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Shortest path poset of Bruhat intervals

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Abstract We define the shortest path poset $SP(u, v)$ of a Bruhat interval $[u, v]$, by considering the shortest u–v paths in the Bruhat graph of a Coxeter group W, where $u, v \in W$. We consider the case of $SP(u, v)$ having a unique rising chain under a reflection order and show that in this case $SP(u, v)$ is a Gorenstein* poset. This allows us to derive the nonnegativity of certain coefficients of the complete **cd**-index. We furthermore show that the shortest path poset of an irreducible, finite Coxeter group exhibits a symmetric chain decomposition.

Keywords Shortest paths · Bruhat graph · Bruhat order · $\widetilde{R}$-polynomials · Complete **cd**-index

1 Introduction

Let (W, S) denote a Coxeter system with set of reflections $T := \{wsw^{-1} : s \in S, w \in W\}$. The *Bruhat graph* of (W, S) is the directed graph $B(W) := (V, E)$ with $V = W$ and $(u, v) \in E$ for $u, v \in W$ if $\ell(u) < \ell(v)$ and there exists $t \in T$ such that $ut = v$. Here, $\ell(\cdot)$ denotes the length function of (W, S). Furthermore, if $u, v \in W$, we denote the set of u–v paths of length (number of edges) k by $B_k(u, v)$, and let $B(u, v) := \bigcup_k B_k(u, v)$. As a convention, we will denote $\Delta \in B_k(u, v)$ in one of two ways:

(i) $\Delta = (x_0 = u < x_1 < \cdots < x_k = v)$, with $x_i \in W$, when we wish to refer to the vertices of Δ, and
(ii) $\Delta = (t_1, t_2, \ldots, t_k)$, with $t_i \in T$ and $x_{i-1}t_i = x_i$, with $i = 1, \ldots, k$, when we wish to refer to the edges that Δ traverses.

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One can define a partial order $\leq$, called the *Bruhat order*, as follows: $u \leq v$ if there exists a directed u–v path in $B(W)$. Bruhat intervals are endowed with many combinatorial properties; for example, they are *Eulerian* posets, that is, they are graded posets in which every nontrivial interval has the same number of elements of even rank as of odd rank.

A *reflection order* $<_T$ is a total order on T such that, for any Coxeter system of the form $(\langle t_1, t_2\rangle, \{t_1, t_2\})$ with $t_1, t_2 \in T$, either

$$t_1 <_T t_1t_2t_1 <_T t_1t_2t_1t_2t_1 <_T \cdots <_T t_2t_1t_2t_1t_2 <_T t_2t_1t_2 < t_2, \quad \text{or}$$

$$t_2 <_T t_2t_1t_2 <_T t_2t_1t_2t_1t_2 <_T \cdots <_T t_1t_2t_1t_2t_1 <_T t_1t_2t_1 < t_1.$$

Reflection orders have been shown to exist and have proven to be an important tool in the study of Coxeter groups (see, e.g., [11], [4, Chap. 5]). An *initial section* A_T of a reflection order $<_T$ is a subset of T satisfying $a <_T b$ for all $a \in A_T$ and $b \in T \setminus A_T$.

For $w \in W$, we define the *negative set* of w, denoted by $N(w)$, to be the set of reflections that shorten the length of w, i.e., $N(w) := \{t \in T \mid \ell(wt) < \ell(w)\}$. Notice that if $s_1 \cdots s_k$ is a reduced expression for w, then $N(w) = \{t_1, \ldots, t_k\}$, where $t_i = s_k \cdots s_{k-i+2}s_{k-i+1}s_{k-i+2} \cdots s_k$ for $i = 1, \ldots, k$. Furthermore, the total order defined by

$$t_k <_w t_{k-1} <_w \cdots <_w t_1$$

is said to be *induced* by the reduced expression $s_1 \cdots s_k$ for w. Dyer showed that finite initial sections are induced by reduced expressions.

Lemma 1 ([11], Lemma 2.11) *A_T is a finite initial section of a reflection order if and only if $A_T = N(w)$ for some $w \in W$. In other words A_T is a finite initial section of a reflection order if and only if it is induced by a reduced expression for some $w \in W$.*

Notice that [4, Proposition 2.3.1(i)] gives the existence of a unique longest-length element w_0 for finite W, that is, $w_0 \geq w$ for all $w \in W$. Moreover, $|N(w_0)| = \ell(w_0) = |T|$ by [4, Proposition 2.3.2(iv)], and so we have the following corollary.

Corollary 1 *If W is finite, then all reflection orders on T are induced by a choice of reduced expression for w_0.*

Definition 1 (i) The poset P is said to be EL-labelable (**E**dge-wise **L**exicographically labelable) if there exists an edge labeling λ of P so that every subinterval $[x, y] \in P$ has a unique maximal chain that is rising. Furthermore, such a chain is lexicographically earlier than any other maximal chain of $[x, y]$.

(ii) The λ above is called an EL-labeling of P.

In [12] and [11], Dyer proved two important consequences that follow from the existence of reflection orders. One such consequence is the following theorem.

Theorem 1 ([11], Proposition 4.3) *Let $[u, v]$ be a Bruhat interval. Then $[u, v]$ is EL-labelable.*

The second consequence is an alternative, non-recursive definition of the so-called $\widetilde{R}$-polynomials, which is discussed in Sect. 4.

Given a reflection order $<_T$ and a path $\Delta = (t_1, t_2, \dots, t_k) \in B_k(u, v)$, the *descent set* $D_{<_T}(\Delta)$ of Δ under $<_T$ is defined as $D_{<_T}(\Delta) := \{i \in [k-1] : t_{i+1} <_T t_i\}$. A path Δ is said to be *$<_T$-rising*, or simply rising, if $D_{<_T}(\Delta) = \emptyset$. While the descent set of a path depends on the choice of reflection order, the number of rising paths is the same (cf. [4, Proposition 5.3.4]). That is,

Proposition 1 *Let $u, v \in W$, with $u \le v$, and let $<_T$, $<'_T$ be two reflection orders. Then*

$$\left|\left\{\Gamma \in B(u, v) : D_{<_T}(\Gamma) = \emptyset\right\}\right| = \left|\left\{\Gamma \in B(u, v) : D_{<'_T}(\Gamma) = \emptyset\right\}\right|.$$

Furthermore there exists at least one rising path in $B_k(u, v)$ whenever $B_k(u, v) \neq \emptyset$.

Proposition 2 ([6], Proposition 3.9) *Let Δ be the lexicographically first path in $B_k(u, v) \neq \emptyset$ under a reflection order $<_T$. Then $D_{<_T}(\Delta) = \emptyset$, i.e., Δ is $<_T$-rising.*

Moreover, since the reverse of a reflection order is also a reflection order, it follows that

$$\left|\left\{\Gamma \in B_k(u, v) : D_{<_T}(\Gamma) = \emptyset\right\}\right| = \left|\left\{\Gamma \in B_k(u, v) : D_{<_T}(\Gamma) = [k-1]\right\}\right|. \tag{1}$$

The remainder of the paper is organized as follows: In Sect. 2 we define the shortest path poset for Bruhat intervals. In Sect. 3 we recast the main result of [5] stating that $SP(W) := SP(e, w_0)$ is the union of Boolean algebras, where W is a finite, irreducible Coxeter group and w_0 is its element of longest length, and prove that it admits a symmetric chain decomposition. In Sect. 4 we prove that $SP(u, v)$ is EL-labelable, in fact Gorenstein*, if there is a unique rising chain under a reflection order. In Sect. 4.2 we derive the nonnegativity of certain coefficients of the complete **cd**-index. Some of our results have appeared, without proof, in a FPSAC extended abstract in [7].

2 Definition of the shortest path poset

We first need to have a notion of "distance" in $B(u, v)$.

Definition 2 (i) Let Δ be a path of $B(u, v)$ and w be a vertex of Δ. The *distance of w on Δ*, denoted by $d_\Delta(u, w)$, is the number of edges in the u–w path of Δ.

(ii) The *shortest distance* of $[u, v]$, denoted by $\ell_s(u, v)$, is the length of the shortest path of $B(u, v)$. That is, $\ell_s(u, v) := \min\{\ell : B_\ell(u, v) \neq \emptyset\}$. When the interval is clear from the context, we simply write ℓ_s.

Lemma 2 *Consider two paths $\Gamma, \Gamma' \in B_{\ell_s}(u, v)$ and let $x \in [u, v]$ be a vertex in both paths. Then $d_\Gamma(u, x) = d_{\Gamma'}(u, x)$.*

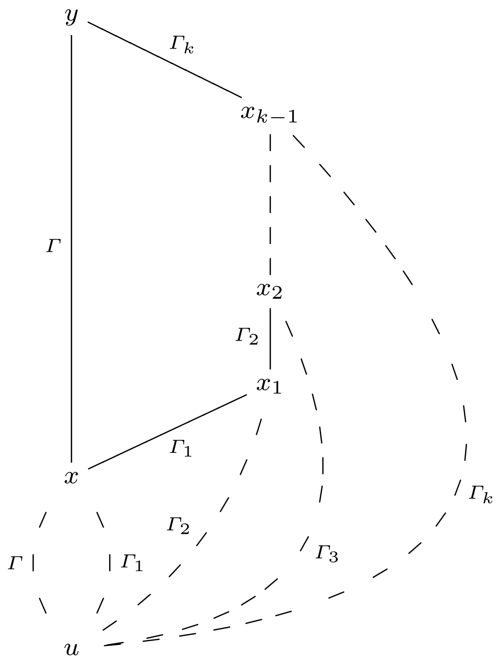

Fig. 1 Illustrating the proof of Proposition 3. Path Γ_i goes through edge $(x_{i-1} < x_i)$ for $1 \le i \le k$

Proof Let $\Gamma = (x_0 = u < x_1 < x_2 < \cdots < x_{\ell_s} = v)$ and $\Gamma' = (x'_0 = u < x'_1 < x'_2 < \cdots < x'_{\ell_s} = v)$. Since x is a vertex of both Γ and Γ', then $x_i = x$ and $x'_j = x$ for some $0 \le i, j \le \ell_s$.

Notice that $d_\Gamma(u, x) = i$ and $d_{\Gamma'}(u, x) = j$. If the two distances are not equal, then one of them is bigger. Suppose without loss of generality that $i < j$. Then $(x_i < x'_{j+1})$ is an edge in the Bruhat graph, and the path $(x_0 = u < \cdots < x_i = x < x'_{j+1} < x'_{j+2} < x'_{\ell_s} = v)$ has length $i + (\ell_s - j) < \ell_s$. This contradicts the definition of ℓ_s. Thus $i = j$. □

Proposition 3 *By ignoring the directions of the edges, $B_{\ell_s}(u, v)$ is the Hasse diagram of a graded poset.*

Proof Since $B(u, v)$ is a directed, acyclic graph, the edges of paths in $B_{\ell_s}(u, v)$ give a partial order $\le_s$ on the elements of $[u, v]$ that are in a u–v path of length ℓ_s. This partial order is defined by $x \le_s y$ if and only if $x = y$ or if there is a path $(x = y_0 < y_1 < \cdots < y_p = y) \in B(x, y)$ such that each edge $(y_{i-1} < y_i)$ is in a shortest u–v path, for $0 < i \le p$.

Let $(x \le_s y)$ be an edge in $B_{\ell_s}(u, v)$. Now, to prove the proposition we need to show that $x \lessdot_s y$. It suffices to show that *there is no* path $(x_0 = x < x_1 < x_2 < \cdots < x_k = y)$ with $k > 1$ such that each edge $(x_{i-1} < x_i)$ is in some path $\Gamma_i \in B_{\ell_s}(u, v)$ for $1 \le i \le k < \ell_s$ (refer to Fig. 1).

Notice that if such a path existed, then $B_{\ell_s}(u, v)$ (when ignoring directions) would not be a Hasse diagram, as there would be edges that would not represent cover

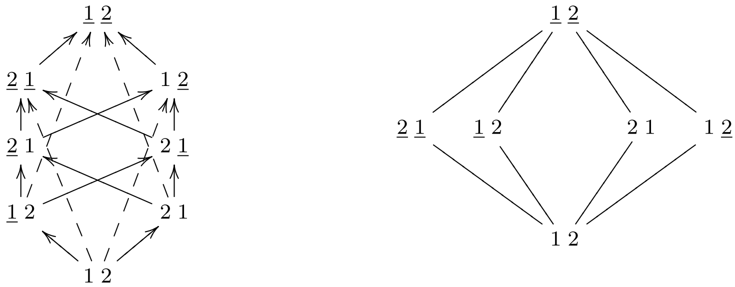

Fig. 2 $B(B_2)$ and $SP(B_2)$

relations. So let us assume for the sake of contradiction that such a path exists. Then

$$d_\Gamma(u, x) = d_{\Gamma_k}(u, x_{k-1}), \tag{2}$$

for otherwise one of them, say $d_\Gamma(u, x)$, is bigger than the other one. Thus there exists a u–v path Γ' formed by the edges of Γ_k up to y and then continue on the edges of Γ. Notice that the length of Γ' is $d_{\Gamma_k}(u, y) + (\ell_s - d_\Gamma(u, y)) < \ell_s$. This contradicts the definition of ℓ_s, and thus $d_\Gamma(u, x) = d_{\Gamma_k}(u, x_{k-1})$. Similarly, we obtain

$$\begin{aligned} d_{\Gamma_{k-1}}(u, x_{k-2}) &= d_{\Gamma_k}(u, x_{k-1}) - 1 \\ d_{\Gamma_{k-2}}(u, x_{k-3}) &= d_{\Gamma_{k-1}}(u, x_{k-2}) - 1 \\ &\vdots \\ d_{\Gamma_1}(u, x) &= d_{\Gamma_2}(u, x_1) - 1. \end{aligned}$$

Hence $d_\Gamma(u, x) = d_{\Gamma_1}(u, x_1) = d_{\Gamma_2}(u, x_2) - 1 = \cdots = d_{\Gamma_k}(u, x_{k-1}) - (k - 1)$. However, since $k > 1$ this contradicts (2). Thus the edges of $B_{\ell_s}(u, v)$ are the cover relations of a poset. Moreover, notice that this poset is graded by $r(x) := d_\Gamma(u, x)$ where $\Gamma \in B_{\ell_s}(u, v)$ contains the vertex x. This is a well-defined rank function by Lemma 2.

Finally, notice that if $(x < y)$ is an edge in $B_{\ell_s}(u, v)$ then there does not exist an x–y path containing an element other than x and y. Thus $x \lessdot y$ by definition. □

We call the poset in Proposition 3 the *shortest path poset* of u, v, which we denote by $SP(u, v)$. We consider the edges of $SP(u, v)$ to be labeled by the corresponding edges in $B_{\ell_s}(u, v)$.

3 $SP(W)$, for finite, irreducible Coxeter groups

Let W be a finite, irreducible Coxeter group. We let $SP(W) := SP(e, w_0)$. Figure 2 depicts B_2 and $SP(B_2)$, respectively. The rank of $SP(B_2)$ is two since that is the length of the shortest paths in $B(B_2)$.

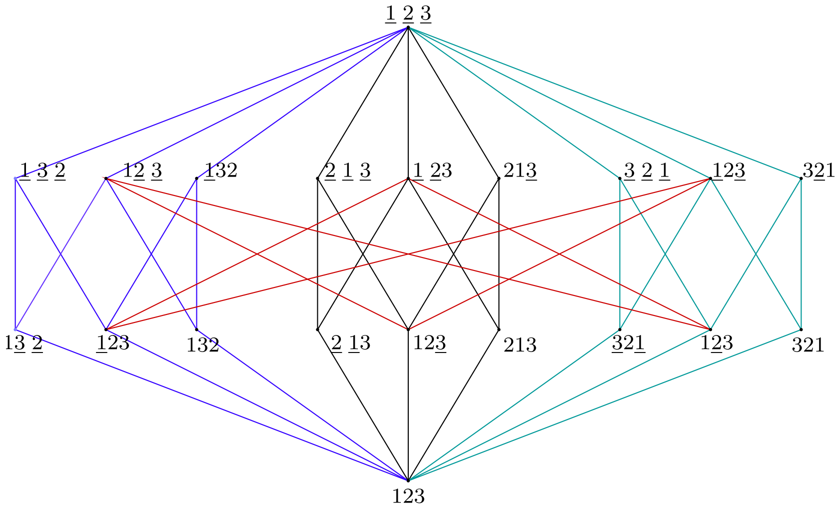

Fig. 3 $SP(B_3)$ has four copies of $\mathcal{B}_3$. Notice that while these copies intersect, each maximal chain is in a unique Boolean poset

For any $w \in W$, one can write $t_1 t_2 \cdots t_n = w$ for some $t_1, t_2, \dots, t_n \in T$. If n is minimal, then we say that w is *T-reduced*, and that the *absolute length* of w is n. The absolute length of w is denoted by $\ell_T(w)$.

Notice that for $w \in W$, if $\ell_T(w) = \ell$, then $t_1 t_2 \cdots t_\ell = w$ for some reflections $t_1, t_2, \dots, t_\ell$ in T, but this *does not* mean that $(t_1, t_2, \dots, t_\ell)$ is a (directed) path in $B(e, w)$. Nevertheless, it is shown in [5] that for finite W and $w = w_0$, $(t_1, t_2, \dots, t_\ell)$, and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \dots, t_{\tau(\ell)})$, $\tau \in S_\ell$, is a path in $B(W)$. To be more specific, one has the following theorem.

Theorem 2 (Theorem 1.1, [5]) *Let W be a finite Coxeter group and $\ell_0 = \ell_T(w_0)$, the absolute length of the longest element of W. Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0.*

We point out that the union of the Boolean posets could share more elements than e and w_0. For instance, consider $SP(B_3)$ depicted in Fig. 3. In addition to 123 and $\underline{1}\,\underline{2}\,\underline{3}$, there are other elements shared by more than one boolean copy; for instance, $12\underline{3}$ and $\underline{1}23$. Nonetheless each maximal chain belongs to a *unique* boolean copy.

3.1 Symmetric chain decomposition

A graded poset P of rank n admits a *symmetric chain decomposition* if it can be partitioned into saturated chains $C_1, C_2, \dots, C_\ell$ that are *centrally symmetric*, that is, the rank of the minimum element equals the corank of the maximum element for each C_i. Furthermore, P is said to admit a *symmetric boolean decomposition* if it can be partitioned into pieces $P_1, P_2, \dots, P_k$ where each P_i is isomorphic to $\mathcal{B}_{n-2k_i}$, the Boolean algebra of rank $n - 2k_i$, with k_i being the rank of the minimum element of P_i. Examples of posets that exhibit symmetric boolean decompositions are the *noncrossing partitions* of type A and B (see [17] and [9]).

The following remark is in order (see, e.g., [15]):

Remark 1 If a poset P admits a symmetric boolean decomposition, then P admits a symmetric chain decomposition.

Let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be family of graded posets of rank n that do not share any elements. Furthermore, denote the minimum and maximum elements of each P_i by $\widehat{0}_i, \widehat{1}_i$, respectively, and let $S(\mathcal{P})$ denote the poset

$$S(\mathcal{P}) := \coprod_{i=1}^{k} (\widehat{0}_i, \widehat{1}_i) \cup \{\widehat{0}, \widehat{1}\},$$

where $\widehat{0}$ and $\widehat{1}$ are defined to be the minimum and maximal element of $\coprod_{i=1}^{k} (\widehat{0}_i, \widehat{1}_i)$, respectively. Here, $(\widehat{0}_i, \widehat{1}_i)$ denotes the poset P_i with its minimum and maximum elements removed. Now we have the following lemma:

Lemma 3 *If P_i, $1 \le i \le k$, admits a symmetric boolean decomposition, then $S(\mathcal{P})$ admits a symmetric chain decomposition.*

Proof Let $C_1, \dots, C_q$ be a symmetric chain decomposition of P_1 and let us assume, without loss of generality, that C_1 contains $\widehat{1}_1$ and $\widehat{0}_1$. Then $C, C_2, \dots, C_q$ is a symmetric chain decomposition of $(\widehat{0}_1, \widehat{1}_1)$, where $C = C_1 \setminus \{\widehat{0}_1, \widehat{0}_1\}$. By Remark 1, each $(\widehat{0}_j, \widehat{1}_j)$, $2 \le j \le k$, has a symmetric chain decomposition. Therefore there exists a symmetric chain decomposition $D_1, D_2, \dots, D_r$ for

$$\coprod_{j=2}^{k} (\widehat{0}_j, \widehat{1}_j).$$

Hence $C \cup \{\widehat{0}, \widehat{1}\}, C_2, \dots, C_q, D_1, \dots, D_r$ is a symmetric chain decomposition of $S(\mathcal{P})$. □

We are now ready to prove the following proposition.

Proposition 4 *$SP(W)$ exhibits a symmetric chain decomposition.*

Proof By Theorem 2, $SP(W)$ is the union of boolean posets. Thus one can remove cover relations from $SP(W)$ to obtain a family of posets $\mathcal{P} = \{P_1, \dots, P_k\}$, each one of them being isomorphic to a Boolean algebra of rank $\ell_T(w_0)$, so that $SP(W) = S(\mathcal{P})$ as sets. Lemma 3 yields the result that $S(\mathcal{P})$ has a symmetric chain decomposition, and therefore so does $SP(W)$. □

One question that arises from the previous proposition is the following: What Bruhat intervals exhibit a symmetric boolean/chain decomposition? Billey [3] provides a sufficient condition for the existence of a symmetric chain decomposition for intervals of the form $[e, w]$, where w is an element of A_n or B_n.

4 Unique rising shortest path

In this section we will show that if there is a unique rising path in $B_{\ell_s}(u, v)$ then $SP(u, v)$ is a Gorenstein* poset. As a consequence, we derive nonnegativity of certain coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. We start with some topological definitions.

A simplicial complex Δ is said to be *Cohen–Macaulay* over $\mathbb{K}$ ($\mathbb{K}$ a field or $\mathbb{Z}$) if the reduced homology

$$\widetilde{H}_i(\operatorname{link}_\Delta F; \mathbb{K}) = 0 \quad \text{for all } F \in \Delta \text{ and } i < \dim \operatorname{link}_\Delta F.$$

Here, $\operatorname{link}_\Delta F := \{G \in \Delta : G \cup F \in \Delta \text{ and } G \cap F = \emptyset\}$. A poset is said to be *Cohen–Macaulay* if its order complex is Cohen–Macaulay. A poset is said to be *Gorenstein** if it is Eulerian and Cohen–Macaulay (cf. [14, Sect. 2.1]).

It turns out that if a poset is EL-labelable, then it is Cohen–Macaulay for all $\mathbb{K}$ [19, Theorem 4.1.9]. So to show that a poset is Gorenstein*, it is enough to show that it is EL-labelable and Eulerian. This is the case for $SP(u, v)$ if there is a unique rising chain.

4.1 $\widetilde{R}$-polynomials

In the study of Coxeter groups, it is common to encounter the $\widetilde{R}$-*polynomials*, which are defined in the proposition below.

Proposition 5 ([4], Proposition 5.3.2) *Let $u, v \in W$ with $u \leq v$ and $\ell(vs) < \ell(v)$. Then there exists a monic polynomial $\widetilde{R}(\alpha)$ of degree $\ell(v) - \ell(u)$ given by*

$$\widetilde{R}_{u,v}(\alpha) = \begin{cases} \widetilde{R}_{us,vs}(\alpha) & \text{if } \ell(us) < \ell(u), \text{ and} \\ \widetilde{R}_{us,vs}(\alpha) + \alpha \widetilde{R}_{u,vs}(\alpha) & \text{otherwise.} \end{cases}$$

The $\widetilde{R}$-polynomials are used, among other things, to define the R-*polynomials*, and these are used to define the *Kazhdan–Lusztig* polynomials from representation theory (see [4]). That is one of the reasons why the $\widetilde{R}$-polynomials are of interest.

Dyer used reflection orders to provide a non-recursive definition of the $\widetilde{R}$-polynomials.

Theorem 3 ([12], Theorem 2.3) *If $u \leq v$, then*

$$\widetilde{R}_{u,v}(\alpha) = \sum_{\substack{\Delta \in B(u,v) \\ D(\Delta) = \emptyset}} \alpha^{\ell(\Delta)}.$$

Dyer's theorem states that the $\widetilde{R}_{u,v}(\alpha)$ is simply the generating function of the rising paths in $B(u, v)$. Using this interpretation, we are able to derive the following inequality.

Theorem 4 *If $u \leq x \leq v$, then $\widetilde{R}_{u,x}(\alpha)\widetilde{R}_{x,v}(\alpha) \leq \widetilde{R}_{u,v}(\alpha)$ (coefficientwise).*

Proof The inequality is equivalent to saying that there are more rising paths in $B(u,v)$ than rising paths in $B(u,x)$ times the number of rising paths of $B(x,v)$. So it is enough to find an injection

$$\varphi_x : \mathcal{R}(u,x) \times \mathcal{R}(x,v) \longrightarrow \mathcal{R}(u,v),$$

where $\mathcal{R}(y,z) = \{\Gamma \in B(y,z) : D(\Gamma) = \emptyset\}$.

Consider a reflection order $<_x$ with initial section $N(x)$. Let $(t_1,\dots,t_p)$ be a $<_x$-rising path of $B(u,x)$ and let $(r_1,\dots,r_q)$ be a $<_x$-rising path of $B(x,v)$. Since $t_p \in N(x)$ and $r_1 \notin N(x)$, it follows that $t_p <_x r_1$. Hence the path $(t_1,\dots,t_p,r_1,\dots r_q)$ is a $<_x$-rising path of $B(u,v)$. By Proposition 1, the number of rising paths is the same under any reflection order. Hence the desired injection φ_x is given by concatenating a $<_x$-rising path in $B(u,x)$ and a $<_x$-rising path in $B(x,v)$. □

We no longer need a specific reflection order, and thus from now on we fix a reflection order $<_T$.

Theorem 4 generalizes the following results due to Brenti. All the inequalities are coefficientwise.

Corollary 2

1. [10, *Theorem* 5.4] *If* $u \le x \le v$, *then*

$$\alpha^{\ell(v)-\ell(x)} \widetilde{R}_{u,x}(\alpha) \le \widetilde{R}_{u,v}(\alpha).$$

2. [10, *Corollary* 5.5] *If* W *is finite and* $u \le x \le y \le v$, *then*

$$\alpha^{\ell(v)-\ell(y)+\ell(x)-\ell(u)} \widetilde{R}_{x,y}(\alpha) \le \widetilde{R}_{u,v}(\alpha).$$

3. [10, *Theorem* 5.6] *Let* $x, y, z \in W$ *be such that* $y \le z$ *in Bruhat order and* $x \le y$ *in* weak Bruhat order (*this is a coarsening of the Bruhat order*). *Then*

$$\alpha^{\ell(y)-\ell(x)} \widetilde{R}_{y,z}(\alpha) \le \widetilde{R}_{x,z}(\alpha).$$

All these inequalities are special cases of Theorem 4. For instance, the first inequality follows immediately from

$$\widetilde{R}_{u,x}(\alpha)\widetilde{R}_{x,v}(\alpha) \le \widetilde{R}_{u,v}(\alpha) \quad \text{and} \quad \alpha^{\ell(v)-\ell(x)} \le \widetilde{R}_{x,v}(\alpha).$$

The second inequality follows from

$$\widetilde{R}_{u,x}(\alpha)\widetilde{R}_{x,y}(\alpha)\widetilde{R}_{y,v}(\alpha) \le \widetilde{R}_{u,v}(\alpha),$$
$$\alpha^{\ell(v)-\ell(y)} \le \widetilde{R}_{y,v}(\alpha), \quad \text{and} \quad \alpha^{\ell(x)-\ell(u)} \le \widetilde{R}_{u,x}(\alpha).$$

Finally, the last inequality follows from

$$\widetilde{R}_{x,y}(\alpha)\widetilde{R}_{y,z}(\alpha) \le \widetilde{R}_{x,z}(\alpha) \quad \text{and} \quad \alpha^{\ell(y)-\ell(x)} \le \widetilde{R}_{x,y}(\alpha).$$

Of special interest for our purposes is the following inequality.

Proposition 6 *If* $u \le x \le y \le v$ *then* $|\mathcal{R}(x, y)| \le |\mathcal{R}(u, v)|$.

Proof Since the interval $[u, v]$ is graded, it is enough to show that the result holds when $u \lessdot x \le y$ or $u \le y \lessdot v$. Either of these cases follow from Theorem 4 since then $\widetilde{R}_{u,x}(\alpha) = 1$ or $\widetilde{R}_{y,v}(\alpha) = 1$, respectively. □

This proposition is a very important ingredient in the proof of the main result of this paper, which now follows.

Theorem 5 *Suppose that* $SP(u, v)$ *has a unique maximal, rising chain. Then* $SP(u, v)$ *is a Gorenstein* poset.*

Proof We verify that $SP(u, v)$ is EL-labelable (cf. Definition 1(i)). Proposition 2 gives the result that any subinterval of $SP(u, v)$ has at least one rising chain: the lexicographically first one. Moreover, Proposition 6 states that the number of rising chains in any subinterval of $SP(u, v)$ can be at most one. Thus any subinterval of $SP(u, v)$ has a unique rising path that is lexicographically first, and so $SP(u, v)$ is EL-labelable.

We just showed that $SP(u, v)$ is Cohen–Macaulay, as it is EL-labelable, and need only show that $SP(u, v)$ is Eulerian. Notice that any interval of rank 2 of $SP(u, v)$ has two atoms, for otherwise there must be more than one rising chain in some interval (of rank 2) by (1). Thus $SP(u, v)$ is thin (as are Bruhat intervals; see e.g., [4, Lemma 2.7.3] and [4, Theorem 2.7.7]). Therefore the poset $P = SP(u, v) \setminus \{u, v\}$ is pure and thin. Hence by [19, Theorem 3.1.12], $SP(u, v)$ is the face poset of a regular CW-decomposition of an $(\ell_s - 2)$-sphere that is homeomorphic to $\Delta(P)$, the order complex of P. In particular $SP(u, v)$ must be Eulerian. Hence, $SP(u, v)$ is a Gorenstein* poset. □

We finish this section with the following two conjectures.

Conjecture 1 *If* $SP(u, v)$ *has a unique rising chain, then* $SP(u, v)$ *is a lattice.*

This conjecture is inspired by an unpublished result due to Dyer [13] stating that if *all* paths of the Bruhat graph of $[u, v]$ have length $\ell(v) - \ell(u)$, then $[u, v]$ is a lattice (in fact, he showed that $[u, v]$ is isomorphic to the face poset of a polytope).

Furthermore, we believe a stronger conjecture is true.

Conjecture 2 *If* $SP(u, v)$ *has a unique rising chain, then* $SP(u, v)$ *is isomorphic to a Bruhat interval.*

We point out that intervals $[u, v]$ for which their shortest path poset $SP(u, v)$ has a unique rising chain appear frequently. For example, there are 37,467 intervals in S_6 having a unique rising chain, with $SP(u, v) \ne [u, v]$ (so the rank of $[u, v]$ is at least three and the rank of $SP(u, v)$ is at least one). For this computer search, we used Stembridge's Maple package [18].

4.2 Nonnegativity consequences for the complete **cd**-index

Billera and Brenti's *complete* ***cd****-index* $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ encodes the distribution of the descent sets of $[u,v]$. The complete **cd**-index is a non-homogeneous polynomial whose terms have degree one less than the lengths of the paths of $B(u,v)$. The terms of highest degree of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ correspond to the **cd**-index of the Eulerian poset $[u,v]$, which encodes the *flag h-vector* of $[u,v]$. For details, the reader is referred to [2], [1, Sect. 4]. The complete **cd**-index provides a combinatorial definition of the Kazhdan–Lusztig polynomials, and its study might shed some light on open problems regarding the coefficients of these polynomials.

It has been conjectured that the coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ are nonnegative [2, Conjecture 6.1]. There is a stronger conjecture for lower intervals, namely $\widetilde{\psi}_{e,v}(1,1) \leq \Phi_{\mathcal{B}_{\ell(v)}}(1,1)$, where $\Phi_{\mathcal{B}_{\ell(v)}}$ denotes the **cd**-index of $[e,v]$ as an Eulerian poset (see [1, Conjecture 3]). Our results allow us to conclude that certain terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ are nonnegative. If $[\mathbf{c}^{\ell_s-1}]\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ denotes the coefficient of $\mathbf{c}^{\ell_s-1}$, where ℓ_s is the shortest distance of $[u,v]$, then we have:

Proposition 7 *If* $[\boldsymbol{c}^{\ell_s-1}]\widetilde{\psi}_{u,v}(\boldsymbol{c},\boldsymbol{d}) = 1$, *the terms of degree* $\ell_s - 1$ *in* $\widetilde{\psi}_{u,v}(\boldsymbol{c},\boldsymbol{d})$ *are nonnegative.*

Proof If $[\mathbf{c}^{\ell_s-1}]\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d}) = 1$, Theorem 5 shows that $SP(u,v)$ is Gorenstein*. Therefore, the terms of degree $\ell_s - 1$ in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ coincide with the **cd**-index of $SP(u,v)$ as an Eulerian poset. Furthermore, the **cd**-index of Gorenstein* posets is nonnegative (see [16]) and therefore the terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ of degree $\ell_s - 1$ are also nonnegative. □

In [5], we obtain a stronger result for finite, irreducible Coxeter groups. Namely, we are able to compute the lowest-degree terms of $\psi_{u,w_0}(\mathbf{c},\mathbf{d})$ in terms of the **cd**-index of Boolean algebras.

5 Further directions

In general, $SP(u,v)$ will have more than one rising chain. We would like to find a procedure that would allow us to partition $SP(u,v)$ into pieces $P_1,\dots,P_k$, each one of which has a unique rising chain. A possible approach would be to "flip" the descents of a chain of $SP(u,v)$ into ascents (see [2, Sect. 6]). If each piece P_i satisfies enough properties, we hope to conclude that the terms of lowest degree of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ are the sum of nonnegative terms contributed by each P_i. There is evidence, both computational and theoretical, supporting this approach (see [8, Chap. 4]).

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