (If $d_0 > 0$, then $(\Phi_d(I))_{\langle j \rangle} = 0$ for all $0 \leq j < d_0$.) □

Borel-fixed ideals For the duration of this paragraph and Proposition 2.5, assume that p is zero or a positive prime number. Given two nonnegative integers a and b, we say that $a \preccurlyeq_p b$ if $\binom{b}{a} \neq 0 \mod p$. Then there is the following characterization of Borel-fixed ideals; for positive characteristic, it was proved by Pardue [11, Proposition II.4]; for details, see [3, Sect. 15.9.3].

Proposition 2.5 [3, Theorem 15.23] *Let $\Bbbk$ be an infinite field of characteristic p. An ideal I of $\Bbbk[x_1, \dots, x_n]$ is Borel fixed if and only I is a monomial ideal and for all $i < j$ and for all monomial minimal generators m of I, $(x_i/x_j)^s m \in I$ for all $s \preccurlyeq_p t$ where t is the largest integer such that $x_j^t \mid m$.*

Conjecture 2.6 [11, Conjecture V.4, p. 43] Let p be a prime number. Let I be a p-Borel-fixed monomial S-ideal. Then the $\mathbb{N}$-graded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$ is independent of $\operatorname{char} \ell$ (equivalently, ℓ) for all fields ℓ (of arbitrary characteristic).

3 Construction

Recall that $S = \mathbb{Z}[x_1, \dots, x_n]$ and that I is a monomial S-ideal. Fix a prime number p and let $\Bbbk$ be any field of characteristic p. We now describe an algorithm that constructs an S-ideal J such that $J(S \otimes_{\mathbb{Z}} \Bbbk)$ is Borel-fixed.

Construction 3.1 Input: A monomial S-ideal I. Set $i = 1$ and $J_0 = I$.

(i) Pick r_i an upper bound for $\operatorname{reg}_{(S \otimes_{\mathbb{Z}} \ell)}(J_{i-1}(S \otimes_{\mathbb{Z}} \ell))$ that is independent of the field ℓ.
(ii) Pick a positive integer e_i such that $p^{e_i} > r_i$. Let $d = (0 < p^{e_i} < 2p^{e_i} < 3p^{e_i} < \cdots)$. Set $J_i = \Phi_d(J_{i-1} + (x_i^{p^{e_i}}))$ with $A = \mathbb{Z}[x_1, \dots, x_i, x_{i+2}, \dots, x_n]$, $z = x_{i+1}$, and $B = S$ (Definition 2.3). Note that we are adding a large power of x_i but modifying the resulting ideal with respect to x_{i+1}.
(iii) If $i = n - 1$, then set $J = J_i$ and exit, else replace i by $i + 1$ and go to Step (i).

Output: A monomial S-ideal J.

Before we state our theorem, we need to identify the region of the $\mathbb{N}^n$-graded Betti table of $J(S \otimes_{\mathbb{Z}} \ell)$ that captures the $\mathbb{N}^n$-graded Betti table of $I(S \otimes_{\mathbb{Z}} \ell)$. Let $\mathcal{A} = \{\mathbf{a} : |\mathbf{a}| \leq r_1\}$ (with r_1 as in Step (i)) and $\mathcal{B} = \{\mathbf{b} : b_j < p^{e_j} - 1\}$.

Theorem 3.2 *The ideal J is p-Borel-fixed. Moreover, there is an injective map $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ such that for all fields ℓ (of arbitrary characteristic), for all $1 \leq i \leq n$, and for all $\mathbf{b} \in \mathcal{B}$,*

$$\beta_{i,\mathbf{b}}^{S\otimes_{\mathbb{Z}}\ell}\big(J(S \otimes_{\mathbb{Z}} \ell)\big) = \begin{cases} \beta_{i,\psi^{-1}(\mathbf{b})}^{S\otimes_{\mathbb{Z}}\ell}(I(S \otimes_{\mathbb{Z}} \ell)) & \textit{if } \mathbf{b} \in \psi, \\ 0 & \textit{otherwise}. \end{cases}$$

Let us make some remarks about the construction. In Step (i), we may, for example, take r_i to be the degree of the least common multiple of the minimal monomial generators of J_{i-1}; that this is a bound for regularity (independent of characteristic) follows from the Taylor resolution. There are stronger bounds, e.g., the largest degree of a minimal generator of the lex-segment ideal with the same Hilbert function as $J_{i-1}(S \otimes_{\mathbb{Z}} \ell)$. Additionally, one may insert a check at Step (iii) whether $J_i(S \otimes_{\mathbb{Z}} \mathbb{Z}/p)$ is Borel-fixed using Proposition 2.5. The algorithm will, then, terminate before or at the stage $i = m - 1$ where $m = \max\{i : x_i$ divides a minimal monomial generator of $I\}$.

The proofs of Theorem 3.2 and Proposition 3.4 hinge on the following lemma. See [3, Sect. A3.12] for mapping cones and [10, Chap. 4] for cellular resolutions. In the proof of the theorem, we first describe the change in the $\mathbb{N}^n$-graded Betti table at Step (ii). The readers familiar with multigraded resolutions will be able to see that the Betti numbers of J in the region $\mathcal{B}$ should be the Betti numbers of the ideal obtained from I by replacing x_i with $x_i^{p^{e_i - 1}}$ and hence contain information of the Betti numbers of I. For the sake of readability, we will abbreviate, for monomial S-ideals $\mathfrak{a}$, $\beta_{i,\mathbf{b}}^{S\otimes_{\mathbb{Z}}\ell}(\mathfrak{a}(S \otimes_{\mathbb{Z}} \ell))$ by $\beta_{i,\mathbf{b}}^{\ell}(\mathfrak{a})$ and $\operatorname{reg}_{(S\otimes_{\mathbb{Z}}\ell)}(\mathfrak{a}(S \otimes_{\mathbb{Z}} \ell))$ by $\operatorname{reg}_\ell(\mathfrak{a})$ from here till the end of the proof of the theorem.

Lemma 3.3 *Let $1 \leq j \leq n$, and ℓ be any field.*

(i) $(J_{j-1} :_S x_j^{p^{e_j}}) = (J_{j-1} :_S x_j^{\infty})$.
(ii) *Let $F_\bullet$ and $F'_\bullet$ be minimal $(S \otimes_{\mathbb{Z}} \ell)$-free resolutions of $(S/J_{j-1}) \otimes_{\mathbb{Z}} \ell$ and $(S/(J_{j-1} :_S x_j^{p^{e_j}})) \otimes_{\mathbb{Z}} \ell$.*

Write $M_\bullet$ for the mapping cone of the comparison map $F'_\bullet(-x_j^{p^{e_j}}) \longrightarrow F_\bullet$ that lifts the injective map $(S/(J_{j-1} :_S x_j^{p^{e_j}})(-x_j^{p^{e_j}}) \xrightarrow{x_j^{p^{e_j}}} S/J_{i-1}) \otimes_{\mathbb{Z}} \ell$. Then for each i, the set of degrees of homogeneous minimal generators of $F'_i(-x_j^{p^{e_j}})$ is disjoint from that of F_i. In particular, $M_\bullet$ is a minimal $(S \otimes_{\mathbb{Z}} \ell)$-free resolution of $(S/(J_{j-1} + (x_j^{p^{e_j}}))) \otimes_{\mathbb{Z}} \ell$.

Proof (i): Follows from the choice of e_j.

(ii): The assertion about generating degrees follows from the choice of e_j. As a consequence, we see that the map $F_i'(-x_j^{p^{e_j}}) \longrightarrow F_i$ is minimal, i.e., if we represent it by a matrix, all the entries are in the homogeneous maximal ideal. Therefore, $M_\bullet$ is minimal, and, hence a minimal resolution of $(S/(J_{j-1} + (x_j^{p^{e_j}}))) \otimes_{\mathbb{Z}} \ell$. □

Proof of the theorem Without loss of generality, we may assume that $\Bbbk$ is infinite. Let $x_1^{a_1} \cdots x_n^{a_n}$ be a minimal monomial generator of J. For all $1 \le i \le n-1$, a_{i+1} is a multiple of p^{e_i} and $x_i^{p^{e_i}} \in J$. Note that for all integers $l \ge 1$, if $m \preccurlyeq_p lp^{e_i}$ for some integer m, then m is a multiple of p^{e_i}. By Proposition 2.5, J is p-Borel-fixed; note that $e_1 < e_2 < \cdots$. The assertion about the Betti numbers $\beta_{i,\mathbf{b}}^\ell(J)$ follows from the discussion below, repeatedly applying (3.2).

Fix $1 \le j \le n-1$. If $|\mathbf{b}| \ge i + p^{e_j}$, then $|\mathbf{b}| > i + \operatorname{reg}_\ell(J_{j-1})$, so the Betti numbers $\beta_{i,\mathbf{b}}^\ell(J_{j-1} + (x_j^{p^{e_j}}))$ are determined by the resolution of $(S/(J_{j-1} :_S x_j^\infty))(-p^{e_j}\mathbf{e}_j)$; hence, in particular, for such $\mathbf{b}$, if $\beta_{i,\mathbf{b}}^\ell(J_{j-1} + (x_j^{p^{e_j}})) \neq 0$, then $b_j \ge i + p^{e_j}$. Putting this together, we obtain the following:

$$\beta_{i,\mathbf{b}}^\ell\big(J_{j-1} + \big(x_j^{p^{e_j}}\big)\big) = \begin{cases} \beta_{i,\mathbf{b}}^\ell(J_{j-1}) & \text{if } b_j < i + p^{e_j}, \\ \beta_{i-1,\mathbf{b}-p^{e_j}\mathbf{e}_j}^\ell(J_{j-1} :_S x_j^\infty) & \text{otherwise.} \end{cases}$$

Proposition 2.4 implies that, for all $\mathbf{b} \in \mathbb{N}^n$,

$$\beta_{i,\mathbf{b}}^\ell(J_j) = \begin{cases} \beta_{i,\mathbf{b}'}^\ell(J_{j-1}) & \text{if } p^{e_j} \mid b_{j+1} \text{ and } b_j < i + p^{e_j}, \\ \beta_{i-1,\mathbf{b}''}^\ell(J_{j-1} :_S x_j^\infty) & \text{if } p^{e_j} \mid b_{j+1} \text{ and } b_j \ge i + p^{e_j}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where we write $\mathbf{b}' = \mathbf{b} - (b_{j+1} - \frac{b_{j+1}}{p^{e_j}})\mathbf{e}_{j+1}$ and $\mathbf{b}'' = \mathbf{b}' - p^{e_j}\mathbf{e}_j$. We can recover the $\mathbb{N}^n$-graded Betti table of J_{j-1} from the $\mathbb{N}^n$-graded Betti table of J_j. To make this precise, suppose that $\beta_{i,\mathbf{b}}^\ell(J_j) \neq 0$. Then the resulting dichotomous situation from (3.1) has the following reinterpretation:

$$\begin{aligned} b_j < i + p^{e_j} \quad &\text{if and only if} \quad \beta_{i,\mathbf{b}}^\ell(J_j) = \beta_{i,\mathbf{b}'}^\ell(J_{j-1}), \\ b_j \ge i + p^{e_j} \quad &\text{if and only if} \quad \beta_{i,\mathbf{b}}^\ell(J_j) = \beta_{i-1,\mathbf{b}'}^\ell\big(J_{j-1} :_S x_j^\infty\big). \end{aligned} \tag{3.2}$$

We will not explicitly construct the map ψ but will observe that it can be done putting together the changes at each stage j. □

Proposition 3.4 *Let p be any prime number, $\Bbbk$ a field of characteristic p, and $R := S \otimes_Z \Bbbk = \Bbbk[x_1, \dots, x_n]$. Let I be any monomial S-ideal, and J be as in Construction 3.1. If IR has a noncellular minimal R-free resolution, then so does JR. In particular, there exists a Borel-fixed R-ideal with a noncellular minimal resolution.*

Proof The second assertion follows from the first since there are monomial ideals that have noncellular minimal resolutions [15]; therefore, we prove that if IR is a noncellular minimal resolution, then so does JR. As proposition does not involve looking at the behavior of I and J in two different characteristics, so, for the duration of this proof, we may assume that Construction 3.1 is done over R instead of S. Hereafter, we assume that I and J are R-ideals.

Note that it suffices to show, inductively, that, in Construction 3.1, if J_{i-1} has a noncellular minimal resolution, then so does J_i. It is immediate that J_i has a cellular minimal resolution if and only if $(J_{i-1} + (x_i^{p^{e_i}}))$ has one; this is because the same CW-complex supports minimal resolutions of $(J_{i-1} + (x_i^{p^{e_i}}))$ and $J_i := \Phi_d(J_{i-1} + (x_i^{p^{e_i}}))$. Therefore, it suffices to show that if J_{i-1} has a noncellular minimal resolution, then so does $(J_{i-1} + (x_i^{p^{e_i}}))$.

This is an immediate consequence of the choice of e_i and of Lemma 3.3. Let $F_\bullet$ be a noncellular minimal resolution of J_{i-1}. Let $F'_\bullet$ be any minimal resolution of $S/(J_{j-1} :_S x_j^{p^{e_j}})$. Then the mapping cone $M_\bullet$ is necessarily noncellular: for, otherwise, if there is a CW-complex X that supports $M_\bullet$, then for $\mathbf{b} = (p^{e_i} - 1, \dots, p^{e_i} - 1)$, $X_{\leq \mathbf{b}}$ supports $F_\bullet$. □

Example 3.5 (Counterexamples to Conjecture 2.6) Note that since graded Betti numbers are upper-semicontinuous functions of characteristic, for an S-ideal J, the $\mathbb{N}$-graded Betti table of $(J(S \otimes_{\mathbb{Z}} \ell))$ depends on char ℓ if and only if the $\mathbb{N}^n$-graded Betti table depends on char ℓ. Let I be any monomial S-ideal such that its $\mathbb{N}^n$-graded Betti table depends on char ℓ. Let p be any prime number, and $\Bbbk$ any field of characteristic p. Let J be the ideal from Construction 3.1. Then $J(S \otimes_{\mathbb{Z}} \ell)$ is Borel-fixed, while its $\mathbb{N}^n$-graded Betti table depends on char ℓ. As a specific example, we consider the minimal triangulation of the real projective plane [2, Sect. 5.3]. We have

$$S = \mathbb{Z}[x_1, \dots, x_6],$$
$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6).$$

With $p = 2$, $e_1 = 3$, $e_2 = 5$, $e_3 = 7$, $e_4 = 9$, and $e_5 = 11$, we obtain

$$J = \big(x_1^8, x_2^{32}, x_1x_2^8x_3^{32}, x_3^{128}, x_1x_2^8x_4^{128}, x_4^{512}, x_1x_3^{32}x_5^{512}, x_2^8x_4^{128}x_5^{512}, x_3^{32}x_4^{128}x_5^{512}, x_5^{2048}, x_2^8x_3^{32}x_6^{2048}, x_1x_4^{128}x_6^{2048}, x_3^{32}x_4^{128}x_6^{2048}, x_1x_5^{512}x_6^{2048}, x_2^8x_5^{512}x_6^{2048}\big).$$

Then the Betti numbers $\beta_{2,2729}^{S\otimes_{\mathbb{Z}}\ell}(J(S \otimes_{\mathbb{Z}} \ell))$ and $\beta_{3,2729}^{S\otimes_{\mathbb{Z}}\ell}(J(S \otimes_{\mathbb{Z}} \ell))$ (which correspond to $\beta_{2,6}^{S\otimes_{\mathbb{Z}}\ell}(I(S \otimes_{\mathbb{Z}} \ell))$ and $\beta_{3,6}^{S\otimes_{\mathbb{Z}}\ell}(I(S \otimes_{\mathbb{Z}} \ell))$, respectively) are nonzero precisely when char $\ell = 2$; otherwise they are zero.

After this paper was posted on the arXiv, Matteo Varbaro asked us whether there are p-Borel-fixed ideals minimally generated in a single degree that exhibit different Betti tables in different characteristics. The answer is positive. For instance, if we take J_1 to be the subideal of the ideal J of the above example generated by the monomials of degree 2725 in J, i.e., $J_1 = J \cap (x_1, \dots, x_6)^{2725}$, then J_1 is p-Borel-fixed as the intersection of two p-Borel-fixed ideals. Moreover, for all i, for all $j > 2725$, and for all fields ℓ, $\beta_{i,i+j}^{S\otimes_{\mathbb{Z}}\ell}(J(S \otimes_{\mathbb{Z}} \ell)) = \beta_{i,i+j}^{S\otimes_{\mathbb{Z}}\ell}(J_1(S \otimes_{\mathbb{Z}} \ell))$.

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