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Degenerate flag varieties: moment graphs and Schröder numbers

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Abstract We study geometric and combinatorial properties of the degenerate flag varieties of type A. These varieties are acted upon by the automorphism group of a certain representation of a type A quiver, containing a maximal torus T. Using the group action, we describe the moment graphs, encoding the zero- and one-dimensional T-orbits. We also study the smooth and singular loci of the degenerate flag varieties. We show that the Euler characteristic of the smooth locus is equal to the large Schröder number and the Poincaré polynomial is given by a natural statistics counting the number of diagonal steps in a Schröder path. As an application we obtain a new combinatorial description of the large and small Schröder numbers and their q-analogues.

Keywords Schröder numbers · Moment graphs · Flag varieties · Quiver Grassmannians

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1 Introduction

For $n \geq 1$, let $\mathcal{F}^a_{n+1}$ be the degenerate flag variety attached to the Lie algebra $\mathfrak{sl}_{n+1}$ (see [13, 14]). This is a flat degeneration of the classical flag variety, defined using the PBW filtration on irreducible representations of $\mathfrak{sl}_{n+1}$ (see [17]). By construction, the $\mathcal{F}^a_{n+1}$ is acted upon by the degenerate Lie group SL^a_{n+1}, which is the semi-direct product of the Borel subgroup B and the abelian group $\mathbb{G}_a^N$, where $\mathbb{G}_a$ is the additive group of the field. In particular, $\mathbb{G}_a^N$ acts on $\mathcal{F}^a_{n+1}$ with an open dense orbit. The degenerate flag varieties are singular normal projective algebraic varieties, sharing many nice properties with their classical analogues. In particular, they enjoy a description in terms of linear algebra as subvarieties inside a product of Grassmann varieties.

It has been observed in [9] that the degenerate flag varieties can be identified with certain quiver Grassmannians of the equioriented quiver of type A_n. More precisely, $\mathcal{F}^a_{n+1}$ is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\dim A}(A \oplus A^*)$, where A and A^* are the path algebra of the equioriented A_n quiver, resp. its dual. This observation was used in two different ways: first, to get a deeper understanding of the geometry and combinatorics of the degenerate flag varieties, and, second, to generalize the results and constructions to a wider class of quiver Grassmannians. In this paper we continue the study of the varieties $\mathcal{F}^a_{n+1}$ using the techniques from the theory of quiver Grassmannians. More concretely, we achieve two things: first, we describe the combinatorial structure of the moment graph of $\mathcal{F}^a_{n+1}$. Second, we describe explicitly the smooth and singular loci of the degenerate flag varieties. Let us give a brief description of our results.

Recall that the notion of the moment graph attached to an algebraic variety X acted upon by an algebraic torus was introduced in [6, 21]. This combinatorial object captures the structure of zero- and one-dimensional orbits of T. It turns out to be very useful for describing various geometric properties of X, such as cohomology and intersection cohomology. Our first task is to describe the moment graph Γ of $\mathcal{F}^a_{n+1}$. We note that the automorphism group $\mathrm{Aut}(A \oplus A^*)$ acts on $\mathcal{F}^a_{n+1}$. The maximal torus T of the automorphism group acts with a finite number of fixed points (this number is equal to the normalized median Genocchi number, see [9, 13, 15]). It is proved in [9] that there exists a codimension one subgroup $\mathfrak{A} \subset \mathrm{Aut}(A \oplus A^*)$ containing the torus T such that $\mathfrak{A}$-orbits through T-fixed points are affine cells that provide a cellular decomposition of $\mathcal{F}^a_{n+1}$. We describe $\mathfrak{A}$ as a quotient of the Borel subgroup of SL_{2n}. Using this description, we prove the following theorem (for a more precise formulation see Sect. 3):

Theorem 1.1 *The number of one-dimensional T-orbits in $\mathcal{F}^a_{n+1}$ is finite. The edges of Γ correspond to the one-parameter subgroups of $\mathfrak{A}$.*

We note that the structure of Γ has many common features with its classical analogue (see [8, 22, 27]).

Our next goal is to describe the smooth locus of the degenerate flag varieties. Since $\mathcal{F}^a_{n+1}$ has a cellular decomposition by $\mathfrak{A}$-orbits of T-fixed points, it suffices to decide which T-fixed points are smooth. We recall that the T-fixed points are labeled

by collections $\mathbf{S} = (S_1, \ldots, S_n)$ of subsets of $\{1, \ldots, n+1\}$ such that $\#S_i = i$ and $S_i \subset S_{i+1} \cup \{i+1\}$. We denote the corresponding T-fixed point by $p_{\mathbf{S}}$.

Theorem 1.2 *A point $p_{\mathbf{S}}$ is smooth if and only if for all $1 \leq j < i \leq n$, the condition $i \in S_j$ implies $j+1 \in S_i$. The number of smooth T-fixed points is given by the large Schröder number r_n.*

We recall (see [20, 26]) that the large Schröder number r_n is equal to the number of Schröder paths, i.e. subdiagonal lattice paths starting at $(0,0)$ and ending at (n,n) with the following steps allowed: $(1,0)$, $(0,1)$ and $(1,1)$. In particular, Theorem 1.2 implies that the Euler characteristic of the smooth locus of $\mathcal{F}^a_{n+1}$ is equal to r_n. Moreover we prove the following theorem:

Theorem 1.3 *The Poincaré polynomial of the smooth locus of $\mathcal{F}^a_{n+1}$ is equal to the (scaled) q-Schröder number $q^{n(n-1)/2} r_n(q)$, where $r_n(q)$ is defined via the statistics on Schröder paths, counting the number of $(1,1)$ steps in a path.*

As an application, we obtain a new proof of the statement that $r_n(q)$ is divisible by $1+q$. The ratio is known to give a q-analogue of the small Schröder numbers.

Let us mention two more results of the paper. First, we prove that, for a general Dynkin type quiver Q and a projective Q-module P and an injective Q-module I, the quiver Grassmannian $\mathrm{Gr}_{\dim P}(P \oplus I)$ is smooth in codimension 2. Second, we prove that the smooth locus of $\mathcal{F}^a_{n+1}$ can be described as the subvariety of points where the desingularization map $R_{n+1} \to \mathcal{F}^a_{n+1}$ (see [16]) is one-to-one.

Finally, we note that all the results of the paper can be generalized to the case of the degenerate partial (parabolic) flag varieties.

Our paper is organized as follows:

In Sect. 1 we introduce the main objects and recall the main definitions and results needed in the rest of the paper.

In Sect. 2 we describe the moment graph of the degenerate flag varieties.

In Sect. 3 we prove a criterion for smoothness of a T-fixed point and compute the Euler characteristics and Poincaré polynomials.

In Appendix A we prove the regularity in codimension 2 of certain quiver Grassmannians.

In Appendix B we describe the smooth locus in terms of the desingularization.

In Appendix C we compute the moment graph for the degenerate flag variety $\mathcal{F}^a_4$.

2 Quiver Grassmannians and degenerate flag varieties

In this section we recall definitions and results on the degenerate flag varieties and quiver Grassmannians to be used in the main body of the paper.

2.1 Degenerate flag varieties

Let $\mathcal{F}_{n+1}$ be the complete flag variety for the group SL_{n+1}, i.e. the quotient SL_{n+1}/B by the Borel subgroup B. This variety has an explicit realization as the subvariety of the product of Grassmannians $\prod_{k=1}^{n} \mathrm{Gr}_k(\mathbb{C}^{n+1})$ consisting of collections

$(V_1, \dots, V_n)$ such that $V_i \subset V_{i+1}$ for all i. In [13, 14] flat degenerations $\mathcal{F}^a_{n+1}$ of the classical flag varieties were introduced. The degenerate flag varieties $\mathcal{F}^a_{n+1}$ are (typically singular) irreducible normal projective algebraic varieties, sharing many nice properties with their classical analogues. In particular, they also have a very explicit description in linear algebra terms. Namely, let W be an $(n+1)$-dimensional vector space with a basis $w_1, \dots, w_{n+1}$. Let $pr_k : W \to W$ be the projection operators defined by $pr_k w_k = 0$ and $pr_k w_i = w_i$ if $i \neq k$. The following lemma is proved in [13], Theorem 2.1.

Lemma 2.1 *The degenerate flag variety $\mathcal{F}^a_{n+1}$ is a subvariety of the product of Grassmannians $\prod_{k=1}^n \mathrm{Gr}_k(W)$, consisting of collections $(V_k)_{k=1}^n$ such that*

$$pr_{k+1} V_k \subset V_{k+1} \quad \text{for all } k = 1, \dots, n-1.$$

Another important property of the varieties $\mathcal{F}^a_{n+1}$ is that they admit a cellular decomposition into a disjoint union of complex cells. Moreover, there exists an algebraic group $\mathfrak{A}$ and a torus $T \subset \mathfrak{A}$ acting on $\mathcal{F}^a_{n+1}$ such that each cell contains exactly one T-fixed point and the $\mathfrak{A}$-orbit through this point coincides with the cell. Let us describe the combinatorics of the cells, postponing the description of the group action to the next subsection. So let $\mathbf{S} = (S_1, \dots, S_n)$ be a collection of subsets of the set $\{1, \dots, n+1\}$ such that each S_i contains i elements. Then the cells in $\mathcal{F}^a_{n+1}$ are labeled by the collections satisfying the following property:

$$S_k \subset S_{k+1} \cup \{k+1\}, \quad k = 1, \dots, n-1. \tag{2.1}$$

We call such collections admissible. The number of admissible collections (and hence the Euler characteristic of $\mathcal{F}^a_{n+1}$) is equal to the normalized median Genocchi number h_{n+1} (see [9, 13, 15]). We note that the correspondence between the admissible collections and T-fixed points is very explicit. Namely, for a collection $\mathbf{S}$ we denote by $p_{\mathbf{S}} \in \mathcal{F}^a_{n+1}$ a point defined by

$$p_{\mathbf{S}} = (V_1, \dots, V_n), \qquad V_k = \mathrm{span}(w_i,\ i \in S_k).$$

Clearly, such a point belongs to $\mathcal{F}^a_{n+1}$ if and only if the collection $\mathbf{S}$ is admissible.

2.2 Quiver Grassmannians

The construction above can be reformulated in the language of quiver Grassmannians (see e.g. [7, 24, 25]). Let Q be the equioriented type A_n quiver with vertices labeled by numbers from 1 to n and arrows $i \to i+1$, $i = 1, \dots, n-1$:

$$Q: \quad \bullet \to \bullet \to \cdots \to \bullet$$

For a representation M of Q we denote by M_k the subspace of M attached to the vertex k. For a pair $1 \le i \le j \le n$ let $R_{i,j}$ be an indecomposable representation of Q supported on the vertices $i, \dots, j$ (i.e. $(R_{i,j})_k = \mathbb{C}$ for $i \le k \le j$ and is trivial otherwise). We have the following immediate lemma.

Lemma 2.2

$$\dim \mathrm{Hom}(R_{i,j}, R_{k,l}) = \begin{cases} 1, & \text{if } k \le i \le l \le j, \\ 0, & \text{otherwise;} \end{cases}$$

$$\dim \mathrm{Ext}^1(R_{i,j}, R_{k,l}) = \begin{cases} 1, & \text{if } i+1 \le k \le j+1 \le l, \\ 0, & \text{otherwise.} \end{cases}$$

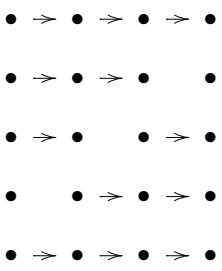
We note that the representations $R_{1,j}$ are injective and the $R_{i,n}$ are projective (note that these are all indecomposable injective and projective representations of Q). We set

$$I_k = R_{1,k}, \qquad P_k = R_{k,n}, \qquad P = \bigoplus_{k=1}^{n} P_k, \qquad I = \bigoplus_{k=1}^{n} I_k.$$

Hence, P is isomorphic to the path algebra of Q and I is isomorphic to its linear dual. For a dimension vector $\mathbf{e} = (e_1, \ldots, e_n)$ and a representation M of Q, we denote by $\mathrm{Gr}_{\mathbf{e}}(M)$ the quiver Grassmannian of $\mathbf{e}$-dimensional subrepresentations of M. Then by definition one gets

$$\mathcal{F}^a_{n+1} \simeq \mathrm{Gr}_{\dim P}(P \oplus I). \tag{2.2}$$

Remark 2.3 The representation $P \oplus I$ can be visualized by the following picture (here $n = 4$). Each fat dot corresponds to a basis vector and two dots corresponding to the vectors u and v are connected by an arrow $u \to v$ if u is mapped to v. The quiver obtained in this way is called the coefficient-quiver of $P \oplus I$.



(2.3)

The isomorphism (2.2) has many important consequences. In particular the automorphism group of the Q-module $P \oplus I$ acts on $\mathcal{F}^a_{n+1}$. The group $\mathrm{Aut}(P \oplus I)$ is of the form $\mathrm{Aut}(P \oplus I) = \left(\begin{smallmatrix} \mathrm{Aut}\, P & \mathrm{Hom}(I,P) \\ \mathrm{Hom}(P,I) & \mathrm{Aut}\, I \end{smallmatrix}\right)$. The part $\mathrm{Hom}(I, P)$ is one-dimensional ($\mathrm{Hom}(I, P) = \mathrm{Hom}(I_n, P_1)$). We denote by $\mathfrak{A} \subset \mathrm{Aut}(P \oplus I)$ the subgroup

$$\mathfrak{A} = \begin{pmatrix} \mathrm{Aut}\, P & 0 \\ \mathrm{Hom}(P, I) & \mathrm{Aut}\, I \end{pmatrix}.$$

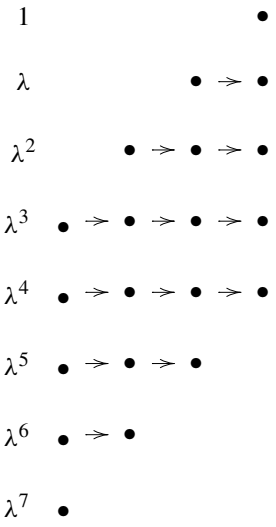
The group $\mathfrak{A}$ contains a torus T isomorphic to $(\mathbb{C}^*)^{2n}$, where each factor scales the corresponding indecomposable summand in $P \oplus I$. The importance of the group $\mathfrak{A}$ comes from the following lemma, proved in [9].

Lemma 2.4 *The group $\mathfrak{A}$ acts on $\mathcal{F}^a_{n+1}$ with a finite number of orbits. Each orbit is a complex affine cell, containing exactly one T-fixed point. The orbits are labeled by admissible collections.*

For an admissible collection $\mathbf{S}$ we denote by $C_{\mathbf{S}}$ the cell containing the T-fixed point $p_{\mathbf{S}}$.

Remark 2.5 We note that T contains a one-parameter subgroup which acts by the identity automorphism on the degenerate flag variety. Hence one gets a $(2n-1)$-dimensional torus acting effectively on $\mathcal{F}^a_{n+1}$, while the maximal torus T^c acting on the classical flag variety $\mathcal{F}_{n+1}$ is n-dimensional. We note that there is a natural embedding $T^c \subset T$. In fact recall that any point of $\mathcal{F}^a_{n+1}$ is of the form $(V_k)_{k=1}^n$, $V_k \subset W \simeq \mathbb{C}^{n+1}$. Hence any diagonal (in the basis w_i) matrix in $SL(W)$ induces an automorphism of the degenerate flag variety. Hence we obtain the embedding $T^c \subset T$.

Finally, we note that the torus T contains a one-dimensional sub-torus T_0 with the following properties: the set of T-fixed points coincides with the set of the T_0-fixed points and the attracting set of a fixed point p coincides the orbit $\mathfrak{A}p$ (which is an affine cell) [9, Theorem 5.1]. The action of the one-dimensional torus can be illustrated as follows ($n = 4$, the scalar $\lambda \in \mathbb{C}^*$ is the parameter of the torus and the power of λ corresponds to the scaling factor of the T_0 action):

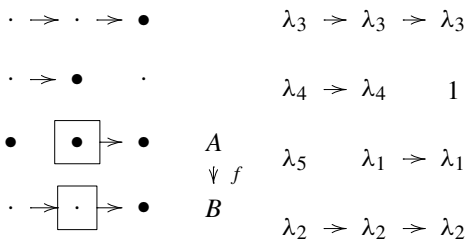


(2.4)

This picture is obtained from the picture (2.3) by putting the P-part on top of the I-part.

We conclude this section by describing the action of the torus T on the tangent space at a T-fixed point $p_{\mathbf{S}}$. Recall that the tangent space at $p_{\mathbf{S}}$ is isomorphic to $\mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$ where $M = P \oplus I$ ([9, Lemma 2.3], [7, 25]). Let θ_M be the coefficient quiver of M (see Remark 2.3) and let $\pi : \theta_M \to Q$ be the natural projec-

tion onto the A_n quiver Q. The coefficient quiver of $M/p_{\mathbf{S}}$ is $\theta_M \setminus \mathbf{S}$. The vector space $\mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$ has a distinguished basis, denoted by $\mathcal{B}$, parameterized by triples (A, f, B) where A is a predecessor-closed connected sub-quiver of $\mathbf{S}$, B is a successor-closed connected sub-quiver of $\theta_M \setminus \mathbf{S}$ and $f : A \to B$ is a quiver isomorphism compatible with π (see [11]). For example, in the left-hand side of the picture below



(2.5)

the fat dots highlight the coefficient-quiver $\mathbf{S}$ of a T-fixed point $p_{\mathbf{S}}$ of $\mathcal{F}_4^a$ and the frames highlight a distinguished basis vector of the tangent space at $p_{\mathbf{S}}$.

Proposition 2.6 *Given a T-fixed point $p_{\mathbf{S}}$ of $\mathcal{F}_{n+1}^a$, the torus T acts on the tangent space at $p_{\mathbf{S}}$ diagonally in the basis $\mathcal{B}$. Moreover the eigenvalues are (generically) distinct.*

Proof Given $\lambda \in T$ and $f \in \mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$, $(\lambda.f)(v) = \lambda.f(\lambda^{-1}.v)$. Now, by definition of T, each connected component R of θ_M has a weight $wt(R)$ and hence a basis vector (A, f, B) receives the weight $wt(B)/wt(A)$. □

To illustrate the previous proposition, let us consider $\mathcal{F}_4^a$ and the action of T depicted in the right-hand side of (2.5). The tangent space at $p_{\mathbf{S}}$ has dimension 7 and the torus acts in the standard basis $\mathcal{B}$ as the diagonal matrix $\mathrm{diag}(\frac{1}{\lambda_3}, \frac{\lambda_3}{\lambda_4}, \frac{\lambda_2}{\lambda_4}, \frac{\lambda_3}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \frac{1}{\lambda_2}, \frac{\lambda_4}{\lambda_5})$. The one-dimensional torus T_0 is given by putting $\lambda_i := \lambda^i$. In particular its action on the tangent space at $p_{\mathbf{S}}$ is given by the diagonal matrix $\mathrm{diag}(\lambda^{-3}, \lambda^{-1}, \lambda^{-2}, \lambda^2, \lambda^1, \lambda^{-2}, \lambda^{-1})$. Notice that the eigenvalues of the T_0 action are *not* distinct.

Corollary 2.7 *The T-fixed one-dimensional vector subspaces of* $\mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$ *are precisely the coordinate ones, i.e. those generated by standard basis vectors.*

2.3 Partial flag varieties

The whole picture described above has a straightforward generalization to the case of partial flag varieties. Namely, given a collection $\mathbf{d} = (d_1, \ldots, d_k)$, where $1 \leq d_1 < d_2 < \cdots < d_k \leq n$, let $\mathcal{F}_{\mathbf{d}}$ be the corresponding partial flag variety for SL_{n+1} ($\mathcal{F}_{\mathbf{d}}$ is a quotient of SL_{n+1} by a parabolic subgroup). Explicitly, $\mathcal{F}_{\mathbf{d}}$ consists of collections $(V_{d_1}, \ldots, V_{d_k})$ of subspaces of an $(n+1)$-dimensional vector space W such that $\dim V_m = m$ and $V_{d_i} \subset V_{d_{i+1}}$. These varieties can be degenerated in the same

way as the complete flag variety (see [13, 14]). As a result one gets a variety $\mathcal{F}^a_{\mathbf{d}}$, consisting of collections of subspaces $(V_{d_1}, \dots, V_{d_k})$ of W such that $\dim V_m = m$ and

$$pr_{d_i+1} \cdots pr_{d_{i+1}} V_{d_i} \subset V_{d_{i+1}}, \quad i = 1, \dots, k-1.$$

These varieties are also certain quiver Grassmannians (see [9]). Namely, consider the equioriented quiver of type A_k. Then the degenerate partial flag variety $\mathcal{F}^a_{\mathbf{d}}$ is isomorphic to

$$\mathrm{Gr}_{(d_1,\dots,d_k)}\big(P_1^{d_1} \oplus P_2^{d_2-d_1} \oplus \cdots \oplus P_k^{d_k-d_{k-1}} \oplus I_1^{d_2-d_1} \oplus \cdots \oplus I_{k-1}^{d_k-d_{k-1}} \oplus I_k^{n+1-d_k}\big), \tag{2.6}$$

where P_i and I_j are projective and injective modules of the A_k quiver. There is a natural surjection $\mathcal{F}^a_{n+1} \to \mathcal{F}^a_{\mathbf{d}}$, sending $(V_i)_{i=1}^n$ to $(V_{d_j})_{j=1}^k$. The group $\mathfrak{A}$ thus acts on $\mathcal{F}^a_{\mathbf{d}}$; the orbits are affine cells containing exactly one T-fixed point. These T-fixed points are parametrized by collections $\mathbf{S} = (S_{d_1}, \dots, S_{d_k})$ of subsets of $\{1, \dots, n+1\}$ subject to the conditions $\#S_{d_i} = d_i$ and

$$S_{d_i} \subset S_{d_{i+1}} \cup \{d_i + 1, \dots, d_{i+1}\}, \quad i = 1, \dots, k-1. \tag{2.7}$$

We call such collections $\mathbf{d}$-admissible. As for the complete flags, the corresponding T-fixed point $p_{\mathbf{S}} = (V_{d_1}, \dots, V_{d_k})$ is given by $V_{d_i} = \mathrm{span}(w_j,\ j \in S_{d_i})$.

3 The moment graph

In this section we study the combinatorics and geometry of the cellular decomposition of the degenerate flag varieties.

3.1 The group action

Recall the group $\mathfrak{A}$ acting on $\mathcal{F}^a_{n+1}$. The following lemma is simple, but important for us. Let $B \subset GL_{2n}$ be the Borel subgroup of lower-triangular matrices and $N \subset B$ be the subgroup of matrices $(a_{i,j})_{i \geq j}$ such that $a_{i,i} = 1$ and $a_{i,j} = 0$ unless $i - j > n$. For example, for $n = 5$ the group N looks as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 3.1 *The group $\mathfrak{A}$ is isomorphic to the quotient group B/N.*

Proof Consider the isomorphism $\mathrm{Aut}(P \oplus I) \simeq \mathrm{Aut}(\bigoplus_{i=1}^{n} P_i \oplus \bigoplus_{k=1}^{n} I_k)$. We note that for any pair of indecomposable summands of $P \oplus I$ the space of homomorphisms between them is either one-dimensional or trivial. More precisely, let us introduce the following notation for the indecomposable summands of $P \oplus I$:

$$R_1 = P_n, R_2 = P_{n-1}, \ldots, R_n = P_1,\ R_{n+1} = I_n, R_{n+2} = I_{n-1}, \ldots, R_{2n} = I_1. \quad (3.1)$$

Then for two indecomposable summands R_i and R_j one has $\dim \mathrm{Hom}(R_i, R_j) = 1$ if and only if $i \le j$ and $j - i \le n$ (see Lemma 2.2). Hence we obtain a surjection of groups $B \to \mathfrak{A}$ and the kernel coincides with N. □

Remark 3.2 Let us fix a non-zero element $\gamma_{i,j} \in \mathrm{Hom}(R_i, R_j)$ for each pair i, j with $i \le j$, $j - i \le n$. Then any element $g \in \mathfrak{A}$ can be uniquely written as a sum $\sum g_{i,j}\gamma_{i,j}$, defining a matrix in B. This produces a section $\mathfrak{A} \to B$.

Remark 3.3 We note that the direct summands R_i in type A_4 are visualized in (2.4). Namely, R_1 is represented by the only fat dot in the upper line, R_2 is represented by the two dots in the next to the upper line, and so on up to R_8. In general, if $i \le n$, then the dimension vector of R_i is $(0, \ldots, 0, 1, \ldots, 1)$ with i units and each non-zero $(R_i)_k$ is spanned by w_{n+1-i}. If $i > n$, then the dimension vector of R_i is $(1, \ldots, 1, 0, \ldots, 0)$ with $2n - i + 1$ units and each non-zero $(R_i)_k$ is spanned by w_{2n-i+2}.

Recall that the T fixed points in $\mathcal{F}^a_{n+1}$ are labeled by the admissible collections. For an admissible collection $\mathbf{S}$ let $p_{\mathbf{S}}$ be the corresponding T-fixed point and $C_{\mathbf{S}}$ be the cell containing $p_{\mathbf{S}}$. We know that $C_{\mathbf{S}} = \mathfrak{A}p_{\mathbf{S}}$. Our goal now is to describe a unipotent subgroup $U_{\mathbf{S}} \subset \mathfrak{A}$ such that the map $U_{\mathbf{S}} \to C_{\mathbf{S}}$ is one-to-one. Let $\mathfrak{a}$ be the Lie algebra of the group $\mathfrak{A}$. Then

$$\mathfrak{a} = \mathrm{Hom}(P, P) \oplus \mathrm{Hom}(I, I) \oplus \mathrm{Hom}(P, I).$$

The Lie algebra $\mathfrak{a}$ is the quotient of the Borel subalgebra $\mathfrak{b} \subset \mathfrak{gl}_{2n}$ of lower-triangular matrices by the ideal $\mathfrak{n}$ consisting of matrices $(a_{j,i})_{j \ge i}$ such that $a_{i,j} = 0$ unless $j - i > n$ (this is exactly the Lie algebra of N). In particular, the one-dimensional hom-spaces $\mathrm{Hom}(R_i, R_j)$, $i \le j$, $j - i \le n$ between two indecomposable summands of $P \oplus I$ correspond to the root vectors of the form $E_{j,i} \in \mathfrak{b}$ ($E_{j,i}$ are matrix units). We have

$$\mathfrak{a} = \mathfrak{t} \oplus \bigoplus_{\substack{1 \le i < j \le 2n \\ j - i \le n}} \mathfrak{a}_{i,j},$$

where $\mathfrak{t}$ is the Lie algebra of the torus T and $\mathfrak{a}_{i,j} = \mathrm{Hom}(R_i, R_j)$.

Consider a pair R_i, R_j of direct summands of $P \oplus I$ such that $\dim \mathrm{Hom}(R_i, R_j) = 1$ and fix a non-zero $\gamma \in \mathrm{Hom}(R_i, R_j)$.

Definition 3.4 A pair of indices (i, j) (a pair of representations R_i, R_j) is called **S**-effective, if $p_{\mathbf{S}} \cap R_i \neq 0$ and $\gamma(p_{\mathbf{S}} \cap R_i)$ does not sit inside $p_{\mathbf{S}}$.

Remark 3.5 **S**-effective pairs have the following geometric interpretation: they are in bijection with standard basis vectors of the tangent space at $p_{\mathbf{S}}$ on which T_0 acts with *positive* weight (see the end of Sect. 2.2). Let us prove this statement. In notation (3.1), we denote by $\mathbf{R}_k$ the coefficient-quiver of R_k. Given an **S**-effective pair (i, j) a non-zero $\gamma \in \mathrm{Hom}(R_i, R_j)$ is determined (up to scalar multiplication) by a (unique) triple (A, f, B). So $A \subset \mathbf{R}_i$ is predecessor-closed, $B \subset \mathbf{R}_j$ is successor closed and $f : A \to B$ is a quiver isomorphism compatible with π (see Sect. 2.2). The sub-representation $\gamma(p_{\mathbf{S}} \cap R_i) \subset R_j$ determines the successor-closed sub-quiver $f(\mathbf{S} \cap A)$ of B. Since by definition $\gamma(p_{\mathbf{S}} \cap R_i)$ does not sit inside $p_{\mathbf{S}}$, $f(\mathbf{S} \cap A)$ strictly contains $\mathbf{S} \cap B$ and the difference $f(\mathbf{S} \cap A) \setminus (\mathbf{S} \cap B)$ is the coefficient quiver of the non trivial quotient $\gamma(p_{\mathbf{S}} \cap R_i)/(\gamma(p_{\mathbf{S}} \cap R_i) \cap p_{\mathbf{S}})$. The map

$$\gamma \mapsto b_\gamma := \big(\mathbf{S} \cap A \setminus f^{-1}(\mathbf{S} \cap B),\, f|_{\mathbf{S}\cap A\setminus f^{-1}(\mathbf{S}\cap B)},\, f(\mathbf{S} \cap A) \setminus (\mathbf{S} \cap B)\big)$$

gives a bijection between **S**-effective pairs and standard basis vectors of the tangent space $T_{p_{\mathbf{S}}}(\mathcal{F}^a_{n+1}) = \mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$ on which T_0 acts with a positive weight. To see this we notice that $\mathbf{S} \cap A$ is predecessor-closed in **S** and $\mathbf{S} \cap B$ is successor closed in B. Then $f^{-1}(\mathbf{S} \cap B)$ is successor closed in $\mathbf{S} \cap A$ and hence $\mathbf{S} \cap A \setminus f^{-1}(\mathbf{S} \cap B)$ is predecessor closed in $\mathbf{S} \cap A$ and hence in **S**. We notice that $\mathbf{S} \cap B$ coincides with $\mathbf{S} \cap \mathbf{R}_j$ (otherwise $\mathbf{S} \cap B$ would not be strictly contained in $f(\mathbf{S} \cap A)$). Since $f(\mathbf{S} \cap A)$ is successor closed in $\mathbf{R}_j$ and $\mathbf{S} \cap B = \mathbf{S} \cap \mathbf{R}_j$, it follows that $f(\mathbf{S} \cap A) \setminus (\mathbf{S} \cap B)$ is successor closed in $\mathbf{R}_j \setminus (\mathbf{S} \cap \mathbf{R}_j)$ and hence in $\theta_M \setminus \mathbf{S}$. The quiver morphism $f|_{\mathbf{S}\cap A\setminus f^{-1}(\mathbf{S}\cap B)}$ is a quiver isomorphism between $\mathbf{S} \cap A \setminus f^{-1}(\mathbf{S} \cap B)$ and $f(\mathbf{S} \cap A) \setminus (\mathbf{S} \cap B)$ compatible with π, since f is so. The image b_γ of γ is hence a standard basis vector of $\mathrm{Hom}(p_{\mathbf{s}}, M/p_{\mathbf{S}})$. The action of T_0 on b_γ is given by $\lambda.b_\gamma = \lambda^{j-i} b_\gamma$. Since $\gamma \neq 0$, then $i \leq j$ and hence b_γ has positive weight. The map is hence well-defined and injective. Let us show that it is surjective. Let $b = (A', f', B')$ be a standard basis vector of $\mathrm{Hom}(p_{\mathbf{S}}, M/p_{\mathbf{S}})$ on which T_0 acts with a positive weight. Then there are indices i and j such that A' is a predecessor-closed sub-quiver of $\mathbf{S} \cap \mathbf{R}_i$, and B' is a successor-closed sub-quiver of $\mathbf{R}_j \setminus (\mathbf{R}_j \cap \mathbf{S})$. The torus T_0 acts on b as $\lambda.b = \lambda^{j-i} b$ and hence $j > i$. We claim that $j - i \leq n$. Indeed if $j - i > n$ then $\pi(\mathbf{R}_j)$ and $\pi(\mathbf{R}_i)$ are disjoint in Q (otherwise $\mathrm{Hom}(R_i, R_j) \neq 0$ against the hypothesis $j - i > n$) and hence the quiver isomorphism $f' : A' \to B'$ could not exist. In view of Lemma 2.2 and the proof of Lemma 3.1, it follows that there is a non-zero standard basis vector $\gamma \in \mathrm{Hom}(R_i, R_j)$ defined by a triple (A, f, B). Notice that $\pi(A) = \pi(B) = \pi(R_i) \cap \pi(R_j) \supset \pi(A') = \pi(B')$. It follows that $A' \subset A$, $B' \subset B$ and $f' = f|_{A'}$. From this we conclude that $p_{\mathbf{S}} \cap R_i \neq 0$ and $\gamma(p_{\mathbf{S}} \cap R_i)$ does not sit inside $p_{\mathbf{S}}$ and hence (i, j) is an **S**-effective pair.

Let $U_{i,j} \subset \mathfrak{A}$ be the one-parameter subgroup with the Lie algebra $\mathfrak{a}_{i,j}$. The importance of effective pairs is explained by the following lemma:

Lemma 3.6 *If a pair (i, j) is not* **S***-effective then $U_{i,j}p_{\mathbf{S}} = p_{\mathbf{S}}$. Otherwise, the map $U_{i,j} \to \mathcal{F}^a_{n+1}$, $g \mapsto gp_{\mathbf{S}}$ is injective.*

Proof Assume that a pair R_i, R_j is not **S**-effective and take a non trivial $\gamma \in \mathrm{Hom}(R_i, R_j)$. By definition, $\gamma p_{\mathbf{S}} \subset p_{\mathbf{S}}$ and hence the exponent of the (scaled) operator γ fixes $p_{\mathbf{S}}$. To prove the second claim we note that

$$\exp(c\gamma)p_{\mathbf{S}} = (\mathrm{Id} + c\gamma)p_{\mathbf{S}}.$$

Hence, if $\gamma p_{\mathbf{S}}$ does not sit inside $p_{\mathbf{S}}$, then all the points $\exp(c\gamma)p_{\mathbf{S}}$, $c \in \mathbb{C}$ are different. □

For an admissible **S** let $\mathfrak{a}_{\mathbf{S}} \subset \mathfrak{a}$ be the subspace defined as the direct sum of one-dimensional spaces $\mathrm{Hom}(R_i, R_j)$ for all **S**-effective pairs R_i, R_j.

Lemma 3.7 *The subspace $\mathfrak{a}_{\mathbf{S}}$ is a Lie subalgebra of $\mathfrak{a}$.*

Proof Let $\gamma_1 \in \mathfrak{a}_{i,j}$ and $\gamma_2 \in \mathfrak{a}_{k,l}$, $i > j$, $k > l$ be two elements such that $[\gamma_1, \gamma_2] \neq 0$. Then either $j = k$ or $i = l$. We work out the first case (the second is very similar). We have $[\gamma_1, \gamma_2] = \gamma_1\gamma_2 \in \mathfrak{a}_{i,l}$. Since γ_2 is **S**-effective, we have

$$\gamma_2(p_{\mathbf{S}} \cap R_l) \supsetneq p_{\mathbf{S}} \cap R_k.$$

Now, since

$$\gamma_1(p_{\mathbf{S}} \cap R_j) \supsetneq p_{\mathbf{S}} \cap R_i$$

and $j = k$, we obtain

$$\gamma_1\gamma_2(p_{\mathbf{S}} \cap R_l) \supsetneq p_{\mathbf{S}} \cap R_i$$

and hence $\gamma_1\gamma_2$ is **S**-effective. □

Let $U_{\mathbf{S}}$ be the Lie group of $\mathfrak{a}_{\mathbf{S}}$, i.e. $U_{\mathbf{S}}$ is generated by all $U_{i,j}$ with **S**-effective (i, j). We note that $U_{\mathbf{S}}$ is invariant with respect to the torus T action by conjugation.

Theorem 3.8 *The map $U_{\mathbf{S}} \to C_{\mathbf{S}}$, $g \mapsto gp_{\mathbf{S}}$ is bijective and T-equivariant.*

Proof First, we note that T-equivariance follows from $Tp_{\mathbf{S}} = p_{\mathbf{S}}$. Now let us prove that the map $U_{\mathbf{S}} \to C_{\mathbf{S}}$ is surjective. Let us write an element $\mathfrak{g} \in \mathfrak{A}$ as $g = g_{\mathbf{S}}g_1h$, where $h \in T$, $g_{\mathbf{S}} \in U_{\mathbf{S}}$ and g_1 belongs to the subgroup of $\mathfrak{A}$, generated by $U_{i,j}$ with non **S**-effective (i, j). Then $gp_{\mathbf{S}} = g_{\mathbf{S}}p_{\mathbf{S}}$ and hence we are done. Finally, let us prove the injectivity. Assume that there exists $g \in U_{\mathbf{S}}$ such that $gp_{\mathbf{S}} = p_{\mathbf{S}}$. We identify g with the corresponding lower-triangular matrix in GL_{2n} with entries $g_{i,j}$ satisfying $g_{i,i} = 1$ and $g_{i,j} = 0$ if $i - j > n$. Our goal is to prove that $g_{i,j} = 0$ for all $i > j$. Let $p(\mathbf{S}) = (V_1, \ldots, V_n)$ and assume that $g_{i,j} \neq 0$ for $i > j$. Since $g \in U_{\mathbf{S}}$, the pair (i, j) is **S**-effective. Consider a non-zero element $\gamma \in \mathfrak{a}_{i,j}$ (so $\gamma \in \mathrm{Hom}(R_i, R_j)$). Let $t = 1, \ldots, n$ be a number such that $V_t \cap R_i \neq 0$ and $\gamma V_t \cap V_t = 0$. Choose a non-zero vector $w \in V_t \cap R_i$. Then $gw \notin V_t$ and hence $gp_{\mathbf{S}} \neq p_{\mathbf{S}}$. □

Remark 3.9 We note that Theorem 3.8 is analogous to the corresponding theorem for classical flag varieties, see e.g. [27], Lemma 3.2.

Proposition 3.10 *The number of* **S***-effective pairs* (i, j) *is equal to the sum* $N_{PI}(\mathbf{S}) + N_{PP}(\mathbf{S}) + N_{II}(\mathbf{S})$ *of three numbers defined by*:

- $N_{PI}(\mathbf{S})$ *is the number of pairs* $1 \le k < l \le n+1$ *such that there exists* t *with* $k \le t < l$ *such that* $k \in S_t$, $l \notin S_t$.
- $N_{PP}(\mathbf{S})$ *is the number of pairs* $1 \le k < l \le n$ *such that there exists* $t \ge l$ *such that* $l \in S_t$, $k \notin S_t$.
- $N_{II}(\mathbf{S})$ *is the number of pairs* $2 \le k < l \le n+1$ *such that there exists* $t < k$ *such that* $l \in S_t$, $k \notin S_t$.

Proof We divide **S**-effective pairs into three parts $R_i, R_j \subset P$, $R_i, R_j \subset I$ and $R_i \subset P$, $R_j \subset I$. We claim that the number of **S**-effective pairs from the first (second, third) part is equal to $N_{PP}(\mathbf{S})$ ($N_{II}(\mathbf{S})$, $N_{PI}(\mathbf{S})$).

(i) The case $R_i \subset P$, $R_j \subset I$. Then $1 \le i \le n < j \le 2n$. Since (i, j) is **S**-effective, there exists an index $t: n+1-i \le t \le 2n+1-j$ such that $n+1-i \in S_t$ and $2(n+1)-j \notin S_t$. Put $k = n+1-i$ and $l = 2(n+1)-j$.

(ii) The case $R_i, R_j \subset P$. Since (i, j) is **S**-effective then $1 \le i < j \le n$ and there is an index $t: t \ge n+1-i > n+1-j$ such that $n+1-i \in S_t$ and $n+1-j \notin S_t$. Put $l = n+1-i$ and $k = n+1-j$.

(iii) The case $R_i, R_j \subset I$. Since (i, j) is **S**-effective then $n+1 \le i < j \le 2n$ and there is an index $t: t \le 2n+1-j < 2n+1-i$ such that $2(n+1)-i \in S_t$ and $2(n+1)-j \notin S_t$. Put $l = 2(n+1)-i$ and $k = 2(n+1)-j$.

□

Corollary 3.11 *The dimension of* $C_{\mathbf{S}}$ *is equal to the sum* $N_{PI}(\mathbf{S}) + N_{PP}(\mathbf{S}) + N_{II}(\mathbf{S})$.

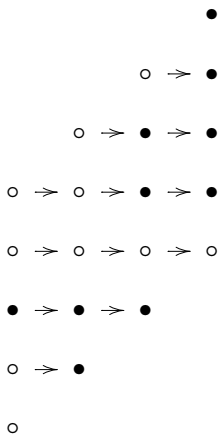
Proof Thanks to Theorem 3.8 the dimension of the cell $C_{\mathbf{S}}$ is equal to the number of **S**-effective pairs R_i, R_j. Now Proposition 3.10 implies the corollary. □

Corollary 3.12 *The Poincaré polynomial of* $\mathcal{F}^a_{n+1}$ *is equal to the sum of the terms* $q^{N_{PI}(\mathbf{S})+N_{PP}(\mathbf{S})+N_{II}(\mathbf{S})}$, *where the sum runs over the set of admissible collections.*

Remark 3.13 In [9, Theorem 5.1] it is shown that although $\mathcal{F}^a_{n+1}$ is not smooth, the one-dimensional sub-torus T_0 of T still produces a Białynicki–Birula type cell decomposition ([3], [10, Theorem 2.4.3]). In other words, the attracting set of a T_0-fixed point $p_{\mathbf{S}}$ is a cell and it has dimension equal to the dimension of the positive part of the tangent space at $p_{\mathbf{S}}$ (the positive part is the vector subspace generated by vectors on which T_0 acts with positive weight). In view of Remark 3.5, this dimension is precisely the number of **S**-effective pairs. Theorem 3.8 provides another and more explicit proof of this fact.

Remark 3.14 From the discussion above (see Corollary 3.11 and Remark 3.5), the dimension of the cell with center $p_{\mathbf{S}}$ can be easily read off from **S**, viewed inside

the coefficient quiver of $P \oplus I$ written in the form (2.4). Indeed in this diagram let us color a vertex black if it belongs to $\mathbf{S}$ and white otherwise. In the ith column (counting from left to right) there are precisely i black vertices. Some of them are sources of $\mathbf{S}$. For every such source $t \in S_i$ let us count the number w_t of white vertices below it. Let c_i be the sum of the w_t's. Then the dimension of the cell with center $p_{\mathbf{S}}$ equals the sum $c_1 + c_2 + \cdots + c_n$. For example let us consider the following T-fixed point of $\mathcal{F}_5^a$:



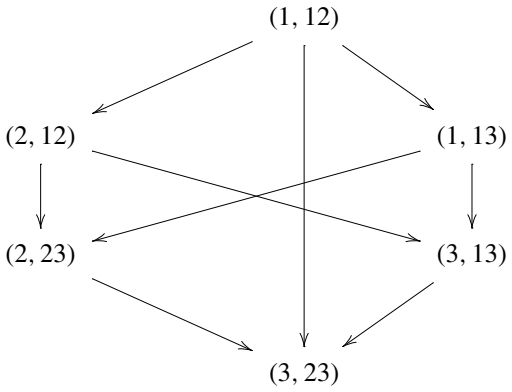
then $c_1 = 2$, $c_2 = 0$, $c_3 = 2$ and $c_4 = 2$. The cell has hence dimension 6.

3.2 Moment graph

We briefly recall the definition of a moment graph (see [6, 21]). Let X be a projective algebraic variety acted upon by a torus $T = (\mathbb{C}^*)^d$ with a fixed one-dimensional sub-torus $\iota : \mathbb{C}^* \subset T$. Assume that the T action on X has finitely many fixed points and one-dimensional orbits and any $\mathbb{C}^*$ fixed point is T-fixed ($X^T = X^{\mathbb{C}^*}$). Assume further that X has a decomposition as a disjoint union of T-invariant affine cells in such a way that each cell C contains exactly one $\mathbb{C}^*$-fixed point p and $C = \{x \in X : \lim_{\lambda \to 0} \iota(\lambda)x = p\}$ (i.e. the cell consists of all points attracted by p, see [3]). We denote this cell by C_p. The moment graph Γ has its set of vertices labeled by the T-fixed points. Two points p_1 and p_2 are connected by an edge in Γ if there exists a one-dimensional T-orbit L such that $\bar{L} = L \sqcup p_1 \sqcup p_2$ (i.e. p_1 and p_2 are the T-fixed points in the closure of L). Thus the edges of Γ are labeled by the one-dimensional T-orbits. We orient Γ by the following rule: for two vertices p_1 and p_2 we say $p_1 \geq p_2$ if $C_{p_2} \subset \bar{C}_{p_1}$. If there is an edge connecting p_1 and p_2 in Γ then we put an arrow $p_1 \to p_2$. Finally, one defines a labeling α_L of the edges L of Γ by the elements $\alpha_L \in \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of the torus T. Namely, for an edge L let $T_x \subset T$ be the stabilizer of a point $x \in L$ (obviously, T_x is independent of $x \in L$).

Then the Lie algebra $\mathfrak{t}_x \subset \mathfrak{t}$ is a hyperplane. We define α_L as a non-zero element in the annihilator of $\mathfrak{t}_x$.

Example 3.15 Here we give an example of the moment graph for the classical flag variety $\mathcal{F}_3 = SL_3/B$. The torus T has six fixed points labeled by pairs (S_1, S_2) of subsets of $\{1, 2, 3\}$ such that $\#S_1 = 1$, $\#S_2 = 2$ and $S_1 \subset S_2$. The moment graph of $\mathcal{F}_3$ looks as follows:



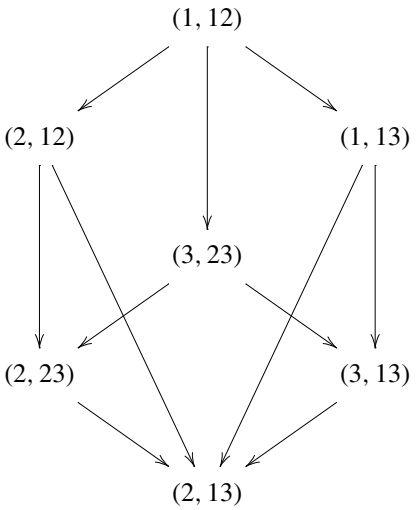
We note that usually the arrows in the moment graph direct from bottom to top. However, for our purposes it is more convenient to draw the vertices from top to bottom, since in the degenerate situation the dense cell corresponds to the point $(1, 12)$, see Example 3.17. This is not important in the classical situation due to the Chevalley involution, but crucial in the degenerate case.

Our goal is to describe the moment graph of the degenerate flag varieties.

Remark 3.16 We note that the moment graphs turn out to be a powerful tool for computing various cohomology groups of algebraic varieties (see [6, 18, 19, 21, 27]). A crucial role is played by the notion of sheaves on moment graphs. In this paper we do not discuss Γ-sheaves, but only describe the combinatorial structure of the graphs. Computation of the (equivariant) cohomology as well as the (equivariant) intersection cohomology of the degenerate flag varieties is an interesting open problem.

Example 3.17 Here we give a picture of the moment graph for the degenerate flag variety $\mathcal{F}_3^a$. Recall that the T-fixed points are labeled by pairs (S_1, S_2) of subsets of

the set $\{1, 2, 3\}$ such that $\#S_1 = 1$, $\#S_2 = 2$ and $S_1 \subset S_2 \cup \{2\}$.



The moment graph for the degenerate flag variety $\mathcal{F}_4^a$ is computed in Appendix C.

We now give an explicit combinatorial description of the moment graph. We identify the Lie algebra $\mathfrak{t}$ of T with the diagonal traceless $2n \times 2n$ matrices. For a pair of indices i, j, $1 \le i < j \le 2n$, we denote by $\alpha_{i,j} \in \mathfrak{t}^*$ the element $\alpha_{i,j}(\mathrm{diag}(x_1, \ldots, x_{2n})) = x_i - x_j$.

Theorem 3.18 *The number of one-dimensional T-orbits in $\mathcal{F}_{n+1}^a$ is finite. The orbits are of the form $U_{i,j}\, p_{\mathbf{S}} \setminus p_{\mathbf{S}}$, where $\mathbf{S}$ is admissible and (i, j) is $\mathbf{S}$-effective. The edge in the moment graph, which corresponds to $U_{i,j}\, p_{\mathbf{S}} \setminus p_{\mathbf{S}}$ is labeled by $\alpha_{i,j}$.*

Proof Thanks to Theorem 3.8, we only need to describe the one-dimensional T-orbits in $U_{\mathbf{S}}$. It is easy to see that these are non-identity elements in $U_{i,j}$. □

Remark 3.19 Theorem 3.18 also follows from Corollary 2.7 and Remark 3.5. Indeed in view of Corollary 2.7, the directions around $p_{\mathbf{S}}$ of the one-dimensional T-orbits containing $p_{\mathbf{S}}$ are precisely the standard basis vectors of the tangent space $T_{p_{\mathbf{S}}}(\mathcal{F}_{n+1}^a)$ at $p_{\mathbf{S}}$. In particular the number of such T-orbits is bigger or equal than $\dim \mathcal{F}_{n+1}^a$ and it is equal if and only if $p_{\mathbf{S}}$ is smooth. Any such curve ℓ consists of three T-orbits $\ell = \{p_{\mathbf{S}}\} \cup \{\ell'\} \cup \{p_{\mathbf{R}}\}$. The direction of ℓ is fixed also by the one-dimensional torus T_0. In particular this standard basis vector of $T_{p_{\mathbf{S}}}(\mathcal{F}_{n+1}^a)$ has either positive or negative T_0 weight. If the weight is positive then $\{p_{\mathbf{S}}\} \cup \{\ell'\}$ sits inside the attracting set of $p_{\mathbf{S}}$ which is the cell $\mathfrak{A} p_{\mathbf{S}}$ and hence $p_{\mathbf{R}}$ (and its attracting cell) is in the closure of this cell. It follows that in the moment graph there is an arrow $p_{\mathbf{S}} \to p_{\mathbf{R}}$. In particular the number of arrows starting from $p_{\mathbf{S}}$ in the moment graph, equals the number of standard basis vector of $T_{p_{\mathbf{S}}}(\mathcal{F}_{n+1}^a)$ on which T_0 acts with positive weight. In view of Remark 3.5 this number equals the number of $\mathbf{S}$-effective pairs.

Corollary 3.20 *The dimension of a cell $C_{\mathbf{S}}$ is equal to the number of edges in the moment graph which are directed outwards the vertex $p_{\mathbf{S}}$.*

The following theorem generalizes the results as above to the case of the degenerate partial flag varieties.

Theorem 3.21 *The number of one-dimensional T orbits on $\mathcal{F}^a_{\mathbf{d}}$ is finite. Each of these orbits is covered by a one-dimensional T-orbit in $\mathcal{F}^a_{n+1}$ via the surjection $\mathcal{F}^a_{n+1} \to \mathcal{F}^a_{\mathbf{d}}$. All the orbits are of the form $U_{i,j}p \setminus p$ for some i, j and a T-fixed $p \in \mathcal{F}^a_{\mathbf{d}}$.*

4 Smooth locus and the Schröder numbers

In this section we describe the smooth locus of the degenerate flag varieties $\mathcal{F}^a_{n+1}$ and compute Euler characteristics and Poincaré polynomials.

4.1 Smooth cells

Take a point $N \in \mathrm{Gr}_{\dim P}(P \oplus I)$. Then N can be split as $N = N_P \oplus N_I$, where $N_P \subset P$ and $N_I \subset I$, such that N_I and P/N_P are of the same dimension vector (see [9, Theorem 1.3]).

Lemma 4.1 *A point N in a quiver Grassmannian $\mathrm{Gr}_{\dim P}(P \oplus I)$ is smooth if and only if $\mathrm{Ext}^1(N_I, P/N_P) = 0$.*

Proof Let $\langle \cdot, \cdot \rangle$ be the Euler form of the quiver Q, given on a pair of dimension vectors $\mathbf{d}$, $\mathbf{e}$ by $\langle \mathbf{d}, \mathbf{e} \rangle = \sum_{i=1}^{n} d_i e_i - \sum_{i=1}^{n-1} d_i e_{i+1}$. Then $\langle \dim X, \dim Y \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$ for arbitrary representations X and Y of Q. By [9, Theorem 1.1], we have

$$\langle \dim P, \dim I \rangle = \dim \mathrm{Gr}_{\dim P}(P \oplus I).$$

By the formula [9, Lemma 2.3] for the dimension of the tangent space T_N to the point $N \in \mathrm{Gr}_{\dim P}(P \oplus I)$, we then have

$$\begin{aligned} \dim T_N &= \dim \mathrm{Hom}(N_I \oplus N_P, P/N_P \oplus I/N_I) \\ &= \langle \dim P, \dim I \rangle - \dim \mathrm{Ext}^1(N_I \oplus N_P, P/N_P \oplus I/N_I). \end{aligned}$$

Since N_P is projective and N/N_I is injective, we obtain

$$\dim \mathrm{Ext}^1(N_I \oplus N_P, P/N_P \oplus I/N_I) = \dim \mathrm{Ext}^1(N_I, P/N_P).$$

Hence, the dimension of the tangent space at a point N is equal to the dimension of the Grassmannian if and only if $\mathrm{Ext}^1(N_I, P/N_P)$ vanishes. □

Recall that the quiver Grassmannian $\mathrm{Gr}_{\dim P}(P \oplus I)$ can be decomposed into the disjoint union of $\mathfrak{A}$-orbits of the form $\mathfrak{A}p_{\mathbf{S}}$. Hence all the points of the orbit are

smooth or singular together with $p_{\mathbf{S}}$. So it suffices to understand what are the conditions for an admissible collection $\mathbf{S}$ that guarantee the smoothness of $p_{\mathbf{S}}$. We use Lemma 4.1 above.

Theorem 4.2 *A point $p_{\mathbf{S}}$ is smooth if and only if for all $1 \le j < i \le n$, the condition $i \in S_j$ implies $j+1 \in S_i$.*

Proof Given an admissible collection $\mathbf{S} = (S_i)_{i=1}^n$, we introduce the following numbers for all $i = 1, \dots, n+1$:

$$k_i = \min\{1 \le k < i : i \in S_k\}, \qquad l_j = \min\{j \le l \le n : j \in S_l\}.$$

Recall the indecomposable representations $R_{k,l}$ with the support on the interval $[k,l]$. A representation $p_{\mathbf{S}}$ is isomorphic to the direct sum $N_I \oplus N_P$, where $N_I \subset I$ and $N_P \subset P$. It is easy to see that

$$N_I = \bigoplus_i R_{k_i, i-1}, \qquad P/N_P = \bigoplus_j R_{j, l_j - 1}.$$

The extension groups between the indecomposables are given by Lemma 2.2. Thus we obtain $0 \neq \mathrm{Ext}^1(N_I, P/N_P)$ if and only if there exist indices i and j such that $k_i + 1 \le j \le i \le l_j - 1$. This holds (writing out the three inequalities) if and only if there exist indices $j \le i$ such that

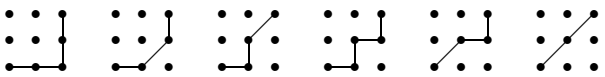
$$\min\{1 \le k < i : i \in S_k\} < j, \qquad \min\{j \le l \le n : j \in S_l\} > i.$$

This translates into the condition that there exist $j \le i$ such that $i \in S_{j-1}$, but $j \notin S_i$. Conversely, this means that the orbit is smooth if and only if for all $1 \le j \le i \le n+1$, if $i \in S_{j-1}$, then $j \in S_i$. Note that this condition is void in case $j = 1$ or $i = n+1$, so that we can replace j by $j-1$, and obtain the assertion of theorem. □

In what follows we call an admissible collection $\mathbf{S}$ smooth iff $p_{\mathbf{S}}$ is a smooth point.

4.2 The large Schröder numbers

Let r_n be the nth large Schröder number, defined as the number of Schröder paths, i.e. subdiagonal lattice paths from $(0,0)$ to (n,n) consisting of the steps $(0,1)$, $(1,0)$ or $(1,1)$. The sequence $r_0, r_1, r_2, \dots$ starts with $1, 2, 6, 22, 90, 394$. Here are the six Schröder paths for $n = 2$:



We note that (see e.g. [1, 5, 26])

$$r_n = r_{n-1} + \sum_{k=0}^{n-1} r_k r_{n-1-k}.$$

The small Schröder numbers s_n are defined as halves of the large ones.

Recall that a collection $\mathbf{S} = (S_a)_{a=1}^n$ of subsets of the set $\{1, \dots, n+1\}$ is smooth if $\#S_a = a$, $S_a \subset S_{a+1}$ and for all $1 \le a < b \le n$ the following condition holds (see Theorem 4.2):

$$\text{if} \quad b \in S_a, \quad \text{then} \quad a+1 \in S_b.$$

Let LS_n be the set of length n smooth collections and $\bar{r}_n$ be the cardinality of LS_n.

Proposition 4.3 *The numbers $\bar{r}_n$ satisfy the recursion*

$$\bar{r}_n = \bar{r}_{n-1} + \sum_{k=0}^{n-1} \bar{r}_k \bar{r}_{n-1-k}.$$

Proof We divide all smooth collections according to the values of S_1. So first, let $S_1 = \{1\}$. Let us show that the number of such smooth collections is equal to $\bar{r}_{n-1}$. Note that all S_a contain 1. For $a = 1, \dots, n-1$ we set

$$S'_a = \{i : i+1 \in S_{a+1}\} \subset \{1, \dots, n\}.$$

We claim that the collection $(S'_a)_{a=1}^{n-1}$ is smooth and all (length $n-1$) smooth collections arise in this way. First, obviously, $\#S'_a = a$ and $S'_a \subset S'_{a+1}$. Now the conditions ($b \in S_a$ implies $a+1 \in S_b$), $2 \le a < b \le n$ are equivalent to the conditions ($b \in S'_a$ implies $a+1 \in S'_b$), $1 \le a < b \le n-1$.

Let $LS_n^k \subset LS_n$ be the set of smooth collections satisfying $S_1 = \{k\}$, $2 \le k \le n+1$. We want to show that the cardinality of LS_n^k is equal to $\bar{r}_{k-2}\bar{r}_{n+1-k}$. To this end we construct a bijection $F : LS_n^k \to LS_{k-2} \times LS_{n+1-k}$. For convenience, we write $F = (f, g)$, where

$$f : LS_n^k \to LS_{k-2}, \qquad g : LS_n^k \to LS_{n+1-k}.$$

First, since $S_1 \subset S_a$ for any a, we have $k \in S_a$, $2 \le a \le n$. Now the conditions $k \in S_a$ for $a = 1, \dots, k-1$ imply

$$\{2, \dots, k\} \subset S_a \quad \text{for all } a \ge k.$$

Given a collection $\mathbf{S} \in LS_n^k$, we define

$$g(\mathbf{S}) = \big(g(\mathbf{S})_1, \dots, g(\mathbf{S})_{n+1-k}\big)$$

as follows:

$$g(\mathbf{S})_a = \begin{cases} \{i : 2 \le i \le n+1-k, i+k-1 \in S_{a+k-1}\}, & \text{if } 1 \notin S_{a+k-1}, \\ \{1\} \cup \{i : 2 \le i \le n+1-k, i+k-1 \in S_{a+k-1}\} & \text{otherwise.} \end{cases} \tag{4.1}$$

We note that the image depends only on the sets S_a with $a \ge k$.

Now, given a collection $\mathbf{S} \in LS_n^k$, we need to define

$$f(\mathbf{S}) = \big(f(\mathbf{S})_1, \dots, f(\mathbf{S})_{k-2}\big).$$

Let $S_k = \{2, \dots, k\} \cup \{i\}$ for some $i = 1, k+1, \dots, n+1$. We note that since $k \in S_a \subset S_k$ for $a < k$, each $S_a \setminus k$ for $2 \le a \le k-1$ is an $(a-1)$-element subset of the fixed set of cardinality $k-1$ (this set is $\{2, \dots, k-1\} \cup \{i\}$). We now define the map f as follows:

$$f(\mathbf{S})_a = \begin{cases} \{i : 1 \le i \le k-2, i+1 \in S_{a+1}\}, & \text{if } S_{a+1} \subset \{2, \dots, k\}, \\ \{i : 1 \le i \le k-2, i+1 \in S_{a+1}\} \cup \{k-1\}, & \text{otherwise.} \end{cases} \tag{4.2}$$

We note (this is important) that $f_1(\mathbf{S})$ depends only on $S_2, \dots, S_{k-1}$.

Our first goal is to show that $f(\mathbf{S}) \in LS_{k-2}$ and $g(\mathbf{S}) \in LS_{n-k+1}$ for any $\mathbf{S} \in LS_n^k$. By definition, $g(\mathbf{S})_a \subset g(\mathbf{S})_{a+1}$ for $1 \le a \le n-k$ and

$$g(\mathbf{S})_a \in \{1, \dots, n-k+2\}, \qquad \#g(\mathbf{S})_a = a \quad \text{for } 1 \le a \le n-k+1.$$

Let us show that for $1 \le a < b \le n-k+1$ the inclusion $b \in g(\mathbf{S})_a$ implies $a+1 \in g(\mathbf{S})_b$. Since $b > 1$, $b \in g(\mathbf{S})_a$ implies $b+k-1 \in S_{a+k-1}$. Since $\mathbf{S}$ is smooth, we obtain $a+k \in S_{b+k-1}$, which gives $a+1 \in g(\mathbf{S})_b$ and we are done. Similarly, one proves that $f(\mathbf{S}) \in LS_{k-2}$.

Finally, we have to prove that the map $F = (f, g) : LS_n^k \to LS_{k-2} \times LS_{n-k+1}$ is one-to-one. Given an element $(\mathbf{S}', \mathbf{S}'') \in LS_{k-2} \times LS_{n-k+1}$, we use formulas (4.1) and (4.1) to reconstruct $\mathbf{S}$ such that $F(\mathbf{S}) = (\mathbf{S}', \mathbf{S}'')$. □

Corollary 4.4 *The Euler characteristic of the smooth locus of $\mathcal{F}_{n+1}^a$ is equal to the nth Schröder number r_n.*

Finally, let us formulate the analogue of Theorem 4.2 for the degenerate partial flag varieties. We omit the proof since it is very close to the proof of Theorem 4.2. Recall that the T-fixed points in $\mathcal{F}_{\mathbf{d}}^a$ are labeled by $\mathbf{d}$-admissible collections $\mathbf{S} = (S_{d_1}, \dots, S_{d_k})$ (see (2.7)).

Theorem 4.5 *A T-fixed point $p_{\mathbf{S}} \in \mathcal{F}_{\mathbf{d}}^a$ is smooth if and only if the following conditions hold: if $b \in S_{d_i}$ and $d_{j+1} \ge b > d_j$ for some $j \ge i$, then $\{d_i + 1, \dots, d_{i+1}\} \subset S_{d_{j+1}}$.*

4.3 Poincaré polynomials

There are several ways to define q-analogues of the Schröder numbers (see [1, 2, 5]). We will need the simplest one (see [5], page 37, polynomials $d_n(q)$). They are called Narayana polynomials there, but in other papers the same polynomials are also referred to as Schröder polynomials, see e.g. [20]). For a Schröder path P let $\mathrm{diag}(P)$ be the number of the diagonal steps in P. Define $r_n(q)$ as the sum of the terms $q^{\mathrm{diag}(P)}$ over the set of Schröder paths P. Here are the first several polynomials

$$r_0(q) = 1, \qquad r_1(q) = 1 + q, \qquad r_2(q) = 2 + 3q + q^2,$$
$$r_3(q) = 5 + 10q + 6q^2 + q^3, \qquad r_4(q) = 14 + 35q + 30q^2 + 10q^3 + q^4.$$

Clearly, $r_n(0)$ is the nth Catalan number. Let $P_n^{\mathrm{sm}}(q)$ be the Poincaré polynomial of the smooth locus of $\mathcal{F}_{n+1}^a$. Our goal here is to prove the following theorem:

Theorem 4.6 $P_n^{\mathrm{sm}}(q) = q^{n(n-1)/2} r_n(q)$.

Recall (see [1, 5]) that

$$r_n(q) = q r_{n-1}(q) + \sum_{k=0}^{n-1} r_k(q) r_{n-1-k}(q). \tag{4.3}$$

Proposition 4.7 *The Poincaré polynomials of the smooth locus satisfy the following recursion*:

$$P_n^{\mathrm{sm}}(q) = q^n P_{n-1}^{\mathrm{sm}}(q) + \sum_{l=0}^{n-1} q^{(l+1)(n-l)-1} P_l^{\mathrm{sm}}(q) P_{n-1-l}^{\mathrm{sm}}(q). \tag{4.4}$$

Proof First, let us consider smooth collections $(S_1, \dots, S_n)$ with $S_1 = 1$. Then the cells labeling such collections are in one-to-one correspondence with smooth collections $\mathbf{S}'$ of length $n-1$: $S_i' = S_{i+1} \setminus \{1\}$. We claim that

$$\dim C_{\mathbf{S}} = \dim C_{\mathbf{S}'} + n. \tag{4.5}$$

We use Proposition 3.10. Clearly, the terms N_{PP} and N_{II} for $p_{\mathbf{S}}$ and $p(\mathbf{S}')$ do coincide and the difference of the terms N_{PI} is equal to n (since $S_1 = \{1\}$, in the definition of N_{PI} we can take $t = 1$, $i = 1$, $j = 2, \dots, n+1$). Now (4.5) produces the first term of the right hand side of (4.4).

Recall the bijection $F = (f, g) : LS_n^k \to LS_{k-2} \times LS_{n-k+1}$, $k \geq 2$, from the set of smooth collections with $S_1 = \{k\}$ to the product $LS_{k-2} \times LS_{n-k+1}$. Our goal is to prove that

$$\dim C_{\mathbf{S}} = \dim C_{f(\mathbf{S})} + \dim C_{g(\mathbf{S})} + (k-1)(n+2-k) - 1 \tag{4.6}$$

(after the shift $l = k-2$ one gets the corresponding term in (4.4)). Recall that since $k \in S_1$ we have

$$\{2, \dots, k\} \subset S_m \quad \text{for all } m \geq k.$$

In particular, $S_k = \{2, \dots, k\} \cup \{r\}$ for some number $r = 1, k+1, \dots, n+1$. We claim that

$$\begin{aligned} &N_{PI}(\mathbf{S}) + N_{PP}(\mathbf{S}) \\ &\quad = N_{PI}\big(f(\mathbf{S})\big) + N_{PI}\big(g(\mathbf{S})\big) + N_{PP}\big(f(\mathbf{S})\big) + N_{PP}\big(g(\mathbf{S})\big) + (k-1)(n+1-k), \end{aligned}$$

and

$$N_{II}(\mathbf{S}) = N_{II}\big(f(\mathbf{S})\big) + N_{II}\big(g(\mathbf{S})\big) + k - 2.$$

First, let us prove the first formula. Assume that $1 = r = S_k \setminus \{2, \dots, k\}$. Then

$$N_{PI}(\mathbf{S}) = N_{PI}\big(f(\mathbf{S})\big) + N_{PI}\big(g(\mathbf{S})\big) + (k-1)(n+1-k),$$
$$N_{PP}(\mathbf{S}) = N_{PP}\big(f(\mathbf{S})\big) + N_{PP}\big(g(\mathbf{S})\big).$$

Here the term $(k-1)(n+1-k)$ comes from the fact that in the definition of $N_{PI}(\mathbf{S})$ one can take $i = 2, \dots, k$, $j = k+1, \dots, n+1$ and $t = k$. These possibilities are not counted in $N_{PI}(f(\mathbf{S})) + N_{PI}(g(\mathbf{S}))$. Now assume that $r > k$. Then one has

$$N_{PI}(\mathbf{S}) = N_{PI}\big(f(\mathbf{S})\big) + N_{PI}\big(g(\mathbf{S})\big) + (k-1)(n-k),$$
$$N_{PP}(\mathbf{S}) = N_{PP}\big(f(\mathbf{S})\big) + N_{PP}\big(g(\mathbf{S})\big) + k - 1.$$

Here the term $(k-1)(n-k)$ comes from the fact that in the definition of $N_{PI}(\mathbf{S})$ one can take $i = 2, \dots, k$, $j \in \{k+1, \dots, n+1\} \setminus r$ and $t = k$. The term $k-1$ in the right hand side of the second equality comes from the fact that in the definition of $N_{PP}(\mathbf{S})$ one can take $i = 1$, $j = 2, \dots, k$ and $t = k$. All these possibilities are lost when computing $N_{PI}(f(\mathbf{S}))$, $N_{PI}(g(\mathbf{S}))$, $N_{PP}(f(\mathbf{S}))$ and $N_{PP}(g(\mathbf{S}))$.

Now let us prove that

$$N_{II}(\mathbf{S}) = N_{II}\big(f(\mathbf{S})\big) + N_{II}\big(g(\mathbf{S})\big) + k - 2.$$

Here the argument is even simpler: the missing $k-2$ comes from the following possibilities for $N_{II}(\mathbf{S})$ missing in $N_{II}(f(\mathbf{S})) + N_{II}(g(\mathbf{S}))$: $i = 2, \dots, k-1$, $j = k$, $t = 1$.

We thus obtain

$$\dim C_{\mathbf{S}} = \dim C_{f(\mathbf{S})} + \dim C_{g(\mathbf{S})} + (k-1)(n+1-k) + (k-2),$$

which implies (4.6) as well as the proposition. □

Corollary 4.8 *Theorem 4.6 holds.*

Proof We note that $P_1^{\mathrm{sm}}(q) = 1 + q = r_1(q)$. Now the induction procedure combined with (4.3) and Proposition 4.7 gives the desired result. □

Remark 4.9 It is natural to define a q,t-version $h_{n+1}(q,t)$ of the normalized median Genocchi numbers as the sum over admissible collections $\mathbf{S}$ of the terms

$$q^{\dim C_{\mathbf{S}}} t^{\dim T_{P_{\mathbf{S}}} \mathcal{F}^a_{n+1}} t^{-n(n+1)/2}.$$

Then the value $h_n(1,1)$ is exactly the normalized median Genocchi number and $h_{n+1}(q,0) = q^{n(n-1)/2} r_n(q)$ is the (scaled) nth Schröder polynomial. Here are first few q,t-Genocchi polynomials:

$$h_2(q,t) = 1 + q, \qquad h_3(q,t) = 2q + 3q^2 + q^3 + t,$$
$$h_4(q,t) = q^3\big(5 + 10q + 6q^2 + q^3\big) + tq\big(2q + 7q^2 + 5q^3\big) + t^2(1+q).$$

4.4 Schröder numbers: from large to small

Recall the polynomials $P_n^{\mathrm{sm}}(q)$, which are equal to $q^{n(n-1)/2} r_n(q)$, $r_n(q)$ being the q-Schröder polynomials. Recall (see [12, 20]) that the polynomials $r_n(q)$ are divisible by $1+q$. The ratios are denoted by $s_n(q)$ (thus $r_n(q) = s_n(q)(1+q)$). These are the small q-Schröder polynomials. (In particular, $s_n(1)$ are the small Schröder numbers.) Our goal here is to show that the divisibility of $r_n(q)$ by $1+q$ has a very simple and concrete explanation within our approach. We give two proofs: one is due to the referee and uses the result from Appendix B. The second proof is based on the existence of a certain involution on the set of smooth cells.

Theorem 4.10 *The polynomials $P_n^{\mathrm{sm}}(q)$ and thus $r_n(q)$ are divisible by $1+q$.*

Proof According to Theorem B.1 there exists an embedding of the smooth locus of $\mathcal{F}_n^a$ into the desingularization R_n (see [16] and Appendix B for more details). Recall that a point of R_n is represented by a collection of subspaces $V_{i,j}$ and the map $(V_{i,j})_{1\le i\le j\le n-1} \mapsto V_{1,n-1}$ is a fibration $R_n \to \mathbb{P}^1$ (recall that $V_{1,n-1}$ is a subspace of the two-dimensional space $\mathrm{span}(w_1, w_n)$). We thus obtain a composition map ρ from the smooth locus of $\mathcal{F}_n^a$ onto $\mathbb{P}^1$, which is SL_2-equivariant, where the group SL_2 acts naturally on the two-dimensional space $\mathrm{span}(w_1, w_n)$. Therefore, the map ρ is a cellular fibration and $P_n^{\mathrm{sm}}(q)$ is divisible by the Poincaré polynomial of $\mathbb{P}^1$, which equals $1+q$. □

We now give the second proof of the theorem above.

Theorem 4.11 *There exists a fixed-point free involution σ on the set of smooth collections. For any smooth collection $\mathbf{S}$ and the corresponding cell $C_{\mathbf{S}}$ one has*

$$\dim C_{\mathbf{S}} = \dim C_{\sigma\mathbf{S}} \pm 1.$$

Proof Consider the map $w : \{1, \dots, n+1\} \to \{1, \dots, n+1\}$, which interchanges 1 and $n+1$ and stabilizes all other elements. Define a map σ by the formula

$$\sigma(S_1, \dots, S_n) = (wS_1, \dots, wS_n).$$

First, we note that σ maps each smooth $\mathbf{S}$ to a smooth collection. Second, since w^2 is the identity map, $\sigma^2 = \mathrm{Id}$. Third, let us show that σ is fixed-point free. In fact, a smooth $\mathbf{S}$ is fixed by σ if and only if for all $k = 1, \dots, n$ the set S_k either contains both 1 and $n+1$ or does not contain any of these elements. We note that $\#S_n = n$ and hence S_n contains at least one of the elements $1, n+1$. If $\sigma\mathbf{S} = \mathbf{S}$, then $S_n \supset \{1, n+1\}$. Now let $1 \le k < n$ be a number such that $\{1, n+1\}$ is contained in S_{k+1} but not in S_{k+1} (since $\#S_1 = 1$ such k does exist). If $\sigma\mathbf{S} = \mathbf{S}$, then we have $1, n+1 \notin S_k$. Since $\mathbf{S}$ is smooth, $S_k \subset S_{k+1}$ and therefore S_{k+1} contains two non-intersecting sets S_k and $\{1, n+1\}$. This contradicts with $\#S_{k+1} = k+1$.

Now let $\mathbf{S}$ be a smooth collection. Let k be a number such that $1 \in S_k \setminus S_{k-1}$ and, similarly, let l be a number such that $n+1 \in S_l \setminus S_{l-1}$. As we proved above, $k \ne l$.

Assume that $k < l$. We claim that

$$\dim C_{\mathbf{S}} = \dim C_{\sigma\mathbf{S}} + 1.$$

Recall that $\dim C_{\mathbf{S}}$ is the sum of three numbers $N_{PI}(\mathbf{S}) + N_{PP}(\mathbf{S}) + N_{II}(\mathbf{S})$ (see Proposition 3.10). First, we note that a pair $i = 1$, $j = n + 1$ adds one to $N_{PI}(\mathbf{S})$, but not to $N_{PI}(\sigma\mathbf{S})$. Second, each pair i, j with $1 < i, j < n + 1$, either shows up for both $\mathbf{S}$ and $\sigma\mathbf{S}$ in the dimension counting as in Proposition 3.10 or does not show up for both cells. Now let us look at other pairs and compute the difference between the dimensions of $C_{\mathbf{S}}$ and that of $C_{\sigma\mathbf{S}}$.

Take m satisfying $k \le m < l$ and consider j such that $j > m$, $j \notin S_m$. Then a pair $i = 1$, j adds one to $N_{PI}(\mathbf{S})$, but not to $N_{PI}(\sigma\mathbf{S})$ (since $1 \in S_m$, but $1 \notin (\sigma S)_m$). However, let us look at a pair $i = m$, $j = n + 1$. Since $n + 1 \in (\sigma S)_m \setminus S_m$, the pair $(m, n + 1)$ adds one to $N_{II}(\sigma\mathbf{S})$, but not to $N_{II}(\mathbf{S})$.

Now take m satisfying $k \le m < l$ and consider i such that $i \le m$, $i \in S_m$. Then a pair $i, j = n + 1$ adds one to $N_{PI}(\mathbf{S})$, but not to $N_{PI}(\sigma\mathbf{S})$ (since $n + 1 \notin S_m$, but $n + 1 \in (\sigma S)_m$). However, let us look at a pair $i = 1$, $j = m$. Since $1 \in S_m \setminus (\sigma S)_m$, the pair $(1, m)$ adds one to $N_{PP}(\sigma\mathbf{S})$, but not to $N_{PP}(\mathbf{S})$.

Summarizing, the difference

$$N_{PI}(\mathbf{S}) + N_{PP}(\mathbf{S}) + N_{II}(\mathbf{S}) - N_{PI}(\sigma\mathbf{S}) - N_{PP}(\sigma\mathbf{S}) - N_{II}(\sigma\mathbf{S})$$

is equal to one (coming from the pair $i = 1$, $j = n + 1$). This implies the second statement of the theorem. □

Corollary 4.12 *The polynomials $P_n^{\mathrm{sm}}(q)$ and $r_n(q)$ are divisible by $1 + q$. The ratio $P_n^{\mathrm{sm}}(q)/(1+q)$ is equal to the sum of the terms $q^{\dim C_{\mathbf{S}}}$ taken over smooth $\mathbf{S}$ satisfying the following conditions for all $m = 1, \ldots, n$: if $1 \in S_m$ then $n + 1 \in S_m$.*

Proof The Theorem above states that $P_n^{\mathrm{sm}}(q)$ is equal to the sum over the orbits of the involution σ of the terms $q^d(1 + q)$, where d is the minimum of the dimensions of the cells corresponding to the collections in the orbit. But we know that $\dim C_{\mathbf{S}} = \dim C_{\sigma\mathbf{S}} - 1$ if there exists m such that $n + 1 \in S_m$, but $1 \notin S_m$. This implies the corollary. □

Let us relabel the smooth collections as follows. To a smooth collection $\mathbf{S}$ we attach a permutation $\pi \in S_{n+1}$ by the formula $\pi(m) = S_m \setminus S_{m-1}$. Then $\mathbf{S}$ is smooth if and only if the corresponding permutation satisfies the following conditions for all $1 \le a < b \le n$:

$$\text{if} \quad \pi^{-1}(b) \le a \quad \text{then} \quad \pi^{-1}(a + 1) \le b.$$

Corollary 4.13 *The number of permutations, corresponding to smooth collections, is equal to the large Schröder number. The number of such permutations satisfying $\pi^{-1}(n + 1) < \pi^{-1}(1)$ is equal to the small Schröder number.*

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Appendix A: Regularity in codimension 2

We consider the Grassmannian $\mathrm{Gr}_{\dim P}(P \oplus I)$ for P a projective and I an injective representation over a Dynkin quiver Q. Recall that a variety X is said to be regular in codimension d if there exists a codimension $d+1$ subvariety $Y \subset X$ such that all points of $X \setminus Y$ are smooth. For example, normal varieties are regular in codimension one. In [9] it is proved that quiver Grassmannians $\mathrm{Gr}_{\dim P}(P \oplus I)$ are normal. We now prove a stronger statement.

Theorem A.1 $\mathrm{Gr}_{\dim P}(P \oplus I)$ *is regular in codimension* 2.

Proof Recall that the group $\mathfrak{A} \subset \mathrm{Aut}(P \oplus I)$ acts on $\mathrm{Gr}_{\dim P}(P \oplus I)$ with orbits parametrized by pairs of representations N_I, Q_P of the same dimension vector such that N_I is a subrepresentation of I and Q_P is a quotient of P. Assume that an orbit, parametrized by a pair (N_I, Q_P) of dimension vector $\mathbf{f}$, and admitting exact sequences

$$0 \to N_I \to I \to Q_I \to 0, \qquad 0 \to N_p \to P \to Q_P \to 0,$$

is a singular codimension 2 stratum. Using the codimension formula of the proof of [9, Theorem 4.5], this means that

$$\langle \mathbf{f}, \mathbf{f} \rangle + [N_I, N_I]^1 + [Q_P, Q_P]^1 = 2 \quad \text{and} \quad [N_I, Q_P]^1 \neq 0$$

(we use the abbreviations $[X, Y] = \dim \mathrm{Hom}(X, Y)$ and $[X, Y]^1 = \dim \mathrm{Ext}^1(X, Y)$). If $\langle \mathbf{f}, \mathbf{f} \rangle = 0$, then $\mathbf{f} = 0$, thus $N_I = 0 = Q_P$, and all extension groups are zero, a contradiction. If $\langle \mathbf{f}, \mathbf{f} \rangle = 2$, then $[N_I, N_I]^1 = 0 = [Q_P, Q_P]^1$, thus both N_I and Q_P are isomorphic to the unique exceptional representation G of dimension vector $\mathbf{f}$. In particular, $[N_I, Q_P]^1 = [G, G]^1 = 0$, a contradiction. Thus we have $\langle \mathbf{f}, \mathbf{f} \rangle = 1$ and (without loss of generality) $[N_I, N_I]^1 = 0$ and $[Q_P, Q_P]^1 = 1$. Thus $\mathbf{f}$ is a root and N_I is the corresponding indecomposable. Q_P is a minimal degeneration of N_I, thus by [4, Theorem 4.5]) there exists a non-split short exact sequence

$$0 \to U \to N_I \to V \to 0$$

such that both U and V are indecomposable, and $Q_P \simeq U \oplus V$. In particular, $[V, U]^1 \neq 0$, thus $[U, V]^1 = 0$ since Dynkin quivers are representation-directed. We thus have $1 = [Q_P, Q_P]^1 = [U \oplus V, U \oplus V]^1 = [V, U]^1$. From $[N_I, N_I]^1 = 0$ it

follows that $[N_I, V]^1 = 0$ using the above exact sequence, thus $0 \neq [N_I, Q_P]^1 = [N_I, U \oplus V]^1 = [N_I, U]^1$. Applying $\mathrm{Hom}(\_, U)$ to the above sequence yields

$$\mathrm{Hom}(U, U) \to \mathrm{Ext}^1(V, U) \to \mathrm{Ext}^1(N_I, U) \to \mathrm{Ext}^1(U, U) = 0.$$

The first two terms in this sequence are both one-dimensional. The connecting map is non-zero since the above exact sequence is non-split, thus it is invertible. This implies that $[N_I, U]^1 = 0$, a contradiction. □

Appendix B: Desingularization and the smooth locus

Let $\pi_{n+1} : R_{n+1} \to \mathcal{F}^a_{n+1}$ be the desingularization of the degenerate flag variety of type A_n of [16]. Our goal here is to prove the following theorem.

Theorem B.1 *$\pi^{-1}_{n+1}(x)$ is a single point iff x is a smooth point of $\mathcal{F}^a_{n+1}$.*

Recall that R_{n+1} can be explicitly realized as follows. Let W be an $(n+1)$-dimensional space with a basis $(w_1, \ldots, w_{n+1})$. For a pair $1 \leq i \leq j \leq n$, let $W^{n+1}_{i,j} = \mathrm{span}(w_1, \ldots, w_i, w_{j+1}, \ldots, w_{n+1})$. Then R_n is the variety of collections $(V_{i,j})_{1 \leq i \leq j \leq n}$ such that $V_{i,j} \in \mathrm{Gr}_i(W^{n+1}_{i,j})$ and $V_{i,j} \subset V_{i+1,j}$ and $pr_{j+1} V_{i,j} \subset V_{i,j+1}$.

Lemma B.2 *R_{n+1} can be embedded into $\mathcal{F}^a_{n+1} \times R_n$ in such a way that π_{n+1} is simply the projection to the first factor.*

Proof We first note that the map $\pi_{n+1} : R_{n+1} \to \mathcal{F}^a_{n+1}$ is explicitly given by $(V_{i,j})_{i \leq j} \mapsto (V_{i,i})_{i=1}^n$. Now consider the forgetful map

$$(V_{i,j})_{1 \leq i \leq j \leq n} \to (V_{i,j})_{1 \leq i < j \leq n}$$

(the diagonal terms $V_{i,i}$ are omitted). We claim that the image is isomorphic to R_n. Namely, for a pair $1 \leq i < j \leq n$, we consider the "shift" map $sh_{i,j} : W^{n+1}_{i,j} \to W^n_{i,j-1}$ given by

$$sh_{i,j} w_k = \begin{cases} w_k, & \text{if } k \leq i, \\ w_{k-1}, & \text{if } k > j. \end{cases}$$

Then for a point $(V_{i,j})_{i \leq j} \in R_{n+1}$, the collection

$$\left(V'_{i,j}\right)_{1 \leq i \leq j \leq n-1} = (sh_{i,j+1} V_{i,j+1})_{1 \leq i \leq j \leq n-1}$$

belongs to R_n. We denote the map $R_{n+1} \to R_n$ by ψ_{n+1}. Now the embedding $R_{n+1} \to \mathcal{F}^a_{n+1} \times R_n$ is given by the map $A = (\pi_{n+1}, \psi_{n+1})$. □

Lemma B.3 *Let $\mathbf{S}$ be a length n smooth collection. Then*

$$\pi_n \psi_{n+1} \pi^{-1}_{n+1} p_{\mathbf{S}} \subset \mathcal{F}^a_n$$

is a single point. Moreover, it is a smooth torus fixed point.

Proof Recall that

$$p_{\mathbf{S}} = \big((p_{\mathbf{S}})_i\big)_{i=1}^{n}, \qquad (p_{\mathbf{S}})_i = \operatorname{span}(w_a : a \in S_i).$$

Our first goal is to prove that there exists a unique way to define spaces $(V_{i,i+1})_{i=1}^{n-1}$ such that there exists a point in R_{n+1} with the diagonal components being $(p_{\mathbf{S}})_i$ and the $(i, i+1)$-st components being $V_{i,i+1}$. In fact, fix some i with $1 \le i \le n-1$. We need $V_{i,i+1}$ such that $\dim V_{i,i+1} = i$ and

$$pr_{i+1}(p_{\mathbf{S}})_i \subset V_{i,i+1} \subset W_{i,i+1}^{n+1} \cap (p_{\mathbf{S}})_{i+1}.$$

If $i+1 \notin S_i$, then $\dim pr_{i+1}(p_{\mathbf{S}})_i = i$ and hence $V_{i,i+1} = pr_{i+1}(p_{\mathbf{S}})_i$. If $i+1 \in S_i$, then since $\mathbf{S}$ is smooth, we have $i+1 \in S_{i+1}$. Therefore the intersection

$$W_{i,i+1}^{n+1} \cap (p_{\mathbf{S}})_{i+1} = \operatorname{span}(w_a : a \neq i+1) \cap \operatorname{span}(w_a : a \in S_{i+1})$$

is i-dimensional and hence $V_{i,i+1}$ is forced to coincide with this intersection. Note that in both cases $V_{i,i+1}$ is the linear span of some basis vectors. We denote by $S_{i,i+1} \subset \{1, \dots, i, i+2, \dots, n+1\}$ the set of indices of these vectors, i.e.

$$V_{i,i+1} = \operatorname{span}(w_a : a \in S_{i,i+1}).$$

We note that $S_{i,i+1} \subset S_{i+1}$ and $S_i \subset S_{i,i+1} \cup \{i+1\}$.

We identify the collection of subspaces $(V_{i,i+1})_{i=1}^{n-1}$ constructed above with the point $(sh_{i,i+1}V_{i,i+1})_{i=1}^{n-1} \in \mathcal{F}_n^a$. As mentioned above, each component of this point is a linear span of basis vectors and thus $(sh_{i,i+1}V_{i,i+1})_{i=1}^{n-1} = p(\bar{\mathbf{S}})$ for some collection $\bar{\mathbf{S}} = (\bar{S}_1, \dots, \bar{S}_{n-1})$. Explicitly,

$$\bar{S}_i = \{a : a \in S_{i,i+1}, a \le i\} \cup \{a-1 : a \in S_{i,i+1}, a > i+1\}.$$

Our goal is to prove that this collection is smooth. In fact, assume $b \in \bar{S}_a$ for some $1 \le a < b \le n-1$. Then since $b > a$ we have $b+1 \in S_{a,a+1}$. We consider two cases: $b+1 \in S_a$ and $b+1 \notin S_a$. If $b+1 \in S_a$, then $a+1 \in S_{b+1}$ ($\mathbf{S}$ is smooth). Since $S_a \subset S_{b+1}$, we have $b+1 \in S_{b+1}$. Therefore, $S_{b,b+1} = S_{b+1} \setminus \{b+1\}$ and, in particular, $a+1 \in S_{b,b+1}$. Since $a+1 \le b$, this implies $a+1 \in \bar{S}_b$. Now assume $b+1 \notin S_a$. Then $S_{a,a+1} \neq S_a$ and hence $a+1 \in S_a$. This implies $a+1 \in S_b$ and so $a+1 \in S_{b,b+1}$ (because $w_{a+1} = pr_{b+1}w_{a+1} \in V_{b,b+1}$). We thus arrive at $a+1 \in \bar{S}_b$, which means that $\bar{\mathbf{S}}$ is smooth. □

Corollary B.4 *The map π_{n+1} is one-to-one over the smooth locus of $\mathcal{F}_{n+1}^a$.*

Proof We note that since the fibers over any two points of a given cell in $\mathcal{F}_{n+1}^a$ are isomorphic, it suffices to prove that the fiber is a single point over a smooth torus fixed point. Let $\mathbf{S}$ be a smooth collection and $p(\bar{\mathbf{S}}) = \pi_n \psi_{n+1} \pi_{n+1}^{-1}$. Since $\bar{\mathbf{S}}$ is smooth, our corollary follows by induction on n. □

To complete the proof of Theorem B.1, we need to show that the fiber over a non-smooth point has positive dimension. It suffices to prove that if a collection $\mathbf{S}$ is not

smooth, then the preimage of $p_{\mathbf{S}}$ has positive dimension. We first prove the following lemma.

Lemma B.5 *Assume that S_a is not a subset of S_{a+1} for some a. Then the dimension of the fiber $\pi_{n+1}^{-1} p_{\mathbf{S}}$ is positive.*

Proof Assume that $p_{\mathbf{S}}$ is the image of $(V_{i,j})_{1\le i\le j\le n}$. Let us look at possible sets $V_{a,a+1}$. We know that

$$pr_{a+1}(p_{\mathbf{S}})_a \subset V_{a,a+1} \subset (p_{\mathbf{S}})_{a+1} \cap \operatorname{span}(w_i : i \neq a+1). \tag{B.1}$$

Since S_a is not a subset of S_{a+1} and $S_a \subset S_{a+1} \cup \{a+1\}$, we obtain $a+1 \in S_a$, $a+1 \notin S_{a+1}$. Therefore,

$$\dim pr_{a+1}(p_{\mathbf{S}})_a = a-1, \qquad \dim (p_{\mathbf{S}})_{a+1} \cap \operatorname{span}(w_i : i \neq a+1) = a+1.$$

Thus the choice of $V_{i,i+1}$ as in (B.1) is equivalent to the choice of a point in $\mathbb{P}^1$. Therefore the preimage $\pi_{n+1}^{-1} p_{\mathbf{S}}$ is at least one-dimensional. □

Corollary B.6 *If $\mathbf{S}$ is not smooth, then the dimension of the fiber $\pi_{n+1}^{-1} p_{\mathbf{S}}$ is positive.*

Proof Let $k \ge 1$ be a minimal number such that there exists a number a, $1 \le a \le n-k$ such that $a+k \in S_a$, but $a+1 \notin S_{a+k}$. We prove our corollary by induction on k. First, we note that the case $k=1$ means that $S_a \notin S_{a+1}$ and we are done by the lemma above. Now let $k > 1$. Since $k > 1$ the sets $S_{a,a+1}$ satisfying

$$S_a \cup \{a+1\} \subset S_{a,a+1} \subset S_{a+1}$$

are defined uniquely. Now define a length $n-1$ collection $\bar{\mathbf{S}}$ as above:

$$\bar{S}_i = \{l : l \in S_{i,i+1}, l \le i\} \cup \{l-1 :\ l \in S_{i,i+1}, l > i+1\}.$$

Since $a+k \in S_a$ and $k > 1$ we obtain $a+k-1 \in \bar{S}_a$. Also, since $a+1 \notin S_{a+k}$, we obtain $a+1 \notin \bar{S}_{a+k}$ and hence $a+1 \notin \bar{S}_{a+k-1}$ (since $k > 1$ we have $S_{a+k-1} \subset S_{a+k}$). This proves that k becomes $k-1$ for $\bar{\mathbf{S}}$. By the inductive assumption we know that the preimage $\pi_n^{-1} p(\bar{\mathbf{S}})$ is positive-dimensional. But $\pi_{n+1}^{-1} p_{\mathbf{S}} = \pi_n^{-1} p(\bar{\mathbf{S}})$ and we are done. □

Appendix C

In this appendix we compute the moment graph of $\mathcal{F}_4^a$. The T-fixed points of $\mathcal{F}_4^a$ are listed in Fig. 1. Recall that such points are parameterized by successor-closed

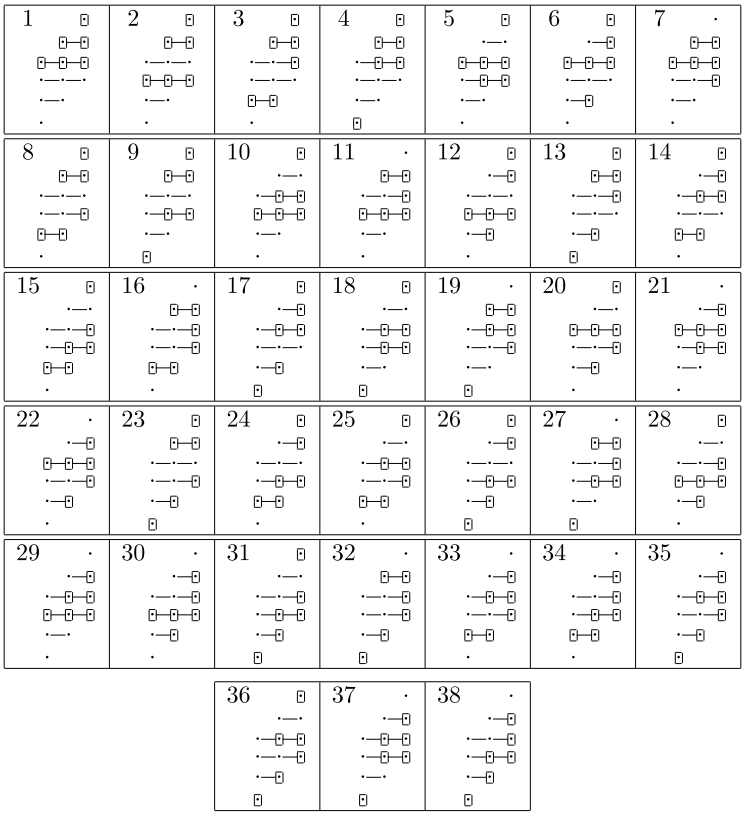

Fig. 1 The T-fixed points of $\mathcal{F}_4^a$

subquivers of the following quiver:

$$\begin{array}{lll} 3 & & \cdot \\ 2 & & \cdot \to \cdot \\ 1 & \cdot \to \cdot \to \cdot & \\ 4 & \cdot \to \cdot \to \cdot & \\ 3 & \cdot \to \cdot & \\ 2 & \cdot & \end{array} \tag{C.1}$$

having one vertex in the first column, two in the second and three vertices in the third column.

Figure 2 shows the moment graph of the degenerate flag variety $\mathcal{F}_4^a$ (we used Bernhard Keller's quiver mutation applet to draw the picture [23]). The 22 smooth torus

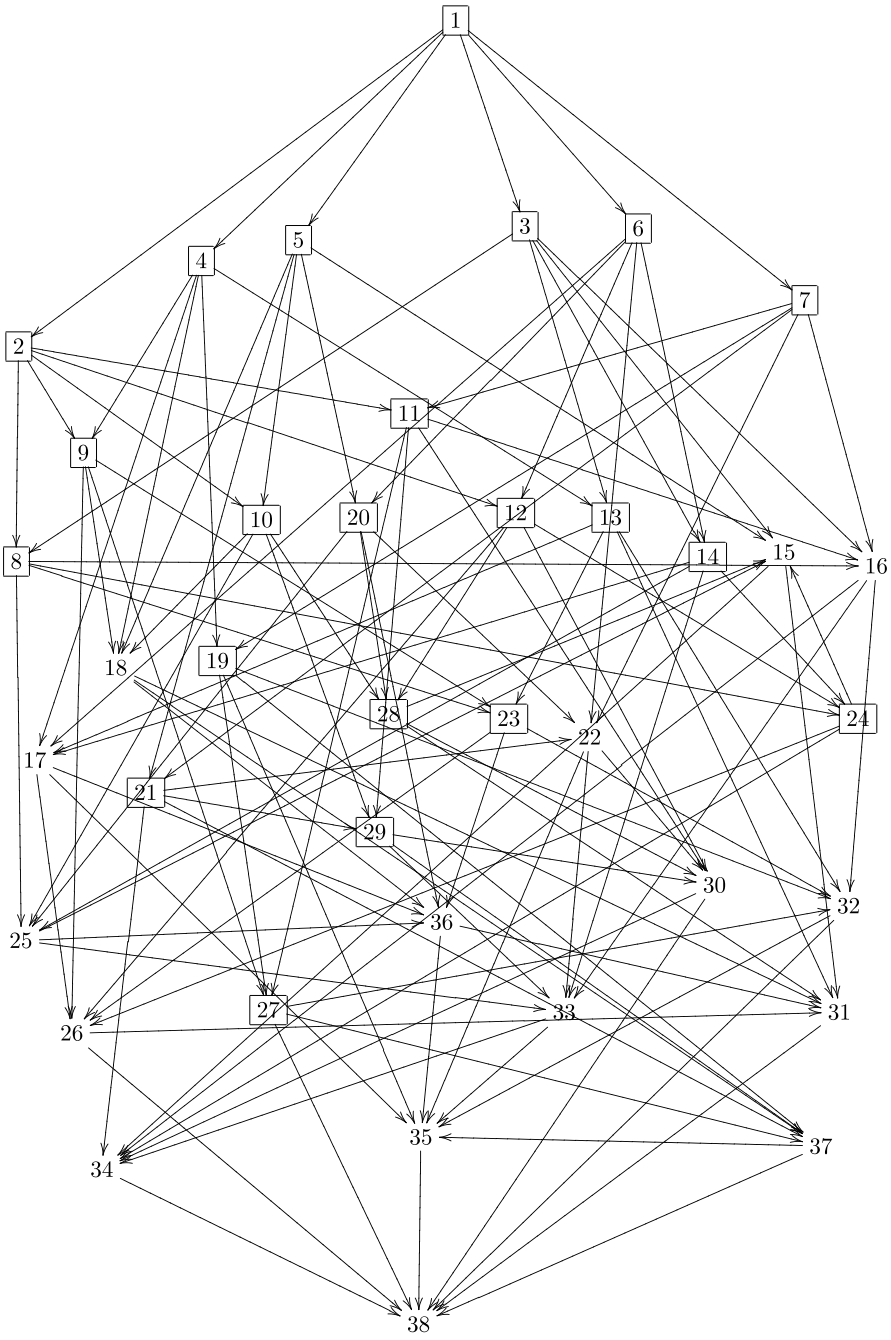

Fig. 2 The moment graph of $\mathcal{F}_4^a$. The vertices are labeled according to Fig. 1. The *highlighted vertices* correspond to the smooth T-fixed points

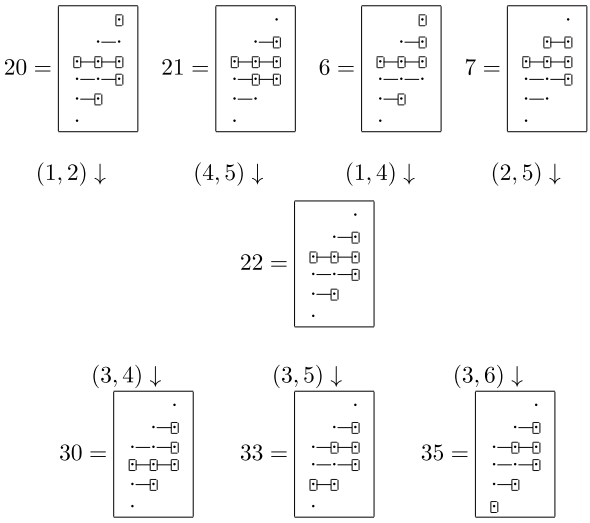

Fig. 3 The moment graph around vertex (22)

fixed points are highlighted by a frame. These are the vertices adjacent to precisely $6 = \dim \mathcal{F}_4^a$ edges. An edge $p_{\mathbf{S}}$–$p_{\mathbf{R}}$ of the moment graph corresponds to a T-fixed curve between $p_{\mathbf{S}}$ and $p_{\mathbf{R}}$ in $\mathcal{F}_4^a$ whose direction around $p_{\mathbf{S}}$ and $p_{\mathbf{R}}$ is given by a standard basis vector of the tangent space at them. The edge is oriented $p_{\mathbf{S}} \to p_{\mathbf{R}}$ if and only if the direction around $p_{\mathbf{S}}$ has positive T_0-weight and it is labeled by the corresponding **S**-effective pair (see Theorem 3.18 and Remark 3.5).

To illustrate, let us describe in detail the graph around vertex (22). There are 7 edges connected to this vertex as depicted in Fig. 3. In particular this T-fixed point is not smooth.

The arrow (20) → (22) corresponds to the following curve (in the basis (C.1))

$$\big(\langle v_1\rangle, \langle v_1, v_3\rangle, \langle v_3 + \lambda v_2, v_1, v_4\rangle\big) \quad \lambda \in \mathbb{P}^1.$$

For $\lambda = 0$ one gets the starting point (20) of α, for $\lambda = \infty$ one gets the end point (22) of α. Its direction around (22) has negative T_0 weight while around (20) it has positive weight. The corresponding (20)-effective pair is $(1, 2)$. The remaining labelings of Fig. 3 are obtained similarly.

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