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Relative hemisystems on the Hermitian surface

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Abstract Let S be a generalized quadrangle of order (q^2, q) containing a subquadrangle S' of order (q, q). Then any line of S either meets S' in $q+1$ points or is disjoint from S'. After Penttila and Williford (J. Comb. Theory, Ser. A 118:502–509, 2011), we call a subset H of the lines disjoint from S' a *relative hemisystem* of S with respect to S', provided that for each point x of $S \setminus S'$, exactly half of the lines through x disjoint from S' lie in H. A new infinite family of relative hemisystems on the generalized quadrangle $\mathcal{H}(3, q^2)$ admitting the linear group $\mathrm{PSL}(2, q)$ as an automorphism group is constructed. The association schemes arising from our construction are not equivalent to those arising from the Penttila–Williford relative hemisystems.

Keywords Generalized quadrangle · Relative hemisystem · Association scheme

1 Introduction and motivation

A *generalized quadrangle of order* (s, t) is an incidence structure of points and lines with the properties that any two points (lines) are incident with at most one line (point), every point is incident with $t+1$ lines, every line is incident with $s+1$ points, and for any point P and line l that are not incident, there is a unique point on l collinear with P. The standard reference is [12].

One of the classical generalized quadrangles is $\mathcal{H}(3, q^2)$, the incidence structure of all points and lines of a nonsingular Hermitian surface in $\mathrm{PG}(3, q^2)$. It is a generalized quadrangle of order (q^2, q) with automorphism group $\mathrm{P}\Gamma\mathrm{U}(4, q^2)$. The dual of $\mathcal{H}(3, q^2)$ is the generalized quadrangle $\mathcal{Q}^-(5, q)$, the incidence structure of all points

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and lines of an elliptic quadric in PG$(5,q)$, a generalized quadrangle of order (q,q^2), with automorphism group PΓO$^-(6,q)$, [12, Theorem 3.2.3].

In his celebrated paper [14], Beniamino Segre introduced the notion of regular system on $\mathcal{H}(3,q^2)$. A *regular system of order* m on $\mathcal{H}(3,q^2)$ is a set $\mathcal{R}$ of lines of $\mathcal{H}(3,q^2)$ with the property that every point lies on exactly m lines of $\mathcal{R}$, $0<m<q+1$. Segre proved that, if q is odd, such a system must necessarily have $m=(q+1)/2$ and called a regular system on $\mathcal{H}(3,q^2)$ of order $(q+1)/2$ a *hemisystem* on $\mathcal{H}(3,q^2)$. He also constructed a hemisystem on $\mathcal{H}(3,9)$ admitting the linear group PSL$(3,4)$ as an automorphism group.

In [3], the nonexistence of regular systems on $\mathcal{H}(3,q^2)$ for q even was established. A simple proof that a regular system on $\mathcal{H}(3,q^2)$ is a hemisystem (and so q is odd) was also given by Thas [15]. Cameron, Goethals, and Seidel [6] adopted a more general approach defining a hemisystem on a generalized quadrangle of order (s,s^2), s odd, to be a set of points meeting every line in $(s+1)/2$ points.

In 1995, Thas [16] conjectured the nonexistence of hemisystems on $\mathcal{H}(3,q^2)$ for $q>3$. In 2005, exactly forty years after Segre's paper appeared, Penttila and Cossidente [7] presented counterexamples to this conjecture. They proved that the generalized quadrangle $\mathcal{H}(3,q^2)$, q odd, has a hemisystem admitting P$\Omega^-(4,q)$ and giving Segre's example for $q=3$. Also they constructed a "sporadic" example on $\mathcal{H}(3,25)$ admitting $3.A_7.2$. The interest in constructing hemisystems on generalized quadrangles is motivated, for instance, by the study of the so-called partial quadrangles. These were introduced by Cameron [5]. A *partial quadrangle* PQ(s,t,μ) is an incidence structure of points and lines with the properties that any two points are incident with at most one line, every point is incident with $t+1$ lines, every line is incident with $s+1$ points, any two noncollinear points are jointly collinear with exactly μ points, and for any point P and line l that are not incident, there is at most one point Q on l collinear with P. However, there are not many constructions of partial quadrangles known, and most of them arise from a generalized quadrangle of order (s,s^2) by deleting a point, all lines on that point, and all points collinear with that point; this gives a PQ$(s-1,s^2,s^2-s)$.

The preceding results imply that a hemisystem on a generalized quadrangle of order (s,s^2) gives a partial quadrangle PQ$((s-1)/2,s^2,(s-1)^2/2)$ (the points of the partial quadrangle being the points of the hemisystem, and the lines of the partial quadrangle being the lines of the generalized quadrangle).

Hemisystems on $\mathcal{H}(3,q^2)$ are also related to strongly regular graphs. Indeed, a hemisystem yields a strongly regular decomposition of the collinearity graph of $\mathcal{Q}^-(5,q)$ in the sense of [9]. Indeed, in [11], strongly regular graphs whose vertices can be partitioned into two subsets of equal size on each of which a strongly regular subgraph is induced are investigated, and it is shown that such graphs must belong either to the two parameter family of Smith graphs, see [6, Sect. 6], or to a special one parameter family of graphs. Notice that the collinearity graph of $\mathcal{Q}^-(5,q)$ is a Smith graph.

Finally, the construction of hemisystems in [7] was extended in a very clever way in the beautiful paper [1] of Bamberg, Giudici, and Royle by proving that every flock generalized quadrangle of order (s^2,s), s odd, contains a hemisystem.

As we have already observed, regular systems on $\mathcal{H}(3,q^2)$ for even q cannot exist. Is there an analogue of hemisystems on $\mathcal{H}(3,q^2)$, q even? The answer is affirmative and is contained in a recent paper by Penttila and Williford [13].

Let S be a generalized quadrangle of order (q^2,q) containing a subquadrangle S' of order (q,q). Then any line of S either meets S' in $q+1$ points or is disjoint from S'. After [13], we call a subset H of the lines disjoint from S' a *relative hemisystem* of S with respect to S', provided that for each point x of $S\setminus S'$, exactly half of the lines through x disjoint from S' lie in H. Of course, q must be even for this to be possible, since every point of $S\setminus S'$ is on q lines disjoint from S'.

It happens that the symplectic generalized quadrangle $\mathcal{W}(3,q)$ that has order (q,q) can always be embedded in the generalized quadrangle $\mathcal{H}(3,q^2)$ as we will show later (Sect. 2.1) and a relative hemisystem of $\mathcal{H}(3,q^2)$ with respect to $\mathcal{W}(3,q)$ produces certain association schemes. Indeed, Penttila and Williford [13] proved that a relative hemisystem of $\mathcal{H}(3,q^2)$ always gives rise to a primitive 3-class cometric association scheme, which is not metric.

Theorem 1.1 [13, Theorem 4] *If $\mathcal{H}(3,q^2)$, $q>2$ has a relative hemisystem with respect to $\mathcal{W}(3,q)$, then a primitive Q-polynomial 3-class scheme can be constructed on the lines of the relative hemisystem with the following relations*:

- R_1: *We have $(l,m)\in R_1$ if and only if l and m are not concurrent and $|O_l\cap O_m| = 1$*;
- R_2: *We have $(l,m)\in R_2$ if and only if l and m are not concurrent and $|O_l\cap O_m| = q+1$*;
- R_3: *We have $(l,m)\in R_3$ if and only if l and m are concurrent*;

where for a fixed line l which does not meet $\mathcal{W}(3,q)$, O_l denotes the set of lines meeting both l and $\mathcal{W}(3,q)$.

Also, they showed that $\mathcal{H}(3,q^2)$, q even, has a relative hemisystem with respect to an embedded $\mathcal{W}(3,q)$ admitting the orthogonal group $\mathrm{P}\Omega^-(4,q)$ [13, Theorem 5] and left as an open question whether or not $\mathcal{H}(3,q^2)$ has any nonisomorphic relative hemisystems. Notice that the Penttila–Williford relative hemisystem could be considered the analogue of the Cossidente–Penttila hemisystem just because they have the same automorphism group.

In this paper we will construct a new infinite family of relative hemisystems of $\mathcal{H}(3,q^2)$, $q=2^m$, $m>2$, admitting the linear group $\mathrm{PSL}(2,q)$ as an automorphism group. Our approach is elementary in nature. We start from the two Penttila–Williford relative hemisystems arising from the action of the group $\mathrm{P}\Omega^-(4,q)$ on generators of $\mathcal{H}(3,q^2)$ disjoint from a given subquadrangle $\mathcal{W}(3,q)$. This means that an elliptic quadric $\mathcal{Q}^-(3,q)$ of $\mathcal{W}(3,q)$ has been fixed. Then, we consider a conic section of $\mathcal{Q}^-(3,q)$ and its group that is isomorphic to $\mathrm{PSL}(2,q)$. Of course, each of the two Penttila–Williford relative hemisystems splits into orbits under the action of $\mathrm{PSL}(2,q)$. The idea is to consider a mild perturbation of the Penttila–Williford relative hemisystems suitably "interchanging" certain $\mathrm{PSL}(2,q)$-orbits.

2 The generalized quadrangle $\mathcal{H}(3, q^2)$

In this section we give some general information on the generalized quadrangle $\mathcal{H}(3, q^2)$.

In $\mathrm{PG}(3, q^2)$ a *nonsingular Hermitian surface* is defined to be the set of all absolute points of a nondegenerate unitary polarity and is denoted by $\mathcal{H}(3, q^2)$.

A Hermitian surface $\mathcal{H}(3, q^2)$ has the following properties, for which [14], [10, Chap. 19] are excellent sources.

1. The number of points on $\mathcal{H}(3, q^2)$ is $(q^2 + 1)(q^3 + 1)$.
2. Any line of $\Sigma = \mathrm{PG}(3, q^2)$ meets $\mathcal{H}(3, q^2)$ in either 1 or $q + 1$ or $q^2 + 1$ points. The latter lines are the *generators* of $\mathcal{H}(3, q^2)$, and there are $(q + 1)(q^3 + 1)$ such lines. The intersections of size $q + 1$ are Baer sublines, whereas lines meeting $\mathcal{H}(3, q^2)$ in one point are called *tangent lines*.
3. Through every point P of $\mathcal{H}(3, q^2)$ there pass exactly $q + 1$ generators, and these generators are coplanar. The plane containing these generators, say π_P, is the polar plane of P with respect to the unitary polarity defining $\mathcal{H}(3, q^2)$. The tangent lines through P are precisely the remaining $q^2 - q$ lines of π_P incident with P, and π_P is called the *tangent plane* to $\mathcal{H}$ at P.
4. Every plane of Σ which is not a tangent plane to $\mathcal{H}(3, q^2)$ meets $\mathcal{H}(3, q^2)$ in a nondegenerate Hermitian curve $\mathcal{H}(2, q^2)$.

2.1 $\mathcal{W}(3, q)$ inside $\mathcal{H}(3, q^2)$

The symplectic group $\mathrm{PSp}(4, q)$ is embedded in $\mathrm{P}\Gamma\mathrm{U}(4, q^2)$ as a subfield subgroup, stabilizing a subquadrangle of $\mathcal{H}(3, q^2)$ isomorphic to $\mathcal{W}(3, q)$. In terms of coordinates, assuming that $\mathcal{H}(3, q^2)$ has the equation $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} + X_4^{q+1} = 0$, where $X_1, \dots, X_4$ are homogeneous projective coordinates in $\mathrm{PG}(3, q^2)$, q even, the set $\{(x, x^q, y, y^q) : x, y \in \mathrm{GF}(q^2)\}$ is the point set of a symplectic Baer subgeometry $\mathcal{W}(3, q)$ embedded in $\mathcal{H}(3, q^2)$ [14]. The stabilizer of $\mathcal{W}(3, q)$ in $\mathrm{P}\Gamma\mathrm{U}(4, q^2)$ has exactly two orbits on points of $\mathcal{H}(3, q^2)$, $\mathcal{W}(3, q)$ and its complement. In particular, every generator of $\mathcal{H}(3, q^2)$ either meets $\mathcal{W}(3, q)$ in a totally isotropic line of $\mathcal{W}(3, q)$ or is disjoint from it. This means that a point of $\mathcal{H}(3, q^2) \setminus \mathcal{W}(3, q)$ lies on a unique totally isotropic line of $\mathcal{W}(3, q)$. The group $\mathrm{PSp}(4, q)$ has two conjugacy classes of subgroups isomorphic to $\mathrm{PSL}(2, q^2) \simeq \mathrm{P}\Omega^-(4, q)$. A group in the first class stabilizes a regular spread of $\mathcal{W}(3, q)$ (i.e., a partition of the point set of $\mathcal{W}(3, q)$ into $q^2 + 1$ totally isotropic lines). A group in the second class stabilizes an elliptic quadric $\mathcal{Q}^-(3, q)$ that is an ovoid of $\mathcal{W}(3, q)$ (i.e., a subset E of $q^2 + 1$ points of $\mathcal{W}(3, q)$ such that every totally isotropic line of $\mathcal{W}(3, q)$ meets E at exactly one point). We are interested in this second class. Fix an elliptic quadric $\mathcal{Q}^-(3, q)$ of $\mathcal{W}(3, q)$ and its group $\mathrm{P}\Omega^-(4, q)$. Let us fix a conic section, say $\mathcal{C}$, of $\mathcal{Q}^-(3, q)$ obtained by intersecting $\mathcal{Q}^-(3, q)$ with a plane, say π. Since q is even, $\mathcal{C}$ has a nucleus N. It turns out that $\pi = N^{\mathcal{A}}$, where $\mathcal{A}$ is the symplectic polarity fixed by $PSp(4, q)$. There are $q + 1$ totally isotropic lines on N in π, and each of them is tangent to $\mathcal{C}$. By Witt's theorem, every isometry of $\mathcal{A}$ extends to a semisimilarity of $\mathcal{U}$, where $\mathcal{U}$ is the unitary polarity fixed by $\mathrm{P}\Gamma\mathrm{U}(4, q^2)$. It follows that the $\mathrm{GF}(q^2)$-span of π is the tangent plane $\pi^{\mathcal{U}}$ to $\mathcal{H}(3, q^2)$ at N. The stabilizer of $\mathcal{C}$ in $\mathrm{P}\Omega^-(4, q)$ is a group isomorphic to $\mathrm{G}=\mathrm{PSL}(2,q)$. The group G stabilizes $q/2$ elliptic quadrics of $\mathcal{W}(3,q)$, all sharing the conic $\mathcal{C}$ [4, Type 3g(ii), p. 262]. Also, let t be the unique nonlinear involutory automorphism fixing every point of $\mathcal{W}(3,q)$. It is easily seen that t commutes with G.

3 The Penttila–Williford relative hemisystem

Here, we briefly recall the construction of the infinite family of relative hemisystems of $\mathcal{H}(3,q^2)$ due to Penttila and Williford [13]. We have already observed that the stabilizer of $\mathcal{W}(3,q)$ in $\mathrm{P\Gamma U}(4,q^2)$ has exactly two orbits on points of $\mathcal{H}(3,q^2)$, namely $\mathcal{W}(3,q)$ and its complement. Also, it acts transitively on totally isotropic lines of $\mathcal{W}(3,q)$. Using the action of the normalizer of a Singer cyclic group of $\mathrm{P\Omega}^-(4,q)$, Penttila and Williford proved that $\mathrm{P\Omega}^-(4,q)$, $q>2$, has exactly two orbits on generators of $\mathcal{H}(3,q^2)$ disjoint from $\mathcal{W}(3,q)$ [13, Theorem 5]. It turns out that each of these orbits, say H_1 and H_2, is a relative hemisystem with respect to $\mathcal{W}(3,q)$. By construction, the involution t introduced above switches the two relative hemisystems H_1 and H_2 [13, Theorem 5].

4 The new family

We adopt the geometric setting of the Penttila–Williford construction. In particular, we start from the two relative hemisystems H_1 and H_2 described in Sect. 3. From now on we assume that $q>2$.

From our discussion in Sect. 2.1 it follows that the group G permutes the $q+1$ generators of $\mathcal{H}(3,q^2)$ on N. Let g be one of such generators. The stabilizer K of g in G has order $q(q-1)$. The subgroup S of K of order q, which is elementary abelian, fixes g pointwise. Hence, each involution in S has g as an axis [8]. A subgroup B of K of order $q-1$ partitions the point set of $g\setminus\{N, g\cap\mathcal{C}\}$ into $q+1$ orbits of size $q-1$, q of which consist of imaginary points of g, and exactly one of them lies on $\mathcal{W}(3,q)$. Call the B-orbits on imaginary points of g, $\Sigma_1,\dots,\Sigma_q$. Since G permutes the generators on N, each orbit Σ_i gives rise to a G-orbit of size q^2-1, say $\overline{\Sigma}_i$, $i=1,\dots,q$, lying on $N^{\mathcal{U}}$. If $P\in\overline{\Sigma}_i$, the stabilizer of P in G is S.

Lemma 4.1 *The* 2*-group* S *acts on the* q *generators of* $\mathcal{H}(3,q^2)$ *on* P *distinct from* g *with two orbits of size* $q/2$.

Proof Since G is a subgroup of the group $\mathrm{P\Omega}^-(4,q)$ stabilizing a Penttila–Williford relative hemisystem, any $\mathrm{P\Omega}^-(4,q)$-orbit on generators through P is the union of G-orbits. Under the action of $\mathrm{P\Omega}^-(4,q)$, the set of q generators on P distinct from g splits into two orbits of size $q/2$. On the other hand, the action of S on points of $\mathcal{H}(3,q^2)\setminus\{g\}$ is semiregular, and hence all S-orbits in $P^{\mathcal{U}}\setminus\{g\}$ have size q. We want to prove that each S-orbit in $P^{\mathcal{U}}\setminus\{g\}$ is an arc.

Let us choose $d\in\mathrm{GF}(q)$ so that the polynomial $x^2+dxy+y^2$ is irreducible over $\mathrm{GF}(q)$. Define $Q:\mathrm{GF}(q)^4\to\mathrm{GF}(q)$ by $Q(x,y,z,w)=xw+y^2+dyz+z^2$. Then

Q is a quadratic form of minus type. Let $\mathcal{Q}^-(3,q)$ be the set of zeroes of Q, so that $\mathcal{Q}^-(3,q)$ is the elliptic quadric $\{(1,s,t,s^2+dst+t^2): s,t\in \mathrm{GF}(q)\}\cup\{(0,0,0,1)\}\in \mathrm{PG}(3,q)$. Let

$$\mathrm{M}_\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda^2 & 0 & d\lambda & 1 \end{bmatrix}, \quad \lambda \in \mathrm{GF}(q).$$

It is easy to see that M_λ is an isometry of Q. Let $\mathrm{S}=\{\mathrm{M}_\lambda : \lambda\in\mathrm{GF}(q)\}$. Then S is an elementary abelian group of order q. The group S fixes the conic $\mathcal{C}=\{(1,s,0,s^2): s\in\mathrm{GF}(q)\}\cup\{(0,0,0,1)\}$ obtained by sectioning $\mathcal{Q}^-(3,q)$ with the plane $z=0$. The conic $\mathcal{C}$ has nucleus $(0,1,0,0)$. Let us consider now the Hermitian form $H((x,y,z,w),(x',y',z',w'))=xw'^q+wx'^q+dyz'^q+dzy'^q$, and let $\mathcal{H}(3,q^2)$ be the associated Hermitian surface. Let $g=\{(0,y,0,w): y,z\in\mathrm{GF}(q^2)\}$. Then g is a generator of $\mathcal{H}(3,q^2)$ on the point $\{(0,0,0,1)\}$. Let $P=(0,1,0,a)$ with $a\in\mathrm{GF}(q^2)\setminus\mathrm{GF}(q)$. Then $P^{\mathcal{U}}$ ($\mathcal{U}$ is the Hermitian polarity associated with $\mathcal{H}(3,q^2)$) has the equation $z=(a^q/d)x$. Restricting S to $P^{\mathcal{U}}=\{(x,y,(a^q/d)x,w): x,y,w\in \mathrm{GF}(q^2)\}$ shows that M_λ acts as

$$\begin{bmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \lambda^2+\lambda a^q & 0 & 1 \end{bmatrix}$$

on the coordinates x,y,w, which has each orbit on $P^{\mathcal{U}}$ not on g, a q-arc. Indeed, the determinant of the matrix

$$\begin{bmatrix} x & y & w \\ x & \lambda_1 x + y & (\lambda_1^2+\lambda_1 a^q)x+w \\ x & \lambda_2 x+y & (\lambda_2^2+\lambda_2 a^q)x+w \end{bmatrix}$$

with $\lambda_1,\lambda_2\neq 0$ and $\lambda_1\neq\lambda_2$ is $x^3(\lambda_1\lambda_2^2+\lambda_1^2\lambda_2)$, which is zero if and only if the point (x,y,w) lies on g or $\lambda_1=\lambda_2$.

Since the isometry group of $\mathcal{Q}^-(3,q)$ is transitive on conic sections of $\mathcal{Q}^-(3,q)$, we can choose $\mathcal{C}$ as we did. Since the group of $\mathcal{C}$ is transitive on points of $\mathcal{C}$, we can always assume that the generator g contains the point $(0,0,0,1)$. Finally, $\mathcal{Q}^-(3,q)$ determines $\mathcal{H}(3,q^2)$. Notice that the group S acts on $P^{\mathcal{U}}$ as an elation group of order q whose centers lie on g. Let A be an S-orbit in $P^{\mathcal{U}}\setminus\{g\}$. Let $X\in$ A, and let ι be an involution in S. Then $\iota(X)\in$ A, and the line $X\iota(X)$ meets g at one point, which is the center of ι. Since A is an arc, the centers of the involutions of S are all distinct. In particular, there is exactly one involution in S (M_λ for $\lambda=a+a^q$) fixing each of the q generators on P distinct from g. Hence, the points of A lie on $q/2$ generators, A meets each of the $q/2$ generators lying on an orbit of the stabilizer of P in $\mathrm{P}\Omega_4^-(q)$ exactly twice, and the $q/2$ generators on P in the stabilizer of P in $\mathrm{P}\Omega^-(4,q)$ are also an orbit of the stabilizer of P in G. □

It follows that G has $2q$ orbits of size $(q^3-q)/2$ on generators meeting $\overline{\Sigma}_i$, $i=1,\dots,q$ (distinct from the generators on N). Of course, all these $q^2(q^2-1)$

generators are disjoint from $\mathcal{W}(3,q)$. The number of generators of $\mathcal{H}(3,q^2)$ disjoint from $\mathcal{W}(3,q)$ is $(q+1)(q^3-q^2)=q^2(q^2-1)$, and hence the $2q$ orbits above cover all generators of $\mathcal{H}(3,q^2)$ disjoint from $\mathcal{W}(3,q)$.

We are ready to construct our new relative hemisystem by perturbating a Penttila–Williford relative hemisystem.

The involution t fixing $\mathcal{W}(3,q)$ fixes no point outside $\mathcal{W}(3,q)$ and hence partitions the set $\{\Sigma_1,\dots,\Sigma_q\}$ into pairs $\{\Sigma_i,t(\Sigma_i)\}$. We recall that the involution t interchanges H_1 and H_2. Each such pair produces four G-orbits on generators of size $(q^3-q)/2$, say X_i and $t(Y_i)$ (arising from $\overline{\Sigma}_i$) and Y_i and $t(X_i)$ (arising from $t(\overline{\Sigma_i})$), where we can assume that X_i and Y_i are in H_1 and hence that $t(X_i)$ and $t(Y_i)$ are in H_2. If $P(X)$ denotes the point set of $\mathcal{H}(3,q^2)$ covered by the generators in the set X, we have that $P(X_i)\cup P(Y_i)$ is the same as $P(t(X_i))\cup P(t(Y_i))$, and both sets are t-invariant. Indeed, X_i is a G-orbit whose generators meet $\overline{\Sigma}_i$. Then $P(X_i)$ is the union of point G-orbits, and one of them is certainly $\overline{\Sigma}_i$. All the other point G-orbits have full size $q(q^2-1)$ (if R is a point of $\mathcal{H}(3,q^2)\setminus\mathcal{W}(3,q)$, then it can be proved that $\mathrm{Stab}_G(R)$ acts as the identity on $\mathcal{H}(3,q^2)\setminus\mathcal{W}(3,q)$). The involution t sends $\overline{\Sigma}_i$ to $t(\overline{\Sigma}_i)$ and either fixes a point G-orbit on $P(X_i)$ or moves a point G-orbit O on $P(X_i)$ to another point G-orbit, say $t(O)$, of course of the same size. Since the generators of H_1 cover the whole of $\mathcal{H}(3,q^2)\setminus\mathcal{W}(3,q)$, $t(O)$ is a point G-orbit on generators belonging again to H_1. Such a point G-orbit must be on $P(Y_i)$ since t sends $\overline{\Sigma}_i$ to $t(\overline{\Sigma}_i)$. On the other hand, t interchanges H_1 and H_2, and hence we have the following lemma.

Lemma 4.2 *The sets $P(X_i)\cup P(Y_i)$ and $P(t(X_i))\cup P(t(Y_i))$ are both t-invariant and cover the same point set of $\mathcal{H}(3,q^2)$*

We have the following theorem.

Theorem 4.3 *There exists an infinite family of relative hemisystems of $\mathcal{H}(3,q^2)$, q even, $q>4$, admitting* $\mathrm{PSL}(2,q)$ *as an automorphism group.*

Proof The key idea is to consider a mild perturbation of one of the two relative hemisystems of Penttila and Williford arising from a given elliptic quadric on $\mathcal{C}$ fixed by $\mathrm{G}=\mathrm{PSL}(2,q)$, by deleting certain G-orbits from H_1 and by adding their images under t belonging to H_2 as in Lemma 4.2. Since $q>4$, there exist (apart from H_1 and H_2) at least q other relative hemisystems fixed by G:

$$\big(H_1\setminus\{X_i\cup Y_i\}\big)\cup\big\{t(X_i)\cup t(Y_i)\big\}\quad\text{for some } i=1,\dots,q.$$

On the other hand, the number of G-invariant relative hemisystems of Penttila and Williford (H_1 and H_2 included) is exactly q. It follows that there exists a relative hemisystem admitting G as an automorphism group. □

Remark 4.4 From the Penttila–Williford construction, once fixed a symplectic subquadrangle $\mathcal{W}(3,q)$ of $\mathcal{H}(3,q^2)$, it follows that any elliptic quadric on the conic $\mathcal{C}$ gives rise to two Penttila–Williford relative hemisystems. Since the number of elliptic quadrics in $\mathcal{W}(3,q)$ on $\mathcal{C}$ stabilized by G is $q/2$, certainly there exist q relative hemisystems of Penttila–Williford type on $\mathcal{H}(3,q^2)$. Assume that $q=4$. In this case we have eight G-orbits on generators of $\mathcal{H}(3,16)$ disjoint from $\mathcal{W}(3,4)$, each of size 60, say X_1, $t(X_1)$, Y_1, $t(Y_1)$, Z_1, $t(Z_1)$, U_1, $t(U_1)$. Since there are two elliptic quadrics of $\mathcal{W}(3,4)$ on $\mathcal{C}$, there are four Penttila–Williford relative hemisystems:

$$\mathrm{H}_1 : X_1 \cup Y_1 \cup Z_1 \cup U_1, \qquad \mathrm{H}_2 : t(X_1) \cup t(Y_1) \cup t(Z_1) \cup t(U_1),$$
$$\mathrm{H}_3 : X_1 \cup Y_1 \cup t(Z_1) \cup t(U_1), \qquad \mathrm{H}_4 : t(X_1) \cup t(Y_1) \cup Z_1 \cup U_1.$$

Let us fix an elliptic quadric, say E_1, of $\mathcal{W}(3,4)$ on $\mathcal{C}$. We get two Penttila–Williford relative hemisystems, suppose H_1 and H_2. From our construction, if we choose, for instance, X_1 and Y_1 and replace them with $t(X_1)$ and $t(Y_1)$, we get the relative hemisystem H_4 whose complement is the relative hemisystem H_3, and these two relative hemisystems are of Penttila–Williford type arising from the second elliptic quadric, say E_2, of $\mathcal{W}(3,q)$ on $\mathcal{C}$. Of course, no other replacement is possible. This is why $q>4$ in the theorem is required. In other words, any $\mathrm{PSL}_2(4)$-relative hemisystem is of Penttila–Williford type.

Remark 4.5 The smallest Hermitian surface $\mathcal{H}(3,4)$ contains 45 points and 27 lines. Fix a symplectic Baer subgeometry $\mathcal{W}(3,2)$ of $\mathcal{H}(3,4)$. The symplectic group $\mathrm{PSp}_4(2) \simeq \mathrm{S}_6$ stabilizing $\mathcal{W}(3,2)$ has two orbits on points of $\mathcal{H}(3,4)$, $\mathcal{W}(3,2)$, and its complement in $\mathcal{H}(3,4)$, and the subgroup A_6 of $\mathrm{PSp}(4,2)$ has three orbits on generators: an orbit of size 15 (the totally isotropic lines of $\mathcal{W}(3,2)$) and two orbits of size 6 on generators disjoint from $\mathcal{W}(3,2)$ forming a double-six. Of course, each orbit of size 6 is trivially a relative hemisystem of $\mathcal{H}(3,4)$ with respect to $\mathcal{W}(3,2)$ admitting A_6 as an automorphism group.

5 The equivalence problem

Our relative hemisystems are not equivalent to those constructed by Penttila and Williford due to the fact that they have nonisomorphic automorphism groups. As already observed, a relative hemisystem of $\mathcal{H}(3,q^2)$ gives rise to a primitive 3-class cometric association scheme, which is not metric. In this section we show that the association schemes arising from our construction are not equivalent to those arising from the Penttila–Williford relative hemisystems. First of all, we need to introduce some definitions.

An ovoid of a generalized quadrangle is a set of points meeting each line in exactly one point. If a generalized quadrangle S of order (s,t) contains a subquadrangle S' of order (s,t'), then it can be seen that $x^\perp \cap S'$ is an ovoid of S' for all points $x \in S \setminus S'$. Following [2], we say that this ovoid is *subtended* by the point x and denote it by $\mathcal{O}_x$. An ovoid of S' is said to be *doubly subtended* if it is subtended by exactly two points of $S \setminus S'$. We say that the generalized quadrangle S' is doubly subtended in S if every subtended ovoid of S' is doubly subtended. We will denote by x' the other point subtending $\mathcal{O}_x$ and refer to x and x' as antipodes. In [2], Brown introduced the notion of doubly subtended subquadrangles S' of order (r,r) in a generalized quadrangle of order (r,r^2). This is, for instance, the case of $\mathcal{Q}(4,q)$, which is the dual of $\mathcal{W}(3,q)$, inside $\mathcal{Q}^-(5,q)$. Brown proved that S' is doubly subtended in S if and only if S

has an involutory automorphism that fixes S' pointwise. This automorphism simply interchanges antipodes while leaving the points of S' fixed. In other words, the point set of $S \setminus S'$ is an algebraic 2-fold cover of the strongly regular graph formed by taking all subtended ovoids of S' with two ovoids adjacent when they intersect in one point. This geometric context enabled Penttila and Williford to form a primitive Q-polynomial scheme. In particular, utilizing a slight modification of the argument of Thas in [15], Penttila and Williford showed that if $S = \mathcal{Q}^-(5,q)$ and $S' = \mathcal{Q}(4,q)$, the set X_1 of half of the points of $S \setminus S'$ such that every line of $S \setminus S'$ meets X_1 in $q/2$ points forms a primitive Q-polynomial subscheme of the Q-polynomial Q-bipartite scheme on the points of $S \setminus S'$, see [13, Theorems 3 and 5] for more details.

In the sequel we will prefer to adopt the dual setting, and hence we will work on the elliptic quadric $\mathcal{Q}^-(5,q)$ instead of $\mathcal{H}(3,q^2)$.

Let $\mathcal{Q} = \mathcal{Q}^-(5,q)$, and let $\mathrm{H}_1, \mathrm{H}_2$ and $\mathrm{G}_1, \mathrm{G}_2$ be two relative hemisystems of $\mathcal{Q}$ with respect to the subquadrangle $\mathcal{Q}' = \mathcal{Q}(4,q)$. For each vertex $x \in \mathrm{H}_1$, let x' denote its antipode in H_2, and for each vertex $x \in \mathrm{G}_1$, let x'' denote its antipode in G_2. Let α be the involution that fixes $\mathcal{Q}'$ pointwise. Since α swaps antipodes of all sets that are relative hemisystems with respect to $\mathcal{Q}'$, this map satisfies $\alpha(\mathrm{H}_1) = \mathrm{H}_2$ and $\alpha(\mathrm{G}_1) = \mathrm{G}_2$. Now suppose that the association schemes afforded by H_1 and G_1 are isomorphic. This is equivalent to saying that there is a map β from H_1 to G_1 such that for all $x, y \in \mathrm{H}_1$, we have that $\beta(x)$ is collinear with $\beta(y)$ if and only if x is collinear to y and that $\beta(x)$ is collinear with $\beta(y)''$ if and only if x is collinear to y'. We will show that β lifts to an automorphism of $\mathcal{Q}$.

We first extend β to a map from $\mathrm{H}_1 \cup \mathrm{H}_2$ to $\mathrm{G}_1 \cup \mathrm{G}_2$ by defining $\beta(x) = \alpha(\beta(\alpha(x)))$ for all $x \in \mathrm{H}_2$. Note that this implies that β commutes with the restriction of α to $\mathcal{Q}\setminus\mathcal{Q}'$. Then $\beta(x)$ is collinear with $\beta(y)$ if and only if x is collinear to y for all $x, y \in \mathrm{H}_2$ as well. Now let $x \in \mathrm{H}_1$ and $y' \in \mathrm{H}_2$. We have that x, y' are collinear if and only if $\beta(x), \beta(y)''$ are collinear, which occurs if and only if $\beta(x)$ is collinear with $\alpha(\beta(y)) = \beta(\alpha(y)) = \beta(y')$. Therefore all points of $\mathcal{Q}\setminus\mathcal{Q}'$ are collinear if and only if their images under β are collinear.

We now extend β to all of $\mathcal{Q}$ by defining it on $\mathcal{Q}'$. We call a collection of $q^2 - q$ lines in $\mathcal{Q}\setminus\mathcal{Q}'$ a *bundle* if the lines meet in a single point in $\mathcal{Q}'$. For each bundle B, let p_B be the point of $\mathcal{Q}'$ where all of the lines meet. Bundles are precisely the sets of $q^2 - q$ lines in $\mathcal{Q}\setminus\mathcal{Q}'$ which do not meet in $\mathcal{Q}\setminus\mathcal{Q}'$ and with the property that no line in $\mathcal{Q}\setminus\mathcal{Q}'$ meets two lines of the set. This implies that bundles can be defined using only the incidence structure $\mathcal{Q}\setminus\mathcal{Q}'$, and so β must map bundles to other bundles. This yields a natural extension of β to $\mathcal{Q}'$. Let $p_B \in \mathcal{Q}'$ and define $\beta(p_B) = p_{\beta(B)}$. Clearly, p_B is collinear with $x \in \mathcal{Q}\setminus\mathcal{Q}'$ if and only if $x \in B$, which occurs if and only if $\beta(x) \in \beta(B)$ if and only if $\beta(p_B)$ is collinear with $\beta(x)$.

Lastly, let $p_C \in \mathcal{Q}'$, where C is a bundle of $\mathcal{Q}\setminus\mathcal{Q}'$. We have that p_B is collinear with p_C if and only if B is disjoint from C, which occurs if and only if $\beta(B)$ is disjoint from $\beta(C)$. The latter occurs if and only if $p_{\beta(B)} = \beta(p_B)$ is collinear with $p_{\beta(C)} = \beta(p_C)$. All cases considered, β is an automorphism of $\mathcal{Q}$ which maps H_1 to G_1, so H_1 and G_1 are equivalent.

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