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Block characters of the symmetric groups

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Abstract A block character of a finite symmetric group is a positive definite function which depends only on the number of cycles in a permutation. We describe the cone of block characters by identifying its extreme rays, and find relations of the characters to descent representations and the coinvariant algebra of $\mathfrak{S}_n$. The decomposition of extreme block characters into the sum of characters of irreducible representations gives rise to certain limit shape theorems for random Young diagrams. We also study counterparts of the block characters for the infinite symmetric group $\mathfrak{S}_\infty$, along with their connection to the Thoma characters of the infinite linear group $GL_\infty(q)$ over a Galois field.

Keywords Symmetric group · Characters · Coinvariant algebra · Limit shape

1 Introduction

Let G be a group. Under a *character* of G, we shall understand a positive-definite class function $\chi : G \to \mathbb{C}$. A character which satisfies $\chi(e) = 1$ will be called *normalized*.

The work on the paper started in 1999 when Sergei Kerov (1946–2000) was visiting the first author at the University of Göttingen.

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For $g \mapsto R_g$, a finite-dimensional matrix representation of G, the trace

$$\chi(g) = \mathrm{Trace}(R_g)$$

is a character. For a finite or, more generally, compact group, the set of normalized characters is a simplex whose extreme points are normalized traces $g \mapsto \chi(g)/\chi(e)$ of irreducible representations of G. For infinite groups, the connection is more delicate since there are many infinite-dimensional representations and the matrix traces are of no use. A classical construction associates *extreme* normalized characters with factor representations of finite von Neumann type, see [8, 34]. Yet another approach exploits spherical representations of the Gelfand pairs, see [26, 27]. The representation theory is mainly focused on the classification of extreme characters and the decomposition of the generic character in a convex sum of the extremes, the latter being as a counterpart of the decomposition of a representation into the irreducible ones. However, the extreme characters may be complicated functions and the set of the extremes may be too large, so it is of interest to study smaller tractable families of reducible characters, for instance, those which have some kind of symmetry or depend on some simple statistic on the group.

In this paper, we study symmetric groups $\mathfrak{S}_n$ and their *block* characters which depend on a permutation $g \in \mathfrak{S}_n$ only through the number of cycles $\ell_n(g)$. Our interest to block characters is motivated by the analogous concept of *derangement* characters of the general linear group $GL_n(q)$ of invertible matrices over a Galois field, as studied in [13]. A derangement character depends on the matrix $h \in GL_n(q)$ only through the dimension of the space $\ker(h - Id)$ of fixed vectors of h. See [29, 35] for a connection of the derangement characters to representation theory of $GL_\infty(q)$ and [10] for a connection of these characters to some random walks. The natural embedding $\mathfrak{S}_n$ in $GL_n(q)$, which maps a permutation g to a permutation matrix h, yields a link between two families of characters. Indeed, it is easily seen that $\ell_n(g)$ is equal to the dimension of the space of fixed vectors of h, so the restriction of a derangement character from $GL_n(q)$ to $\mathfrak{S}_n$ is a block character.

The convex set of normalized block characters of $\mathfrak{S}_n$ is a simplex whose extreme points are normalized versions of the characters $\tau_1^n, \dots, \tau_n^n$ introduced by Foulkes [9]. It should be stressed that, a priori, there is no general reason for the set of normalized block characters to be a simplex. To compare, the set of normalized derangement characters of $GL_n(q)$ is a simplex for some n, and not a simplex for other [13]. Characters τ_k^n are related to the descent statistics of permutations. In particular, $\tau_k^n(e)$ coincides with the Eulerian number, which counts permutations with $k-1$ descents. Using a decomposition of the *coinvariant algebra*, we will find representations of $\mathfrak{S}_n$ whose traces are the τ_k^n's. Versions of decompositions of the coinvariant algebra is a classical topic (see [32, Proposition 4.11], [23], [28, Sect. 8.3]) which has been studied recently in [1] in connection with *descent representations* [30].

Foulkes [9] defined the τ_k^n's by summing 'rim hook' characters which are not block functions at all. In this paper, we take a more straightforward approach, starting with a collection of block characters associated with a natural action of $\mathfrak{S}_n$ on words. On this way, we derive 'from scratch' a number of known results on decomposition and branching of the Foulkes characters, as found in [6, 19].

Extending the finite-n case, we shall consider the infinite symmetric group $\mathfrak{S}_\infty$ of bijections $g:\mathbb{N}\to\mathbb{N}$ satisfying $g(j)=j$ for all sufficiently large j. The counterparts of the block characters of $\mathfrak{S}_\infty$ are the characters depending on the permutation through its *decrement* defined by $c(g):=n-\ell_n(g)$, with any large enough n. We will show that the set of normalized block characters of $\mathfrak{S}_\infty$ is a Choquet simplex with extreme points $\sigma_z^\infty(g):=z^{c(g)}$ where $z\in\mathbb{V}=\{0,\pm 1,\pm 1/2,\pm 1/3,\dots\}$ (the instance, $z=0$ is understood as the delta function at e). Recall that the characteristic property of a Choquet simplex is the uniqueness of decomposition of the generic point in a convex mixture of extremes [15].

The extreme normalized characters of $\mathfrak{S}_\infty$ were parameterized in a seminal paper by Thoma [36] (see also [21, 25, 37]) via two infinite sequences

$$\alpha_1\ge\alpha_2\ge\cdots\ge 0,\qquad \beta_1\ge\beta_2\ge\cdots\ge 0,\qquad \sum_i(\alpha_i+\beta_i)\le 1. \tag{1}$$

It turns out that the extreme normalized block characters of $\mathfrak{S}_\infty$ are extreme among all normalized characters, with $\sigma_{1/k}^\infty$ corresponding to the parameters $\alpha_1=\alpha_2=\cdots=\alpha_k=1/k$ and $\sigma_{-1/k}^\infty$ to the parameters $\beta_1=\beta_2=\cdots=\beta_k=1/k$ (where $k>0$). The character σ_0^∞ has $\alpha_j\equiv\beta_j\equiv 0$ and corresponds to the regular representation of $\mathfrak{S}_\infty$.

The block characters also possess an additional extremal property. Observe that the set (1) of parameters of extreme normalized characters of $\mathfrak{S}_\infty$ is a simplex by itself. The block characters $\sigma_z^\infty(g)$ correspond precisely to all extreme points of this simplex.

Every normalized character χ^n of the symmetric group $\mathfrak{S}_n$ defines a probability measure on the set $\mathbb{Y}_n$ of Young diagrams with n boxes. Indeed, recall that irreducible representations of $\mathfrak{S}_n$ are parameterized by the elements of $\mathbb{Y}_n$ and decompose χ^n into the linear combination of their (conventional) characters χ^λ:

$$\chi^n(\cdot)=\sum_{\lambda\in\mathbb{Y}_n}p^n(\lambda)\frac{\chi^\lambda(\cdot)}{\chi^\lambda(e)}.$$

The numbers $p^n(\lambda)$ are non-negative and sum up to 1, thus, they define a probability distribution on $\mathbb{Y}_n$ or *random Young diagram* Y^{χ^n}. As $n\to\infty$, these random Young diagrams may possess intriguing properties depending on the sequence of characters χ_n.

Kerov, Vershik [38] and, independently, Logan, Shepp [24] proved in the 1970s that if we choose χ^n to be the character of the regular representation of $\mathfrak{S}_n$ then after a proper rescaling the boundary of the Young diagram Y^{χ^n} converges to a deterministic smooth curve called the *limit shape*. We will prove a similar result for the extreme block characters τ_k^n. More precisely, if $n\to\infty$ and $k\sim c\sqrt{n}$ then the (rescaled) boundary of the Young diagram $Y^{\tau_k^n}$ converges to the deterministic limit shape depending on c; similar result holds if $n-k\sim c\sqrt{n}$. Our limit shapes are the same as those obtained by Biane [4] in the context of tensor representations of $\mathfrak{S}_n$. This fact could have been predicted since the characters considered by Biane coincide with restrictions of σ_z^∞ on finite symmetric groups $\mathfrak{S}_n$ and extreme block characters τ_k^n approximate σ_z^∞ as n tends to infinity.

From another probabilistic viewpoint, the characters $\sigma_{1/k}^{\infty}$ ($k > 0$) have been studied in [18] and [20, Sect. III.3]; it was shown that the probability measures on the set of Young diagrams which they define are related to the distributions of eigenvalues of random matrices.

Like 'supercharacters' of Diaconis and Isaacs [7], the block characters are constant on big blocks of conjugacy classes ('superclasses'). The latter feature motivated our choice of the name for this family of functions. However, the block characters do not fit in the theory of 'supercharacters' since the extremes τ_k^n (hence their mixtures) are not disjoint in their decomposition over the irreducible traces. The same distinction applies to the derangement characters of finite linear groups as well.

The rest of the paper is organized as follows. In Sect. 2, we introduce families of block characters of $\mathfrak{S}_n$ and derive their properties. In Sect. 3, we prove that the set of normalized block characters of $\mathfrak{S}_n$ is a simplex and identify its extreme points (the normalized Foulkes characters). In Sect. 4, we study block characters τ_k^n and their relation with the coinvariant algebra of $\mathfrak{S}_n$. In Sect. 5, we prove the limit shape theorem for extreme block characters of $\mathfrak{S}_n$. In Sect. 6, we derive the branching rule for the characters τ_k^n's as n varies. In Sect. 7, we prove that the set of normalized block characters of $\mathfrak{S}_\infty$ is a simplex and identify its extreme points. Finally, in Sect. 8, we comment on the relation between the block characters and the derangement characters of $GL_n(q)$ for $n \le \infty$.

2 The block characters

Let $\mathfrak{S}_n$ denote the group of permutations of $\{1, \dots, n\}$ and let $\ell_n(g)$ be the number of cycles of permutation $g \in \mathfrak{S}_n$. A *block function* on $\mathfrak{S}_n$ is a function which depends on g only through $\ell_n(g)$.

In general, a *character* of a group G is a complex-valued function χ on G which is

1. Central, i.e., a class function: $\chi(a^{-1}ba) = \chi(b)$.
2. Positive definite, i.e., for any finite collection (g_i) of elements of G the matrix with entries $\chi(g_i g_j^{-1})$ is a Hermitian non-negative definite matrix.

If $g \mapsto R_g$ is a finite-dimensional matrix representation of a group G, then its trace $\chi(g) = \mathrm{Trace}(R_g)$ is a character.

A *block character* of $\mathfrak{S}_n$ is a positive-definite block function. A conjugacy class of $g \in \mathfrak{S}_n$ is determined by the partition of n into parts equal to the cycle-sizes of g, hence every block function is central, and every block character is indeed a character.

Our starting point is an elementary construction of a family of block characters. Fix an integer $k > 0$ and let A_k^n be the set of all words of length n in the alphabet $\{1, 2, \dots, k\}$. The group $\mathfrak{S}_n$ naturally acts in A_k^n by permuting *positions* of letters, and this action defines a unitary representation of the group in $\mathcal{L}_2(A_k^n)$ by the formula:

$$(gF)(x) = F\big(g^{-1}x\big), \quad F \in \mathcal{L}_2\big(A_k^n\big), g \in \mathfrak{S}_n.$$

Let R_k^n denote this representation and let σ_k^n be its character. Furthermore, let $\widehat{S}^n$ be the one-dimensional sign representation of $\mathfrak{S}_n$, which has the block character

$g \mapsto (-1)^{n-\ell_n(g)}$, and let $\widehat{R}_k^n$ be the tensor product of the representations $\widehat{S}^n$ and R_k^n:

$$\widehat{R}_k^n = \widehat{S}_1^n \otimes R_k^n.$$

Let $\widehat{\sigma}_k^n$ be the matrix trace of $\widehat{R}_k^n$.

Proposition 2.1 *The functions σ_k^n and $\widehat{\sigma}_k^n$ are block characters of $\mathfrak{S}_n$. Explicitly,*

$$\sigma_k^n(g) = k^{\ell_n(g)},$$
$$\widehat{\sigma}_k^n(g) = (-1)^n(-k)^{\ell_n(g)}.$$

Proof The trace of $R_k^n(g)$ is equal to the number of words in A_n^k fixed by g. A word is fixed if within each set of positions comprising a cycle the letters occupying these positions are the same, whence the first formula. The second formula follows from the multiplication rule for traces of tensor products. □

The possible values of the function ℓ_n on $\mathfrak{S}_n$ are integers $1, \dots, n$. On the other hand, the determinant of the matrix $(k^\ell)_{k,\ell\in\{1,\dots,n\}}$ is a nonzero Vandermonde determinant; therefore, the n characters $\sigma_1^n, \dots, \sigma_n^n$ comprise a basis of the linear space of block functions.

We are mostly interested in the *extreme rays* of the cone of block characters. Now we define another family of block characters which (as we will see later) generate these rays.

Definition 2.2 *For $k = 1, \dots, n$ define*

$$\tau_k^n := \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} \sigma_{k-j}^n. \tag{2}$$

Clearly, τ_k^n is a block function, although it is not obvious whether it is a character. The formula (2.2) can be inverted as follows:

Proposition 2.3 *We have*

$$\sigma_k^n = \sum_{j=0}^{k-1} \binom{n+j}{j} \tau_{k-j}^n. \tag{3}$$

Proof The inversion formula is equivalent to the identity (for $\alpha > 0$)

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{\alpha+j-1}{j} \binom{\alpha}{m-j} = 1 \quad (m = 0)$$

(where and henceforth $1(\cdots)$ is 1 when $\cdots$ is true and 0 otherwise). The identity is derived by substituting the generating function

$$(1+x)^{-\alpha} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j-1}{j} x^j$$

in $(1-x)^{-\alpha}(1-x)^{\alpha} - 1 = 0$ and equating the coefficients to 0. $\square$

Our next aim is to decompose σ_k^n and τ_k^n into linear combinations of irreducible characters of $\mathfrak{S}_n$. Let us introduce some notations first.

A *partition* of n is a finite nondecreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of positive integers such that $|\lambda| := \sum \lambda_i = n$. We identify partition λ with its *Young diagram* defined as the set $\{(i, j) \in \mathbb{Z}_+^2 : 1 \le j \le \lambda_i\}$. We call an element $x = (i, j) \in \lambda$ a *box*, and draw it as a unit square at location (i, j) (with the English convention that $(1, 1)$ is at the top left and the first coordinate is vertical). Let $\mathbb{Y}_n$ denote the set of all partitions of n and let $\mathbb{Y}_n^k$ denote the set of all partitions of n with at most k non-zero parts (i.e., such that $\ell \le k$).

It is well-known that irreducible representations of $\mathfrak{S}_n$ are enumerated by the elements of $\mathbb{Y}_n$. For $\lambda \in \mathbb{Y}_n$ we denote V^λ and χ^λ the irreducible representation corresponding to λ and the character (matrix trace) of this representation, respectively.

A *Young tableau* T of shape λ is a map assigning to boxes of the Young diagram λ positive integer entries which are non-decreasing along rows and columns. We denote $T(x)$ the entry assigned to box x. A Young tableau is *semistandard* if the entries strictly increase along the columns. A Young tableau T of shape λ is *standard* if the set of entries of T is $\{1, 2, \dots, |\lambda|\}$. For a standard Young tableau T of shape λ, a *descent* is an integer $0 < i < |\lambda|$ such that the entry $i+1$ appears in T below entry i, that is to say, the vertical coordinate of $T^{-1}(i+1)$ is greater than that of $T^{-1}(i)$. The number of descents is denoted $d(T)$. Figure 1 gives an example.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array}$$

Fig. 1 Standard Young tableau of shape $(4, 3, 1)$ with the set of descents $\{3, 5, 6\}$

Now we proceed to the decomposition of the characters σ_k^n and τ_k^n.

Proposition 2.4 *We have*

$$\sigma_k^n = \sum_{\lambda \in \mathbb{Y}_n} s_k(\lambda) \chi^\lambda,$$

where $s_k(\lambda)$ *is the number of semistandard Young tableaux of shape* λ *with entries belonging to the set* $\{1, \dots, k\}$.

Proof Let W_k be a k-dimensional vector space with basis $w_1, \dots, w_k$. The symmetric group $\mathfrak{S}_n$ acts on the tensor power $W_k^{\otimes n}$ by permuting the factors. Note that the basis of $W_k^{\otimes n}$ is enumerated by the elements of A_k^n, thus the representation in $W_k^{\otimes n}$ is equivalent to R_k^n. The decomposition of the representation in $W_k^{\otimes n}$ into irreducibles is a well-known fact related to the Schur–Weyl duality (see, e.g., [39]):

$$W_k^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Y}_n^{\urcorner k}} \mathrm{Dim}_k(\lambda) \cdot V^\lambda, \tag{4}$$

where $\mathrm{Dim}_k(\lambda)$ is equal to the number of semistandard Young tableaux with entries from $\{1, \dots, k\}$ (which, in turn, equals to the dimension of the irreducible representation, corresponding to λ, of the group of unitary matrices of size k). The proposition easily follows. □

Proposition 2.5 *We have*

$$\tau_k^n = \sum_{\lambda \in \mathbb{Y}_n} m_k(\lambda) \chi^\lambda,$$

where $m_k(\lambda)$ is the number of standard Young tableaux of shape λ with $k-1$ descents.

Proof Proposition 2.4 and Definition 2.2 imply that

$$\tau_k^n = \sum_{\lambda \in \mathbb{Y}_n} h_k(\lambda) \chi^\lambda,$$

where

$$h_k(\lambda) = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} s_{k-j}(\lambda).$$

Thus, it remains to prove that $h_k(\lambda) = m_k(\lambda)$, which amounts to showing that for every $\lambda \in \mathbb{Y}_n$ and every $k = 1, 2, \dots, n$ we have

$$m_k(\lambda) = \sum_{j=0}^{k-1} (-1)^j \binom{n+1}{j} s_{k-j}(\lambda). \tag{5}$$

Similarly to the proof of Proposition 2.3, Eq. (5) is equivalent to the inversion formula

$$s_k(\lambda) = \sum_{j=0}^{k-1} \binom{n+j}{j} m_{k-j}(\lambda), \tag{6}$$

which will be shown combinatorially (see also [31, Eq. (7.96)] for the proof of a more general fact).

Fix a standard Young tableau T of shape λ. We call a non-decreasing integer sequence $X = (X_1, \dots, X_n)$ T-admissible if $1 \le X_1$, $X_n \le k$, and $X_i < X_{i+1}$ for every descent i of T. Given a pair (T, X), where X is T-admissible, we define a semistandard tableau $Y_{T,X}$ of the same shape λ by setting $Y_{T,X}(T^{-1}(j)) = X_j$ for $j = 1, \dots, n$. An example of such procedure is shown in Fig. 2.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & \hat{3} & 4 & \hat{5} & \hat{6} & 7 & 8 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ \hline\end{array}$$

Fig. 2 Standard Young tableau T of shape $\lambda = (4, 3, 1)$ with 3 descents (*top-left*), numbers $\{1, \dots, 8\}$ with descents of T marked by hats (*top-right*), lexicographically minimal T-admissible sequence X (*bottom-right*) and corresponding semistandard Young tableau $Y_{T,X}$ (*bottom-left*)

Observe that for a given standard Young tableau T with $k - j - 1$ descents there are $\binom{n+j}{j}$ ways to choose a T-admissible sequence X. One easily proves that the map $(T, X) \to Y_{T,X}$ is a bijection between the set of pairs (T, X) and the set of semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$. The number of pairs (T, X), where T is a standard Young tableau of shape λ and X is a T-admissible sequence, is equal to the right-hand side of (6), while the number of semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$ is $s_k(\lambda)$, so we are done. $\square$

Corollary 2.6 *Block functions τ_k^n are traces of some representations of $\mathfrak{S}_n$, in particular, they are characters.*

Proof By Proposition 2.5, τ_k^n is the matrix trace of the representation

$$\pi_k^n \cong \bigoplus_{\lambda \in \mathbb{Y}_n} m_k(\lambda) V^\lambda. \tag{7}$$

$\square$

There is a certain duality among the characters τ_k^n, as described in the following proposition.

Proposition 2.7 *Let π_k^n be the representation with character τ_k^n. As above, let $\widehat{S}^n$ be the one-dimensional sign representation of $\mathfrak{S}_n$. Then the representation $\pi_k^n \otimes \widehat{S}^n$ is equivalent to π_{n+1-k}^n, in particular, its character coincides with τ_{n+1-k}^n.*

Proof We multiply (7) by $\widehat{S}^n$ and use the fact that $V^\lambda \otimes \widehat{S}^n$ is equivalent to $V^{\lambda'}$, where λ' is the transposed diagram obtained by reflecting λ about the main diagonal, so that the row lengths of λ' become the column lengths of λ. We obtain:

$$\pi_k^n \otimes \widehat{S}^n \cong \bigoplus_{\lambda \in \mathbb{Y}_n} m_k(\lambda') V^\lambda.$$

We claim that $m_k(\lambda') = m_{n+1-k}(\lambda)$. Indeed, if T is a standard Young tableau of shape λ, then every $i = 1, \dots, n-1$ is either a descent in T or a descent in T', where T' is the tableau of shape λ' defined by $T'(i,j) = T(j,i)$. In particular, $d(T) + d(T') = n-1$. $\square$

Corollary 2.8 *We have*

$$\widehat{\sigma}^n_k = \sum_{j=0}^{k-1} \binom{n+j}{j} \tau^n_{n+1-k+j}. \tag{8}$$

We shall give now two further examples of block characters. Formula (3) generalizes as

$$\theta^{\ell_n(g)} = \sum_{j=1}^{n} \binom{\theta+n-j}{n} \tau^n_j(g), \tag{9}$$

where the generalized binomial coefficient involves parameter $\theta > 0$. This is a polynomial identity which may be shown by extrapolating from the integer values $\theta \in \{1, \dots, n\}$. For $\theta \geq 0$ the function $g \mapsto p\theta^{\ell_n(g)}$, where $p^{-1} = \theta(\theta+1)\cdots(\theta+n-1)$, is a probability on $\mathfrak{S}_n$, known as Ewens' distribution; see, e.g., [3, 33]. Ewens' distribution was first discovered in the context of population genetics and has become central in many contexts of pure and applied probability. It is also intensively used in the harmonic analysis on $\mathfrak{S}_\infty$, see [22].

As a simple corollary of the results in the next section, the function $\theta^{\ell_n(g)}$ is a character for $\theta \geq n-1$, but it is not positive definite for non-integer $0 < \theta < n-1$.

Another natural series of characters is obtained by splitting the set A^n_k of words on k letters in subsets invariant under $\mathfrak{S}_n$. Taking the set of words with *exactly* k letters, the character of the corresponding representation is equal to the number of such words fixed by g, which by the inclusion–exclusion principle is equal to

$$\psi^n_k := \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \sigma^n_{k-j} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \sigma^n_j. \tag{10}$$

From Proposition 2.4 and an inversion formula, one finds

$$\psi^n_k = \sum_{j=1}^{k} \binom{n-j}{k-j} \tau^n_j.$$

Both ψ- and τ-characters can be obtained by the iterated differencing of the sequence $\sigma^n_\cdot$ (with $\sigma^n_k = 0$ for $k \leq 0$). Specifically, introducing the forward and backward difference operators acting as $\nabla(x_\cdot)_k = x_k - x_{k-1}$ and $\Delta(x_\cdot)_k = x_{k+1} - x_k$, respectively, we have $\tau^n_k = \nabla^n(\sigma^n_\cdot)_k$ and $\psi^n_k = \Delta^k(\sigma^n_\cdot)_0$.

3 The simplex of normalized block characters of $\mathfrak{S}_n$

The set of all block characters is a convex cone of non-negative linear combinations of the extreme normalized block characters. In this section, we prove that the normalized block characters form a simplex whose extreme points are the normalized versions of characters τ_k^n. A closely related result can be found in Sect. 2 of [19].

Because the functions $\{\sigma_k^n\}_{k=1,\dots,n}$ form a basis of the linear space of block functions, and because the systems of functions $\{\tau_k^n\}$ and $\{\sigma_k^n\}$ are related by a triangular linear transform, it follows that every block function φ on $\mathfrak{S}_n$ can be uniquely written in the basis of characters $\tau_1^n, \dots, \tau_n^n$ as a linear combination

$$\varphi = \sum_{k=1}^{n} a_k \tau_k^n.$$

On the other hand, the traces of irreducible representations χ^λ, $\lambda \in \mathbb{Y}_n$, comprise an orthonormal basis in the space of central functions on $\mathfrak{S}_n$ endowed with the scalar product

$$\langle \varphi, \psi \rangle = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \varphi(g)\psi(g).$$

Now, immediately from Proposition 2.5, we have

$$a_k = \frac{\langle \varphi, \chi^{\rho_k^n} \rangle}{\binom{n-1}{k-1}},$$

for ρ_k^n the hook diagram with k rows: $\rho_k^n = (n-k+1, 1^{k-1})$. Indeed, there are $\binom{n-1}{k-1}$ standard Young tableaux of shape ρ_k^n, and every such tableau has exactly $k-1$ descents.

Corollary 3.1 *A block function φ is a character if and only if $a_k \geq 0$ for $k = 1, \dots, n$.*

Proof If $a_k \geq 0$ for $k = 1, \dots, n$, then

$$\varphi = \sum_{\lambda \in \mathbb{Y}_n} b(\lambda)\chi^\lambda, \tag{11}$$

with non-negative coefficients $b(\lambda)$. Since the traces of irreducible representation χ^λ are positive-definite, so is φ.

If φ is positive-definite, then in the decomposition (11) all coefficients $b(\lambda)$ are non-negative, in particular, $b(\rho_k^n) \geq 0$ for $k = 1, \dots, n$. It follows that $a_k \geq 0$ for $k = 1, \dots, n$. □

Corollary 3.2 *The set of normalized block characters is a simplex whose extreme points are the normalized characters $\tau_k^n(\cdot)/\tau_k^n(e)$, $k = 1, \dots, n$.*

Corollary 3.3 *Suppose θ is not an integer. Then $\theta^{\ell_n(g)}$ is a character if and only if $\theta > n-1$.*

Proof Indeed, all the coefficients in (9) are non-negative if and only if $\theta > n-1$. □

4 Characters τ_k^n and the coinvariant algebra

In this section, we construct representations π_k^n with traces τ_k^n. We start with computation of the dimension $\tau_k^n(e)$. We give first a purely combinatorial proof, an alternative representation-theoretic proof will be given at the end of this section, and yet another proof based on the branching rule will be retained for the next section.

Let $g = (g(1), \dots, g(n)) \in \mathfrak{S}_n$ be a permutation written in the one-row notation. The *descent number* $d(g)$ counts descents, that is, positions j such that $g(j+1) < g(j)$. In particular, the descent number of the identity permutation $e = (1, 2, \dots, n)$ is 0, while the descent number of the reverse permutation $(n, \dots, 2, 1)$ is $n-1$. The Eulerian number $\left\langle {n \atop k} \right\rangle$ counts permutations from $\mathfrak{S}_n$ whose descent number is $k-1$.

Proposition 4.1 *The dimension of the representation of $\mathfrak{S}_n$ with character τ_k^n is the Eulerian number*

$$\tau_k^n(e) = \left\langle {n \atop k} \right\rangle.$$

Proof We refer to [11] for the Robinson–Schensted–Knuth (RSK) correspondence between permutations and pairs of standard Young tableaux. The RSK has the following property: $g(j+1) < g(j)$ in permutation g if and only if entry j is a descent in the recording tableau (see [11]).

It follows that the RSK sends permutations with k descents to pairs of Young tableaux such that the recording tableau has k descents. Thus

$$\left\langle {n \atop k} \right\rangle = \sum_{\lambda \in \mathbb{Y}_n} m_k(\lambda) \dim(\lambda), \tag{12}$$

where $\dim(\lambda) = \chi^\lambda(e)$ is the dimension of the corresponding irreducible representation, equal to the number of standard Young tableaux of shape λ. From the latter classical fact and comparing (12) with Proposition 2.5, the proof of the proposition is concluded. □

Now we proceed to the construction of representations π_k^n.

Proposition 4.2 *The regular representation R_{reg}^n of $\mathfrak{S}_n$ can be decomposed as*

$$R_{\mathrm{reg}}^n = \pi_1^n \oplus \pi_2^n \oplus \cdots \oplus \pi_n^n, \tag{13}$$

where representation π_k^n has trace τ_k^n.

Proof Indeed, we have

$$\sum_{k=1}^{n} \tau_k^n = \sum_{k=1}^{n} \sum_{\lambda \in \mathbb{Y}_n} m_k(\lambda) \chi^\lambda = \sum_{\lambda \in \mathbb{Y}_n} \dim(\lambda) \chi^\lambda = \chi^{\mathrm{reg}},$$

where $\dim(\lambda) = \sum_k m_k(\lambda)$ is the dimension of π^λ, that is, the number of standard Young tableaux of shape λ. Here $\chi^{\mathrm{reg}}(g)$ is the character of the regular representation of $\mathfrak{S}_n$. □

In the remaining part of this section, we focus on an explicit construction of the decomposition (13).

Let us denote $\mathcal{R}_n$ the algebra of polynomials in variables $x_1, \dots, x_n$. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{R}_n$ by permuting the variables. Let $\mathcal{R}_n^{\mathfrak{S}_n}$ denote the subalgebra of invariants of this action, which is the algebra of symmetric polynomials. *The coinvariant algebra* $\mathcal{R}_n^*$ is the quotient-algebra

$$\mathcal{R}_n^* = \mathcal{R}_n / I_n,$$

where I_n is the ideal in $\mathcal{R}_n$ spanned by the symmetric polynomials without constant term. The elementary symmetric polynomials

$$e_k := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

is the set of algebraic generators of $\mathcal{R}_n^{\mathfrak{S}_n}$, thus I_n is an ideal spanned by the polynomials $e_1, \dots, e_n$. We denote by P_n the canonical projection:

$$P_n : \mathcal{R}_n \to \mathcal{R}_n^*.$$

Observe that $\mathcal{R}_n^*$ inherits from $\mathcal{R}_n$ the structure of a $\mathfrak{S}_n$-module, and that the projection P_n is an intertwining operator. It is known that the representation of $\mathfrak{S}_n$ in $\mathcal{R}_n^*$ is equivalent to the left regular representation; see, e.g., [5] or [17, Sect. 3.6].

Given a multidegree $p = (p_1, \dots, p_n)$, let $\lambda(p)$ be a partition obtained by rearranging the coordinates of p in a non-increasing order. In what follows, we write $\lambda < \mu$ for two partitions λ and μ if $(\lambda_1, \lambda_2, \dots)$ precedes $(\mu_1, \mu_2, \dots)$ in the lexicographic order. For a polynomial

$$f = \sum b_p x^p \in \mathcal{R}_n,$$

its *partition degree* $\mathrm{pdeg}(f)$ is a minimal partition μ such that $b_p = 0$ each time $\lambda(p) > \mu$.

We need the following filtration $\mathcal{F}_n(k)$, $k = 0, 1, \dots$ of the algebra $\mathcal{R}_n$:

$$\mathcal{F}_n(k) = \{ f \in \mathcal{R}_n : \mathrm{pdeg}(f)_1 \le k \},$$

where index 1 refers to the largest part of the partition. Note that $\mathcal{F}_n(k)$ is a $\mathfrak{S}_n$ submodule of $\mathcal{R}_n$. Also, denote

$$\mathcal{F}_n^*(k) = P_n(\mathcal{F}_n(k))$$

and observe that $\mathcal{F}_n^*(k)$ is a $\mathfrak{S}_n$ submodule of $\mathcal{R}_n^*$.

Next, we want to introduce the so-called Garsia–Stanton descent basis in $\mathcal{R}_n^*$, which agrees with filtration $\mathcal{F}_n^*(k)$.

For a permutation $g \in \mathfrak{S}_n$, let $D(g)$ be the set of its descents and let $d_i(g) = |D(g) \cap \{i, i+1, \dots, n\}|$. For a standard Young tableau T of shape λ with $|\lambda| = n$, let $D(T)$ be the set of its descents and let $d_i(T) = |D(T) \cap \{i, i+1, \dots, n\}|$.

The descent monomial u_g of permutation g is defined as

$$u_g = x_{g(1)}^{d_1(g)} x_{g(2)}^{d_2(g)} \cdots x_{g(n)}^{d_n(g)}.$$

In our notation, $\operatorname{pdeg}(u_g) = (d_1(g), \dots, d_n(g))$.

Theorem 4.3 [1, 2, 12] *Classes $u_g + I_n$, $g \in \mathfrak{S}_n$, form a linear basis of $\mathcal{R}_n^*$. If a polynomial f belongs to $u_g + I_n$, then* $\operatorname{pdeg}(f) \geq \operatorname{pdeg}(u_g)$.

Theorem 4.3 implies the following description of $\mathcal{F}_n^*(k)$. For a class $f^* \in \mathcal{R}_n^*$, we denote by $\operatorname{pdeg}(f^*)$ the minimum value of pdeg on all polynomials of the class f^*. Then

$$\mathcal{F}_n^*(k) = \left\{ f^* \in \mathcal{R}_n^* : \operatorname{pdeg}\left(f^*\right)_1 \leq k \right\},$$

and also

$$\mathcal{F}_n^*(k) = \operatorname{span}\left\{ u_g : d_1(g) \leq k \right\}.$$

The following theorem explains the relevance of the filtration $\mathcal{F}_n^*(k)$ to the study of block characters.

Theorem 4.4 *Let π_k^n be the representation of $\mathfrak{S}_n$ with character τ_k^n. We have the following isomorphism of $\mathfrak{S}_n$-modules*:

$$\mathcal{F}_n^*(k) \cong \bigoplus_{i=1}^{k} \pi_i^n$$

and

$$\pi_k^n \cong \mathcal{F}_n^*(k) / \mathcal{F}_n^*(k-1).$$

Remark This theorem gives another way to prove that the dimension of π_k^n is $\left\langle {n \atop k} \right\rangle$.

Proof of Theorem 4.4 Let $\bar{q} = (q_1, q_2, q_3, \dots)$ be a sequence of formal variables. For a permutation $s \in \mathfrak{S}_n$, let $\operatorname{Tr}_{\mathcal{R}_n^*}^{\bar{q}}(s)$ denote its graded trace in the representation in $\mathcal{R}_n^*$:

$$\operatorname{Tr}_{\mathcal{R}_n^*}^{\bar{q}}(s) = \sum_{g \in \mathfrak{S}_n} \left\langle s(u_g), u_g \right\rangle q_1^{d_1(g)} \cdots q_n^{d_n(g)},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{R}_n^*$, such that the elements u_g form an orthonormal basis. The following formula for the above graded trace was obtained in

Sect. 4.2 of [1]:

$$\mathrm{Tr}^{\bar{q}}_{\mathcal{R}^*_n}(s) = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda(s) \sum_{T\in \mathrm{SYT}(\lambda)} \prod_{i=1}^n q_i^{d_i(T)},$$

where SYT(λ) is the set of all standard Young tableau of shape λ. Now setting $q_1 = q$, $q_2 = \cdots = q_n = 1$, we obtain the identity

$$\sum_{g\in\mathfrak{S}_n} \langle s(u_g), u_g\rangle q^{d_1(g)} = \sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda(s) \sum_{T\in \mathrm{SYT}(\lambda)} q^{d_1(T)}. \tag{14}$$

Note that

$$\sum_{\lambda\in\mathbb{Y}_n} \chi^\lambda(s) \sum_{T\in \mathrm{SYT}(\lambda)} q^{d_1(T)} = \sum_{k=1}^n q^{k-1} \sum_\lambda \chi^\lambda(s) m_k(\lambda) \tag{15}$$

and

$$\sum_{g\in\mathfrak{S}_n} \langle s(u_g), u_g\rangle q^{d_1(g)} = \sum_{k=1}^n q^{k-1} \widetilde{\tau}_k^n(s), \tag{16}$$

where $\widetilde{\tau}_k^n$ is the matrix trace of the representation of $\mathfrak{S}_n$ in $\mathcal{F}^*_n(k)/\mathcal{F}^*_n(k-1)$. Combining (14), (15), and (16), we get

$$\sum_{k=1}^n q^{k-1} \sum_\lambda \chi^\lambda(s) m_k(\lambda) = \sum_{k=1}^n q^{k-1} \widetilde{\tau}_k^n(s).$$

Therefore,

$$\widetilde{\tau}_k^n(s) = \sum_\lambda \chi^\lambda(s) m_k(\lambda). \tag{17}$$

Comparing (17) with Proposition 2.5, we conclude that $\tau_k^n = \widetilde{\tau}_k^n$. □

5 Limit shapes

In this section, we prove that the decomposition of the characters τ_n^k into the linear combination of the characters of irreducible representations of $\mathfrak{S}_n$ leads to certain limit shape theorems.

Following [38] and [4], given a Young diagram λ, we construct a piecewise-linear function $f_\lambda(x)$, $x \in \mathbb{R}$ with slopes ± 1 as shown in Fig. 3.

Now let χ be a character of $\mathfrak{S}_n$. Recall that irreducible representations of $\mathfrak{S}_n$ are parameterized by the Young diagrams with n boxes and write

$$\chi(\cdot) = \sum_{\lambda\in\mathbb{Y}_n} c(\lambda)\chi^\lambda(\cdot).$$

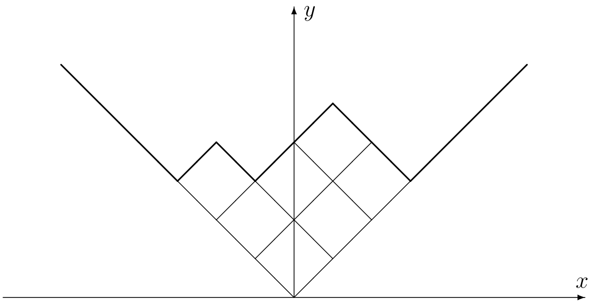

Fig. 3 Young diagram $\lambda = (3, 3, 1)$ and the graph $y = f_\lambda(x)$ of the corresponding piecewise-linear function

Define

$$P(\lambda) := \frac{c(\lambda)\chi^\lambda(e)}{\chi(e)}$$

and note that $P(\lambda)$ are non-negative numbers summing up to 1. Therefore, $P(\lambda)$ defines a probability distribution on the set $\mathbb{Y}_n$ or, equivalently, on piecewise-linear functions. Let us denote by $f^\chi(\cdot)$ the resulting *random* piecewise-linear function.

Biane [4] proved the following concentration theorem about the behavior of random piecewise-linear functions corresponding to characters σ_k^n.

Theorem 5.1 *Let $n, k \to \infty$ in such a way that $k/\sqrt{n} \to w > 0$, then for any $\varepsilon > 0$*

$$\operatorname{Prob}\Big\{\sup_x \big| f^{\sigma_k^n}(x\sqrt{n})/\sqrt{n} - g_w(x)\big| > \varepsilon\Big\} \to 0.$$

Here $g_w(x)$ is a deterministic function (depending on w).

The explicit formulas for the functions $g_w(x)$ are quite involved, they can be found in [4, Sect. 3]. As $w \to \infty$, the curves $g_w(x)$ approach the celebrated Vershik–Kerov–Logan–Shepp curve, which is a limit shape for the Plancherel random Young diagrams, see [24, 38].

It turns out that the limit behavior of the random functions corresponding to the extreme block characters τ_k^n is described by the very same curves $g_w(x)$.

Theorem 5.2 *If $n, k \to \infty$ in such a way that $k/\sqrt{n} \to w > 0$, then for any $\varepsilon > 0$*

$$\operatorname{Prob}\Big\{\sup_x \big| f^{\tau_k^n}(x\sqrt{n})/\sqrt{n} - g_w(x)\big| > \varepsilon\Big\} \to 0.$$

If $n, k \to \infty$ in such a way that $(n-k)/\sqrt{n} \to w > 0$, then for any $\varepsilon > 0$

$$\operatorname{Prob}\Big\{\sup_x \big| f^{\tau_k^n}(x\sqrt{n})/\sqrt{n} - g_w(-x)\big| > \varepsilon\Big\} \to 0.$$

The proof of Theorem 5.2 is based on the following observation.

Proposition 5.3 *The following estimate holds*:

$$\sum_{\lambda\in\mathbb{Y}_n}\left|\mathrm{Prob}\{f^{\tau_k^n}=f_\lambda\}-\mathrm{Prob}\{f^{\sigma_k^n}=f_\lambda\}\right|<c(k,n),$$

where $c(k,n)\to 0$ *as* $n,k\to\infty$ *in such a way that* $k/\sqrt{n}\to w>0$.

Proof Propositions 2.5 and 4.1 imply that

$$\mathrm{Prob}\{f^{\tau_k^n}=f_\lambda\}=\frac{m_k(\lambda)\dim(\lambda)}{\left\langle {n \atop k}\right\rangle},$$

while Proposition 2.4 yields

$$\mathrm{Prob}\{f^{\sigma_k^n}=f_\lambda\}=\frac{s_k(\lambda)\dim(\lambda)}{k^n}.$$

Then Eq. (5) implies that

$$\mathrm{Prob}\{f^{\tau_k^n}=f_\lambda\}=\sum_{j=0}^{k-1}c_{j,k,n}\,\mathrm{Prob}\{f^{\sigma_{k-j}^n}=f_\lambda\},$$

$$c_{j,k,n}=\frac{(-1)^j\binom{n+1}{j}(k-j)^n}{\left\langle {n \atop k}\right\rangle}. \tag{18}$$

Let us analyze the coefficients $c_{j,k,n}$. We have

$$|c_{j,k,n}|=c_{0,k,n}\binom{n+1}{j}(1-j/k)^n<c_{0,k,n}(n+1)^j(1-1/k)^{jn}.$$

Now if $n,k\to\infty$ in such a way that $k/\sqrt{n}\to w>0$ and $j<k$, then for large enough n,

$$(n+1)^j(1-1/k)^{jn}=\big((n+1)(1-1/k)^n\big)^j<\exp\left(-\frac{1}{2w}\sqrt{n}\,j\right).$$

On the other hand, summing (18) over all $\lambda\in\mathbb{Y}_n$, we conclude that

$$\sum_{j=0}^{k-1}c_{j,k,n}=1.$$

Therefore, for large enough n, we have

$$|c_{0,k,n}-1|<\exp\left(-\frac{1}{3w}\sqrt{n}\right),$$

and for $0<j<k$ we have

$$|c_{j,k,n}|<\exp\left(-\frac{1}{3w}\sqrt{n}\right).$$

Hence,

$$\sum_{\lambda\in\mathbb{Y}_n}\left|\mathrm{Prob}\left\{f^{\tau_k^n}=f_\lambda\right\}-\mathrm{Prob}\left\{f^{\sigma_k^n}=f_\lambda\right\}\right|$$
$$<\exp\left(-\frac{1}{3w}\sqrt{n}\right)\sum_{j=0}^{k-1}\sum_{\lambda\in\mathbb{Y}_n}\mathrm{Prob}\left\{f^{\sigma_j^n}=f_\lambda\right\}=k\exp\left(-\frac{1}{3w}\sqrt{n}\right),$$

which vanishes as $n\to\infty$. □

Now Theorem 5.2 becomes a simple corollary of Theorem 5.1.

Proof of Theorem 5.2 First, suppose that $n,k\to\infty$ in such a way that $k/\sqrt{n}\to w>0$, then for any $\varepsilon>0$

$$\mathrm{Prob}\left\{\sup_x\left|f^{\tau_k^n}(x\sqrt{n})/\sqrt{n}-g_w(x)\right|>\varepsilon\right\}$$
$$\le\mathrm{Prob}\left\{\sup_x\left|f^{\sigma_k^n}(x\sqrt{n})/\sqrt{n}-g_w(x)\right|>\varepsilon\right\}$$
$$+\sum_{\lambda\in\mathbb{Y}_n}\left|\mathrm{Prob}\left\{f^{\tau_k^n}=f_\lambda\right\}-\mathrm{Prob}\left\{f^{\sigma_k^n}=f_\lambda\right\}\right|\to 0.$$

Next, suppose that $n,k\to\infty$ in such a way that $(n-k)/\sqrt{n}\to w>0$. Proposition 2.7 yields that the distributions of $f^{\tau_k^n}(x)$ and $f^{\tau_{n-k}^n}(-x)$ coincide. Hence we get the second claim of Theorem 5.2. □

6 Branching rules

Let us embed $\mathfrak{S}_{n-1}$ into $\mathfrak{S}_n$ as the subgroup of permutations fixing n. For a central function χ on $\mathfrak{S}_n$ the restriction $\mathrm{Res}_{n-1}(\chi)$ on $\mathfrak{S}_{n-1}$ is a central function on $\mathfrak{S}_{n-1}$. If χ is a block function, then so is the restriction $\mathrm{Res}_{n-1}\chi$. The following two propositions describe what happens with the characters σ_k^n,τ_k^n by restricting them to the subgroup; we call these formulas *the branching rules*.

Proposition 6.1 *For* $1\le k\le n$ *we have*

$$\mathrm{Res}_{n-1}\sigma_k^n=k\sigma_k^{n-1},$$
$$\mathrm{Res}_{n-1}\widehat{\sigma}_k^n=k\widehat{\sigma}_k^{n-1}.$$

In terms of the normalized characters, the relations can be rewritten as:

$$\mathrm{Res}_{n-1}\left(\frac{\sigma_k^n}{\sigma_k^n(e)}\right)=\frac{\sigma_k^{n-1}}{\sigma_k^{n-1}(e)},$$
$$\mathrm{Res}_{n-1}\left(\frac{\widehat{\sigma}_k^n}{\widehat{\sigma}_k^n(e)}\right)=\frac{\widehat{\sigma}_k^{n-1}}{\widehat{\sigma}_k^{n-1}(e)}.$$

Proof This immediately follows from $\ell_{n+1}(g) = \ell_n(g) + 1$. □

Proposition 6.2 *For* $1 < k < n$ *we have*

$$\operatorname{Res}_{n-1} \tau_k^n = k\tau_k^{n-1} + (n-k+1)\tau_{k-1}^{n-1};$$
$$\operatorname{Res}_{n-1} \tau_1^n = \tau_1^{n-1}, \qquad \operatorname{Res}_{n-1} \tau_n^n = \tau_{n-1}^{n-1}.$$

Proof Observe the binomial coefficients identity

$$\binom{n+1}{k-j} j = k\binom{n}{k-j} - (n-k+1)\binom{n}{k-j-1}.$$

Indeed, this is equivalent to

$$\begin{aligned} &\frac{n!}{(k-j)!(n+1-k+j)!}(n+1)j \\ &\quad= \frac{n!}{(k-j)!(n+1-k+j)!}\big(k(n+1-k+j) - (n-k+1)(k-j)\big), \end{aligned}$$

which is easy to check.

Then using the definition of τ_k^n and the branching rule for σ_k^n, we obtain

$$\begin{aligned} \operatorname{Res}_{n-1} \tau_k^n &= \sum_{j=1}^{k}(-1)^{k-j}\binom{n+1}{k-j}\operatorname{Res}_{n-1}\big(\sigma_j^n\big) = \sum_{j=1}^{k}(-1)^{k-j}\binom{n+1}{k-j} j\sigma_j^{n-1} \\ &= \sum_{j=1}^{k}(-1)^{k-j}\left(k\binom{n}{k-j} - (n-k+1)\binom{n}{k-j-1}\right)\sigma_j^{n-1} \\ &= k\tau_k^{n-1} + (n-k+1)\tau_{k-1}^{n-1}, \end{aligned}$$

as wanted. □

Remark Proposition 6.2 suggests another way to prove Proposition 4.1. Indeed, evaluating at e, we obtain the relation

$$\tau_k^n(e) = k\tau_k^{n-1}(e) + (n-k+1)\tau_{k-1}^{n-1}(e),$$

which coincides with the well-known recursion for Eulerian numbers.

7 The Choquet simplex of block characters of $\mathfrak{S}_\infty$

We proceed with a classification of block characters of the infinite symmetric group $\mathfrak{S}_\infty$. Let us recall the definition of the group. Let $\mathbb{Z}_+$ denote the set of positive integers. Realize the group $\mathfrak{S}_n$ as the group of bijections $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ which may only move the first n integers, that is, satisfy $g(j) = j$ for $j > n$. This yields a natural embedding $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$ and allows one to introduce the infinite symmetric group as an inductive limit

$$\mathfrak{S}_\infty := \bigcup_{n \geq 1} \mathfrak{S}_n,$$

whose elements are the bijections $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ satisfying $g(j) = j$ for all sufficiently large j.

The notion of a block character for the group $\mathfrak{S}_\infty$ needs to be adapted. We call the statistic $c(g) := n - \ell_n(g)$ the *decrement* of permutation $g \in \mathfrak{S}_n$. Note that the concept is consistent with embeddings, that is, considering g as an element of $\mathfrak{S}_{n+1}$, the decrement remains the same: $n - \ell_n(g) = n + 1 - \ell_{n+1}(g)$. Thus, $c(g)$ is a well defined function on $\mathfrak{S}_\infty$.

A block function on $\mathfrak{S}_\infty$ is defined as a function which depends on a permutation through its decrement. A positive definite normalized block function will be called normalized block character of $\mathfrak{S}_\infty$. Two trivial examples are the unit character and the delta function at e.

We introduce next the analogues of characters σ_k^n and $\widehat{\sigma}_k^n$. Denote

$$\mathbb{V} = \{0, \pm 1, \pm 1/2, \pm 1/3, \pm 1/4, \ldots\}$$

and set

$$\sigma_z^\infty(g) := z^{c(g)}, \quad \text{for } z \in \mathbb{V}, \tag{19}$$

where σ_0^∞ is understood as the delta-function at e

$$\sigma_0^\infty(g) = \begin{cases} 1 & g = e, \\ 0 & \text{otherwise.} \end{cases}$$

Keep in mind that in our parametrization $\sigma_{1/k}^\infty$ is a counterpart of σ_k^n and $\sigma_{-1/k}^\infty$ is an analogue of $\widehat{\sigma}_k^n$, for $k = 1, 2, \ldots$.

Proposition 7.1 *The functions σ_z^∞, $z \in \mathbb{V}$, are normalized block characters of $\mathfrak{S}_\infty$.*

Proof For σ_0^∞ the statement is trivial. As for $\sigma_{\pm 1/k}^\infty$ they are, clearly, central normalized functions on $\mathfrak{S}_\infty$. Note that

$$\sigma_{1/k}^\infty(g) = \frac{\sigma_k^n(g)}{\sigma_k^n(e)} \tag{20}$$

for $g \in \mathfrak{S}_n$, and similarly for $\sigma_{-1/k}^\infty$. Therefore, the positive-definiteness of σ_k^n and $\widehat{\sigma}_k^n$ implies the positive-definiteness of $\sigma_{\pm 1/k}^\infty$. □

The characters σ_z^∞ can be associated with certain infinite-dimensional representations of $\mathfrak{S}_\infty$. One way to establish the connection is to realize σ_z^∞ as the trace of a finite von Neumann factor representation, which may be seen as a substitute of irreducible representation, see [34, 36]. Another approach is to consider spherical representations of the Gelfand pair $(\bar{G}, \bar{K})$, where $\bar{G}$ and $\bar{K}$ are certain extensions of the groups $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$ and $\mathfrak{S}_\infty$, respectively, see [26, 27] and also [22] for details and references. By both approaches, the character σ_0^∞ corresponds to the regular representation of $\mathfrak{S}_\infty$.

It follows readily from a multiplicative property of the extremes [36] that the characters σ_z^∞, $z \in \mathbb{V}$, are in fact extreme points in the set of *all* normalized characters of $\mathfrak{S}_\infty$. In Thoma's parametrization (see [36] and [21, 25, 37]) of the extremes by two nonincreasing sequences (α, β), $\sigma_{1/k}^\infty$ corresponds to $\alpha_1 = \cdots = \alpha_k = 1/k, \beta = 0$, while $\widehat{\sigma}_{1/k}^\infty$ corresponds to the pair $\alpha = 0, \beta_1 = \cdots = \beta_k = 1/k$.

Note that for a finite n the situation is different: as we saw, the representations with traces σ_k^n and $\widehat{\sigma}_k^n$ are reducible, so their normalized traces are not extreme characters. A priori, there is no reason for the list $\{\sigma_z^\infty, z \in \mathbb{V}\}$ to exhaust *all* extreme normalized block characters of $\mathfrak{S}_\infty$. The following theorem asserts that this is indeed the case.

Theorem 7.2 *Let Ω be the infinite-dimensional simplex of non-negative sequences $(\gamma_z, z \in \mathbb{V})$ with $\sum_{1/z \in \mathbb{Z} \cup \{\infty\}} \gamma_z = 1$. Endowed with the topology of point-wise convergence, the set of normalized block characters of $\mathfrak{S}_\infty$ is a Choquet simplex. The correspondence*

$$(\gamma_z) \to \sum_{z \in \mathbb{V}} \gamma_z \sigma_z^\infty$$

is an affine homeomorphism between Ω and the Choquet simplex of normalized block characters of $\mathfrak{S}_\infty$. In particular, the set of extreme normalized block characters of $\mathfrak{S}_\infty$ is $\{\sigma_z^\infty, z \in \mathbb{V}\}$.

Proof Let χ be a block character of $\mathfrak{S}_\infty$ and let χ^n denote the restriction of χ on the subgroup $\mathfrak{S}_n$. Then χ^n is a normalized block character of $\mathfrak{S}_n$, hence, by Proposition 3.2, it can be uniquely decomposed as a convex combination as follows:

$$\chi^n(\cdot) = \sum_{k=1}^n a_k^n \frac{\tau_k^n(\cdot)}{\tau_k^n(e)},$$

where the array of coefficients satisfies

$$a_k^n \geq 0, \qquad \sum_{k=1}^n a_k^n = 1. \tag{21}$$

By Proposition 6.2, the coefficients a_k^n satisfy the following backward recursion: for $1 \leq k \leq n,\ n = 1, 2, \ldots$:

$$a_k^n = k \frac{\left\langle {n \atop k} \right\rangle}{\left\langle {n+1 \atop k} \right\rangle} a_k^{n+1} + (n-k+1) \frac{\left\langle {n \atop k} \right\rangle}{\left\langle {n+1 \atop k+1} \right\rangle} a_{k+1}^{n+1}. \tag{22}$$

Moreover, the correspondence $\chi \leftrightarrow (a_k^n)$ is an affine homeomorphism between normalized block characters and arrays (a_k^n) satisfying (21) and (22) (Condition (22) implies that in (21) it is enough to require $a_1^1 = 1$).

The fact that the set of arrays (a_k^n) satisfying (21) and (22) is a Choquet simplex is just a particular case of a very general result; see, e.g., [15, Proposition 11.6]. As for the extreme points of this simplex, it was shown in [14] that all *extreme* solutions to the recursion(22) subject to the constraints (21) are of one of the types

$$a_k^n = \frac{\binom{n+K-k}{n}}{K^n}\left\langle {n \atop k} \right\rangle, \quad K = 1, 2, \dots, \tag{23}$$

$$a_k^n = \frac{\binom{K+k-1}{n}}{K^n}\left\langle {n \atop k} \right\rangle, \quad K = 1, 2, \dots, \tag{24}$$

$$a_k^n = \frac{1}{n!}\left\langle {n \atop k} \right\rangle. \tag{25}$$

It is possible to write all three types as a single factorial formula (see [14], Eq. (2)).

Calculating with (23), we arrive, in view of Proposition 2.3, at

$$\sum_{k=1}^{n} a_k^n \frac{\tau_k^n(\cdot)}{\tau_k^n(e)} = K^{-n} \sum_{k=1}^{n} \binom{n+K-k}{K-k} \tau_k^n(\cdot) = \sum_{j=0}^{K-1} \binom{n+j}{j} \tau_{K-j}^n(\cdot) = \frac{\sigma_K^n(\cdot)}{\sigma_K^n(e)},$$

which means that the array (23) corresponds to σ_z^∞ with $1/z = K \in \{1, 2, \dots\}$. Similarly, with (24) and (8) the convex combination is $\widehat{\sigma}_K^n$, so we arrive at σ_z^∞ with $1/z = -K \in \{-1, -2, \dots\}$. Finally, with the array (25) the convex combination obtained is the normalized character of the regular representation of $\mathfrak{S}_n$, as it follows from (13), and this corresponds to σ_0^∞. Thus all extreme block characters of $\mathfrak{S}_\infty$ have been identified with (19). □

8 Connection to the characters of the linear groups over the Galois fields

The group $GL_\infty(q)$ is the group of infinite matrices of the kind

$$g = \begin{pmatrix} h & 0 \\ 0 & 1_\infty \end{pmatrix}$$

where h is a finite square matrix with coefficients from the Galois field $\mathbb{F}_q$ with q elements, 1_∞ denotes the infinite unit matrix, and 0's are zero matrices of suitable dimensions. From this definition it is clear that the group has the structure of inductive limit $GL_\infty(q) = \bigcup_{n\ge 1} GL_n(q)$. Thoma [35] conjectured that all extreme normalized characters of $GL_\infty(q)$ are of the form

$$\chi(g) = \epsilon(\det g) q^{-mc(g)}, \quad m \in \mathbb{Z}_{\ge 0} \cup \{\infty\}, \tag{26}$$

where ϵ is a one-dimensional character of the cyclic group $\mathbb{F}_q^*$, and $c(g)$ is the rank of the matrix $g - Id$. The conjecture was proved by Skudlarek [29]. Thoma's characters $g \mapsto q^{-mc(g)}$ belong to the class of *derangement* characters introduced in [13] as the characters of $GL_n(q)$ (respectively, $GL_\infty(q)$) which only depend on $c(g)$.

As mentioned in the Introduction, a connection between block characters and derangement characters is established via the natural embedding $\mathfrak{S}_\infty \hookrightarrow GL_\infty(q)$ obtained by writing a permutation as a permutation matrix. The decrement of permutation is equal to the rank of the matrix $g - Id$. Furthermore, the determinant of the permutation matrix is ± 1 depending on the parity of the permutation.

Each extreme character (26) restricts from $GL_\infty(q)$ to $\mathfrak{S}_\infty$ as one of the characters σ_z^∞ with $z = \pm q^{-m}$, $m \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$, unless q is a power of 2. It q is a power of 2, then $-1 = 1$ in $\mathbb{F}_q$, hence the negative values are excluded. For other values of z, the function $g \mapsto z^{c(g)}$ is not positive definite on $GL_\infty(q)$.

For $\mathfrak{S}_n$ we constructed n extreme characters τ_k^n from the basic characters in just one step, by differencing the series $\sigma_1^n, \dots, \sigma_n^n$. For $GL_n(q)$ the situation is more involved: one first needs to construct q-analogues of characters (10), then proceed with further differencing and search for the extremes. Moreover, the set of normalized derangement characters of $GL_n(q)$ is known to be a simplex for $n \in \{1, 2, 3, 4, 5, 6, 8, 9, 11, 12\}$ and it is *not* a simplex (has more than n extremes) for $n \in \{7, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22\}$ (see [13]). Another, more substantial, distinction occurs on the level of infinite groups. For $GL_\infty(q)$ the extreme normalized derangement characters exhaust all extreme normalized characters of the group (up to tensoring with a linear character, and literally all for the special linear group $SL_\infty(q)$). For $\mathfrak{S}_\infty$ the extreme normalized block characters comprise only a countable subset of the infinite-dimensional Thoma simplex of all extreme normalized characters.

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