

The topology of restricted partition posets

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Abstract For each composition \vec{c} we show that the order complex of the poset of pointed set partitions $\Pi_{\vec{c}}^{\bullet}$ is a wedge of spheres of the same dimension with the multiplicity given by the number of permutations with descent composition \vec{c} . Furthermore, the action of the symmetric group on the top homology is isomorphic to the Specht module S^B where B is a border strip associated to the composition. We also study the filter of pointed set partitions generated by a knapsack integer partition and show the analogous results on homotopy type and action on the top homology.

Keywords Border strip Specht module · Complex of ordered set partitions · Descent set statistics · Knapsack integer partitions · Poset of pointed set partitions

1 Introduction

The study of partitions with restrictions on their block sizes began in the dissertation by Sylvester [19], who studied the poset Π_n^2 of partitions of $\{1, 2, \dots, n\}$ where every block has even size. He proved that the Möbius function of this poset is given by $\mu(\Pi_n^2 \cup \{\hat{0}\}) = (-1)^{n/2} \cdot E_{n-1}$, where E_n denotes the n th Euler number. Recall that the n th Euler number enumerates alternating permutations, that is, permutations $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group \mathfrak{S}_n such that $\alpha_1 < \alpha_2 > \alpha_3 < \alpha_4 > \cdots$. Stanley [16] generalized this result to the d -divisible partition lattice Π_n^d , that is, the collection of partitions of $\{1, 2, \dots, n\}$ where each block size is divisible by d . He found that the Möbius function $\mu(\Pi_n^d \cup \{\hat{0}\})$ is, up to the sign $(-1)^{n/d}$, the number of permutations in \mathfrak{S}_{n-1} with descent set $\{d, 2d, \dots, n-d\}$; in other words, the number of permutations with descent composition $(d, \dots, d, d-1)$.

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The enumerative results for the d -divisible partition lattice were extended homologically. Calderbank, Hanlon and Robinson [4] considered the action of the symmetric group \mathfrak{S}_{n-1} on the top homology group of the order complex of $\Pi_n^d - \{\hat{1}\}$. They showed this action is the Specht module on the border strip corresponding to the composition $(d, \dots, d, d-1)$. Wachs [20] showed that the other reduced homology groups vanish. She presented an EL -labeling for the d -divisible partition lattice and hence as a corollary found that the homotopy type is a wedge of spheres of dimension $n/d - 2$. She then gave a more constructive proof of the representation of the top homology of $\Delta(\Pi_n^d - \{\hat{1}\})$ by exhibiting an explicit isomorphism. She identified cycles in the complex $\Delta(\Pi_n^d - \{\hat{1}\})$ which are the barycentric subdivisions of cubes and associated with them polytabloids in the Specht module.

So far we see that the d -divisible partition lattice is closely connected with permutations having the descent composition $(d, \dots, d, d-1)$. We explain this phenomenon in this paper by introducing pointed partitions. They are partitions where one block is considered special, called the pointed block. We obtain such a partition by removing the element n from its block and making this block the pointed block. We now extend the family of posets under consideration. For each composition $\vec{c} = (c_1, \dots, c_k)$ of n we define a poset $\Pi_{\vec{c}}^{\bullet}$ such that the Möbius function $\mu(\Pi_{\vec{c}}^{\bullet} \cup \{\hat{0}\})$ is the sign $(-1)^k$ times the number of permutations with descent composition \vec{c} . Furthermore, we show the order complex of $\Pi_{\vec{c}}^{\bullet} - \{\hat{1}\}$ is homotopy equivalent to a wedge of spheres of dimension $k - 2$. Finally, we show the action of the symmetric group on the top homology group $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^{\bullet} - \{\hat{1}\}))$ is the Specht module corresponding to the composition \vec{c} .

Our techniques differ from Wachs' method for studying the d -divisible partition lattice [20] as we do not obtain an EL -labeling of $\Pi_{\vec{c}}^{\bullet} \cup \{\hat{0}\}$. Instead we apply Quillen's Fiber Lemma and transform the question into studying a subcomplex $\Delta_{\vec{c}}$ of the complex of ordered partitions. This subcomplex is in fact the order complex of a rank-selected Boolean algebra. Hence $\Delta_{\vec{c}}$ is shellable and its homotopy type is a wedge of spheres. Furthermore, we use an equivariant version of Quillen's Fiber Lemma to conclude that the reduced homology groups $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^{\bullet} - \{\hat{1}\}))$ and $\tilde{H}_{k-2}(\Delta_{\vec{c}})$ are isomorphic as \mathfrak{S}_n -modules. Finally, to show that the top homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$ is isomorphic to the Specht module S^B , we follow Wachs' footsteps by giving an explicit isomorphism between these two \mathfrak{S}_n -modules, that is, given a polytabloid e_r in the Specht module S^B we give an explicit cycle g_{α} spanning the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$.

Ehrenborg and Readdy [5] introduced the notion of a knapsack partition, which is an integer partition such that no integer can be written as a sum of parts of the partition in two different ways. They considered the filter in the pointed partition lattice where the generators of the filter have their type given by a knapsack partition. They obtained the Möbius function of this filter is given by a sum of descent set statistics. We extend their results topologically by showing that the associated order complex is a wedge of spheres. The proof follows the same outline as the previous study except that we use discrete Morse theory to determine the homotopy type of the associated complexes of ordered set partitions. Furthermore we find that the action of the symmetric group on the top homology is a direct sum of Specht modules.

We end the paper with open questions for future research.

2 Preliminaries

For basic notions concerning partially ordered sets (posets), see Stanley’s book [17]. For topological background, see Björner’s article [2] and Kozlov’s book [12]. For representation theory of the symmetric group, see Sagan [15]. Finally, a good reference for all three areas is Wachs’ article [22].

Let $[n]$ denote the set $\{1, 2, \dots, n\}$ and for $i \leq j$ let $[i, j]$ denote the interval $\{i, i + 1, \dots, j\}$. A *pointed set partition* π of the set $[n]$ is a pair (σ, Z) , where Z is a subset of $[n]$ and $\sigma = \{B_1, B_2, \dots, B_k\}$ is a partition of the set difference $[n] - Z$. We will write the pointed partition π as

$$\pi = \{B_1, B_2, \dots, B_k, \underline{Z}\},$$

where we underline the set Z and we write $1358|4|267$ as shorthand for $\{\{1, 3, 5, 8\}, \{4\}, \{2, 6, 7\}\}$. Let Π_n^\bullet denote the set of all pointed set partitions on the set $[n]$. The set Π_n^\bullet has a natural poset structure. The cover relation is given by two relations

$$\begin{aligned} \{B_1, B_2, \dots, B_k, \underline{Z}\} &< \{B_1 \cup B_2, \dots, B_k, \underline{Z}\}, \\ \{B_1, B_2, \dots, B_k, \underline{Z}\} &< \{B_2, \dots, B_k, \underline{B_1 \cup Z}\}. \end{aligned}$$

Lemma 2.1 *The poset Π_n^\bullet is the intersection lattice of the hyperplane arrangement*

$$\begin{cases} x_i = x_j & 1 \leq i < j \leq n, \\ x_i = 0 & 1 \leq i \leq n. \end{cases}$$

Proof For the pointed partition $\pi = \{B_1, B_2, \dots, B_k, \underline{Z}\}$ construct a subspace satisfying the equalities $x_{j_1} = x_{j_2}$ if j_1 and j_2 belong to the same block of π , and let $x_j = 0$ if j belongs to the block Z . It is straightforward to see that this is a bijection and thus proving the lemma. □

As a corollary to this claim we see that the poset Π_n^\bullet is a lattice. Moreover, we call the set Z the *zero set* or the *pointed block*. The first name is motivated by the fact that the set Z corresponds to the variables set to be zero in an element in the intersection lattice.

The lattice Π_n^\bullet is isomorphic to the partition lattice Π_{n+1} by the bijection

$$\{B_1, \dots, B_k, \underline{Z}\} \mapsto \{B_1, \dots, B_k, Z \cup \{n + 1\}\}.$$

However, it is to our advantage to work with pointed set partitions.

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group \mathfrak{S}_n define its *descent set* to be the set

$$\{i \in [n - 1] : \alpha_i > \alpha_{i+1}\}.$$

Subsets of $[n - 1]$ are in a natural bijective correspondence with compositions of n . Hence we define the *descent composition* of the permutation α to be the composition

$$\text{Des}(\alpha) = (s_1, s_2 - s_1, s_3 - s_2, \dots, s_{k-1} - s_{k-2}, n - s_{k-1}),$$

where the descent set of α is the set $\{s_1 < s_2 < \dots < s_{k-1}\}$. We define a *pointed integer composition* $\vec{c} = (c_1, \dots, c_k)$ to be a list of positive integers c_1, \dots, c_{k-1} and a non-negative integer c_k with $c_1 + \dots + c_k = n$. Note that the only part allowed to be 0 is the last part. When the last part is positive we refer to \vec{c} as a *composition*. Let $\beta(\vec{c})$ denote the number of permutations α in \mathfrak{S}_n with descent composition \vec{c} for $c_k > 0$. If $c_k = 0$, let $\beta(\vec{c}) = 0$ for $k \geq 2$ and $\beta(\vec{c}) = 1$ for $k = 1$.

Define the (right) weak Bruhat order on the symmetric group \mathfrak{S}_n by the cover relation

$$\alpha < \alpha \circ (i, i + 1)$$

if $\alpha_i < \alpha_{i+1}$. Observe that the smallest element is the identity element $12\dots n$ and the largest is $n\dots 21$.

On the set of compositions of n we define an order relation by letting the cover relation be adding adjacent entries, that is,

$$(c_1, \dots, c_i, c_{i+1}, \dots, c_k) < (c_1, \dots, c_i + c_{i+1}, \dots, c_k).$$

Observe that this poset is isomorphic to the Boolean algebra B_n on n elements and the maximal and minimal elements are the two compositions (n) and $(1, \dots, 1, 0)$.

An *integer partition* λ of a non-negative integer n is a multiset of positive integers whose sum is n . We will indicate multiplicities with a superscript. Thus $\{5, 3, 3, 2, 1, 1, 1\} = \{5, 3^2, 2, 1^3\}$ is a partition of 16. A *pointed integer partition* (λ, m) of n is a pair where m is a non-negative integer and λ is a partition of $n - m$. We write this as $\{\lambda_1, \dots, \lambda_p, \underline{m}\}$ where $\lambda = \{\lambda_1, \dots, \lambda_p\}$ is the partition and m is the pointed part. This notion of pointed integer partition is related to pointed set partitions by defining the type of a pointed set partition $\pi = \{B_1, B_2, \dots, B_k, \underline{Z}\}$ to be the pointed integer partition

$$\text{type}(\pi) = \{|B_1|, |B_2|, \dots, |B_k|, \underline{|Z|}\}.$$

Similarly, the type of a composition $\vec{c} = (c_1, \dots, c_k)$ is the pointed integer partition

$$\text{type}(\vec{c}) = \{c_1, \dots, c_{k-1}, \underline{c_k}\}.$$

Recall that the *Möbius function* of a poset P is defined by the initial condition $\mu(x, x) = 1$ and the recursion $\mu(x, z) = -\sum_{x \leq y < z} \mu(x, y)$. When the poset P has a minimal element $\hat{0}$ and a maximal element $\hat{1}$, we call the quantity $\mu(P) = \mu(\hat{0}, \hat{1})$ the Möbius function of the poset P .

For a poset P define its *order complex* to be the simplicial complex $\Delta(P)$ where the vertices of the complex $\Delta(P)$ are the elements of the poset P and the faces are the chains in the poset. In other words, the order complex of P is given by

$$\Delta(P) = \{\{x_1, x_2, \dots, x_k\} : x_1 < x_2 < \dots < x_k, x_1, \dots, x_k \in P\}.$$

As a consequence of Hall’s Theorem [17, Proposition 3.8.5] we find that the reduced Euler characteristic of the order complex $\Delta(P)$, that is, $\tilde{\chi}(P)(\Delta(P))$, is given by the Möbius function $\mu(P \cup \{\hat{0}, \hat{1}\})$, where $P \cup \{\hat{0}, \hat{1}\}$ denotes the poset P with new minimal and maximal elements adjoined.

Shelling and discrete Morse theory are two powerful tools for determining the homotopy type of simplicial complexes. See [7, 8, 12] for more details. We begin by a review of shellings. A simplicial complex is *pure* if all its facets (maximal faces) have the same dimension. A pure simplicial complex is *shellable* if either it is a collection of disjoint vertices or there is an ordering on the facets F_1, \dots, F_m such that for $2 \leq i \leq m$ the intersection $F_i \cap (F_1 \cup \dots \cup F_{i-1})$ is a pure subcomplex of F_i of dimension $\dim(F_i) - 1$. A facet F_i is called *spanning* if the intersection $F_i \cap (F_1 \cup \dots \cup F_{i-1})$ is the boundary of F_i .

Theorem 2.2 *A shellable simplicial complex of dimension d is homotopy equivalent to a wedge of b d -dimensional spheres where b is the number of spanning facets in the shelling.*

Next we review discrete Morse theory.

Definition 2.3 A *partial matching* in a poset P is a partial matching in the underlying graph of the Hasse diagram of P , that is, a subset $M \subseteq P \times P$ such that $(x, y) \in M$ implies $x < y$ and each $x \in P$ belongs to at most one element of M . For a pair (x, y) in the matching M we write $x = d(y)$ and $y = u(x)$, where d and u stand for down and up, respectively.

Definition 2.4 A partial matching M on P is *acyclic* if there does not exist a cycle

$$z_1 < u(z_1) > z_2 < u(z_2) > \dots > z_n < u(z_n) > z_1,$$

of elements in P with $n \geq 2$ and all $z_i \in P$ distinct. Given an acyclic matching, the unmatched elements are called *critical*.

We need the following version of the main theorem of discrete Morse theory.

Theorem 2.5 *Let Γ be a simplicial complex with an acyclic matching on its face poset, where the empty face (set) is included. Assume that there are b critical cells and that they all have the same dimension k . Then the simplicial complex Γ is homotopy equivalent to a wedge of b spheres of dimension k .*

For the remainder of this section, we restrict ourselves to considering compositions of n where the last part is positive. This restriction will also hold in Sects. 6, 7, 9 and 10. Such a composition lies in the interval from $(1, \dots, 1)$ to (n) . This interval is isomorphic to the Boolean algebra B_{n-1} which is a complemented lattice. Hence for such a composition \vec{c} there exists a complementary composition \vec{c}^c such that $\vec{c} \wedge \vec{c}^c = (1, \dots, 1)$ and $\vec{c} \vee \vec{c}^c = (n)$. As an example, the complement of the composition $(1, 3, 1, 1, 4) = (1, 1 + 1 + 1, 1, 1, 1 + 1 + 1 + 1)$ is obtained by exchanging commas and plus signs, that is, $(1 + 1, 1, 1 + 1 + 1 + 1, 1, 1, 1) = (2, 1, 4, 1, 1, 1)$. Note that the complementary composition has $n - k + 1$ parts.

For a composition $\vec{c} = (c_1, \dots, c_k)$ define the intervals R_1, \dots, R_k by $R_i = [c_1 + \dots + c_{i-1} + 1, c_1 + \dots + c_i]$. Define the subgroup $\mathfrak{S}_{\vec{c}}$ of the symmetric group \mathfrak{S}_n by

$$\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{R_1} \times \dots \times \mathfrak{S}_{R_k}.$$

Let K_1, \dots, K_{n-k+1} be the corresponding intervals for the complementary composition \vec{c}^c . Define the subgroup $\mathfrak{S}_{\vec{c}}^c$ by

$$\mathfrak{S}_{\vec{c}}^c = \mathfrak{S}_{\vec{c}^c} = \mathfrak{S}_{K_1} \times \dots \times \mathfrak{S}_{K_{n-k+1}}.$$

A *border strip* is a connected skew shape which does not contain a 2 by 2 square [18, Sect. 7.17]. For each composition \vec{c} there is a unique border strip B that has k rows and the i th row from below consists of c_i boxes. If we label the n boxes of the border strip from southwest to northeast, then the intervals R_1, \dots, R_k correspond to the rows and the intervals K_1, \dots, K_{n-k+1} correspond to the columns. Furthermore, the group $\mathfrak{S}_{\vec{c}}$ is the row stabilizer and the group $\mathfrak{S}_{\vec{c}}^c$ is the column stabilizer of the border strip B .

3 Two subsets of the pointed partition lattice

We now define the first poset central to this paper.

Definition 3.1 For \vec{c} a composition of n , let $\Pi_{\vec{c}}^\bullet$ be the subset of the pointed partition lattice Π_n^\bullet described by

$$\Pi_{\vec{c}}^\bullet = \{\pi \in \Pi_n^\bullet : \exists \vec{d} \geq \vec{c}, \text{ type}(\pi) = \text{type}(\vec{d})\}.$$

In other words, the poset $\Pi_{\vec{c}}^\bullet$ consists of all pointed set partitions such that their type is the type of some composition \vec{d} which is greater than or equal to the composition \vec{c} in the composition order.

Example 3.2 Consider the composition $\vec{c} = (d, \dots, d, d - 1)$ of the integer $n = dk - 1$. For a composition to be greater than or equal to \vec{c} , all of its parts must be divisible by d except the last part which is congruent to $d - 1$ modulo d . Hence $\Pi_{\vec{c}}^\bullet$ consists of all pointed set partitions where the block sizes are divisible by d except the zero block whose size is congruent to $d - 1$ modulo d . Hence the poset $\Pi_{\vec{c}}^\bullet$ is isomorphic to the d -divisible partition lattice Π_{n+1}^d .

Example 3.3 We note that $\Pi_{\vec{c}}^\bullet \cup \{\hat{0}\}$ is in general not a lattice. Consider the composition $\vec{c} = (1, 1, 2, 1)$ and the four pointed set partitions

$$\pi_1 = 1|2|34|\underline{5}, \quad \pi_2 = 2|5|34|\underline{1}, \quad \pi_3 = 34|\underline{125} \quad \text{and} \quad \pi_4 = 2|\underline{1345}$$

in $\Pi_{(1,1,2,1)}^\bullet$. In the pointed partition lattice Π_5^\bullet we have $\pi_1, \pi_2 < 2|34|\underline{15} < \pi_3, \pi_4$. Since the pointed set partition $2|34|\underline{15}$ does not belong to $\Pi_{(1,1,2,1)}^\bullet$, we conclude that $\Pi_{(1,1,2,1)}^\bullet \cup \{\hat{0}\}$ is not a lattice.

We now turn our attention to filters in the pointed partition lattice Π_n^\bullet that are generated by a pointed knapsack partition. These filters were introduced in [5].

Recall that we view an integer partition λ as a multiset of positive integers. Let $\lambda = \{\lambda_1^{e_1}, \dots, \lambda_q^{e_q}\}$ be an integer partition, where we assume that the λ_i 's are distinct. If all the $(e_1 + 1) \cdots (e_q + 1)$ integer linear combinations

$$\left\{ \sum_{i=1}^q f_i \cdot \lambda_i : 0 \leq f_i \leq e_i \right\}$$

are distinct, we call λ a *knapsack partition*. A pointed integer partition $\{\lambda, \underline{m}\}$ is called a *pointed knapsack partition* if the partition λ is a knapsack partition.

This definition was introduced by Ehrenborg–Readdy [5]. Their motivation was to compute the Möbius function of filters generated by knapsack partitions in the pointed partition lattice; see Corollary 8.7. However, earlier Kozlov [11] introduced the same notion under the name no equal-subsets sums (NES). His motivation was the same, except he studied the topology of filters in the partition lattice.

Definition 3.4 For a pointed knapsack partition $\{\lambda, \underline{m}\} = \{\lambda_1, \lambda_2, \dots, \lambda_k, \underline{m}\}$ of n define the subposet $\Pi_{\{\lambda, \underline{m}\}}^\bullet$ to be the filter of Π_n^\bullet generated by all pointed set partitions of type $\{\lambda, \underline{m}\}$.

Example 3.5 Observe that $\lambda = \{d, d, \dots, d\}$ is a knapsack partition. Hence all the block sizes in the filter $\Pi_{\{\lambda, \underline{m}\}}^\bullet$ are divisible by d , except the pointed block. Hence for the pointed knapsack partition $\{d, d, \dots, d, \underline{d-1}\}$ we obtain the d -divisible partition lattice again, as in Example 3.2.

Note that $\Pi_{\{\lambda, \underline{m}\}}^\bullet \cup \{\hat{0}\}$ is indeed a lattice since it inherits the join operation from Π_n^\bullet . The fact that the meet exists is due to Proposition 3.3.1 in [17].

Example 3.6 To see the difference between the two subposets defined in this section, consider the pointed knapsack partition $\{3, 1, 1, \underline{0}\}$ and the composition $(3, 1, 1, 0)$. Observe that the pointed set partition $\pi = 1/2/\underline{345}$ belongs to the filter $\Pi_{\{3, 1, 1, \underline{0}\}}^\bullet$. However, π does not belong to the subposet $\Pi_{(3, 1, 1, 0)}^\bullet$. To observe this fact, note that the cardinality of the pointed block is 3 and there is no composition \vec{d} greater than or equal to $(3, 1, 1, 0)$ whose last part is 3.

4 The simplicial complex of ordered set partitions

An *ordered set partition* τ of a set S is a list of blocks (C_1, C_2, \dots, C_m) where the blocks are subsets of the set S satisfying:

- (i) All blocks except possibly the last block are non-empty, that is, $C_i \neq \emptyset$ for $1 \leq i \leq m - 1$.
- (ii) The blocks are pairwise disjoint, that is, $C_i \cap C_j = \emptyset$ for $1 \leq i < j \leq m$.
- (iii) The union of the blocks is S , that is, $C_1 \cup \dots \cup C_m = S$.

To distinguish from pointed partitions we write 36-127-8-45 for $(\{3, 6\}, \{1, 2, 7\}, \{8\}, \{4, 5\})$. The *type* of an ordered set partition, $\text{type}(\tau)$, is the composition $(|C_1|, |C_2|, \dots, |C_m|)$.

Let Δ_n denote the collection of all ordered set partitions of the set $[n]$. We view Δ_n as a simplicial complex. The ordered set partition $\tau = (C_1, C_2, \dots, C_m)$ forms an $(m - 2)$ -dimensional face. It has $m - 1$ facets, which are $(C_1, \dots, C_i \cup C_{i+1}, \dots, C_m)$ for $1 \leq i \leq m - 1$. The empty face corresponds to the ordered partition $([n])$. The complex Δ_n has $2^n - 1$ vertices that are of the form (C_1, C_2) where $C_1 \neq \emptyset$. Moreover there are $n!$ facets corresponding to permutations in the symmetric group \mathfrak{S}_n , that is, for a permutation $\alpha = \alpha_1 \cdots \alpha_n$, the associated facet is $(\{\alpha_1\}, \{\alpha_2\}, \dots, \{\alpha_n\}, \emptyset)$.

The permutahedron is the $(n - 1)$ -dimensional polytope obtained by taking the convex hull of the $n!$ points $(\alpha_1, \dots, \alpha_n)$ where $\alpha = \alpha_1 \cdots \alpha_n$ ranges over all permutations in the symmetric group \mathfrak{S}_n . Let P_n denote the boundary complex of the dual of the $(n - 1)$ -dimensional permutahedron. Since the permutahedron is a simple polytope the complex P_n is a simplicial complex homeomorphic to an $(n - 2)$ -dimensional sphere. Another view is that P_n is the barycentric subdivision of the boundary of the n -dimensional simplex. Note that the link of the vertex $([n], \emptyset)$ in the complex Δ_n is the complex P_n . In fact, the complex Δ_n is the cone of P_n .

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group \mathfrak{S}_n and a composition $\vec{c} = (c_1, \dots, c_k)$ of n , define the ordered partition

$$\begin{aligned} \sigma(\alpha, \vec{c}) &= (\{\alpha_j : j \in R_i\})_{1 \leq i \leq k} \\ &= (\{\alpha_1, \dots, \alpha_{c_1}\}, \{\alpha_{c_1+1}, \dots, \alpha_{c_1+c_2}\}, \dots, \{\alpha_{c_1+\dots+c_{k-1}+1}, \dots, \alpha_n\}). \end{aligned}$$

We write $\sigma(\alpha)$ when it is clear from the context what the composition \vec{c} is.

For a composition \vec{c} define the subcomplex $\Delta_{\vec{c}}$ to be

$$\Delta_{\vec{c}} = \{\tau \in \Delta_n : \vec{c} \leq \text{type}(\tau)\}.$$

This complex has all of its facets of type \vec{c} . Especially, each facet has the form $\sigma(\alpha, \vec{c})$ for some permutation α . As an example, note that $\Delta_{(1,1,\dots,1)}$ is the complex P_n . Two more examples are shown in Figs. 1 and 2.

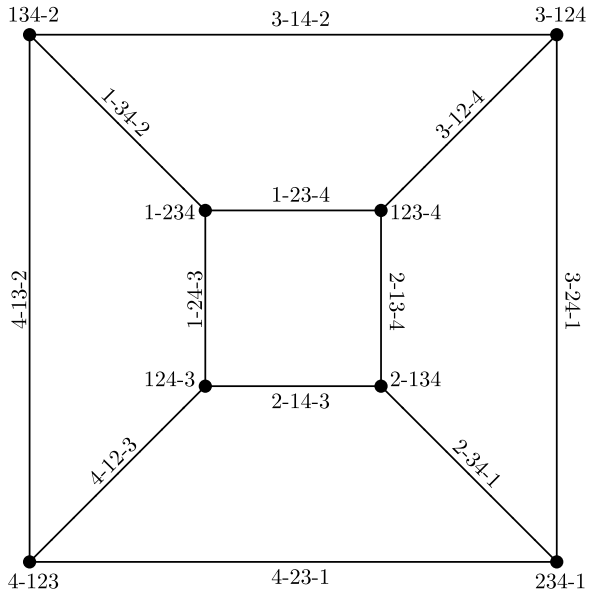
Lemma 4.1 *If the pointed composition $\vec{c} = (c_1, \dots, c_k)$ ends with 0, then the simplicial complex $\Delta_{\vec{c}}$ is a cone over the complex $\Delta_{(c_1,\dots,c_{k-1})}$ with apex $([n], \emptyset)$ and hence contractible.*

However, for a facet F in $\Delta_{\vec{c}}$ there are $\vec{c}! = c_1! \cdots c_k!$ permutations that map to it by the function σ . Hence let $\sigma^{-1}(F)$ denote the smallest permutation α with respect to the weak Bruhat order that gets mapped to the facet F . This permutation satisfies the inequalities

$$\alpha_{c_1+\dots+c_i+1} < \cdots < \alpha_{c_1+\dots+c_{i+1}}$$

for $0 \leq i \leq k - 1$. Furthermore, the descent composition of the permutation $\sigma^{-1}(F)$ is greater than or equal to the composition \vec{c} , that is, $\text{Des}(\sigma^{-1}(F)) \geq \vec{c}$.

Fig. 1 The simplicial complex $\Delta_{(1,2,1)}$ of ordered partitions. Note that the ordered partition 1234 corresponds to the empty face



Theorem 4.2 *Let \vec{c} be a composition not ending with a zero. Then the simplicial complex $\Delta_{\vec{c}}$ is shellable. The spanning facets are of the form $\sigma(\alpha)$ where α ranges over all permutations in the symmetric group \mathfrak{S}_n with descent composition \vec{c} , that is, $\text{Des}(\alpha) = \vec{c}$. Hence the complex $\Delta_{\vec{c}}$ is homotopy equivalent to a wedge of $\beta(\vec{c})$ spheres of dimension $k - 2$.*

Proof Let S be the subset of $[n - 1]$ associated with the composition \vec{c} , that is, $S = \{c_1, c_1 + c_2, \dots, c_1 + \dots + c_{k-1}\}$. A facet (C_1, \dots, C_k) of the complex $\Delta_{\vec{c}}$ corresponds to the maximal chain

$$\emptyset \subseteq C_1 \subseteq C_1 \cup C_2 \subseteq \dots \subseteq C_1 \cup \dots \cup C_{k-1} \subseteq [n]$$

of the S -rank-selected Boolean algebra $B_n(S) = \{x \in B_n : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Hence $\Delta_{\vec{c}}$ is the order complex $\Delta(B_n(S) - \{\hat{0}, \hat{1}\})$. The Boolean algebra has an EL -labeling, which implies that the rank-selected poset $B_n(S)$ has a shellable order complex; see [1, Theorem 4.1]. In fact, the rank-selected poset $B_n(S)$ is CL -shellable; see [3, Theorem 8.1] or [22, Theorem 3.4.1]. Furthermore, it follows from Björner’s construction that the spanning facets are exactly of the above form. \square

Finally, we note the following consequence.

Corollary 4.3 *If the pointed composition $\vec{c} = (c_1, \dots, c_k)$ ends with 0, then the simplicial complex $\Delta_{\vec{c}}$ is shellable.*

Proof The statement follows from the fact that the complex is the cone over a shellable complex. \square

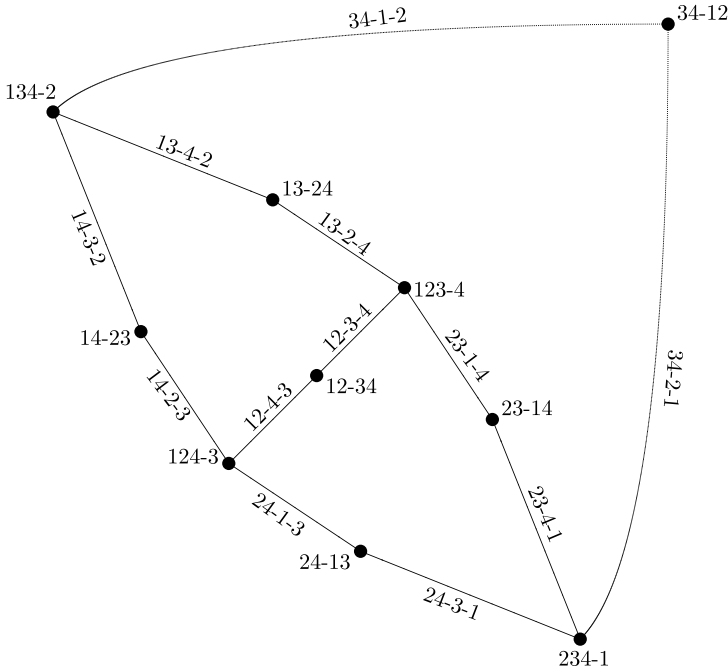


Fig. 2 The simplicial complex $\Delta_{(2,1,1)}$ of ordered partitions

5 The homotopy type of the poset $\Pi_{\vec{c}}^{\bullet}$

We now will use Quillen’s Fiber Lemma to show that the chain complex $\Delta(\Pi_{\vec{c}}^{\bullet} - \{\hat{1}\})$ is homotopy equivalent to the simplicial complex $\Delta_{\vec{c}}$. Recall that a *simplicial map* f from a simplicial complex Γ to a poset P sends vertices of Γ to elements of P and faces of the simplicial complex to chains of P . We have the following result due to Quillen [14]. See also [2, Theorem 10.5] and [22, Theorem 5.2.1].

Theorem 5.1 (Quillen’s Fiber Lemma) *Let f be a simplicial map from the simplicial complex Γ to the poset P such that for all x in P , the complex $\Delta(f^{-1}(P_{\geq x}))$, that is, the subcomplex of Γ induced by $f^{-1}(P_{\geq x})$, is contractible. Then the order complex $\Delta(P)$ and the simplicial complex Γ are homotopy equivalent.*

Recall that the *barycentric subdivision* of a simplicial complex Γ is the simplicial complex $sd(\Gamma)$ whose vertices are the non-empty faces of Γ and whose faces are subsets of chains of faces in Γ ordered by inclusion. It is well-known that Γ and $sd(\Gamma)$ are homeomorphic since they have the same geometric realization and hence are homotopy equivalent.

Consider the map $\phi : \Delta_n \rightarrow \Pi_n^{\bullet}$ that sends an ordered set partition (C_1, C_2, \dots, C_k) to the pointed partition $\{C_1, C_2, \dots, C_{k-1}, \underline{C_k}\}$. We call this map the *forgetful* map since it forgets the order between the blocks except it keeps the last

part as the pointed block. Observe that the inverse image of the pointed partition $\{C_1, C_2, \dots, C_{k-1}, \underline{C_k}\}$ consists of $(k - 1)!$ ordered set partitions.

Lemma 5.2 *Let π be the pointed partition $\{B_1, \dots, B_{m-1}, \underline{B_m}\}$ in $\Pi_{\vec{c}}^\bullet$ where $m \geq 2$. Let Ω be the subcomplex of the complex $\Delta_{\vec{c}}$ whose faces are given by the inverse image*

$$\phi^{-1}((\Pi_{\vec{c}}^\bullet - \{\hat{1}\})_{\geq \pi}).$$

Then the complex Ω is a cone over the apex $([n] - B_m, B_m)$ and hence is contractible.

Proof Let $\zeta = (C_1, C_2, \dots, C_r)$ be an ordered partition in the complex Ω . Observe that the pointed block B_m is contained in the last block C_r . Let Ω' be the subcomplex of Ω consisting of all faces (C_1, C_2, \dots, C_r) such that the last block C_r strictly contains the pointed block B_m . Then the complex Ω is a cone over the complex Ω' with the apex $([n] - B_m, B_m)$. □

Theorem 5.3 *The order complex $\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\})$ is homotopy equivalent to the barycentric subdivision $\text{sd}(\Delta_{\vec{c}})$ and hence $\Delta_{\vec{c}}$.*

Proof Note that

$$\phi^{-1}((\Pi_{\vec{c}}^\bullet - \{\hat{1}\})_{\geq \pi}) = \text{sd}(\Omega) \simeq \Omega,$$

which is contractible by Lemma 5.2. Hence Quillen’s Fiber Lemma applies and we conclude that $\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\})$ is homotopy equivalent to $\text{sd}(\Delta_{\vec{c}})$. □

By considering the reduced Euler characteristic of the complex $\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\})$, we have the following corollary.

Corollary 5.4 *The Möbius function of the poset $\Pi_{\vec{c}}^\bullet \cup \{\hat{0}\}$ is given by $(-1)^k \cdot \beta(\vec{c})$.*

We note that a combinatorial proof can be given for this corollary, which avoids the use of Quillen’s Fiber Lemma.

6 Cycles in the complex $\Delta_{\vec{c}}$

In this section and the next we assume that the last part of the composition \vec{c} is non-zero, since in the case $c_k = 0$ the top homology group is the trivial group; see Lemma 4.1.

For α a permutation in the symmetric group \mathfrak{S}_n , define the subcomplex Σ_α of the complex $\Delta_{\vec{c}}$ to be the simplicial complex whose facets are given by

$$\{\sigma(\alpha \circ \gamma) : \gamma \in \mathfrak{S}_c^c\},$$

where σ is defined in Sect. 4.

Lemma 6.1 *The subcomplex Σ_α is isomorphic to the join of the duals of the permutohedra*

$$P_{|K_1|} * \cdots * P_{|K_{n-k+1}|}$$

and hence it is a sphere of dimension $k - 2$.

Proof Represent the complex $P_{|K_i|}$ by ordered partitions on the set K_i . A face of the join $P_{|K_1|} * \cdots * P_{|K_{n-k+1}|}$ is then an $(n - k + 1)$ -tuple $(\tau_1, \dots, \tau_{n-k+1})$ where $\tau_i = (D_{i,1}, \dots, D_{i,j_i}) \in P_{|K_i|}$. We obtain a face of Σ_α by gluing these ordered partitions together, that is,

$$(D_{1,1}, \dots, D_{1,j_1} \cup D_{2,1}, \dots, D_{2,j_2} \cup D_{3,1}, \dots, D_{n-k+1,j_{n-k+1}}).$$

It is straightforward to see this is a bijective correspondence proving the first claim. Since the join of an m -sphere and an n -sphere is an $(m + n + 1)$ -sphere, we obtain a sphere of dimension $(|K_1| - 2) + \cdots + (|K_{n-k+1}| - 2) + n - k = k - 2$. \square

Observe that the facets of $\Delta_{\vec{c}}$ are in bijection with permutations α such that $\text{Des}(\alpha) \geq \vec{c}$ in the composition order.

Lemma 6.2 *Let α be a permutation in the symmetric group \mathfrak{S}_n with descent composition \vec{c} and let γ belong to the column stabilizer $\mathfrak{S}_{\vec{c}}^c$. Then we have the inequality $\alpha \circ \gamma \leq \alpha$ in the weak Bruhat order.*

Proof Write γ as $(\gamma_1, \dots, \gamma_{n-k+1})$ where γ_i belongs to \mathfrak{S}_{K_i} . Then γ_i acts on the interval $K_i = [u, v]$. Note that $v(v - 1) \cdots u$ is the largest permutation in the Bruhat order on \mathfrak{S}_{K_i} . Hence $\alpha \circ \gamma$ restricted to K_i is a smaller element in the Bruhat order. The inequality follows by concatenating these partial permutations into the permutation $\alpha \circ \gamma$. \square

We note that this lemma has a dual version.

Lemma 6.3 *Let α be a permutation in the symmetric group \mathfrak{S}_n with descent composition \vec{c} and let γ belong to the row stabilizer $\mathfrak{S}_{\vec{c}}$. Then we have the inequality $\alpha \circ \gamma \geq \alpha$ in the weak Bruhat order.*

Directly we have the next lemma.

Lemma 6.4 *Let F be a facet of Σ_α . Then the inequality $\sigma^{-1}(F) \leq \alpha$ holds in the weak Bruhat order.*

Proof There is an element γ in the column stabilizer $\mathfrak{S}_{\vec{c}}^c$ such that $F = \sigma(\alpha \circ \gamma)$. Since $\sigma^{-1}(F)$ is the smallest permutation in the Bruhat order that maps to F , we have $\sigma^{-1}(F) \leq \alpha \circ \gamma \leq \alpha$. \square

Recall that the boundary map of the face $\sigma = (C_1, \dots, C_r)$ in the chain complex of $\Delta_{\vec{c}}$ is defined by

$$\partial((C_1, \dots, C_r)) = \sum_{i=1}^{r-1} (-1)^{i-1} \cdot (C_1, \dots, C_i \cup C_{i+1}, \dots, C_r).$$

The next lemma follows from Lemma 6.1, apart from the signs.

Lemma 6.5 *For $\alpha \in \mathfrak{S}_n$, the element*

$$g_\alpha = \sum_{\gamma \in \mathfrak{S}_{\vec{c}}} (-1)^\gamma \cdot \sigma(\alpha \circ \gamma)$$

in the chain group $C_{k-2}(\Delta_{\vec{c}})$ belongs to the kernel of the boundary map and hence to the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$.

Proof Apply the boundary map to the above element g_α in the chain group and exchange the order of the two sums. The inner sum is then

$$\sum_{\gamma \in \mathfrak{S}_{\vec{c}}} (-1)^\gamma \cdot (\dots, \{\dots, \alpha_{\gamma_{c_1+\dots+c_i}}, \alpha_{\gamma_{c_1+\dots+c_i+1}}, \dots\}, \dots).$$

Observe that the term corresponding to γ cancels with the term corresponding to γ composed with the transposition $(c_1 + \dots + c_i, c_1 + \dots + c_i + 1)$ which belongs to the column stabilizer $\mathfrak{S}_{\vec{c}}$ and thus the sum vanishes. □

Theorem 6.6 *The cycles g_α , where α ranges over all permutations with descent composition \vec{c} , form a basis for the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$.*

Proof The complex $\Delta_{\vec{c}} - \{\sigma(\alpha) : \text{Des}(\alpha) = \vec{c}\}$ is contractible by the shelling in the proof of Theorem 4.2. Contract this complex to a point and then attach the cells $\sigma(\alpha)$ to this point to obtain a wedge of spheres, denoted by X . Call this contraction map f , that is, we have the continuous function $f : \Delta_{\vec{c}} \rightarrow X$ and hence the homomorphism $f_* : \tilde{H}_{k-2}(\Delta_{\vec{c}}) \rightarrow \tilde{H}_{k-2}(X)$. Since f is a part of a homotopy equivalence, we know that f_* is an isomorphism. Lastly, let h_α denote the cycle corresponding to the sphere $\sigma(\alpha)$ in the homology group $\tilde{H}_{k-2}(X)$.

It is clear that the cycles h_α , where α ranges over all permutations with descent composition \vec{c} , form a basis for the homology group $\tilde{H}_{k-2}(X)$. We now motivate that the images $f_*(g_\alpha)$ also form a basis for the homology group $\tilde{H}_{k-2}(X)$.

Take two permutations α and α' with descent composition \vec{c} . By Lemma 6.4 we find that the coefficient of $h_{\alpha'}$ in $f_*(g_\alpha)$ is zero unless $\alpha' \leq \alpha$ in the weak Bruhat order. Furthermore, the coefficient of h_α in $f_*(g_\alpha)$ is 1. Hence the relationship between the basis $\{h_{\alpha'}\}_{\alpha'}$ and the set $\{f_*(g_\alpha)\}_\alpha$ is triangular and hence invertible. Thus the set $\{f_*(g_\alpha)\}_\alpha$ forms a basis for the homology group $\tilde{H}_{k-2}(X)$. Finally, since f_* is an isomorphism, the cycles g_α form a basis for the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$. □

7 Representation of the symmetric group

The symmetric group \mathfrak{S}_n acts naturally on the poset $\Pi_{\vec{c}}^\bullet$ by relabeling the elements. Hence it also acts on the order complex $\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\})$. Lastly, the symmetric group acts on the top homology group $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\}))$. We show in this section that this action is a Specht module of the border strip B corresponding to the composition \vec{c} . For an overview on the representation theory of the symmetric group, we refer the reader to Sagan’s book [15] and Wachs’ article [22].

Let G be a group which acts on the simplicial complex Γ and on the poset P . We call the simplicial map f a G -simplicial map if the map f commutes with this action. The equivariant version of Quillen’s Fiber Lemma is as follows [14]. See also [22, Sect. 5.2].

Theorem 7.1 (Equivariant homology version of Quillen’s Fiber Lemma) *Let f be a G -simplicial map from the simplicial complex Γ to the poset P such that for all x in P , the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is acyclic. Then the two homology groups $\tilde{H}_r(\Delta(P))$ and $\tilde{H}_r(\Gamma)$ are isomorphic as G -modules.*

The forgetful map ϕ from $\text{sd}(\Delta_{\vec{c}})$ to the order complex of the poset $\Pi_{\vec{c}}^\bullet - \{\hat{1}\}$ commutes with the action of the symmetric group \mathfrak{S}_n . Hence we conclude the next result.

Proposition 7.2 *The two homology groups $\tilde{H}_{k-2}(\text{sd}(\Delta_{\vec{c}}))$ and $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\}))$ are isomorphic as \mathfrak{S}_n -modules.*

It is clear that $\tilde{H}_{k-2}(\text{sd}(\Delta_{\vec{c}}))$ and $\tilde{H}_{k-2}(\Delta_{\vec{c}})$ are isomorphic as \mathfrak{S}_n -modules. Hence in the remainder of this section we will study the action of the symmetric group \mathfrak{S}_n on $\Delta_{\vec{c}}$ and its action on the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$. This is in the spirit of Wachs’ work [20].

Let B be the border strip that has k rows where the i th row consists of c_i boxes. Recall that a tableau is a filling of the boxes of the shape B with the integers 1 through n . A standard Young tableau is a tableau where the rows and columns are increasing. A tabloid is an equivalence class of tableaux under the relation of permuting the entries in each row. To distinguish tabloids from tableaux, only the horizontal lines are drawn in a tabloid. See [15, Sect. 2.1] for details.

Observe that there is a natural bijection between tabloids of shape B and facets of the complex $\Delta_{\vec{c}}$ by letting the elements in each row form a block and letting the order of the blocks go from lowest to highest row. See Fig. 3 for an example. Let M^B be the permutation module corresponding to shape B , that is, the linear span of all tabloids of shape B . Notice that the above bijection induces a \mathfrak{S}_n -module isomorphism between the permutation module M^B and the chain group $C_{k-2}(\Delta_{\vec{c}})$.

Furthermore, there is a bijection between tableaux of shape B and permutations by reading the elements in the northeast direction from the border strip. Recall that the group $\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{n-k+1}}$ is the column stabilizer of the border strip B . Let t be a tableau and α its associated permutation. Hence the polytabloid \mathbf{e}_t corresponding to the tableau t is the element g_α presented in Lemma 6.5; see [15, Definition 2.3.2]. Since the Specht module S^B is the submodule of M^B spanned by all

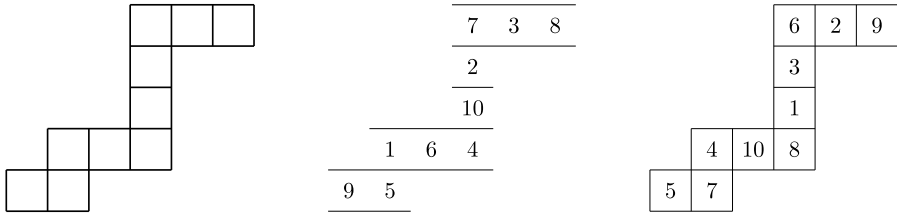


Fig. 3 (a) Border strip corresponding to composition (2, 3, 1, 1, 3). (b) The tabloid corresponding to the face 5 9-1 4 6-10-2-3 7 8. (c) The tableau corresponding to the permutation 5 7 4 10 8 1 3 6 2 9

polytabloids, Lemma 6.5 proves that the Specht module S^B is isomorphic to a submodule of the kernel of the boundary map ∂_{k-2} . Since the kernel is the top homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$, and the Specht module S^B and the homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$ have the same dimension $\beta(\vec{c})$, we conclude that they are isomorphic. To summarize we have the following.

Proposition 7.3 *The top homology group $\tilde{H}_{k-2}(\Delta_{\vec{c}})$ is isomorphic to the Specht module S^B as \mathfrak{S}_n -modules.*

By combining Propositions 7.2 and 7.3, the main result of this section follows.

Theorem 7.4 *The top homology group $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^{\bullet} - \{\hat{1}\}))$ is isomorphic to the Specht module S^B as \mathfrak{S}_n -modules.*

8 Filters generated by knapsack partitions

We now turn our attention to filters in the pointed partition lattice Π_n^{\bullet} that are generated by a pointed knapsack partition, that is, the poset $\Pi_{\{\lambda, \underline{m}\}}^{\bullet}$, which was introduced in Sect. 3. In order to study this poset we need the corresponding collection of ordered set partitions.

Definition 8.1 For a pointed knapsack partition $\{\lambda, \underline{m}\} = \{\lambda_1, \lambda_2, \dots, \lambda_k, \underline{m}\}$ of n define the subcomplex $\Lambda_{\{\lambda, \underline{m}\}}$ of the complex of ordered set partitions Δ_n by

$$\Lambda_{\{\lambda, \underline{m}\}} = \{(C_1, \dots, C_{r-1}, C_r) \in \Delta_n : \{C_1, \dots, C_{r-1}, \underline{C}_r\} \in \Pi_{\{\lambda, \underline{m}\}}^{\bullet}\}.$$

For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n define F to be the filter in the poset of compositions of n generated by compositions \vec{c} such that $\text{type}(\vec{c}) = \{\lambda, \underline{m}\}$. Now define $V(\lambda, \underline{m})$ to be the collection of all pointed compositions $\vec{c} = (c_1, c_2, \dots, c_r)$ in the filter F such that each $c_i, 1 \leq i \leq r - 1$, is a sum of distinct parts of the partition λ and $c_r = m$. As an example, for $\lambda = \{1, 1, 3, 7\}$ we have $(4, 8, m) \in V(\lambda, \underline{m})$ but $(2, 10, m) \notin V(\lambda, \underline{m})$.

For a composition \vec{d} in $V(\lambda, \underline{m})$ define $\epsilon(\vec{d})$ to be the composition of type $\{\lambda, \underline{m}\}$, where each entry d_i of \vec{d} has been replaced with a decreasing list of parts of λ , that

is,

$$\epsilon(\vec{d}) = (\lambda_{1,1}, \dots, \lambda_{1,t_1}, \dots, \lambda_{s,1}, \dots, \lambda_{s,t_s}, m),$$

where $\lambda_{i,1} > \lambda_{i,2} > \dots > \lambda_{i,t_i}$, $\sum_{j=1}^{t_i} \lambda_{i,j} = d_i$ and

$$\{\lambda, \underline{m}\} = \{\lambda_{1,1}, \dots, \lambda_{1,t_1}, \dots, \lambda_{s,1}, \dots, \lambda_{s,t_s}, \underline{m}\}.$$

As an example, for the pointed knapsack partition $\lambda = \{2, 1, \underline{1}\}$ we have $\epsilon((3, 1)) = (2, 1, 1)$, $\epsilon((2, 1, 1)) = (2, 1, 1)$ and $\epsilon((1, 2, 1)) = (1, 2, 1)$. Also note $\epsilon(\vec{d}) \leq \vec{d}$ in the partial order of compositions.

Similar to Theorem 4.2 we have the following topological conclusion. However, this time the tool is not shelling, but discrete Morse theory.

Theorem 8.2 *There is a Morse matching on the simplicial complex $\Lambda_{\{\lambda, \underline{m}\}}$ such that the only critical cells are of the form $\sigma(\alpha, \epsilon(\vec{d}))$ where \vec{d} ranges in the set $V(\lambda, \underline{m})$ and α ranges over all permutations in the symmetric group \mathfrak{S}_n with descent composition \vec{d} . Hence, the simplicial complex $\Lambda_{\{\lambda, \underline{m}\}}$ is homotopy equivalent to a wedge of $\sum_{\vec{d} \in V(\lambda, \underline{m})} \beta(\vec{d})$ spheres of dimension $k - 1$.*

For a pointed knapsack partition $\{\lambda_1, \lambda_2, \dots, \lambda_k, \underline{m}\}$ of n , define a function κ on the domain

$$D = \left\{ \sum_{i \in S} \lambda_i : \emptyset \neq S \subseteq [k] \right\},$$

that is, all non-zero sums of parts of the partition, excluding the pointed part. Now $\kappa : D \rightarrow \mathbb{P}$ is given by

$$\kappa(\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_p}) = \min(\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_p}).$$

Observe that κ is well-defined since λ is a knapsack partition.

For an ordered set partition $\tau = (C_1, C_2, \dots, C_r)$ in $\Lambda_{\{\lambda, \underline{m}\}}$, consider the following $2r - 1$ conditions:

- Let \mathbf{A}_i , for $1 \leq i \leq r - 2$, be the condition that $\max(C_i) < \min(C_{i+1})$ and $|C_i| \leq \kappa(|C_{i+1}|)$.
- Let \mathbf{A}_{r-1} be the condition that $\max(C_{r-1}) < \min(C_r)$.
- Let \mathbf{B}_i , for $1 \leq i \leq r - 1$, be the condition that $\kappa(|C_i|) < |C_i|$.
- Let \mathbf{B}_r be the condition that $|C_r| > m$.

Note that condition \mathbf{A}_i concerns comparing the i th and $(i + 1)$ st blocks of the set partition τ . We also notice that if \mathbf{B}_i is true and $i \leq r - 1$ then the cardinality $|C_i|$ is the sum of at least two parts of the partition λ . Similarly, if \mathbf{B}_r is true then the cardinality $|C_r|$ is the sum of at least two parts of λ and the integer m .

We use these conditions to construct a discrete Morse matching for $\Lambda_{\{\lambda, \underline{m}\}}$. We match the ordered set partition $\tau = (C_1, C_2, \dots, C_r)$ in $\Lambda_{\{\lambda, \underline{m}\}}$ as follows:

- If conditions \mathbf{A}_j and \mathbf{B}_j are false for $1 \leq j \leq i - 1$ and $i \leq r - 1$ but condition \mathbf{B}_i is true, then let X be the $\kappa(|C_i|)$ smallest elements of C_i and Y be the $|C_i| - \kappa(|C_i|)$ largest elements of C_i . Let $u(\tau)$ be given by

$$u(\tau) = (C_1, C_2, \dots, C_{i-1}, X, Y, C_{i+1}, \dots, C_r)$$

and let the type of the edge $(\tau, u(\tau))$ be i .

- If conditions \mathbf{A}_j and \mathbf{B}_j are false for $1 \leq j \leq r - 1$ but condition \mathbf{B}_r is true, then let X be the $\kappa(|C_r| - m)$ smallest elements of C_r and Y be the $|C_r| - \kappa(|C_r| - m)$ largest elements of C_r . Let $u(\tau)$ be given by

$$u(\tau) = (C_1, C_2, \dots, C_{r-1}, X, Y)$$

and let the type of the edge $(\tau, u(\tau))$ be r .

- If conditions \mathbf{A}_j and \mathbf{B}_j are false for $1 \leq j \leq i - 1$ and condition \mathbf{B}_i is false but condition \mathbf{A}_i is true, then let

$$d(\tau) = (C_1, C_2, \dots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \dots, C_r)$$

and let the type of the edge $(d(\tau), \tau)$ be i .

Lemma 8.3 *Let τ and τ' be two different ordered set partitions satisfying the condition $\tau < u(\tau) > \tau' < u(\tau')$. Then this condition implies that the type of $(\tau, u(\tau))$ is greater than the type of $(\tau', u(\tau'))$. Hence the matching is acyclic.*

Proof Consider the following three ordered set partitions:

$$\begin{aligned} \tau &= (C_1, C_2, \dots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \dots, C_r), \\ u(\tau) &= (C_1, C_2, \dots, C_r), \\ \tau' &= (C_1, C_2, \dots, C_{j-1}, C_j \cup C_{j+1}, C_{j+2}, \dots, C_r), \end{aligned}$$

for $i \neq j$. Note that the type of the edge $(\tau, u(\tau))$ is i . If $i < j$, the ordered set partition τ' should be matched to

$$d(\tau') = (C_1, C_2, \dots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \dots, C_{j-1}, C_j \cup C_{j+1}, C_{j+2}, \dots, C_r),$$

contradicting the assumption that τ' was matched upwards with $u(\tau')$. Hence, we conclude that $i > j$ and,

$$u(\tau') = (C_1, C_2, \dots, C_{j-1}, X, Y, C_{j+2}, \dots, C_r)$$

for $X \cup Y = C_j \cup C_{j+1}$ such that $\max(X) < \min(Y)$. Since $\max(C_j) > \min(C_{j+1})$, we have $u(\tau) \neq u(\tau')$ and the type of $(\tau', u(\tau'))$ is j . Therefore, we have $\text{type}(\tau, u(\tau)) = i > j = \text{type}(\tau', u(\tau'))$.

Assume now that this matching is not acyclic. Then we would reach a contradiction by following the inequalities of types around a cycle. □

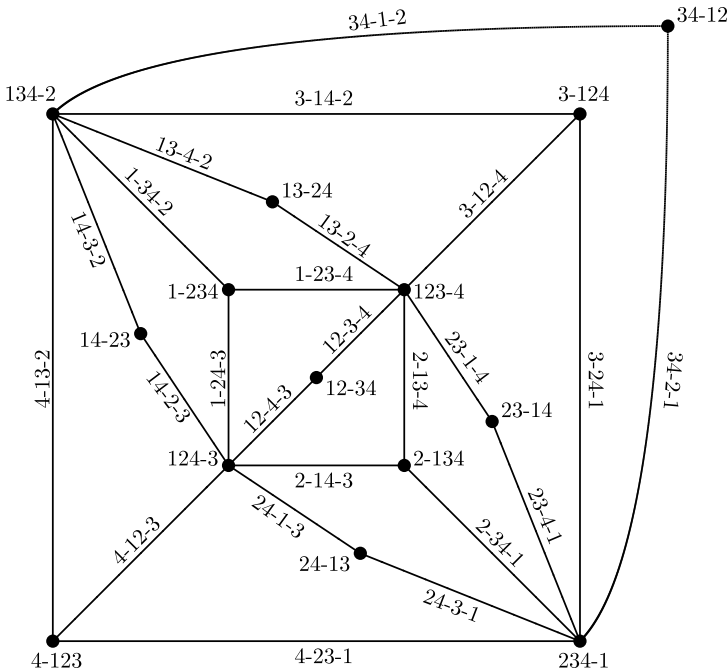


Fig. 4 The simplicial complex $\Lambda_{[2,1,\underline{1}]}$, corresponding to the knapsack partition $\{2, 1, \underline{1}\}$. Notice that this complex is the union of two complexes $\Delta_{(1,2,1)}$ and $\Delta_{(2,1,1)}$, appearing in Figs. 1 and 2

Lemma 8.4 *The critical cells of the above matching of the face poset of $\Lambda_{\{\lambda, \underline{m}\}}$ are of the form $\sigma(\alpha, \epsilon(\vec{d}))$ where \vec{d} ranges in the set $V(\lambda, \underline{m})$ and α ranges over all permutations in the symmetric group \mathfrak{S}_n with descent composition \vec{d} .*

Proof The unmatched cells of the matching presented on the face poset of $\Lambda_{\{\lambda, \underline{m}\}}$ have the form (C_1, C_2, \dots, C_r) such that all the conditions \mathbf{A}_i and \mathbf{B}_i are false for $1 \leq i \leq r - 1$ and \mathbf{B}_r is false.

Since the condition \mathbf{B}_i is false for $i \leq r - 1$ we have $|C_i| = \kappa(|C_i|)$, that is, $|C_i|$ is a part of the partition λ for $i \leq r - 1$. Also \mathbf{B}_r is false implies that $|C_r| = m$. Hence $\{|C_1|, \dots, |C_{r-1}|, |C_r|\}$ is the pointed partition $\{\lambda, \underline{m}\}$. Since the condition \mathbf{A}_i is false, we have $\max(C_i) > \min(C_{i+1})$ or $|C_i| > \kappa(|C_{i+1}|) = |C_{i+1}|$ for $1 \leq i \leq r - 2$. Finally, \mathbf{A}_{r-1} is false, implying $\max(C_{r-1}) > \min(C_r)$.

For the unmatched cell (C_1, \dots, C_r) , let α be the permutation obtained by writing each block of the critical cell in increasing order. Furthermore, let \vec{d} be the descent composition of the permutation α . Observe that the composition \vec{d} belongs to the set $V(\lambda, \underline{m})$ since an entry in \vec{d} is a sum of distinct parts of λ . Furthermore the composition $(|C_1|, \dots, |C_{r-1}|, |C_r|)$ is the composition $\epsilon(\vec{d})$. Hence the unmatched cell is $\sigma(\alpha, \epsilon(\vec{d}))$. □

Example 8.5 Consider the pointed knapsack partition $\{\lambda, \underline{m}\} = \{2, 1, \underline{1}\}$ whose associated complex $\Lambda_{[2,1,\underline{1}]}$ is shown in Fig. 4. Note that $V(\lambda, \underline{m}) = \{(1, 2, 1), (2, 1, 1), (3, 1)\}$. The critical cells of the complex $\Lambda_{[2,1,\underline{1}]}$ are as follows:

\vec{d}	$\beta(\vec{d})$	$\epsilon(\vec{d})$	$W(\vec{d})$	Critical cells
(1, 2, 1)	5	(1, 2, 1)	{(1, 2, 1)}	2-14-3, 3-14-2, 3-24-1, 4-13-2, 4-23-1
(2, 1, 1)	3	(2, 1, 1)	{(2, 1, 1)}	14-3-2, 24-3-1, 34-2-1
(3, 1)	3	(2, 1, 1)	{(1, 2, 1), (2, 1, 1)}	12-4-3, 13-4-2, 23-4-1

Note that $\Lambda_{\{2,1,1\}}$ is homotopy equivalent to a wedge of 11 circles. The notion $W(\vec{d})$ will be defined in the beginning of the next section.

Proof of Theorem 8.2 By Lemma 8.3 the matching presented is a Morse matching and Lemma 8.4 describes the critical cells, proving the theorem. \square

Now by the same reasoning as in Sect. 5, that is, using the forgetful map ϕ and Quillen’s Fiber Lemma, we obtain a homotopy equivalence between the order complex of pointed partitions $\Pi_{\{\lambda, \underline{m}\}}^\bullet - \{\hat{1}\}$ and the simplicial complex of ordered set partitions $\Lambda_{\{\lambda, \underline{m}\}}$. Since the proof follows the same outline, it is omitted.

Theorem 8.6 *The order complex $\Delta(\Pi_{\{\lambda, \underline{m}\}}^\bullet - \{\hat{1}\})$ is homotopy equivalent to the barycentric subdivision $\text{sd}(\Lambda_{\{\lambda, \underline{m}\}})$ and hence the simplicial complex $\Lambda_{\{\lambda, \underline{m}\}}$.*

As a corollary we obtain the Möbius function of the poset $\Pi_{\{\lambda, \underline{m}\}}^\bullet \cup \{\hat{0}\}$; see [5].

Corollary 8.7 (Ehrenborg–Readdy) *The Möbius function of the poset $\Pi_{\{\lambda, \underline{m}\}}^\bullet \cup \{\hat{0}\}$ is given by*

$$\mu(\Pi_{\{\lambda, \underline{m}\}}^\bullet \cup \{\hat{0}\}) = (-1)^k \cdot \sum_{\vec{d} \in V(\lambda, \underline{m})} \beta(\vec{d}).$$

9 Cycles in the complex $\Lambda_{\{\lambda, \underline{m}\}}$

Observe that when the pointed part m is equal to zero, the complex $\Lambda_{\{\lambda, \underline{m}\}}$ is contractible. Hence we tacitly assume that m is positive in this and the next section.

For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n and $\vec{d} \in V(\lambda, \underline{m})$, let $W(\vec{d})$ be the set

$$W(\vec{d}) = \{\vec{c} \in V(\lambda, \underline{m}) : \vec{c} \leq \vec{d}, \text{type}(\vec{c}) = \{\lambda, \underline{m}\}\}.$$

Especially we have $\epsilon(\vec{d}) \in W(\vec{d})$. For the case when the pointed knapsack partition is $\{2, 1, \underline{1}\}$, see Example 8.5.

For α a permutation in the symmetric group \mathfrak{S}_n and \vec{d} a composition of $V(\lambda, \underline{m})$, define the subcomplex $\Sigma_{\alpha, \vec{d}}$ of the complex $\Lambda_{\{\lambda, \underline{m}\}}$ to be the simplicial complex whose facets are given by

$$\{\sigma(\alpha \circ \gamma, \vec{c}) : \vec{c} \in W(\vec{d}), \gamma \in \mathfrak{S}_d^c\}.$$

This means the types of facets in $\Sigma_{\alpha, \vec{d}}$ belong to the set $W(\vec{d})$.

For $\vec{d} = (d_1, \dots, d_r)$, if d_i splits into t_i parts in a composition in $W(\vec{d})$, then the group $\mathfrak{S}_{t_1} \times \dots \times \mathfrak{S}_{t_r}$ acts on $W(\vec{d})$ by permuting the t_i parts d_i splits into. Given $\vec{c} \in W(\vec{d})$ there exists a permutation $\rho \in \mathfrak{S}_{t_1} \times \dots \times \mathfrak{S}_{t_r}$ so $\rho(\epsilon(\vec{d})) = \vec{c}$. Define the sign of \vec{c} , that is, $(-1)^{\vec{c}}$ to be the sign $(-1)^\rho$. Especially, we have $(-1)^{\epsilon(\vec{d})} = 1$.

Similar to Lemma 6.1 we have the next result.

Lemma 9.1 *The subcomplex $\Sigma_{\alpha, \vec{d}}$ is isomorphic to the join of the duals of the permutahedra*

$$P_{|K_1|} * \dots * P_{|K_{n-r+1}|} * P_{t_1} * \dots * P_{t_r}$$

and hence it is a sphere of dimension $k - 1$, where k is the number of parts in the partition λ .

Since this lemma is not necessary for the remainder of this paper, the proof is omitted. However, let us verify the dimension of the sphere. First the dimension of the $(n - r)$ -fold join $P_{|K_1|} * \dots * P_{|K_{n-r+1}|}$ is $(|K_1| - 2) + \dots + (|K_{n-r+1}| - 2) + (n - r) = n - 2 \cdot (n - r + 1) + n - r = r - 2$. Similarly, the dimension of $P_{t_1} * \dots * P_{t_r}$ is given by $(t_1 - 2) + \dots + (t_r - 2) + (r - 1) = (k + 1) - 2 \cdot r + r - 1 = k - r$. Thus the dimension of the sphere in the lemma is $(r - 2) + (k - r) + 1 = k - 1$.

Define the element $g_{\alpha, \vec{d}}$ in the chain group $C_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$ to be

$$g_{\alpha, \vec{d}} = \sum_{\gamma \in \mathfrak{S}_{\vec{d}}^c} \sum_{\vec{c} \in W(\vec{d})} (-1)^\gamma \cdot (-1)^{\vec{c}} \cdot \sigma(\alpha \circ \gamma, \vec{c}).$$

When α has descent composition \vec{d} , note the critical cell $\sigma(\alpha, \epsilon(\vec{d}))$ has sign 1 in $g_{\alpha, \vec{d}}$.

Lemma 9.2 *For $\alpha \in \mathfrak{S}_n$, the element $g_{\alpha, \vec{d}}$ in the chain group $C_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$ belongs to the kernel of the boundary map and hence the homology group $\tilde{H}_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$.*

Proof Apply the boundary map ∂ to $g_{\alpha, \vec{d}}$ in $C_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$ and exchange the order of the three sums to obtain

$$\begin{aligned} \partial(g_{\alpha, \vec{d}}) &= \sum_{i=1}^k (-1)^{i-1} \cdot \sum_{\gamma \in \mathfrak{S}_{\vec{d}}^c} \sum_{\vec{c} \in W(\vec{d})} (-1)^\gamma \cdot (-1)^{\vec{c}} \\ &\quad \cdot (\dots, \{\dots, \alpha_{\gamma_{c_1+\dots+c_i}}, \alpha_{\gamma_{c_1+\dots+c_i+1}}, \dots\}, \dots). \end{aligned}$$

If $\sum_{s=1}^i c_s = \sum_{s=1}^j d_s$ for some j , the term corresponding to γ cancels with the term corresponding to $\gamma \circ (c_1 + \dots + c_i, c_1 + \dots + c_i + 1)$. If there does not exist such an index j , we find the smallest integer ℓ such that $\sum_{s=1}^i c_s < \sum_{s=1}^\ell d_s$. Note that d_ℓ splits into $t_\ell \geq 2$ parts. Now consider the composition $\vec{c}' = (\dots, c_{i-1}, c_{i+1}, c_i, c_{i+2}, \dots)$ where we switched the i th and the $(i + 1)$ st parts of \vec{c} . Note that \vec{c}' also belongs to $W(\vec{d})$ and the sign differs from the sign of \vec{c} , that is, $(-1)^{\vec{c}'} = -(-1)^{\vec{c}}$. Then the term corresponding to \vec{c} cancels with the term corresponding to \vec{c}' . Hence, the sum vanishes. □

Lemma 9.3 *If the critical cell $\sigma(\alpha', \epsilon(\vec{d}'))$ belongs to the cycle $g_{\alpha, \vec{d}}$, then we have the inequality $\alpha' \leq \alpha$ in the weak Bruhat order. Furthermore, if the two permutations α and α' are equal, then the two compositions \vec{d} and \vec{d}' are equal.*

Proof Since $\sigma(\alpha', \epsilon(\vec{d}'))$ belongs to $g_{\alpha, \vec{d}}$, we have $\sigma(\alpha', \epsilon(\vec{d}')) = \sigma(\alpha \circ \gamma, \vec{c})$ for some $\gamma \in \mathfrak{S}'_{\vec{d}}$ and $\vec{c} \in W(\vec{d})$. Note that $\epsilon(\vec{d}') = \vec{c}$. Hence a permutation α' corresponding to the critical cell $\sigma(\alpha', \epsilon(\vec{d}'))$ is obtained by writing each block of $\sigma(\alpha \circ \gamma, \vec{c})$ in increasing order and this implies $\alpha' \leq \alpha \circ \gamma$. Furthermore, $\alpha \circ \gamma \leq \alpha$ in the weak Bruhat order by Lemma 6.2. Hence, we have $\alpha' \leq \alpha$.

Finally, if the two permutations α and α' are equal, we have $\vec{d} = \text{Des}(\alpha) = \text{Des}(\alpha') = \vec{d}'$. □

Using Lemma 9.3 we have the next result. Observe that the proof is similar to that of Theorem 6.6 and hence omitted.

Theorem 9.4 *For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n , the cycles $g_{\alpha, \vec{d}}$ where the composition \vec{d} ranges over the set $V(\lambda, \underline{m})$ and α ranges over all permutations with descent composition \vec{d} , form a basis for the homology group $\tilde{H}_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$.*

10 The group action on the top homology

By the equivariant homology version of Quillen’s Fiber Lemma, Theorem 7.1, we have the following isomorphism.

Theorem 10.1 *The two homology groups $\tilde{H}_{k-1}(\Delta(\Pi_{\{\lambda, \underline{m}\}}^\bullet - \{\hat{1}\}))$ and $\tilde{H}_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$ are isomorphic as \mathfrak{S}_n -modules.*

Hence it remains to make the connection between the action on ordered set partitions and Specht modules.

Theorem 10.2 *The direct sum of Specht modules*

$$\bigoplus_{\vec{d} \in V(\lambda, \underline{m})} S^{B(\vec{d})}$$

is isomorphic to the top homology group $\tilde{H}_{k-1}(\Lambda_{\{\lambda, \underline{m}\}})$ as \mathfrak{S}_n -modules.

Proof We define a homomorphism

$$\Psi: \bigoplus_{\vec{d} \in V(\lambda, \underline{m})} M^{B(\vec{d})} \longrightarrow C_{k-1}(\Lambda_{\{\lambda, \underline{m}\}}),$$

by sending a tabloid s of shape $B(\vec{d})$ to the element $\sum_{\vec{c} \in W(\vec{d})} (-1)^{\vec{c}} \sigma(\alpha, \vec{c})$ in the chain group, where α is the permutation obtained from the tabloid s by reading the

elements in each row in increasing order. Observe that Ψ is a \mathfrak{S}_n -module homomorphism.

Now consider the restriction of the homomorphism Ψ to the direct sum of Specht modules. Note that the group \mathfrak{S}_d^c is the column stabilizer of the border strip $B(\vec{d})$. Let t be a tableau and α its associated permutation. The homomorphism Ψ applied to the polytabloid \mathbf{e}_t (see reference [15, Definition 2.3.2]) is as follows:

$$\Psi(\mathbf{e}_t) = \sum_{\gamma \in \mathfrak{S}_d^c} (-1)^\gamma \cdot \sum_{\vec{c} \in W(\vec{d})} (-1)^{\vec{c}} \cdot \sigma(\alpha \circ \gamma, \vec{c}) = g_{\alpha, \vec{d}},$$

which belongs to the kernel of the boundary map. Hence Ψ maps the directed sum of the Specht modules to the homology group $\tilde{H}_{k-1}(\Delta_{\{\lambda, \underline{m}\}})$.

Since $g_{\alpha, \vec{d}}$ lies in the image of the restriction of Ψ and the elements $g_{\alpha, \vec{d}}$ span the homology group, the restriction is surjective. Furthermore, since the two \mathfrak{S}_n -modules have the same dimension, we conclude that they are isomorphic. \square

Hence we conclude

Theorem 10.3 *The top homology group $\tilde{H}_{k-1}(\Delta(\Pi_{\{\lambda, \underline{m}\}}^\bullet - \{\hat{1}\}))$ and the direct sum of Specht modules*

$$\bigoplus_{\vec{d} \in V(\lambda, \underline{m})} S^{B(\vec{d})}$$

are isomorphic as \mathfrak{S}_n -modules.

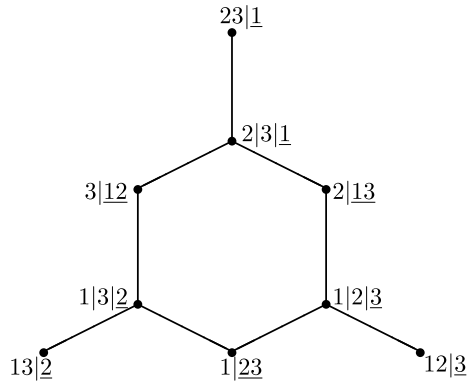
11 Concluding remarks

We have not dealt with the question whether the poset Π_c^\bullet is *EL*-shellable. Recall that Wachs proved that the d -divisible partition lattice $\Pi_n^d \cup \{\hat{0}\}$ has an *EL*-labeling. Ehrenborg and Readdy gave an extension of this labeling to prove that $\Pi_{(d, \dots, d, m)}^\bullet$ is *EL*-shellable [6]. Woodrooffe [23] has shown that the order complex $\Delta(\Pi_n^d - \{\hat{1}\})$ has a convex ear decomposition. This is not true in general for $\Delta(\Pi_c^\bullet - \{\hat{1}\})$. See for instance $\Delta(\Pi_{(1,1,1)}^\bullet - \{\hat{1}\})$ in Fig. 5.

Can the order complex $\Delta(\Pi_c^\bullet - \{\hat{1}\})$ be shown to be shellable, using the fact that the complex $\Delta_{\vec{c}}$ is shellable? That is, can a shelling of $\Delta_{\vec{c}}$ be lifted, similar to using Quillen’s Fiber Lemma? Would this shelling relationship also extend to the two complexes $\Delta_{\{\lambda, \underline{m}\}}$ and $\Delta(\Pi_{\{\lambda, \underline{m}\}} - \{\hat{1}\})$?

Kozlov [11] introduced the notion of a poset to be *EC*-shellable. He showed that the filter Π_λ in the partition lattice generated by a knapsack partition λ is *EC*-shellable [11, Theorem 4.1]. Is it possible to show that Π_c^\bullet and $\Pi_{\{\lambda, \underline{m}\}}^\bullet$ are *EC*-shellable? Furthermore, Kozlov computes the Möbius function of Π_λ . See [11, Corollary 8.5]. Can his answer be expressed in terms of permutation statistics, such as the descent set statistic?

Fig. 5 The order complex of the poset $\Pi_{(1,1,1)}^\bullet - \{\hat{1}\}$



Can these techniques be used for studying other subposets of the partition lattice? One such subposet is the odd partition lattice, that is, the collection of all partitions where each block size is odd. More generally, what can be said about the case when all the block sizes are congruent to r modulo d ? These posets have been studied in [4] and [21]. Moreover, what can be said about the poset $\Pi_{\{\lambda, \underline{m}\}}^\bullet$ when $\{\lambda, \underline{m}\}$ is not a pointed knapsack partition?

Another analogue of the partition lattice is the Dowling lattice. Subposets of the Dowling lattice have been studied in [6] and [9, 10]. Here the first question to ask is: what is the right analogue of ordered set partitions?

Wachs gave a basis for the top homology of the order complex of the d -divisible partition lattice [20, Sect. 2]. Each cycle in her basis is the barycentric subdivision of the boundary of a cube. We can similarly describe a basis for the order complex of $\Pi_{\vec{c}}^\bullet$. The major difference is that the cycles are the barycentric subdivision of a different polytope depending on the composition \vec{c} . In order to describe this polytope, recall that the n -dimensional root polytope R_n (of type A) is defined as the intersection of the $(n + 1)$ -dimensional crosspolytope $\text{conv}(\{\pm 2\mathbf{e}_i\}_{1 \leq i \leq n+1})$ and the hyperplane $x_1 + x_2 + \dots + x_{n+1} = 0$. Equivalently, the root polytope R_n can be defined as the convex hull of the set $\{\mathbf{e}_i - \mathbf{e}_j\}_{1 \leq i, j \leq n+1}$. Lastly, let S_n denote the n -dimensional simplex.

We state the following theorem without proof.

Theorem 11.1 *Given a composition \vec{c} , there is a basis for $\tilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\}))$ where each basis element is the barycentric subdivision of the boundary of the Cartesian product:*

$$\begin{aligned} R_{|K_1|-1} \times R_{|K_2|-1} \times \dots \times R_{|K_{n-k}|-1} \times S_{|K_{n-k+1}|-1} & \text{ if } c_1 \neq 1, \\ S_{|K_1|-1} \times R_{|K_2|-1} \times \dots \times R_{|K_{n-k}|-1} \times S_{|K_{n-k+1}|-1} & \text{ if } c_1 = 1. \end{aligned}$$

Note that when all the parts of the composition \vec{c} are greater than 1, this polytope reduces to the $(k - 1)$ -dimensional cube.

In a recent preprint Miller [13] has extended the definition of the partition poset $\Pi_{\vec{c}}^\bullet$ from type A to all real reflection groups and the complex reflection groups known

as Shephard groups. Can his techniques also extend our results for the filter $\Pi_{\{\lambda, \underline{m}\}}^\bullet$ where $\{\lambda, \underline{m}\}$ is a knapsack partition?

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