J Algebr Comb (2013) 38:491–497
DOI 10.1007/s10801-012-0412-y

The finite edge-primitive pentavalent graphs

Song-Tao Guo · Yan-Quan Feng · Cai Heng Li

Received: 13 December 2011 / Accepted: 17 November 2012 / Published online: 30 November 2012

Abstract A graph is edge-primitive if its automorphism group acts primitively on edges. Weiss (in J. Comb. Theory Ser. B 15, 269–288, 1973) determined edge-primitive cubic graphs. In this paper, we classify edge-primitive pentavalent graphs. The same classification of those of valency 4 is also done.

Keywords Edge-primitive graph · Symmetric graph · s-Transitive graph

1 Introduction

Let G be a group acting on a set Ω. Denote by G_α the subgroup of G fixing the point α. G is said to be *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$, and G is said to be regular if G is transitive and semiregular. A non-empty subset Δ of Ω is called a *block* for G if for each $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. Clearly, the set Ω and the singletons $\{\alpha\}\,(\alpha \in \Omega)$ are blocks for G, called the *trivial* blocks. Any other block is said to be *non-trivial*. Suppose that Δ is a non-trivial block for G. Then $\{\Delta^g \mid g \in G\}$ is the *system of imprimitivity* of G containing Δ. A transitive group G is *primitive* if G has no non-trivial blocks on Ω.

Throughout this paper, we consider undirected finite graphs without loops or multiple edges. As usual, the notation $X = (V, E)$ denotes a graph with vertex set V and

S.-T. Guo · Y.-Q. Feng (✉)
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China
e-mail: yqfeng@bjtu.edu.cn

S.-T. Guo
e-mail: gsongtao@gmail.com

C.H. Li
School of Mathematics and Statistics, The University of Western Australia, Crawley 6009, WA, Australia
e-mail: cai.heng.li@uwa.edu.au

edge set E, and $\mathrm{Aut}(X)$ denotes its automorphism group. If two vertices $u, v \in V$ are adjacent, $\{u, v\}$ denotes the edge between u and v. By $X_1(v)$, we mean the *neighborhood* of a vertex v in X, consisting of vertices which are adjacent to v.

Let $X = (V, E)$ be a graph and $G \leq \mathrm{Aut}(X)$. Then X is said to be *G-locally primitive* if the vertex stabilizer G_v acts primitively on $X_1(v)$ for each $v \in V$. A graph X is said to be *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V or E, respectively. If G is replaced by $\mathrm{Aut}(X)$, the graph X is simply said to be *vertex-transitive* or *edge-transitive*.

An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \dots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A 0-*arc* is a vertex and a 1-arc is also called an *arc* for short. A graph X is said to be *(G, s)-arc-transitive* if $G \leq \mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. A (G, s)-arc-transitive graph is said to be *(G, s)-transitive* if it is not $(G, s+1)$-arc-transitive. A graph X is said to be *s-arc-transitive* or *s-transitive* if the graph is $(\mathrm{Aut}(X), s)$-arc-transitive or $(\mathrm{Aut}(X), s)$-transitive. A graph X is *G-edge-primitive* if $G \leq \mathrm{Aut}(X)$ acts primitively on the set of edges of X, and X is *edge-primitive* if it is $\mathrm{Aut}(X)$-edge-primitive.

Weiss [9] determined all edge-primitive cubic graphs, which are the complete bipartite graph $K_{3,3}$, the Heawood graph of order 14, the Biggs–Smith cubic distance-transitive graph of order 102 and the Tutte–Coxeter graph of order 30 (also known as Tutte's 8-cage or the Levi graph). Giudici and Li [3] systematically analyzed edge-primitive graphs via the O'Nan–Scott Theorem to determine the possible edge and vertex actions of such graphs, and determined all G-edge-primitive graphs for G an almost simple group with socle $\mathrm{PSL}(2, q)$, where q is a prime power and $q \neq 2, 3$. Recently, the authors [4] classified edge-primitive tetravalent graphs, which are the complete graph K_5, the co-Heawood graph of order 14 (the complement graph of the Heawood graph with respect to the complete bipartite graph $K_{7,7}$), the complete bipartite graph $K_{4,4}$, and three coset graphs defined on the almost simple groups $\mathrm{Aut}(\mathrm{PSL}(3,3))$, $\mathrm{Aut}(\mathrm{M}_{12})$ and $\mathrm{Aut}(G_2(3))$, respectively. In [6], edge-primitive 4-arc-transitive graphs are classified. In this paper, we give a classification of edge-primitive graphs of valency 5.

Theorem 1.1 *Let X be an edge-primitive pentavalent graph with an edge $e = \{u, v\}$ and let $A = \mathrm{Aut}(X)$. Then X is s-transitive with $s \geq 2$, and X, s, A, A_v and A_e are listed in Table 1. Furthermore, such a graph X is uniquely determined by its number of vertices.*

From Theorem 1.1, we have the following corollary.

Corollary 1.2 *All finite edge-primitive pentavalent graphs are 2-arc-transitive.*

Remark Let X be an edge-primitive graph with an edge $e = \{u, v\}$ and let $A = \mathrm{Aut}(X)$. Weiss classified such graphs of valency 3 in 1973. However, since then there is no much progress in this line for small valencies. In this paper, we first reduce A to an almost simple group when X has valency 5 and Theorem 1.1 follows from the classification of finite primitive groups with solvable stabilizers given in [6]. The method does not work for valency greater than 5 because A_e can be non-solvable.

Table 1 *s*-transitive edge-primitive pentavalent graphs

X	s	A	A_v	A_e
Complete graph K_6	2	S_6	S_5	$S_4 \times \mathbb{Z}_2$
PSL(2, p)-graph (Example 3.1)	2	PSL(2, p)	A_5	S_4
PGL(2, p)-graph (Example 3.2)	2	PGL(2, p)	A_5	S_4
Complete bipartite graph $K_{5,5}$	3	$S_5 \text{ wr } S_2$	$S_5 \times S_4$	$S_4 \text{ wr } S_2$
$J_3.\mathbb{Z}_2$-graph (Example 3.3)	4	$J_3.\mathbb{Z}_2$	$\mathbb{Z}_2^4 \rtimes \Gamma L(2,4)$	$(\mathbb{Z}_2^4 \rtimes (A_4 \rtimes S_3)).\mathbb{Z}_2$
$\mathrm{PSL}(3,4).D_{12}$-graph (Example 3.4)	4	$\mathrm{PSL}(3,4).D_{12}$	$\mathbb{Z}_2^4 \rtimes \Gamma L(2,4)$	$(\mathbb{Z}_2^4 \rtimes (A_4 \rtimes S_3)).\mathbb{Z}_2$
$\mathrm{PSp}(4,4).\mathbb{Z}_4$-graph (Example 3.5)	5	$\mathrm{PSp}(4,4).\mathbb{Z}_4$	$\mathbb{Z}_2^6 \rtimes \Gamma L(2,4)$	$(\mathbb{Z}_2^6 \rtimes (A_4 \rtimes S_3)).\mathbb{Z}_2$

2 A reduction

Let $X = (V, E)$ be a G-edge-primitive graph of valency 5 with an edge $e = \{u, v\}$. Then $2|E| = 5|V|$, and G is a primitive permutation group on E. By [3, Lemmas 3.1 and 3.4], X is connected and G-arc-transitive. Thus, $5 \mid |G_v|$, but $5^2 \nmid |G_v|$. In particular, X is G-locally primitive. Let $N = \mathrm{Soc}(G) = T^k$, the socle of G. Then T is a simple group, N is transitive on E, and hence N has at most two orbits on V. If N has two orbits on V, denote by V_1 and V_2 these orbits. In this case, X is bipartite with V_1 and V_2 as its bipartition sets.

Lemma 2.1 *The socle N is a minimal normal subgroup of G and is not semiregular on V, and the graph X is N-locally primitive. If $X \neq K_{5,5}$, T is non-abelian simple and if further $k \geq 2$, T is semiregular on V.*

Proof Let $1 \neq M \lhd G$. Suppose that M is semiregular on V. Then $M_v = 1$ and M is transitive on E, implying that M has at most two orbits on V. Thus, $|V| = |M|$ or $2|M|$. The edge-primitivity of G implies that M is transitive on E. It follows that $|E| \mid |M|$ and so $|E| \mid |V|$, which is impossible because $|E| = \frac{5|V|}{2}$. Thus, M is not semiregular on V. Note that $|X_1(v)| = 5$. Since $M \lhd G$ and X is G-arc-transitive, M_v is transitive on $X_1(v)$, and hence primitive on $X_1(v)$. Further, X is M-locally primitive. In particular, by taking $M = N$ we see that N is not semiregular on V and X is N-locally primitive. If G has two distinct minimal normal subgroups, say N_1 and N_2, then $N_1 \times N_2 \leq G$. By taking $M = N_1$ or N_2, X is N_1- and N_2-locally primitive. This implies that $5 \mid |(N_1)_v|$ and $|(N_2)_v|$, forcing that $5^2 \mid |G_v|$, a contradiction. Thus, N is a minimal normal subgroup of G.

To prove the second part, let $X \neq K_{5,5}$. Suppose T is abelian. Then N is abelian and hence regular on E. It follows that $|N| = |E| = \frac{5|V|}{2}$. Recall that N is not semiregular. If N has one orbit on V then N is regular on V, a contradiction. It follows that N has two orbits on V, that is, V_1 and V_2, and for $v \in V_1$, we have $N_v \neq 1$. Since N is abelian, N_v fixes every vertex in V_1, forcing $X = K_{5,5}$, a contradiction. Thus, N is non-abelian. To finish the proof, we further let $k \geq 2$. Suppose that T is not semiregular on V. Write $N = T \times L$, where $L = T^{k-1}$. By the minimality of N in G, L is not semiregular on V.

Assume that N is transitive on V. Since T is not semiregular on V, $T_w \neq 1$ for every $w \in V$, and by the minimality of N in G, $L_w \neq 1$. By the vertex-transitivity and locally primitivity of N, we have $5 \mid |T_w|$ and $5 \mid |L_w|$. It follows that $5^2 \mid |N_w|$, which is impossible.

Now assume that N has two orbits on V. We may let $v \in V_1$ and $u \in V_2$. Suppose that $5 \nmid |T_v|$ and $5 \nmid |T_u|$. Then $5 \nmid |T_w|$ and $5 \nmid |T_x|$ for every $w \in V_1$ and every $x \in V_2$. Since X is N-locally primitive, T_w and T_x fix $X_1(w)$ and $X_1(x)$ pointwise, respectively. The connectivity of X implies that T_u and T_v fix every vertex in V. Then $T_u = T_v = 1$ and hence $T_w = T_x = 1$ for every $w \in V_1$ and every $x \in V_2$, contrary to the assumption that T is not semiregular on V. Thus, we may assume that $5 \mid |T_v|$ (note that we cannot deduce $5 \mid |T_v|$ by the locally primitivity of N when $T_v \neq 1$ because N has two orbits). By the minimality of N, $5 \mid |L_u|$.

Consider the orbits of L. Let $\{B_1, B_2, \dots, B_m\}$ and $\{C_1, C_2, \dots, C_n\}$ be the sets of orbits of L on V_1 and V_2, respectively. We may assume that $v \in B_1$ and $u \in C_1$. Note that B_i and C_i are blocks of N. Since $5 \mid |L_u|$, L_u is transitive on $X_1(u)$. Thus, $X_1(u) \subseteq B_1$, and $X_1(x) \subseteq B_1$ for every $x \in C_1$ because B_1 and C_1 are orbits of L. The edge-transitivity of N implies that if there is an edge between C_i and B_1 then $X_1(x) \subseteq B_1$ for every $x \in C_i$. The connectivity of X implies that $m = 1$, that is, L is transitive on V_1. Thus, $T_v = T_w$ for every vertex $w \in V_1$ because L commutes with T, which forces that $X = K_{5,5}$, a contradiction. □

Lemma 2.2 *Let $X \neq K_{5,5}$. Then G_v is non-solvable and G is 2-arc-transitive.*

Proof Since G is arc-transitive, let X be (G, s)-arc-transitive for some $s \geq 1$. Note that a transitive permutation group of prime degree is either solvable or 2-transitive. To prove the lemma, we only need to show that G_v is non-solvable. Suppose to the contrary that G_v is solvable. Then the primitive permutation group $G_v^{X_1(v)}$ of degree 5 is isomorphic to $\mathbb{Z}_5$, D_{10} or F_{20}, and hence $G_{vu}^{X_1(v)}$ is a 2-group (the identity subgroup is also viewed as a 2-group). Let $G_v^{[1]}$ be the kernel of G_v acting on $X_1(v)$. By [6, Theorem 1.3], G_v is non-solvable for $s \geq 4$. Thus, $s \leq 3$ and by [10, Theorems 4.6–4.7], $G_{uv}^{[1]} = G_u^{[1]} \cap G_v^{[1]} = 1$. Then $G_u^{[1]} \times G_v^{[1]} = G_u^{[1]} G_v^{[1]} \lhd G_{vu}$, and $G_v^{[1]} \cong G_u^{[1]} \cong G_u^{[1]}/(G_u^{[1]} \cap G_v^{[1]}) \cong (G_u^{[1]})^{X_1(v)} \lhd G_{vu}^{X_1(v)}$, implying that $G_v^{[1]}$ is a 2-group. It follows that G_{vu} is a 2-group and hence G_e is a 2-group. The maximality of G_e in G implies that G_e is a Sylow 2-subgroup of G. Thus, $|E|$ is odd and so is $\frac{1}{2}|V|$. Moreover, if $N = T^k$ with $k \geq 2$ then by Lemma 2.1, T is semiregular on V, which is impossible. Thus $k = 1$, and G is almost simple. By [11, Theorem], if the stabilizer of an arc-transitive automorphism group of a graph with prime valency p is solvable then its order is a divisor of $p(p-1)^2$. Thus, $|G_v| \mid 80$, which forces that $|G_e| \mid 32$. Since $|G|$ is divisible by 5, by [6, Tables 14–20] we have $G = \mathrm{PGL}(2, 9)$, M_{10}, or $\mathrm{PSL}(2, 31)$, which is also impossible by the Atlas [1]. □

Let $X \neq K_{5,5}$ and $s \leq 3$. Note that $G_v^{[1]}$ is a $\{2, 3\}$-group and hence solvable. By Lemma 2.2, $G_v^{X_1(v)}$ is non-solvable and so $G_v^{X_1(v)} = \mathrm{A}_5$ or S_5, which implies that $G_{vu}^{X_1(v)} = \mathrm{A}_4$ or S_4, respectively. If $G_v^{[1]} = 1$ then $G_v = \mathrm{A}_5$ or S_5. Now assume $G_v^{[1]} \neq 1$. Since $G_v^{[1]} \lhd G_{vu}$ and $G_v^{[1]} \cap G_u^{[1]} = 1$, $G_v^{[1]}$ is transitive on $X_1(u) \setminus \{v\}$.

We claim that G_v has a normal subgroup A_5 or S_5 and $G_v^{[1]} = \mathrm{A}_4$ or S_4. To prove it, let $H = \langle G_z^{[1]} \mid z \in X_1(v) \rangle$. Then $H \lhd G_v$. Since $G_v^{[1]} \cap G_u^{[1]} = 1$, the action of $G_v^{[1]}$ on $X_1(u)\backslash\{v\}$ is non-trivial and so H has a non-trivial action on $X_1(v)$. Since $H^{X_1(v)} \trianglelefteq G_v^{X_1(v)} = \mathrm{A}_5$ or S_5, we have $H^{X_1(v)} = \mathrm{A}_5$ or S_5. Then $H_{vu}^{X_1(v)} = \mathrm{A}_4$ or S_4, and so H contains a non-identity element h of order 3-power such that $h \in H_{vuw}$ for some $w \in X_1(u)$ with $w \neq v$. On the other hand, it is easy to show that $[H, G_v^{[1]}] \leq G_v^{[1]} \cap G_u^{[1]} = 1$, which implies that $H \cap G_v^{[1]} \leq Z(G_v^{[1]})$, the center of $G_v^{[1]}$. It follows that H commutes with $G_v^{[1]}$ and hence h fixes $X_1(u)$ pointwise because $G_v^{[1]}$ is transitive on $X_1(u)\backslash\{v\}$. Thus, $3 \mid |G_u^{[1]}|$ and $3 \mid |G_v^{[1]}|$. Since $G_v^{[1]} \cong G_v^{[1]}/G_{uv}^{[1]} \trianglelefteq G_{vu}^{X_1(v)} = \mathrm{A}_4$ or S_4, we have $G_v^{[1]} = \mathrm{A}_4$ or S_4. Furthermore, $H \cap G_v^{[1]} \leq Z(G_v^{[1]}) = 1$, implying that $G_v^{[1]}H = G_v^{[1]} \times H$ and H is faithful on $X_1(v)$. Thus, $H = \mathrm{A}_5$ or S_5, as claimed.

Now it is easy to see that $|G_v : G_v^{[1]} \times H| = 1$ or 2 and we have the following lemma.

Lemma 2.3 *Suppose that $X \neq K_{5,5}$ and X is connected (G, s)-transitive with $s \leq 3$. Then either*

(1) $s = 2$, *and* $G_v = \mathrm{A}_5$ *or* S_5, *or*
(2) $s = 3$, *and* $G_v = \mathrm{A}_4 \times \mathrm{A}_5$, $\mathrm{S}_4 \times \mathrm{S}_5$, *or* $(\mathrm{A}_4 \times \mathrm{A}_5).\mathbb{Z}_2$.

In particular, this lemma tells us that G_v does not have a subnormal subgroup $\mathbb{Z}_5$.

Lemma 2.4 *Suppose that $X \neq K_{5,5}$ and X is connected (G, s)-transitive with $s \leq 3$. Then G is almost simple.*

Proof Suppose that $1 \neq M \lhd N$ is regular on E. Then X is M-edge-transitive, and hence M has at most two orbits on V. Thus, $|M| = |E| = \frac{5|V|}{2}$ is divisible by $|V|$ or $\frac{1}{2}|V|$, forcing that M has exactly two orbits on V, X is bipartite, and $|M_v| = 5$. It follows that $\mathbb{Z}_5 \cong M_v \lhd N_v \lhd G_v$, which is impossible by Lemma 2.3. Thus, N does not have a normal subgroup which is regular on E, and by O'Nan–Scott's theorem [2, Theorem 4.1A], G is almost simple, or of product action on E.

Suppose that G is of product action on E. Then by O'Nan–Scott's theorem, $N_e = T_e^k$, $T_e \neq 1$, and $k \geq 2$. Since $X \neq K_{5,5}$, by Lemma 2.1, $T_v = 1$. It follows that $T_e = \mathbb{Z}_2$ and $N_e = T_e^k = \mathbb{Z}_2^k$, which is impossible because $3 \mid |N_e|$ (Lemma 2.3). Thus, G is an almost simple group. □

From Lemma 2.4 we find that if $X \neq K_{5,5}$ then the group G is almost simple, and the edge stabilizer G_e is a maximal subgroup, and $G_e = \mathrm{A}_4.\mathbb{Z}_2$, $\mathrm{S}_4.\mathbb{Z}_2$, $(\mathrm{A}_4 \times \mathrm{A}_4).\mathbb{Z}_2$, $(\mathrm{S}_4 \times \mathrm{S}_4).\mathbb{Z}_2$, or $((\mathrm{A}_4 \times \mathrm{A}_4).\mathbb{Z}_2).\mathbb{Z}_2$.

3 Classification

In this section, we prove Theorem 1.1. First, we introduce the so-called coset graph. Let G be a finite group, H a subgroup of G and $D = D^{-1}$ a union of several double-cosets of the form HgH with $g \notin H$. The *coset graph* $X = \mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have vertex set $V = [G : H]$, the set of the right cosets of H in G, and edge set $E=\{\{Hg,Hdg\}\mid g\in G,d\in D\}$. Then X is well defined and has valency $|D|/|H|$. Furthermore, X is connected if and only if D generates G. Note that G acts on V by right multiplication and so we can view G/H_G as a subgroup of Aut(X), where H_G is the largest normal subgroup of G contained in H. We may show that G acts transitively on the arcs of X if and only if $D=HgH$ for some $g\in G\backslash H$ (see [7, 8]). The following two examples were described in [3, Sect. 8].

Example 3.1 Let p be a prime and let $G=\mathrm{PSL}(2,p)$ with $p\equiv\pm1,\pm9 \pmod{40}$. Then by [3, Proposition 8.5], G has a subgroup $H=\mathrm{A}_5$ and one conjugacy class of maximal subgroups $K=\mathrm{S}_4$ such that $K\cap H=\mathrm{A}_4$. Take an involution $g\in K\backslash H$. Define the pentavalent PSL(2, p)-graph as Cos(G,H,HgH). Then the PSL(2, p)-graph is edge-primitive and has automorphism group PSL(2, p). Furthermore, any connected G-edge-primitive pentavalent graph is isomorphic to the PSL(2, p)-graph.

Example 3.2 Let p be a prime and let $G=\mathrm{PGL}(2,p)$ with $p\equiv\pm11,\pm19 \pmod{40}$. Then by [3, Proposition 8.5], G has a subgroup $H=\mathrm{A}_5$ and one conjugacy class of maximal subgroups $K=\mathrm{S}_4$ such that $K\cap H=\mathrm{A}_4$. Take an involution $g\in K\backslash H$. Define the pentavalent PGL(2, p)-graph as Cos(G,H,HgH). Then the PGL(2, p)-graph is an edge-primitive graph and has automorphism group PGL(2, p). Furthermore, any connected G-edge-primitive pentavalent graph is isomorphic to the PGL(2, p)-graph.

Now we construct an edge-primitive graph, which was given by Weiss [12].

Example 3.3 Let $G=\mathrm{Aut}(\mathrm{J}_3)=\mathrm{J}_3.\mathbb{Z}_2$. Then by the Atlas [1], G has maximal subgroups $H=\mathbb{Z}_2^4\rtimes\Gamma\mathrm{L}(2,4)$ and $K=(\mathbb{Z}_2^4\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)).\mathbb{Z}_2$ such that $K\cap H=\mathbb{Z}_2^4\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)$. Define the pentavalent $\mathrm{J}_3.\mathbb{Z}_2$-graph as Cos($G,H,HgH$), where $g\in K\backslash H$. Then this is a 4-transitive edge-primitive graph, and has automorphism group $\mathrm{J}_3.\mathbb{Z}_2$. Furthermore, any connected G-edge-primitive pentavalent graph is isomorphic to the $\mathrm{J}_3.\mathbb{Z}_2$-graph.

The following two edge-primitive pentavalent graphs are extracted from [5, Sect. 2].

Example 3.4 Let $G=\mathrm{Aut}(\mathrm{PSL}(3,4))=\mathrm{PSL}(3,4).D_{12}$. By the Atlas [1], G has a maximal subgroup $K=(\mathbb{Z}_2^4\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)).\mathbb{Z}_2$ and a subgroup $H=\mathbb{Z}_2^4\rtimes\Gamma\mathrm{L}(2,4)$ such that $K\cap H=\mathbb{Z}_2^4\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)$. Define the pentavalent $\mathrm{PSL}(3,4).D_{12}$-graph as Cos(G,H,HgH), where $g\in K\backslash H$. Then this is a 4-transitive edge-primitive graph and has automorphism group $\mathrm{PSL}(3,4).D_{12}$. Furthermore, any connected G-edge-primitive pentavalent graph is isomorphic to the $\mathrm{PSL}(3,4).D_{12}$-graph.

Example 3.5 Let $G=\mathrm{Aut}(\mathrm{PSp}(4,4))=\mathrm{PSp}(4,4).\mathbb{Z}_4$. By the Atlas [1], G has a maximal subgroup $K=(\mathbb{Z}_2^6\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)).\mathbb{Z}_2$ and a subgroup $H=\mathbb{Z}_2^6\rtimes\Gamma\mathrm{L}(2,4)$ such that $K\cap H=\mathbb{Z}_2^6\rtimes(\mathrm{A}_4\rtimes\mathrm{S}_3)$. Define the pentavalent $\mathrm{PSp}(4,4).\mathbb{Z}_4$-graph as Cos($G,H,HgH$), where $g\in K\backslash H$. Then this is a 5-transitive edge-primitive graph and has automorphism group $\mathrm{PSp}(4,4).\mathbb{Z}_4$. Furthermore, any connected G-edge-primitive pentavalent graph is isomorphic to the $\mathrm{PSp}(4,4).\mathbb{Z}_4$-graph.

Proof of Theorem 1.1 The graph X has an edge $e=\{v,u\}$, and is A-edge-primitive, where $A=\mathrm{Aut}(X)$. Clearly, $K_{5,5}$ is 3-transitive and edge-primitive. Now assume $X \neq K_{5,5}$. By [3, Lemmas 3.1 and 3.4], X is a connected (A,s)-transitive graph for $s \geq 1$.

Let $s \leq 3$. By Lemma 2.4, $T=\mathrm{Soc}(A)$ is a non-abelian simple group. Note that A_e is a $\{2,3\}$-group and hence solvable. By [6, Theorem 1.1], A has a normal subgroup B which is minimal under the condition that $B_e = B \cap A_e$ is maximal in B, and the pairs (B, B_e) are given in [6, Tables 14–20]. Since $B \trianglelefteq A$, the edge-primitivity of A implies that B is edge-transitive and hence edge-primitive by the maximality of B_e in B. Again by [3, Lemma 3.4], X is B-arc-transitive, and by Lemma 2.2, B is 2- or 3-transitive and B_v is non-solvable. Clearly, $\mathrm{Soc}(B)=T$. By Lemma 2.3, $B_e = \mathrm{A}_4.\mathbb{Z}_2$, $\mathrm{S}_4.\mathbb{Z}_2$, $(\mathrm{A}_4 \times \mathrm{A}_4).\mathbb{Z}_2$, $(\mathrm{S}_4 \times \mathrm{S}_4).\mathbb{Z}_2$, or $((\mathrm{A}_4 \times \mathrm{A}_4).\mathbb{Z}_2).\mathbb{Z}_2$. Checking the pairs (B, B_e) listed in [6, Tables 14–20], we have $T=\mathrm{A}_6=\mathrm{PSL}(2,9)$, $\mathrm{PSL}(2,p)$ with p a prime ($p \equiv \pm 1 \pmod 8$ or $p \equiv \pm 11, \pm 19 \pmod{40}$), or $\mathrm{PSL}(3,2)$. Since $5 \nmid |\mathrm{PSL}(3,2)|$, we have $T=\mathrm{PSL}(2,9)$ or $\mathrm{PSL}(2,p)$ ($p \equiv \pm 1 \pmod 8$, or $p \equiv \pm 11, \pm 19 \pmod{40}$). Since X has valency 5, by [3, Theorem 1.3], X is isomorphic to K_6, the $\mathrm{PSL}(2,p)$-graph or the $\mathrm{PGL}(2,p)$-graph.

Let $s \geq 4$. By [6, Theorem 1.3], X is isomorphic to the $\mathrm{J}_3.\mathbb{Z}_2$-graph, the $\mathrm{PSL}(3,4).D_{12}$-graph or the $\mathrm{PSp}(4,4).\mathbb{Z}_4$-graph. □

Acknowledgements This work was partially supported by the National Natural Science Foundation of China (11171020, 11231008) and also by an ARC Discovery Project grant.

References

1. Conway, H.J., Curtis, R.T., Norton, S.P., Parker, R.A., Wilson, R.A.: Atlas of Finite Groups. Oxford University Press, Oxford (1985)
2. Dixon, J.D., Mortimer, B.: Permutation Groups. Springer, New York (1996)
3. Giudici, M., Li, C.H.: On finite edge-primitive and edge-quasiprimitive graphs. J. Comb. Theory, Ser. B **100**, 275–298 (2010)
4. Guo, S.T., Feng, Y.Q., Li, C.H.: Edge-primitive tetravalent graphs. J. Comb. Theory Ser. B (submitted)
5. Li, C.H.: The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$. Trans. Am. Math. Soc. **353**, 3511–3529 (2001)
6. Li, C.H., Zhang, H.: The finite primitive groups with soluble stabilizers, and edge-primitive s-arc transitive graphs. Proc. Lond. Math. Soc. **103**, 441–472 (2011)
7. Lorimer, P.: Vertex-transitive graphs: Symmetric graphs of prime valency. J. Graph Theory **8**, 55–68 (1984)
8. Sabidussi, G.: Vertex-transitive graphs. Monatshefte Math. **68**, 426–438 (1964)
9. Weiss, R.M.: Kantenprimitive Graphen vom Grad drei. J. Comb. Theory, Ser. B **15**, 269–288 (1973)
10. Weiss, R.M.: s-Transitive graphs. Algebr. Methods Graph Theory **2**, 827–847 (1981)
11. Weiss, R.M.: An application of p-factorization methods to symmetric graphs. Math. Proc. Camb. Philos. Soc. **85**, 43–48 (1979)
12. Weiss, R.M.: A characterization and another construction of Janko's group J_3. Trans. Am. Math. Soc. **298**, 621–633 (1986)