

Jucys–Murphy elements and a presentation for partition algebras

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Abstract We give a new presentation for the partition algebras. This presentation was discovered in the course of establishing an inductive formula for the partition algebra Jucys–Murphy elements defined by Halverson and Ram (Eur. J. Comb. 26:869–921, 2005). Using Schur–Weyl duality we show that our recursive formula and the original definition of Jucys–Murphy elements given by Halverson and Ram are equivalent. The new presentation and inductive formula for the partition algebra Jucys–Murphy elements given in this paper are used to construct the seminormal representations for the partition algebras in a separate paper.

Keywords Partition algebras · Jucys–Murphy elements · Central elements · Presentation

1 Introduction

The partition algebras $A_k(n)$, for $k, n \in \mathbb{Z}_{\geq 0}$, are a family of algebras defined in the work of Martin and Jones in [4–6] in connection with the Potts model and higher dimensional statistical mechanics. By [4], the partition algebra $A_k(n)$ is in Schur–Weyl duality with the symmetric group \mathfrak{S}_n acting diagonally on the k -fold tensor product $V^{\otimes k}$ of its n -dimensional permutation representation V . In [7], Martin defined the partition algebras $A_{k+\frac{1}{2}}(n)$, for $k, n \in \mathbb{Z}_{\geq 0}$, as the centralisers of the subgroup

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$\mathfrak{S}_{n-1} \subseteq \mathfrak{S}_n$ acting on $V^{\otimes k}$. Including the algebras $A_{k+\frac{1}{2}}(n)$ in the tower

$$A_0(n) \subseteq A_{\frac{1}{2}}(n) \subseteq A_1(n) \subseteq A_{1+\frac{1}{2}}(n) \subseteq \cdots \quad (1.1)$$

allowed for the simultaneous analysis of the whole tower of algebras using the Jones Basic construction by Martin [7] and Halverson and Ram [3].

Halverson and Ram [3] and East [1] have given a presentation for the partition algebras in terms of Coxeter generators for the symmetric group and certain contractions. Halverson and Ram [3] used Schur–Weyl duality to show that certain diagrammatically defined elements in the partition algebras $A_k(n)$ and $A_{k+\frac{1}{2}}(n)$ play an analogous role to the classical Jucys–Murphy elements in the group algebra of the symmetric group \mathfrak{S}_k .

The Jucys–Murphy elements in the group algebra of the symmetric group respect the inclusions $\mathfrak{S}_{k-1} \subseteq \mathfrak{S}_k$ and are simultaneously diagonalisable in any irreducible representation of the symmetric group. The seminormal representations of the symmetric group are the irreducible matrix representations with respect to a basis on which the Jucys–Murphy elements act diagonally. These may be constructed inductively (see [8], for example).

This paper provides a new presentation for the partition algebras. This presentation was discovered in the course of establishing an inductive formula for the Jucys–Murphy elements for partition algebras given by Halverson and Ram. In a separate paper, we use this new presentation to construct seminormal representations for the partition algebras [2], solving a problem highlighted in the introduction to the paper of Halverson and Ram [3].

In Sect. 2 we recall the presentation of the partition algebras given by Halverson and Ram [3] and East [1]. In Sect. 3 we state a recursion defining a family of operators

$$(L_i, L_{i+\frac{1}{2}} : i = 0, 1, \dots) \quad (1.2)$$

and establish that the operators (1.2) form a commuting family with properties analogous to the Jucys–Murphy elements that arise in the representation theory of the symmetric group. Simultaneously, we establish some basic commutativity results for certain operators denoted

$$(\sigma_i, \sigma_{i+\frac{1}{2}} : i = 1, 2, \dots) \quad (1.3)$$

which arose in the recursive definition of the Jucys–Murphy elements (1.2). In Sect. 4 we show that the elements of (1.3) are involutions which are related to the Coxeter generators for the symmetric group in a precise way. Using the relation between the involutions (1.3) and the Coxeter generators for the symmetric group, and the properties established in Sect. 3, we derive a new presentation for the partition algebras. In Sect. 5 we provide formulae for the actions of the Jucys–Murphy elements (1.2) and the involutions (1.3) on tensor space. Using Schur–Weyl duality, we demonstrate that the recursive definition of Jucys–Murphy elements given in Sect. 3 is equivalent to the definition of Jucys–Murphy elements given by Halverson and Ram [3].

2 The partition algebras

In this section we follow the exposition by Halverson and Ram in [3]. For $k = 1, 2, \dots$, let

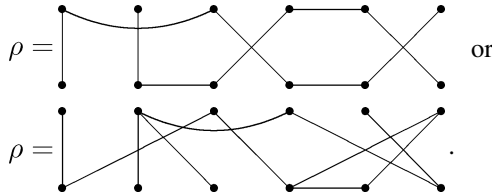
$$A_k = \{ \text{set partitions of } \{1, 2, \dots, k, 1', 2', \dots, k'\} \}, \quad \text{and}$$

$$A_{k-\frac{1}{2}} = \{ d \in A_k \mid k \text{ and } k' \text{ are in the same block of } d \}.$$

Any element $\rho \in A_k$ may be represented as a graph with k vertices in the top row, labelled from left to right, by $1, 2, \dots, k$ and k vertices in the bottom row, labelled from left to right by $1', 2', \dots, k'$, with vertex i joined to vertex j if i and j belong to the same block of ρ . The representation of a partition by a diagram is not unique; for example the partition

$$\rho = \{ \{1, 1', 3, 4', 5', 6\}, \{2, 2', 3', 4, 5, 6'\} \}$$

can be represented by the diagrams



If $\rho_1, \rho_2 \in A_k$, then the composition $\rho_1 \circ \rho_2$ is the partition obtained by placing ρ_1 above ρ_2 and identifying each vertex in the bottom row of ρ_1 with the corresponding vertex in the top row of ρ_2 and deleting any components of the resulting diagram which contains only elements from the middle row. The composition product makes A_k into an associative monoid with identity



Let z be an indeterminant and $R = \mathbb{Z}[z]$. The partition algebra $\mathcal{A}_k(z)$ is the R -module freely generated by A_k , equipped with the product

$$\rho_1 \rho_2 = z^\ell \rho_1 \circ \rho_2, \quad \text{for } \rho_1, \rho_2 \in A_k,$$

where ℓ is the number of blocks removed from the middle row in constructing the composition $\rho_1 \circ \rho_2$. Let $\mathcal{A}_{k-\frac{1}{2}}(z)$ denote the subalgebra of $\mathcal{A}_k(z)$ generated by $A_{k-\frac{1}{2}}$. A presentation for $\mathcal{A}_k(z)$ has been given by Halverson and Ram [3] and East [1].

Theorem 2.1 (Theorem 1.11 of [3]) *If $k = 1, 2, \dots$, then the partition algebra $\mathcal{A}_k(z)$ is the unital associative R -algebra presented by the generators*

$$P_1, P_{1+\frac{1}{2}}, P_2, P_{2+\frac{1}{2}}, \dots, P_k, S_1, S_2, \dots, S_{k-1},$$

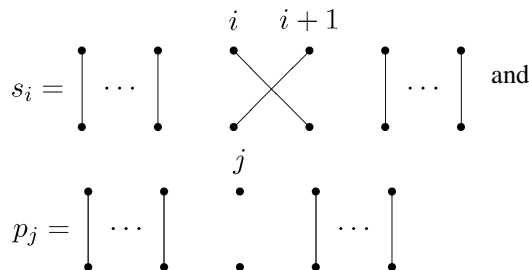
and the relations

- (1) (Coxeter relations)
 - (i) $s_i^2 = 1$, for $i = 1, \dots, k - 1$.
 - (ii) $s_i s_j = s_j s_i$, if $j \neq i + 1$.
 - (iii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $i = 1, \dots, k - 2$.
- (2) (Idempotent relations)
 - (i) $p_i^2 = z p_i$, for $i = 1, \dots, k$.
 - (ii) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}$, for $i = 1, \dots, k - 1$.
 - (iii) $s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i = p_{i+\frac{1}{2}}$, for $i = 1, \dots, k - 1$.
 - (iv) $s_i p_i p_{i+1} = p_i p_{i+1} s_i = p_i p_{i+1}$, for $i = 1, \dots, k - 1$.
- (3) (Commutation relations)
 - (i) $p_i p_j = p_j p_i$, for $i = 1, \dots, k$ and $j = 1, \dots, k$.
 - (ii) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}$, for $i = 1, \dots, k - 1$ and $j = 1, \dots, k - 1$.
 - (iii) $p_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_i$, for $j \neq i, i + 1$.
 - (iv) $s_i p_j = p_j s_i$, for $j \neq i, i + 1$.
 - (v) $s_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} s_i$, for $j \neq i - 1, i + 1$.
 - (vi) $s_i p_i s_i = p_{i+1}$, for $i = 1, \dots, k - 1$.
 - (vii) $s_i p_{i-\frac{1}{2}} s_i = s_{i-1} p_{i+\frac{1}{2}} s_{i-1}$, for $i = 2, \dots, k - 1$.
- (4) (Contraction relations)
 - (i) $p_{i+\frac{1}{2}} p_j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$, for $j = i, i + 1$.
 - (ii) $p_i p_{j-\frac{1}{2}} p_i = p_i$, for $j = i, i + 1$.

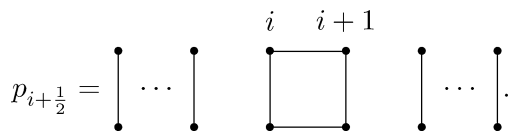
The above relations also imply that

$$\begin{aligned}
 p_{i+\frac{1}{2}} s_i s_{i+1} p_{i+\frac{1}{2}} &= p_{i+\frac{1}{2}} p_{i\pm 1+\frac{1}{2}}, \\
 p_i s_i p_i &= p_i p_{i+1} = p_{i+1} s_i p_{i+1}, \\
 p_i p_{i+\frac{1}{2}} p_{i+1} &= p_i s_i, \\
 p_{i+1} p_{i+\frac{1}{2}} p_i &= p_{i+1} s_i.
 \end{aligned}$$

In Theorem 2.1, the following identifications have been made:



and



We also let \mathfrak{S}_k denote the symmetric group on k letters which is generated by s_1, \dots, s_{k-1} . If $u \in \mathfrak{S}_k \subset \mathcal{A}_k(z)$, and $\rho \in \mathcal{A}_k(z)$, we will sometimes write $\rho^u = u\rho u^{-1}$. Let $*$: $\mathcal{A}_k(z) \rightarrow \mathcal{A}_k(z)$ denote the algebra anti-involution which, given $\rho \in \mathcal{A}_k$, for $i = 1, \dots, k$, interchanges i and i' in ρ . Then $*$ reflects each element of the diagram basis for $\mathcal{A}_k(z)$ in the horizontal axis, and satisfies

$$u^* = u^{-1} \quad (\text{for } u \in \mathfrak{S}_k)$$

and

$$p_i^* = p_i \quad (\text{for } i = 1, \dots, k) \quad \text{and} \quad p_{j+\frac{1}{2}}^* = p_{j+\frac{1}{2}} \quad (\text{for } j = 1, \dots, k - 1).$$

Restricting the map $*$ from $\mathcal{A}_k(z)$ to $\mathcal{A}_{k-\frac{1}{2}}(z)$, gives an algebra anti-involution of $\mathcal{A}_{k-\frac{1}{2}}(z)$ which we also denote by $*$.

3 Jucys–Murphy elements

In this section we recursively define a family of Jucys–Murphy elements in $\mathcal{A}_k(z)$ and $\mathcal{A}_{k+\frac{1}{2}}(z)$. It will be shown in Sect. 5 that the recursive formula given below is equivalent to the combinatorial definition of Jucys–Murphy elements given by Halverson and Ram [3].

Let $(\sigma_i : i = 1, 2, \dots)$ and $(L_i : i = 0, 1, \dots)$ be given by

$$L_0 = 0, \quad L_1 = p_1, \quad \sigma_1 = 1, \quad \text{and} \quad \sigma_2 = s_1,$$

and, for $i = 1, 2, \dots$,

$$L_{i+1} = -s_i L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} + s_i L_i s_i + \sigma_{i+1}, \tag{3.1}$$

where, for $i = 2, 3, \dots$,

$$\begin{aligned} \sigma_{i+1} = & s_{i-1} s_i \sigma_i s_i s_{i-1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\ & - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i. \end{aligned} \tag{3.2}$$

Define $(\sigma_{i+\frac{1}{2}} : i = 1, 2, \dots)$ and $(L_{i+\frac{1}{2}} : i = 0, 1, \dots)$ by

$$L_{\frac{1}{2}} = 0, \quad \sigma_{\frac{1}{2}} = 1, \quad \text{and} \quad \sigma_{1+\frac{1}{2}} = 1,$$

and, for $i = 1, 2, \dots$,

$$L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}} s_i + \sigma_{i+\frac{1}{2}}, \tag{3.3}$$

where, for $i = 2, 3, \dots$,

$$\begin{aligned} \sigma_{i+\frac{1}{2}} = & s_{i-1} s_i \sigma_{i-\frac{1}{2}} s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\ & - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i. \end{aligned} \tag{3.4}$$

Rewriting the last summand in (3.4) as

$$\begin{aligned} s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_i p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\ &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_i p_{i+1} p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\ &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} s_{i-1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\ &= s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1}, \end{aligned}$$

the expression (3.4) becomes

$$\begin{aligned} \sigma_{i+\frac{1}{2}} = & s_{i-1} s_i \sigma_{i-\frac{1}{2}} s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} + s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} \\ & - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ & - s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1}. \end{aligned} \tag{3.5}$$

Using induction, it follows that if $i = 0, 1, \dots$, then $\sigma_{i+\frac{1}{2}} \in \mathcal{A}_{i+\frac{1}{2}}(z)$, and $L_{i+\frac{1}{2}} \in \mathcal{A}_{i+\frac{1}{2}}(z)$. Observe that if $i = 0, 1, \dots$, then $(L_i)^* = L_i$ and $(\sigma_{i+1})^* = \sigma_{i+1}$. The fact that $(L_{i+\frac{1}{2}})^* = L_{i+\frac{1}{2}}$ and $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$ will be shown in Proposition 3.3.

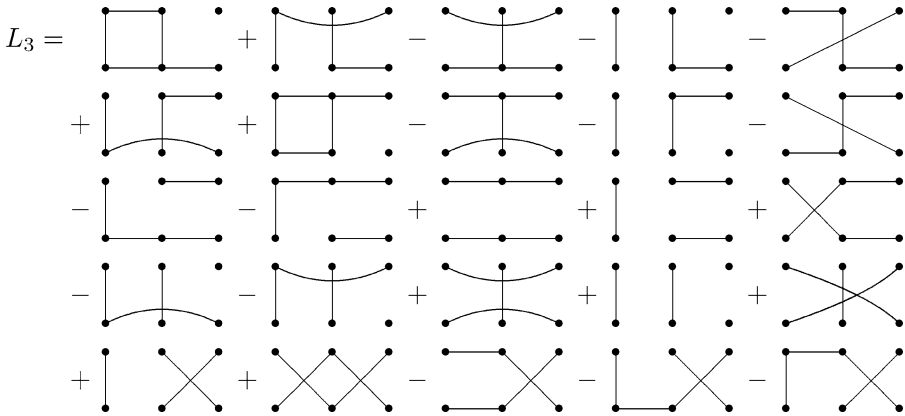
Example 3.1 In terms of the diagram presentation for $\mathcal{A}_3(z)$, we have

$$L_2 = - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \bullet \quad \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

and

$$\sigma_3 = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}$$

Using the above expressions for L_2 and σ_3 , we can write L_3 as



The relations given in the next proposition are fundamental for subsequent calculations.

Proposition 3.2 For $i = 1, 2, \dots$, the following statements hold:

- (1) $\sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$,
- (2) $s_{i+1} \sigma_{i+1} p_{i+\frac{3}{2}} = p_{i+\frac{1}{2}} s_{i+1} \sigma_{i+1}$,
- (3) $\sigma_{i+1} p_i p_{i+\frac{1}{2}} = s_i L_i p_{i+\frac{1}{2}}$,
- (4) $\sigma_{i+1} p_{i+1} p_{i+\frac{1}{2}} = L_i p_{i+\frac{1}{2}}$,
- (5) $p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$.

Proof (1) We consider each of the terms on the right hand side of the definition

$$\begin{aligned} \sigma_{i+1} p_{i+\frac{1}{2}} &= s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} \\ &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &\quad - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &\quad - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}}, \end{aligned}$$

beginning with the argument by induction:

$$s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+\frac{1}{2}} = s_{i-1} s_i \sigma_i p_{i-\frac{1}{2}} s_i s_{i-1} = s_{i-1} s_i p_{i-\frac{1}{2}} s_i s_{i-1} = p_{i+\frac{1}{2}}.$$

From the fact that $p_{i+\frac{1}{2}}$ commutes with L_{i-1} , $s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}} \cdot L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$ and $p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$, while

$$\begin{aligned} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \end{aligned}$$

and, from the relation $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} = p_{i-\frac{1}{2}} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i$, we obtain

$$\begin{aligned}
 p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_{i-1} p_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.
 \end{aligned}$$

Substituting the terms obtained above into the definition of $\sigma_{i+1} p_{i+\frac{1}{2}}$, we observe that all terms vanish except for $p_{i+\frac{1}{2}}$, which completes the proof of (1).

(2) The definition (3.2) gives

$$\begin{aligned}
 s_{i+1} \sigma_{i+1} p_{i+\frac{3}{2}} &= s_{i+1} s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+\frac{3}{2}} + s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} \\
 &\quad + s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} \\
 &\quad - s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} \\
 &\quad - s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}}.
 \end{aligned}$$

Now consider each of the terms on the right hand side of the above equality. Firstly,

$$\begin{aligned}
 s_{i+1} s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+\frac{3}{2}} &= s_{i+1} s_{i-1} s_i s_{i+1} \sigma_i s_{i+1} s_i s_{i-1} p_{i+\frac{3}{2}} \\
 &= s_{i-1} s_{i+1} s_i s_{i+1} \sigma_i s_{i+1} s_i p_{i+\frac{3}{2}} s_{i-1} \\
 &= s_{i-1} s_i s_{i+1} s_i \sigma_i p_{i+\frac{1}{2}} s_{i+1} s_i s_{i-1} \\
 &= s_{i-1} s_i s_{i+1} p_{i-\frac{1}{2}} s_i \sigma_i s_{i+1} s_i s_{i-1} \quad (\text{by induction}) \\
 &= s_{i-1} s_i p_{i-\frac{1}{2}} s_{i+1} s_i \sigma_i s_{i+1} s_i s_{i-1} \\
 &= p_{i+\frac{1}{2}} s_{i-1} s_i s_{i+1} s_i \sigma_i s_{i+1} s_i s_{i-1} \\
 &= p_{i+\frac{1}{2}} s_{i-1} s_{i+1} s_i s_{i+1} \sigma_i s_{i+1} s_i s_{i-1} \\
 &= p_{i+\frac{1}{2}} s_{i+1} s_{i-1} s_i \sigma_i s_i s_{i-1}.
 \end{aligned}$$

Next, use the fact that $s_i p_{i+\frac{3}{2}} s_i$ and $p_{i-\frac{1}{2}}$ commute to observe that

$$\begin{aligned}
 s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} &= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) s_i \\
 &= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i
 \end{aligned}$$

$$\begin{aligned}
 &= s_{i+1}s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i,
 \end{aligned}$$

and that

$$\begin{aligned}
 s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} &= s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} \\
 &= s_{i+1} p_{i-\frac{1}{2}} L_{i-1} (s_i p_{i+\frac{3}{2}} s_i) s_i p_{i-\frac{1}{2}} \\
 &= s_{i+1} p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_{i+1} (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} s_{i+1} s_i s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}.
 \end{aligned}$$

Since $p_{i+\frac{3}{2}}$ commutes with $\mathcal{A}_{i+\frac{1}{2}}(z)$, we see that

$$\begin{aligned}
 s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} &= s_{i+1} s_i p_{i+\frac{3}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} s_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}.
 \end{aligned}$$

For the last term, we use the fact that $(s_i p_{i+\frac{3}{2}} s_i)$ and $p_{i-\frac{1}{2}}$ commute to see that

$$\begin{aligned}
 &s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} \\
 &= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} (s_i p_{i+\frac{3}{2}} s_i) s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} (s_i p_{i+\frac{3}{2}} s_i) p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} (s_i p_{i+\frac{3}{2}} s_i) L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} (s_{i-1} s_i p_{i+\frac{3}{2}} s_i s_{i-1}) s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} (s_{i-1} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{3}{2}} s_i s_{i-1}) s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} (s_{i-1} s_i p_{i+\frac{3}{2}} s_i s_{i-1}) p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= s_{i+1} p_{i-\frac{1}{2}} s_i p_{i+\frac{3}{2}} s_i s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_{i+1} s_i p_{i+\frac{3}{2}} s_i s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i+1} s_i^2 s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= p_{i+\frac{1}{2}} s_{i+1} p_{i-\frac{1}{2}} s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i.
 \end{aligned}$$

Putting the above together, we have

$$\begin{aligned}
 s_{i+1}\sigma_{i+1}p_{i+\frac{3}{2}} &= p_{i+\frac{1}{2}}s_{i+1}s_{i-1}s_i\sigma_i s_i s_{i-1} + p_{i+\frac{1}{2}}s_{i+1}s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
 &\quad + p_{i+\frac{1}{2}}s_{i+1}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}} \\
 &\quad - p_{i+\frac{1}{2}}s_{i+1}s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\
 &\quad - p_{i+\frac{1}{2}}s_{i+1}p_{i-\frac{1}{2}}s_{i-1}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i = p_{i+\frac{1}{2}}s_{i+1}\sigma_{i+1},
 \end{aligned}$$

which completes the proof of (2).

(3) We consider the terms on the right hand side of the equality

$$\begin{aligned}
 \sigma_{i+1}p_i p_{i+\frac{1}{2}} &= s_{i-1}s_i\sigma_i s_i s_{i-1}p_i p_{i+\frac{1}{2}} + s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_i p_{i+\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} - s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} \\
 &\quad - p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_i p_{i+\frac{1}{2}},
 \end{aligned}$$

beginning with

$$\begin{aligned}
 s_{i-1}s_i\sigma_i s_i s_{i-1}p_i p_{i+\frac{1}{2}} &= s_{i-1}s_i\sigma_i p_{i-1}s_i s_{i-1}p_{i+\frac{1}{2}} \\
 &= s_{i-1}s_i\sigma_i p_{i-1}p_{i-\frac{1}{2}}s_i s_{i-1} \\
 &= s_{i-1}s_i s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i s_{i-1} \quad (\text{by induction}) \\
 &= s_i s_{i-1}s_i L_{i-1}s_i s_{i-1}p_{i+\frac{1}{2}} \\
 &= s_i s_{i-1}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}.
 \end{aligned}$$

For the second term, we have

$$\begin{aligned}
 s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_i p_{i+\frac{1}{2}} &= s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+1}s_i p_{i+\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}}L_{i-1}p_i s_i p_{i-\frac{1}{2}}s_i p_{i+\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}},
 \end{aligned}$$

and for the third,

$$\begin{aligned}
 p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} \quad (\text{by induction}) \\
 &= p_{i-\frac{1}{2}}s_i\sigma_i p_{i+\frac{1}{2}} \\
 &= s_i\sigma_i p_{i+\frac{1}{2}} \quad (\text{by item (2)}).
 \end{aligned}$$

Using the relation $p_i p_{i-\frac{1}{2}}p_i = p_i$, we see that the fourth term satisfies

$$s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}} = s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}},$$

while, for the fifth term,

$$\begin{aligned}
 p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_i p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} s_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} p_{i+1} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_i p_{i+1} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &= s_i s_{i-1} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &= s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.
 \end{aligned}$$

Combining the above terms, we obtain

$$\begin{aligned}
 \sigma_{i+1} p_i p_{i+\frac{1}{2}} &= s_i s_{i-1} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + s_i \sigma_i p_{i+\frac{1}{2}} \\
 &\quad - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} - s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} = s_i L_i p_{i+\frac{1}{2}},
 \end{aligned}$$

which completes the proof of (3).

(4) We consider each of the terms on the right hand side of the expression

$$\begin{aligned}
 \sigma_{i+1} p_{i+1} p_{i+\frac{1}{2}} &= s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+1} p_{i+\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &\quad - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &\quad - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}}.
 \end{aligned}$$

Firstly,

$$\begin{aligned}
 s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i+1} p_{i+\frac{1}{2}} &= s_{i-1} s_i \sigma_i p_i s_i s_{i-1} p_{i+\frac{1}{2}} \\
 &= s_{i-1} s_i \sigma_i p_i p_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_{i-1} s_i L_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} \quad (\text{by induction}) \\
 &= s_{i-1} s_i L_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} \\
 &= s_{i-1} L_{i-1} s_{i-1} p_{i+\frac{1}{2}}.
 \end{aligned}$$

For the second term,

$$\begin{aligned}
 s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} &= s_i p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \quad (\text{by induction}) \\
 &= s_i s_i \sigma_i p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \quad (\text{by item (2)}) \\
 &= \sigma_i p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned} &= \sigma_i p_{i+\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}} \\ &= \sigma_i p_{i+\frac{1}{2}}. \end{aligned}$$

For the third term,

$$\begin{aligned} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} &= p_{i-\frac{1}{2}} L_{i-1} p_i s_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}. \end{aligned}$$

For the fourth term, we use the relation $s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} = s_i p_{i-\frac{1}{2}} p_{i-1} \cdot p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$ in

$$\begin{aligned} &s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} p_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \quad (\text{by induction}) \\ &= s_i p_{i-\frac{1}{2}} s_i \sigma_i p_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= s_i s_i \sigma_i p_{i+\frac{1}{2}} p_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \quad (\text{by item (2)}) \\ &= \sigma_i p_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \quad (\text{by item (3)}). \end{aligned}$$

For the final term, we use the relation $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} = p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} s_i$ in

$$\begin{aligned} &p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \quad (\text{by induction}) \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} s_i \sigma_i p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \quad (\text{by item (2)}) \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} s_i \sigma_i p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} p_{i-1} \sigma_i p_{i+\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} \quad (\text{by item (3)}). \end{aligned}$$

Putting the above together,

$$\begin{aligned} \sigma_{i+1} p_{i+1} p_{i+\frac{1}{2}} &= s_{i-1} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} + \sigma_i p_{i+\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\ &\quad - s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} = L_i p_{i+\frac{1}{2}}, \end{aligned}$$

which proves (4).

(5) Parts (1) and (3) give

$$p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} s_i L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_{i+1} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}},$$

as required. □

Proposition 3.3 *If $i = 1, 2, \dots$, then*

- (1) $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$,
- (2) $(L_{i+\frac{1}{2}})^* = L_{i+\frac{1}{2}}$.

Proof (1) We show that the summand $p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}$ in (3.4) is fixed under the $*$ anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$, using Proposition 3.2, as follows:

$$\begin{aligned} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} &= p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_i s_{i-1} p_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_i p_{i-1} p_{i-\frac{1}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}}. \end{aligned}$$

(2) Given that $(\sigma_{i+\frac{1}{2}})^* = \sigma_{i+\frac{1}{2}}$, it suffices to show that the summand $p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}}$ in (3.3) is fixed under the $*$ anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$. Using Proposition 3.2,

$$\begin{aligned} p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} &= p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+1} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i+1} s_i L_i p_{i+\frac{1}{2}} \\ &= p_{i+\frac{1}{2}} s_i p_i L_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_i L_i p_{i+\frac{1}{2}}, \end{aligned}$$

gives the required result. □

Proposition 3.4 *If $i = 1, 2, \dots$, then $\sigma_{i+\frac{1}{2}} s_i = s_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1}$.*

Proof After checking the case $i = 1$, the statement follows from induction and the equality

$$p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}},$$

which was established in the proof of Proposition 3.3. □

The following observation is made for later reference.

Lemma 3.5 *If $i = 1, 2, \dots$, then*

$$s_i s_{i+1} \sigma_{i+1} s_{i+1} s_i p_{i+\frac{1}{2}} = s_{i+1} p_{i+\frac{1}{2}} L_i s_i p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} s_{i+1} p_{i+\frac{1}{2}}.$$

Proof On the one hand,

$$\begin{aligned} s_i s_{i+1} \sigma_{i+1} s_{i+1} s_i p_{i+\frac{1}{2}} &= s_i s_{i+1} \sigma_{i+1} s_{i+1} p_{i+\frac{1}{2}} = s_i s_{i+1} p_{i+\frac{3}{2}} \sigma_{i+1} s_{i+1} \\ &= s_i p_{i+\frac{3}{2}} \sigma_{i+1} s_{i+1} = s_i \sigma_{i+1} s_{i+1} p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} s_{i+1} p_{i+\frac{1}{2}}, \end{aligned}$$

and on the other,

$$\begin{aligned} s_{i+1} p_{i+\frac{1}{2}} L_i s_i p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} &= s_{i+1} p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+1} s_i p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_{i+1} p_{i+\frac{1}{2}} p_{i+1} \sigma_{i+\frac{1}{2}} p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} = s_{i+1} p_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{3}{2}} \sigma_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_i p_{i+\frac{3}{2}} s_i s_{i+1} p_{i+1} p_{i+\frac{3}{2}} \sigma_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} = s_i p_{i+\frac{3}{2}} s_i p_{i+2} p_{i+\frac{3}{2}} \sigma_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_i p_{i+\frac{3}{2}} s_i \sigma_{i+\frac{1}{2}} p_{i+2} p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} = s_i p_{i+\frac{3}{2}} \sigma_{i+1} p_{i+2} p_{i+\frac{3}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_i p_{i+\frac{3}{2}} \sigma_{i+1} s_{i+1} p_{i+1} p_{i+\frac{1}{2}} = s_i \sigma_{i+1} s_{i+1} p_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= s_i \sigma_{i+1} s_{i+1} p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}} s_{i+1} p_{i+\frac{1}{2}}, \end{aligned}$$

as required. □

We are now in a position to prove the first commutativity result of this paper.

Theorem 3.6 *The elements σ_{i+1} and $\sigma_{i+\frac{1}{2}}$ satisfy the following commutativity relations:*

- (1) $\sigma_{i+1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i+1} = p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}$ for $i = 2, 3, \dots$
- (2) $\sigma_{i+1} p_{i-1} = p_{i-1} \sigma_{i+1} = s_{i-1} \sigma_i p_{i+1} s_i s_{i-1}$ for $i = 2, 3, \dots$
- (3) $\sigma_{i+1} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}} \sigma_{i+1}$ for $i = 3, 4, \dots$
- (4) $\sigma_{i+1} s_{i-2} = s_{i-2} \sigma_{i+1}$ for $i = 3, 4, \dots$
- (5) $\sigma_{i+1} p_{i-2} = p_{i-2} \sigma_{i+1}$ for $i = 3, 4, \dots$
- (6) $\sigma_{i+\frac{1}{2}} p_{i-1} = p_{i-1} \sigma_{i+\frac{1}{2}}$ for $i = 2, 3, \dots$
- (7) $\sigma_{i+\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}} \sigma_{i+\frac{1}{2}}$ for $i = 3, 4, \dots$
- (8) $\sigma_{i+\frac{1}{2}} s_{i-2} = s_{i-2} \sigma_{i+\frac{1}{2}}$ for $i = 3, 4, \dots$
- (9) $\sigma_{i+\frac{1}{2}} p_{i-2} = p_{i-2} \sigma_{i+\frac{1}{2}}$ for $i = 3, 4, \dots$

Proof (1) Using Lemma 3.5,

$$\begin{aligned} \sigma_{i+1} p_{i-\frac{1}{2}} &= -s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \\ &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_{i-1} s_i \sigma_i s_{i-1} p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= -p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \\
 &= -p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= -p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= -p_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= -p_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
 &\quad + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + s_i p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= -p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i+1},
 \end{aligned}$$

as required.

(2) We first show that

- (i) $s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1}$,
- (ii) $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} = p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1}$,
- (iii) $s_{i-1} s_i \sigma_i s_i s_{i-1} p_{i-1} = s_{i-1} s_i \sigma_i p_{i+1} s_i s_{i-1} = p_{i-1} s_{i-1} s_i \sigma_i s_i s_{i-1}$.

The left hand side of (i) gives

$$\begin{aligned}
 s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-1} &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1},
 \end{aligned}$$

which is the right hand side of (i). Using the relation $p_{i+\frac{1}{2}} s_{i-1} s_i = s_{i-1} s_i p_{i-\frac{1}{2}}$, the left hand side of (ii) gives

$$\begin{aligned}
 p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-1} s_i \\
 &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i s_{i-1} s_i \\
 &= p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i p_{i-1} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} p_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1},
 \end{aligned}$$

as required. The item (iii) follows from the relation $s_{i-1}s_i p_{i+1} = p_{i-1}s_{i-1}s_i$. Next, using the definition (3.2),

$$\begin{aligned} \sigma_{i+1} p_{i-1} &= s_{i-1}s_i \sigma_i s_i s_{i-1} p_{i-1} + s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1} + p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1} \\ &\quad - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-1} - p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} \\ &= s_{i-1}s_i \sigma_i s_i s_{i-1} p_{i-1}. \end{aligned}$$

Since the right hand side of the last expression is manifestly fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$, the proof of (2) is complete.

(3) We first show that

- (i) $s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} \cdot p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}},$
- (ii) $s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} \cdot s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}},$
- (iii) $s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}},$
- (iv) $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} \cdot s_i p_{i-\frac{3}{2}},$
- (v) $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} s_{i-2} \cdot p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}},$
- (vi) $p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} \cdot p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}},$
- (vii) $p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}},$
- (viii) $p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}},$
- (ix) $s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}},$
- (x) $s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}}.$
- (xi) $s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} = s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{3}{2}} \cdot p_{i-\frac{1}{2}}.$
- (xii) $s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-1} \sigma_{i-\frac{3}{2}} \cdot s_{i-1}.$

For the left hand side of (i),

$$\begin{aligned} & s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\ &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}, \end{aligned}$$

and for the right hand side of (i),

$$\begin{aligned} & s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\ &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} s_{i-2} p_{i-\frac{1}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}.
 \end{aligned}$$

For the left hand side of (ii),

$$\begin{aligned}
 &s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} s_{i-2} p_{i-\frac{1}{2}} L_{i-2} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}},
 \end{aligned}$$

and for the right hand side of (ii),

$$\begin{aligned}
 &s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} \\
 &= s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.
 \end{aligned}$$

For the left hand side of (iii),

$$\begin{aligned}
 &s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= s_i \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = \sigma_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= \sigma_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-\frac{1}{2}},
 \end{aligned}$$

and for the right hand side of (iii),

$$\begin{aligned} & s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\ &= s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} \\ &= s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} = s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-\frac{1}{2}}. \end{aligned}$$

For the left hand side of (iv),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}}, \end{aligned}$$

and, for the right hand side of (iv),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}}. \end{aligned}$$

For the left hand side of (v),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i, \end{aligned}$$

and, for the right hand side of (v),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i. \end{aligned}$$

For the left hand side of (vi),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} p_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{3}{2}} s_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \end{aligned}$$

$$\begin{aligned}
 &= s_i s_{i-1} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_i L_{i-2} p_{i-\frac{3}{2}} s_i \\
 &= s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}},
 \end{aligned}$$

and, for the right hand side of (vi),

$$\begin{aligned}
 & s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} s_i p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} s_i \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} s_i \\
 &= s_{i-1} s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_i = s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}}.
 \end{aligned}$$

For the left hand side of (vii)

$$p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}}$$

and for the right hand side of (vii)

$$\begin{aligned}
 p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} &= p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}}.
 \end{aligned}$$

For the left hand side of (viii),

$$\begin{aligned}
 p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} &= p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}},
 \end{aligned}$$

and for the right hand side of (viii),

$$\begin{aligned}
 p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} &= p_{i-\frac{1}{2}} \sigma_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.
 \end{aligned}$$

For the left hand side of (ix),

$$\begin{aligned}
 s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i
 \end{aligned}$$

and for the right hand side of (ix),

$$s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i.$$

For the left hand side of (x),

$$s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} = s_i p_{i-\frac{1}{2}} \sigma_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}},$$

and for the right hand side of (x),

$$\begin{aligned}
 s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i - 2 s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i - 2 p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}.
 \end{aligned}$$

Starting with the left hand side of (xi),

$$\begin{aligned}
 & s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i-2} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{3}{2}} p_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} s_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i \sigma_{i-1} s_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-\frac{1}{2}},
 \end{aligned}$$

as required. Considering the left hand side of (xii),

$$\begin{aligned}
 & s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-1} s_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-2} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} p_{i-2} p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} s_{i-2} p_{i-1} \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} \\
 &= s_{i-1} s_i s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1} p_{i-\frac{3}{2}} \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} \\
 &= p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-1} \sigma_{i-\frac{3}{2}} s_{i-1},
 \end{aligned}$$

as required. Now, after substituting the expressions

$$L_{i-1} = -p_{i-\frac{3}{2}}L_{i-2}s_{i-2} - s_{i-2}L_{i-2}p_{i-\frac{3}{2}} + p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}} + s_{i-2}L_{i-2}s_{i-2} + \sigma_{i-1}$$

and

$$\sigma_i = s_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2} + s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}}s_{i-1} + p_{i-\frac{3}{2}}L_{i-2}s_{i-1}p_{i-\frac{3}{2}} - s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{3}{2}} - p_{i-\frac{3}{2}}p_{i-1}p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}$$

into the definition (3.2), and simplifying the resulting expression using items (i)–(x), we obtain the equality

$$\begin{aligned} \sigma_{i+1}p_{i-\frac{3}{2}} &= s_{i-1}s_i s_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_i s_{i-1}p_{i-\frac{3}{2}} \\ &+ p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_i p_{i-\frac{1}{2}} \\ &+ s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_i p_{i-\frac{1}{2}}s_i \\ &- s_{i-1}s_i s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{3}{2}}s_i s_{i-1}p_{i-\frac{3}{2}} \\ &- s_{i-1}s_i p_{i-\frac{3}{2}}p_{i-1}p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}s_i s_{i-1}p_{i-\frac{3}{2}}, \end{aligned}$$

in which the two terms with negative coefficients survive from the expansion of $s_{i-1}s_i\sigma_i s_{i-1}p_{i-\frac{3}{2}}$. By items (xi) and (xii), the two terms with negative coefficients in the last expression are interchanged by the $*$ map on $\mathcal{A}_{i+1}(z)$. Since

$$s_{i-1}s_i s_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_i s_{i-1}p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}s_{i-1}s_i s_{i-2}s_{i-1}\sigma_{i-1}s_{i-1}s_{i-2}s_i s_{i-1},$$

the right hand side of the above expression for $\sigma_{i+1}p_{i-\frac{3}{2}}$ is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$. This completes the proof of (3).

(4) We show that after substituting the expression

$$L_{i-1} = -p_{i-\frac{3}{2}}L_{i-2}s_{i-2} - s_{i-2}L_{i-2}p_{i-\frac{3}{2}} + p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}} + s_{i-2}L_{i-2}s_{i-2} + \sigma_{i-1}$$

into the definition (3.2), conjugation by s_{i-2} permutes the summands of σ_{i+1} as follows:

- (i) $(s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})^{s_{i-2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}} \cdot p_i p_{i-\frac{1}{2}},$
- (ii) $(s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})^{s_{i-2}} = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}p_{i-\frac{3}{2}}L_{i-2} \cdot s_{i-2}p_{i-\frac{1}{2}}s_i,$
- (iii) $(s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}p_{i-1}p_{i-\frac{3}{2}}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})^{s_{i-2}} = s_i p_{i-\frac{1}{2}}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}s_i \cdot p_{i-\frac{1}{2}}s_i,$
- (iv) $(s_i p_{i-\frac{1}{2}}s_{i-2}L_{i-2}s_{i-2}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}})^{s_{i-2}} = s_{i-1}s_i s_{i-1}p_{i-\frac{3}{2}}L_{i-2}s_{i-2}p_{i-\frac{1}{2}} \cdot p_{i-1}p_{i-\frac{3}{2}}s_i s_{i-1},$

- (v) $(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}},$
- (vi) $(p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}})^{s_i-2} = p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}},$
- (vii) $(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i)^{s_i-2} = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1},$
- (viii) $(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i)^{s_i-2} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}},$
- (ix) $(s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1})^{s_i-2} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}},$
- (x) $(s_{i-1} s_i s_{i-2} s_{i-1} \sigma_{i-1} s_{i-1} s_{i-2} s_i s_{i-1})^{s_i-2} = s_{i-1} s_i s_{i-2} s_{i-1} \sigma_{i-1} s_{i-1} s_{i-2} s_i s_{i-1}.$

The item (i) follows from the relation $s_{i-2} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}}$ and

$$\begin{aligned} & s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\ &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}. \end{aligned}$$

For the left hand side of (ii),

$$\begin{aligned} & (s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\ &= s_{i-2} s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= s_{i-2} s_i p_{i-\frac{1}{2}} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= s_{i-2} s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= p_{i-\frac{1}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \sigma_{i-1} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= p_{i-\frac{1}{2}} s_i s_{i-1} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} s_{i-2} \\ &= p_{i-\frac{1}{2}} s_i \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2}, \end{aligned}$$

and, for the right hand side of (ii),

$$\begin{aligned} & p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-2} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} p_{i-2} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} p_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} p_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\ &= p_{i-\frac{1}{2}} p_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \end{aligned}$$

$$\begin{aligned}
 &= p_{i-\frac{1}{2}} p_{i-1} s_i - 2s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} p_{i-1} s_i - 2s_{i-1} p_{i+\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i \\
 &= p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} s_i - 2s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i = p_{i-\frac{1}{2}} s_i - 2s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-1} s_i \\
 &= p_{i-\frac{1}{2}} s_i - 2s_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} \sigma_{i-1} = p_{i-\frac{1}{2}} s_i - 2s_i p_{i-\frac{1}{2}} \sigma_{i-1} = p_{i-\frac{1}{2}} s_i \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2},
 \end{aligned}$$

where the last equality follows from $s_{i-2} p_{i-\frac{1}{2}} \sigma_{i-1} = \sigma_{i-1} p_{i-\frac{1}{2}} s_{i-2}$.

For the left hand side of (iii),

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} p_{i-1} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i p_{i-1} p_{i-\frac{3}{2}} p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} p_i p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{3}{2}} L_{i-2} p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-2} \\
 &= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-2} \\
 &= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2},
 \end{aligned}$$

and for the right hand side of (iii),

$$\begin{aligned}
 &s_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} L_{i-2} s_i - 2s_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} s_i p_{i-\frac{3}{2}} L_{i-2} s_i - 2p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_i - 2p_{i-\frac{1}{2}} s_i = p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i = p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i \\
 &= p_{i-\frac{3}{2}} p_{i-1} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} s_i
 \end{aligned}$$

$$\begin{aligned}
 &= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} s_i \\
 &= p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i s_{i-2} s_i = p_{i-\frac{3}{2}} p_{i-1} p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_{i-2}.
 \end{aligned}$$

From the left hand side of (iv), we obtain

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\
 &= (s_i p_{i-\frac{1}{2}} s_{i-2} s_{i-1} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\
 &= (s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} s_{i-2} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i s_{i-2} p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_{i-1} p_{i+\frac{1}{2}} p_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} \\
 &= s_i s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{3}{2}} s_{i-1} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{3}{2}} s_{i-1} \\
 &= s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-2} p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1}
 \end{aligned}$$

which is identical to the right hand side of (iv).

From the left hand side of (v), we obtain

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^{s_i-2} \\
 &= s_{i-2} s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_{i-2} \\
 &= s_i \sigma_{i-\frac{3}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-2} p_{i-\frac{1}{2}} p_{i-\frac{3}{2}} \\
 &= p_{i-\frac{1}{2}} \sigma_{i-1} p_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_{i-2} L_{i-2} p_{i-\frac{3}{2}} s_i p_{i-\frac{1}{2}},
 \end{aligned}$$

which is identical to the right hand side of (v).

The item (vi) follows immediately from the relation $s_{i-2} p_{i-\frac{3}{2}} = p_{i-\frac{3}{2}} s_{i-2} = p_{i-\frac{3}{2}}$.

From the left hand side of (vii), we obtain

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i)^{s_i-2} \\
 &= s_{i-2} s_i s_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_{i-2} s_i p_{i-\frac{1}{2}} s_i s_{i-2}
 \end{aligned}$$

$$\begin{aligned}
 &= s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_i - 2 s_i p_{i-\frac{1}{2}} s_i s_{i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_i - 2 s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} \\
 &= s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} L_{i-2} s_i s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i = s_i s_{i-1} p_{i-\frac{3}{2}} s_i L_{i-2} s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} s_i \\
 &= s_i s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1} s_i = s_{i-1} s_i s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_{i-1} s_i s_{i-1},
 \end{aligned}$$

which is identical to the right hand side of (vii).

From the left hand side of (viii),

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}} s_i)^{s_i-2} \\
 &= s_{i-2} s_i p_{i-\frac{1}{2}} \sigma_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} \\
 &= s_{i-2} s_i \sigma_{i-1} s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} = \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_{i-\frac{3}{2}} p_{i+\frac{1}{2}} s_{i-1} s_{i-2} \\
 &= \sigma_{i-\frac{3}{2}} s_i s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{3}{2}} s_{i-1} s_{i-2} = \sigma_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-1} s_{i-1} s_{i-2} p_{i-\frac{1}{2}} \\
 &= \sigma_{i-\frac{3}{2}} p_{i-\frac{1}{2}} s_i s_{i-2} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i-\frac{3}{2}} s_{i-2} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} \sigma_{i-1} s_i p_{i-\frac{1}{2}},
 \end{aligned}$$

which is identical to the right hand side of (viii).

From the left hand side of (ix),

$$\begin{aligned}
 &(s_{i-1} s_i p_{i-\frac{3}{2}} L_{i-2} s_{i-1} p_{i-\frac{3}{2}} s_i s_{i-1})^{s_i-2} \\
 &= s_{i-2} s_{i-1} p_{i-\frac{3}{2}} L_{i-2} s_i s_{i-1} s_i p_{i-\frac{3}{2}} s_{i-1} s_{i-2} \\
 &= p_{i-\frac{1}{2}} s_{i-2} s_{i-1} L_{i-2} s_i s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_{i-1} s_i s_{i-1} s_i s_{i-1} s_{i-2} p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_{i-2} L_{i-2} s_i s_{i-2} p_{i-\frac{1}{2}},
 \end{aligned}$$

which is identical to the right hand side of (ix).

The equality (x) follows from the fact that $s_{i-2} s_{i-1} s_i s_{i-2} s_{i-1} = s_{i-1} s_i s_{i-2} s_{i-1} s_i$.

(5) By (2) and (4), σ_{i+1} commutes with $\langle s_{i-2}, p_{i-1} \rangle$, and so with $p_{i-2} = s_{i-2} p_{i-1} s_{i-2}$.

(6) By Proposition 3.4, and (2),

$$\sigma_{i+\frac{1}{2}} p_{i-1} = s_i \sigma_{i+1} p_{i-1} = p_{i-1} s_i \sigma_{i+1} = p_{i-1} \sigma_{i+\frac{1}{2}}.$$

(7)–(9) Can be proved using the same argument as part (6). □

Theorem 3.7 *The elements L_{i+1} and $L_{i+\frac{1}{2}}$ satisfy the following commutation relations:*

- (1) $L_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} L_{i+1} = p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}}$, for $i = 1, 2, \dots$
- (2) $L_{i+1} p_i = p_i L_{i+1}$, for $i = 1, 2, \dots$

- (3) $L_{i+1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i+1}$, for $i = 2, 3, \dots$
- (4) $L_{i+1}s_{i-1} = s_{i-1}L_{i+1}$, for $i = 2, 3, \dots$
- (5) $L_{i+1}p_{i-1} = p_{i-1}L_{i+1}$, for $i = 2, 3, \dots$
- (6) $L_{i+\frac{1}{2}}p_i = p_iL_{i+\frac{1}{2}}$, for $i = 1, 2, \dots$
- (7) $L_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i+\frac{1}{2}}$, for $i = 2, 3, \dots$
- (8) $L_{i+\frac{1}{2}}s_{i-1} = s_{i-1}L_{i+\frac{1}{2}}$, for $i = 2, 3, \dots$
- (9) $L_{i+\frac{1}{2}}p_{i-1} = p_{i-1}L_{i+\frac{1}{2}}$, for $i = 2, 3, \dots$

Proof (1) Using Proposition 3.2,

$$\begin{aligned} L_{i+1}p_{i+\frac{1}{2}} &= -s_iL_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}L_i s_i p_{i+\frac{1}{2}} \\ &\quad + p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} + s_iL_i s_i p_{i+\frac{1}{2}} + \sigma_{i+1}p_{i+\frac{1}{2}} \\ &= -s_iL_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} + s_iL_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} \\ &= -s_iL_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} + s_iL_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} \\ &= p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}}, \end{aligned}$$

as required.

(2) From Proposition 3.2, we obtain $\sigma_{i+1}p_i = \sigma_{i+1}p_i p_{i+\frac{1}{2}} p_i = s_iL_i p_{i+\frac{1}{2}} p_i$. Thus

$$\begin{aligned} L_{i+1}p_i &= -s_iL_i p_{i+\frac{1}{2}} p_i - p_{i+\frac{1}{2}}L_i s_i p_i + p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} p_i + s_iL_i s_i p_i + \sigma_{i+1}p_i \\ &= (\sigma_{i+1}p_i - s_iL_i p_{i+\frac{1}{2}} p_i) + (p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} p_i - p_{i+\frac{1}{2}}L_i s_i p_i) \\ &\quad + s_iL_i p_{i+1} s_i \\ &= (\sigma_{i+1}p_i - \sigma_{i+1}p_i) + (p_{i+\frac{1}{2}}L_i s_i p_i - p_{i+\frac{1}{2}}L_i s_i p_i) + s_iL_i p_{i+1} s_i, \end{aligned}$$

and the statement now follows from the fact that the right hand side of the above expression is fixed under the $*$ anti-involution on $\mathcal{A}_{i+1}(z)$.

(3) We first show that

- (i) $p_{i+\frac{1}{2}}L_i s_i p_{i-\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_i s_i$,
- (ii) $s_iL_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}}L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,
- (iii) $p_{i+\frac{1}{2}}L_i p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}}L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,
- (iv) $\sigma_{i+1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}\sigma_{i+1}$.

(i) The definition (3.1) gives

$$\begin{aligned} p_{i+\frac{1}{2}}L_i s_i p_{i-\frac{1}{2}} &= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}s_i p_{i-\frac{1}{2}} \\ &\quad + p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}} \\ &\quad + p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}s_i p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}}\sigma_i s_i p_{i-\frac{1}{2}} \\ &= -p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i+\frac{1}{2}}s_{i-1}s_i \end{aligned}$$

$$\begin{aligned}
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_i + p_{i+\frac{1}{2}} \sigma_i s_i p_{i-\frac{1}{2}} \\
 = &- p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i \\
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i + p_{i+\frac{1}{2}} \sigma_i s_i p_{i-\frac{1}{2}} \\
 = &p_{i+\frac{1}{2}} \sigma_i s_i,
 \end{aligned}$$

where the last equality follows from Proposition 3.2.

(ii) The definition (3.1) gives

$$\begin{aligned}
 s_i L_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = &- s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} - s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &+ s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &+ s_i s_{i-1} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} + s_i \sigma_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 = &- s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} - s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &+ s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + s_i s_{i-1} L_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 = &s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}
 \end{aligned}$$

since $p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$.

(iii) The definition (3.1) gives

$$\begin{aligned}
 p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} = &- p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &- p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} \sigma_i p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 = &- p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &- p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 = &- p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}}.
 \end{aligned}$$

(iv) was demonstrated in Theorem 3.6.

Now, using

$$\begin{aligned}
 s_i L_{i-1} s_{i-1} p_{i-\frac{1}{2}} &= -s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} \\
 &+ s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} \\
 &+ s_i s_{i-1} L_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} + s_i \sigma_i s_i p_{i-\frac{1}{2}} \\
 &= -s_i s_{i-1} L_{i-1} p_{i-\frac{1}{2}} - p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i \\
 &+ s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \\
 &+ s_i s_{i-1} L_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}} \sigma_i s_i,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 L_{i+1} p_{i-\frac{1}{2}} &= -s_i L_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} - p_{i+\frac{1}{2}} L_i s_i p_{i-\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &+ s_i L_i s_i p_{i-\frac{1}{2}} + \sigma_{i+1} p_{i-\frac{1}{2}} \\
 &= -s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} \sigma_i s_i \\
 &+ p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\
 &+ s_i L_i s_i p_{i-\frac{1}{2}} + \sigma_{i+1} p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} + s_i s_{i-1} L_{i-1} s_{i-1} s_i p_{i-\frac{1}{2}} + \sigma_{i+1} p_{i-\frac{1}{2}}.
 \end{aligned}$$

Since the right hand side of the last expression is fixed under the * anti-involution on $A_{i+1}(z)$, the proof of (3) is complete.

(4) We show that after substituting the expression

$$L_i = -p_{i-\frac{1}{2}} L_{i-1} s_{i-1} - s_{i-1} L_{i-1} p_{i-\frac{1}{2}} + p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} + s_{i-1} L_{i-1} s_{i-1} + \sigma_i$$

into the definition (3.1), conjugation by s_{i-1} permutes the summands of L_{i+1} as follows:

- (i) $s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i s_{i-1} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i$,
- (ii) $s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}}$,
- (iii) $s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} s_i s_{i-1} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+1} p_{i+\frac{1}{2}}$,
- (iv) $s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1} s_i s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i$,
- (v) $s_{i-1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}$,
- (vi) $s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} s_{i-1} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}}$,
- (vii) $s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} s_i$,

- (viii) $s_{i-1} p_{i+\frac{1}{2}} \sigma_i p_{i+1} p_{i+\frac{1}{2}} s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i,$
- (ix) $s_{i-1} s_i s_{i-1} L_{i-1} s_{i-1} s_i s_{i-1} = s_i s_{i-1} L_{i-1} s_{i-1} s_i,$
- (x) $s_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_{i-1} = p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}.$

For item (i),

$$s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i s_{i-1} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i s_{i-1} s_i = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i.$$

For item (ii),

$$\begin{aligned} & s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}}. \end{aligned}$$

For item (iii),

$$\begin{aligned} & s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} s_i s_{i-1} \\ &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i s_i s_{i-1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+1} p_{i+\frac{1}{2}}. \end{aligned}$$

For item (iv),

$$\begin{aligned} & s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1} s_i s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} s_{i-1} s_i s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i s_{i-1} s_i s_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} s_i. \end{aligned}$$

For the left hand side of item (v),

$$s_{i-1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} = s_{i-1} \sigma_i s_i p_{i-\frac{1}{2}} s_{i-1} = \sigma_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}},$$

and for the right hand side of item (v),

$$\begin{aligned} & s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} \\ &= s_i p_{i-\frac{1}{2}} s_i \sigma_i s_{i-1} p_{i-1} s_i p_{i-\frac{1}{2}} \\ &= s_i s_i \sigma_i p_{i+\frac{1}{2}} s_{i-1} s_i p_{i-1} p_{i-\frac{1}{2}} = \sigma_i s_{i-1} s_i p_{i-\frac{1}{2}} p_{i-1} p_{i-\frac{1}{2}} = \sigma_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}}. \end{aligned}$$

The item (vi) follows from the relation $s_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}$. For item (vii),

$$\begin{aligned} & s_{i-1}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}p_{i+1}p_{i+\frac{1}{2}}s_{i-1} \\ &= s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i+1}s_i p_{i-\frac{1}{2}}s_i = s_i p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}s_i. \end{aligned}$$

For item (viii),

$$\begin{aligned} & s_{i-1}p_{i+\frac{1}{2}}\sigma_i p_{i+1}p_{i+\frac{1}{2}}s_{i-1} \\ &= s_{i-1}\sigma_i s_i p_{i-\frac{1}{2}}s_i p_{i+1}p_{i+\frac{1}{2}}s_{i-1} = \sigma_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1} \\ &= \sigma_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}}p_i s_{i-1}s_i p_{i-\frac{1}{2}}s_i = \sigma_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}}p_{i-1}s_i p_{i-\frac{1}{2}}s_i \\ &= \sigma_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i \\ &= \sigma_{i-\frac{1}{2}}s_i p_{i-\frac{1}{2}}s_i p_{i-1}p_{i-\frac{1}{2}}s_i = \sigma_{i-\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}}s_{i-1}p_{i-1}p_{i-\frac{1}{2}}s_i \\ &= s_{i-1}p_{i+\frac{1}{2}}s_{i-1}\sigma_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}s_i \\ &= s_i p_{i-\frac{1}{2}}s_i \sigma_{i-\frac{1}{2}}p_{i-1}p_{i-\frac{1}{2}}s_i = s_i p_{i-\frac{1}{2}}s_i L_{i-1}p_{i-\frac{1}{2}}s_i. \end{aligned}$$

(ix) follows from the Coxeter relations, and (x) from the relation $s_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}s_{i-1} = p_{i-\frac{1}{2}}$.

(5) By parts (2) and (4), L_{i+1} commutes with $\langle p_i, s_{i-1} \rangle$, and so with $p_{i-1} = s_{i-1}p_i s_{i-1}$.

(6) From item (3) of Proposition 3.2, $\sigma_{i+\frac{1}{2}}p_i = s_i \sigma_{i+1} p_i p_{i+\frac{1}{2}} p_i = L_i p_{i+\frac{1}{2}} p_i$, and

$$\begin{aligned} L_{i+\frac{1}{2}}p_i &= -p_{i+\frac{1}{2}}L_i p_i - L_i p_{i+\frac{1}{2}}p_i + p_{i+\frac{1}{2}}L_i p_i p_{i+\frac{1}{2}}p_i + s_i L_{i-\frac{1}{2}}s_i p_i + \sigma_{i+\frac{1}{2}}p_i \\ &= -p_{i+\frac{1}{2}}L_i p_i - L_i p_{i+\frac{1}{2}}p_i + p_{i+\frac{1}{2}}L_i p_i + s_i L_{i-\frac{1}{2}}p_{i+1}s_i + L_i p_{i+\frac{1}{2}}p_i \\ &= s_i L_{i-\frac{1}{2}}p_{i+1}s_i. \end{aligned}$$

Since $s_i L_{i-\frac{1}{2}}p_{i+1}s_i = s_i p_{i+1}L_{i-\frac{1}{2}}s_i$, this completes the proof of (6).

(7) We show that

$$\begin{aligned} L_{i+\frac{1}{2}}p_{i-\frac{1}{2}} &= -p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}} \\ &\quad + s_i s_{i-1}L_{i-\frac{3}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i + s_i s_{i-1}\sigma_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}s_i. \end{aligned} \tag{3.6}$$

From the definition (3.3),

$$L_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = (-L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}L_i + p_{i+\frac{1}{2}}L_i p_i p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}}s_i + \sigma_{i+\frac{1}{2}})p_{i-\frac{1}{2}}. \tag{3.7}$$

Using part (1) the first three summands in the right hand side of (3.7) are transformed as

$$L_i p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}L_{i-1}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}} \quad \text{and}$$

$$P_{i+\frac{1}{2}}L_iP_{i-\frac{1}{2}} = P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}P_iP_{i-\frac{1}{2}},$$

and

$$\begin{aligned} P_{i+\frac{1}{2}}L_iP_iP_{i+\frac{1}{2}}P_{i-\frac{1}{2}} &= P_{i+\frac{1}{2}}P_iL_iP_{i-\frac{1}{2}}P_{i+\frac{1}{2}} = P_{i+\frac{1}{2}}P_iP_{i-\frac{1}{2}}L_{i-1}P_iP_{i-\frac{1}{2}}P_{i+\frac{1}{2}} \\ &= P_{i+\frac{1}{2}}P_iL_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} = L_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}. \end{aligned}$$

Next,

$$\begin{aligned} s_iL_{i-\frac{1}{2}}s_iP_{i-\frac{1}{2}} &= -s_iL_{i-1}P_{i-\frac{1}{2}}s_iP_{i-\frac{1}{2}} - s_iP_{i-\frac{1}{2}}L_{i-1}s_iP_{i-\frac{1}{2}} \\ &\quad + s_iP_{i-\frac{1}{2}}L_{i-1}P_{i-1}P_{i-\frac{1}{2}}s_iP_{i-\frac{1}{2}} \\ &\quad + s_i s_{i-1}L_{i-\frac{3}{2}}s_{i-1}s_iP_{i-\frac{1}{2}} + s_i\sigma_{i-\frac{1}{2}}s_iP_{i-\frac{1}{2}} \\ &= -s_iL_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} - s_iP_{i-\frac{1}{2}}L_{i-1}s_iP_{i-\frac{1}{2}} \\ &\quad + s_iP_{i-\frac{1}{2}}L_{i-1}P_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} \\ &\quad + s_i s_{i-1}L_{i-\frac{3}{2}}s_{i-1}s_iP_{i-\frac{1}{2}} + s_i s_{i-1}\sigma_{i-\frac{1}{2}}s_{i-1}s_iP_{i-\frac{1}{2}} \\ &= -L_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} - s_iP_{i-\frac{1}{2}}L_{i-1}s_iP_{i-\frac{1}{2}} + P_{i-\frac{1}{2}}L_{i-1}P_{i-1}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} \\ &\quad + s_i s_{i-1}L_{i-\frac{3}{2}}P_{i+\frac{1}{2}}s_{i-1}s_i + s_i s_{i-1}\sigma_{i-\frac{1}{2}}P_{i+\frac{1}{2}}s_{i-1}s_i. \end{aligned}$$

Substituting each of the above into (3.7), and using part (1) of Theorem 3.6 gives (3.6). The right hand side of (3.6) being fixed under the * anti-involution on $\mathcal{A}_{i+\frac{1}{2}}(z)$, the proof of (7) is complete.

(8) We show that, after substituting the expression

$$L_{i-\frac{1}{2}} = -P_{i-\frac{1}{2}}L_{i-1} - L_{i-1}P_{i-\frac{1}{2}} + P_{i-\frac{1}{2}}L_{i-1}P_{i-1}P_{i-\frac{1}{2}} + s_{i-1}L_{i-\frac{1}{2}}s_{i-1} + \sigma_{i-\frac{1}{2}}$$

into the definition (3.3), conjugation by s_{i-1} permutes the summands of $L_{i+\frac{1}{2}}$ as follows:

- (i) $s_{i-1}(P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}s_{i-1})s_{i-1} = P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}P_iP_{i-\frac{1}{2}}P_iP_{i+\frac{1}{2}},$
- (ii) $s_{i-1}P_{i+\frac{1}{2}}s_{i-1}L_{i-1}P_{i-\frac{1}{2}}s_{i-1} = s_iP_{i-\frac{1}{2}}L_{i-1}s_iP_{i-\frac{1}{2}},$
- (iii) $s_{i-1}P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}P_iP_{i-\frac{1}{2}}s_{i-1} = P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}P_iP_{i-\frac{1}{2}},$
- (iv) $s_{i-1}(P_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1})s_{i-1} = s_iP_{i-\frac{1}{2}}L_{i-1}s_i,$
- (v) $s_{i-1}P_{i+\frac{1}{2}}\sigma_i s_{i-1} = \sigma_i P_{i+\frac{1}{2}},$
- (vi) $s_{i-1}P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}s_{i-1}P_iP_{i+\frac{1}{2}}s_{i-1} = P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}s_{i-1}P_iP_{i+\frac{1}{2}},$
- (vii) $s_{i-1}P_{i+\frac{1}{2}}s_{i-1}L_{i-1}P_{i-\frac{1}{2}}P_iP_{i+\frac{1}{2}}s_{i-1} = s_iP_{i-\frac{1}{2}}P_iP_{i+\frac{1}{2}}s_{i-1}L_{i-1}P_{i-\frac{1}{2}}s_i,$
- (viii) $s_{i-1}P_{i+\frac{1}{2}}s_{i-1}L_{i-1}s_{i-1}P_iP_{i+\frac{1}{2}}s_{i-1} = s_iP_{i-\frac{1}{2}}L_{i-1}P_{i-1}P_{i-\frac{1}{2}}s_i,$
- (ix) $s_{i-1}P_{i-\frac{1}{2}}L_{i-1}s_{i-1}P_{i+\frac{1}{2}}P_iP_{i-\frac{1}{2}}s_{i-1} = P_{i-\frac{1}{2}}L_{i-1}s_{i-1}P_{i+\frac{1}{2}}P_iP_{i-\frac{1}{2}}.$

The left hand side of (i) is

$$s_{i-1}(P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}s_{i-1})s_{i-1} = P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}L_{i-1}$$

and the right hand side of (i) is

$$p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1}.$$

The left hand side of (ii) is

$$s_{i-1} p_{2+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_{i-1} = s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} = s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}},$$

which is the same as the right hand side of (ii). The left hand side of (iii) is

$$s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}} s_{i-1} = p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_i p_{i-\frac{1}{2}},$$

which is the same as the right hand side of (iii). The left hand side of (iv) is

$$s_{i-1} (p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1}) s_{i-1} = s_i p_{i-\frac{1}{2}} s_i L_{i-1} = s_i p_{i-\frac{1}{2}} L_{i-1} s_i,$$

which is the same as the right hand side of (iv). The left hand side of (v) is

$$s_{i-1} p_{i+\frac{1}{2}} \sigma_i s_{i-1} = s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} = s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} = \sigma_i p_{i+\frac{1}{2}},$$

which is the same as the right hand side of (v). The left hand side of (vi) is

$$\begin{aligned} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i \\ &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-1} s_i p_{i-\frac{1}{2}} s_i \\ &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-1} p_{i-\frac{1}{2}}, \end{aligned}$$

which is the same as the right hand side of (vi). The left hand side of (vii) is

$$\begin{aligned} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} = s_i p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} s_i p_i p_{i+\frac{1}{2}} s_{i-1} \\ &= s_i p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i = s_i p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i \\ &= s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i = s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i, \end{aligned}$$

which is the same as the right hand side of (vii). The left hand side of (viii) is

$$\begin{aligned} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} s_{i-1} p_i p_{i+\frac{1}{2}} s_{i-1} &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} s_{i-1} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i s_{i-1} p_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i \\ &= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i-1} p_{i-\frac{1}{2}} s_i, \end{aligned}$$

which is the same as the right hand side of (viii). Since the statement (ix) is evident from the relation $s_{i-1} p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} s_{i-1} = p_{i-\frac{1}{2}}$, the proof of part (8) is complete.

(9) By parts (6) and (8), $L_{i+\frac{1}{2}}$ commutes with $\langle p_i, s_{i-1} \rangle$, and so with $p_{i-1} = s_{i-1} p_i s_{i-1}$. □

Theorem 3.8 *If $i = 1, 2, \dots$, then*

- (1) L_{i+1} commutes with $\mathcal{A}_{i+\frac{1}{2}}(z)$, and σ_{i+1} commutes with $\mathcal{A}_{i-\frac{1}{2}}(z)$,
- (2) $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_i(z)$, and $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$.

Consequently, the family of elements $(L_i, L_{i+\frac{1}{2}} : i = 0, 1, \dots)$ is pairwise commutative.

Proof (1) Observe that L_1 commutes with $\mathcal{A}_{1-\frac{1}{2}}(z)$ and L_2 commutes with $\mathcal{A}_{1+\frac{1}{2}}(z)$, while σ_2 commutes with $\mathcal{A}_{1-\frac{1}{2}}(z)$ and σ_3 commutes with $\mathcal{A}_{2-\frac{1}{2}}(z)$. Since

$$L_{i+1} \text{ commutes with } \langle s_{i-1}, p_{i-1}, p_i, p_{i-\frac{1}{2}}, p_{i+\frac{1}{2}} \rangle, \quad \text{if } i \geq 2, \quad \text{and}$$

$$\sigma_{i+1} \text{ commutes with } \langle s_{i-2}, p_{i-2}, p_{i-1}, p_{i-\frac{3}{2}}, p_{i-\frac{1}{2}} \rangle, \quad \text{if } i \geq 3,$$

it suffices to show that

- (i) L_{i+1} commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, if $i \geq 2$, and
- (ii) σ_{i+1} commutes with $\mathcal{A}_{i-\frac{5}{2}}(z)$, if $i \geq 3$.

If $i \geq 2$, then induction on i shows that L_{i+1} commutes with $\mathcal{A}_{i-2}(z)$, while, if $i \geq 3$, the fact that L_{i+1} commutes with $p_{i-\frac{3}{2}}$ follows from induction on i , and the fact that σ_{i+1} commutes with $p_{i-\frac{3}{2}}$. Similarly, if $i \geq 3$, then induction on i shows that σ_{i+1} commutes with $\mathcal{A}_{i-3}(z)$, and that, if $i \geq 4$, then σ_{i+1} commutes with $p_{i-\frac{5}{2}}$.

(2) Observe that $L_{0+\frac{1}{2}}$ commutes with $\mathcal{A}_0(z)$ and $L_{1+\frac{1}{2}}$ commutes with $\mathcal{A}_1(z)$, while $\sigma_{1+\frac{1}{2}}$ commutes with $\mathcal{A}_0(z)$ and $\sigma_{2+\frac{1}{2}}$ commutes with $\mathcal{A}_1(z)$. Since

$$L_{i+\frac{1}{2}} \text{ commutes with } \langle s_{i-1}, p_{i-1}, p_i, p_{i-\frac{1}{2}} \rangle, \quad \text{if } i \geq 2, \quad \text{and}$$

$$\sigma_{i+\frac{1}{2}} \text{ commutes with } \langle s_{i-2}, p_{i-2}, p_{i-1}, p_{i-\frac{3}{2}} \rangle, \quad \text{if } i \geq 3,$$

it suffices to show that

- (i) $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, if $i \geq 2$, and
- (ii) $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-\frac{5}{2}}(z)$, if $i \geq 3$.

If $i \geq 2$, then induction on i shows that $L_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-2}(z)$, while, if $i \geq 3$, the fact that $L_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{3}{2}}$ follows from induction on i , and the fact that $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{3}{2}}$. Similarly, if $i \geq 3$, then induction on i shows that $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-3}(z)$, and that, if $i \geq 4$, then $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i-\frac{5}{2}}$. □

Proposition 3.9 *For $i = 1, 2, \dots$, the following statements hold:*

- (1) $(L_{i+\frac{1}{2}} + L_{i+1})p_{i+1} = p_{i+1}(L_{i+\frac{1}{2}} + L_{i+1}) = zp_{i+1}$,

- (2) $(L_i + L_{i+\frac{1}{2}})p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}(L_i + L_{i+\frac{1}{2}}) = zp_{i+\frac{1}{2}}$,
- (3) $(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})s_i = s_i(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})$.

Proof (1) It suffices to observe that

- (i) $L_i p_{i+\frac{1}{2}} p_{i+1} = \sigma_{i+1} p_{i+1}$,
- (ii) $p_{i+\frac{1}{2}} L_i p_{i+1} = p_{i+\frac{1}{2}} L_i p_{i+1} p_{i+\frac{1}{2}} p_{i+1}$,
- (iii) $p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} p_{i+1} = p_{i+\frac{1}{2}} L_i s_i p_{i+1}$,
- (iv) $s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1} = zp_{i+1}$,
- (v) $\sigma_{i+\frac{1}{2}} p_{i+1} = s_i L_i p_{i+\frac{1}{2}} p_{i+1}$.

With the exception of (iv), each of the statements above is evident from the defining relations or from what we have already shown. Since $(L_{\frac{1}{2}} + L_1)p_1 = zp_1$,

$$s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1} = s_i (L_{i-\frac{1}{2}} p_i + L_i p_i) s_i = z s_i p_i s_i = zp_{i+1}, \tag{3.8}$$

gives (iv) by induction.

(2) Using the expression (3.8), we have

$$\begin{aligned} zp_{i+\frac{1}{2}} &= zp_{i+\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \\ &= p_{i+\frac{1}{2}} (s_i L_{i-\frac{1}{2}} s_i p_{i+1} + s_i L_i s_i p_{i+1}) p_{i+\frac{1}{2}} \\ &= p_{i+\frac{1}{2}} L_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i s_i p_{i+1} p_{i+\frac{1}{2}} \\ &= L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\ &= L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}}, \end{aligned}$$

which yields

$$\begin{aligned} (L_i + L_{i+\frac{1}{2}})p_{i+\frac{1}{2}} &= L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}} L_i p_{i+\frac{1}{2}} - L_i p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\ &\quad + s_i L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + \sigma_{i+\frac{1}{2}} p_{i+\frac{1}{2}} \\ &= -p_{i+\frac{1}{2}} + zp_{i+\frac{1}{2}} - L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + L_{i-\frac{1}{2}} p_{i+\frac{1}{2}} + p_{i+\frac{1}{2}} \\ &= zp_{i+\frac{1}{2}}. \end{aligned}$$

(3) Since $\sigma_{i+1} s_i = \sigma_{i+\frac{1}{2}} = s_i \sigma_{i+1}$, we have

$$\begin{aligned} (L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1})s_i &= L_{i-\frac{1}{2}} s_i + L_i s_i - p_{i+\frac{1}{2}} L_i s_i - L_i p_{i+\frac{1}{2}} \\ &\quad + p_{i+\frac{1}{2}} L_i p_i p_{i+\frac{1}{2}} \\ &\quad + s_i L_{i-\frac{1}{2}} + \sigma_{i+\frac{1}{2}} s_i - p_{i+\frac{1}{2}} L_i - s_i L_i p_{i+\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &+ p_{i+\frac{1}{2}}L_i p_{i+1}p_{i+\frac{1}{2}} + s_i L_i + \sigma_{i+1}s_i \\
 &= s_i(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1}),
 \end{aligned}$$

as required. □

Theorem 3.10 *If $k = 0, 1, \dots$, then*

- (1) *the element $z_{k+\frac{1}{2}} = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \dots + L_{k+\frac{1}{2}}$ is central in $\mathcal{A}_{k+\frac{1}{2}}(z)$,*
- (2) *the element $z_{k+1} = L_{\frac{1}{2}} + L_1 + L_{1+\frac{1}{2}} + \dots + L_{k+1}$ is central in $\mathcal{A}_{k+1}(z)$.*

Proof (1) We first show that $z_{k+\frac{1}{2}}$ commutes with p_1, \dots, p_k and $p_{1+\frac{1}{2}}, \dots, p_{k+\frac{1}{2}}$. If $i = 1, \dots, k$, then p_i commutes with z_{i-1} and with $L_{i+\frac{1}{2}} + L_{i+1} + \dots + L_{k+\frac{1}{2}}$. Since p_i also commutes with $L_{i-\frac{1}{2}} + L_i$, it follows that p_i commutes with $z_{k+\frac{1}{2}}$. Similarly, since $p_{i+\frac{1}{2}}$ commutes with $z_{i-\frac{1}{2}}, L_i + L_{i+\frac{1}{2}}$, and with $L_{i+1} + L_{i+\frac{3}{2}} + \dots + L_{k+\frac{1}{2}}$, it follows that $p_{i+\frac{1}{2}}$ commutes with $z_{k+\frac{1}{2}}$. Next, suppose that $i = 1, \dots, k - 1$. Since s_i commutes with $z_{i-1}, L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1}$, and with $L_{i+\frac{3}{2}} + L_{i+2} + \dots + L_{k+\frac{1}{2}}$, it is evident that s_i commutes with $z_{k+\frac{1}{2}}$. As $\mathcal{A}_{k+\frac{1}{2}}(z)$ is generated by $p_1, \dots, p_k, p_{1+\frac{1}{2}}, \dots, p_{k+\frac{1}{2}}$, and s_1, \dots, s_{k-1} , we conclude that $z_{k+\frac{1}{2}}$ is central in $\mathcal{A}_{k+\frac{1}{2}}(z)$.

(2) Given that $z_{k+\frac{1}{2}}$ is central in $\mathcal{A}_{k+\frac{1}{2}}(z)$, it suffices to observe that s_k commutes with z_{k+1} . □

Proposition 3.11 *For $i = 1, 2, \dots$, the following statements hold:*

- (1) $(L_i + L_{i+\frac{1}{2}} + L_{i+1})\sigma_{i+1} = \sigma_{i+1}(L_i + L_{i+\frac{1}{2}} + L_{i+1})$,
- (2) $(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}})\sigma_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}(L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}})$.

Proof (1) Since σ_{i+1} commutes with

$$z_{i+1} = z_{i-1} + L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}} + L_{i+1},$$

and σ_{i+1} also commutes with $z_{i-1} + L_{i-\frac{1}{2}} \in \mathcal{A}_{i-\frac{1}{2}}(z)$, it follows that σ_{i+1} commutes with $L_i + L_{i+\frac{1}{2}} + L_{i+1}$.

(2) Given that $\sigma_{1+\frac{1}{2}} = 1$, we may suppose that $i = 2, 3, \dots$. Since $\sigma_{i+\frac{1}{2}}$ commutes with

$$z_{i+\frac{1}{2}} = z_{i-\frac{3}{2}} + L_{i-1} + L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}},$$

and $\sigma_{i+\frac{1}{2}}$ also commutes with $z_{i-\frac{3}{2}} + L_{i-1} \in \mathcal{A}_{i-1}(z)$, it follows that $\sigma_{i+\frac{1}{2}}$ commutes with $L_{i-\frac{1}{2}} + L_i + L_{i+\frac{1}{2}}$. □

4 A presentation for partition algebras

In this section, we rewrite the presentation for $\mathcal{A}_k(z)$ given by [3] in Theorem 2.1 in terms of the elements $\sigma_i, \sigma_{i+\frac{1}{2}}, p_i, p_{i+\frac{1}{2}}$.

Theorem 4.1 *If $k = 1, 2, \dots$, then $\mathcal{A}_k(z)$ is the unital associative algebra presented by the generators*

$$P_1, P_{1+\frac{1}{2}}, P_2, \dots, P_{k-\frac{1}{2}}, P_k, \sigma_2, \sigma_{2+\frac{1}{2}}, \sigma_3, \dots, \sigma_{k-\frac{1}{2}}, \sigma_k,$$

and the relations:

(1) *(Involutions)*

(a) $\sigma_{i+\frac{1}{2}}^2 = 1, \text{ for } i = 2, \dots, k - 1.$

(b) $\sigma_{i+1}^2 = 1, \text{ for } i = 1, \dots, k - 1.$

(2) *(Braid-like relations)*

(a) $\sigma_{i+1}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+1}, \text{ if } j \neq i + 1.$

(b) $\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } j \neq i + 1.$

(c) $\sigma_{i+\frac{1}{2}}\sigma_{j+\frac{1}{2}} = \sigma_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}, \text{ if } j \neq i + 1.$

(d) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for } i = 1, \dots, k - 2, \text{ where}$

$$s_\ell = \begin{cases} \sigma_{\ell+1}, & \text{if } \ell = 1, \\ \sigma_{\ell+\frac{1}{2}}\sigma_{\ell+1}, & \text{if } \ell = 2, \dots, k - 1, \end{cases}$$

are the Coxeter generators for \mathfrak{S}_k .

(3) *(Idempotent relations)*

(a) $p_i^2 = z p_i, \text{ for } i = 1, \dots, k.$

(b) $p_{i+\frac{1}{2}}^2 = p_{i+\frac{1}{2}}^2, \text{ for } i = 1, \dots, k - 1.$

(c) $\sigma_{i+1} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_{i+1} = p_{i+\frac{1}{2}}, \text{ for } i = 1, \dots, k - 1.$

(d) $\sigma_{i+\frac{1}{2}} p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}, \text{ for } i = 1, \dots, k - 1.$

(e) $\sigma_{i+\frac{1}{2}} p_i p_{i+1} = \sigma_{i+1} p_i p_{i+1}, \text{ for } i = 1, \dots, k - 1.$

(f) $p_i p_{i+1} \sigma_{i+\frac{1}{2}} = p_i p_{i+1} \sigma_{i+1}, \text{ for } i = 1, \dots, k - 1.$

(4) *(Commutation relations)*

(a) $p_i p_j = p_j p_i, \text{ for } i, j = 1, \dots, k.$

(b) $p_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} p_{i+\frac{1}{2}}, \text{ for } i, j = 1, \dots, k - 1.$

(c) $p_{i+\frac{1}{2}} p_j = p_j p_{i+\frac{1}{2}}, \text{ for } j \neq i, i + 1.$

(d) $\sigma_i p_j = p_j \sigma_i \text{ if } j \neq i - 1, i.$

(e) $\sigma_i p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} \sigma_i, \text{ if } j \neq i.$

(f) $\sigma_{i+\frac{1}{2}} p_j = p_j \sigma_{i+\frac{1}{2}}, \text{ if } j \neq i, i + 1.$

(g) $\sigma_{i+\frac{1}{2}} p_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} \sigma_{i+\frac{1}{2}}, \text{ if } j \neq i - 1.$

(h) $\sigma_{i+\frac{1}{2}} p_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1} p_{i+1} \sigma_{i+1}, \text{ for } i = 1, \dots, k - 1.$

(i) $\sigma_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \sigma_{i+\frac{1}{2}} = \sigma_i p_{i+\frac{1}{2}} \sigma_i, \text{ for } i = 2, \dots, k - 1.$

(5) *(Contraction relations)*

(a) $p_{i+\frac{1}{2}} p_j p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}, \text{ for } j = i, i + 1.$

(b) $p_i p_{j-\frac{1}{2}} p_i = p_i, \text{ for } j = i, i + 1.$

Proof We first show that the relations given in the statement above are a consequence of the presentation given by [3] in Theorem 2.1.

(1a), (1b) follow from Proposition 4.2.

(2a) To see that σ_{i+1} and $\sigma_{i+\frac{1}{2}}$ commute, we use Propositions 3.4 and 4.2 to obtain

$$\sigma_{i+1} = \sigma_{i+\frac{1}{2}}s_i = s_i\sigma_{i+\frac{1}{2}} \quad \text{and} \quad \sigma_{i+\frac{1}{2}}\sigma_{i+1} = \sigma_{i+1}\sigma_{i+\frac{1}{2}} = s_i$$

(for $i = 1, \dots, k - 1$) (4.1)

where s_i is a Coxeter generator for $\mathfrak{S}_k \subset \mathcal{A}_k(z)$. Theorem 3.8 shows that σ_j commutes with $\mathcal{A}_{j-\frac{3}{2}}(z)$, and hence that σ_j commutes with $\sigma_{1+\frac{1}{2}}, \dots, \sigma_{j-\frac{3}{2}}$. From Theorem 3.8, $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$, and hence with $\sigma_2, \dots, \sigma_{i-1}$. In general, however, σ_i and $\sigma_{i+\frac{1}{2}}$ do not commute.

(2b), (2c) Theorem 3.8 shows that σ_{i+1} commutes with $\sigma_2, \dots, \sigma_{i-1}$ and that $\sigma_{i+\frac{1}{2}}$ commutes with $\sigma_{1+\frac{1}{2}}, \dots, \sigma_{i-\frac{3}{2}}$.

(2d) Follows from (4.1), whereby for $j = 1, \dots, k - 1$, each product $\sigma_{j+\frac{1}{2}}\sigma_{j+1} = s_j$ is a Coxeter generator for $\mathfrak{S}_k \subset \mathcal{A}_k(z)$.

(3a), (3b) Are included in the set of relations given by [3].

(3c), (3d) That $\sigma_{i+1}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+1} = p_{i+\frac{1}{2}}$ is given in Proposition 3.2. Proposition 3.4 shows that $\sigma_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = s_i\sigma_{i+1}p_{i+\frac{1}{2}} = s_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$.

(3e), (3f) Propositions 3.4 and 4.2 show that

$$p_i p_{i+1} = s_i p_i p_{i+1} = \sigma_{i+\frac{1}{2}}\sigma_{i+1} p_i p_{i+1} \quad \text{and} \quad \sigma_{i+\frac{1}{2}} p_i p_{i+1} = \sigma_{i+1} p_i p_{i+1}.$$

(4a)–(4c) are included in the set of relations given by [3].

(4d), (4e) By Theorem 3.8, σ_i commutes with $\mathcal{A}_{i-\frac{3}{2}}(z)$, and hence with p_1, \dots, p_{i-2} and with $p_{1+\frac{1}{2}}, \dots, p_{i-\frac{3}{2}}$. Proposition 3.2 shows that σ_i commutes with $p_{i-\frac{1}{2}}$.

(4f), (4g) By Theorem 3.8, $\sigma_{i+\frac{1}{2}}$ commutes with $\mathcal{A}_{i-1}(z)$, and hence with p_1, \dots, p_{i-1} and with $p_{1+\frac{1}{2}}, \dots, p_{i-\frac{3}{2}}$. From (3d), it follows that $\sigma_{i+\frac{1}{2}}$ commutes with $p_{i+\frac{1}{2}}$.

(4h) Propositions 3.4 and 4.2 show that

$$p_i = s_i p_{i+1} s_i = \sigma_{i+\frac{1}{2}}\sigma_{i+1} p_{i+1} \sigma_{i+1} \sigma_{i+\frac{1}{2}} \quad \text{and} \quad \sigma_{i+\frac{1}{2}} p_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1} p_{i+1} \sigma_{i+1}.$$

(4i) Proposition 3.2 shows that $p_{i-\frac{1}{2}}s_i\sigma_i = s_i\sigma_i p_{i+\frac{1}{2}}$. Propositions 3.4 and 4.2, together with the fact that σ_{i+1} commutes with $p_{i-\frac{1}{2}}$ give

$$p_{i-\frac{1}{2}}\sigma_{i+1}\sigma_{i+\frac{1}{2}}\sigma_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}\sigma_i p_{i+\frac{1}{2}} \quad \text{and} \quad p_{i-\frac{1}{2}}\sigma_{i+\frac{1}{2}}\sigma_i = \sigma_{i+\frac{1}{2}}\sigma_i p_{i+\frac{1}{2}}.$$

Multiplying both sides of the last expression by $\sigma_i\sigma_{i+\frac{1}{2}}$ on the left, and using Proposition 4.2 once more shows that

$$\sigma_i\sigma_{i+\frac{1}{2}}p_{i-\frac{1}{2}}\sigma_{i+\frac{1}{2}}\sigma_i = \sigma_i\sigma_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}}\sigma_i p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}} \quad \text{and}$$

$$\sigma_{i+\frac{1}{2}}p_{i-\frac{1}{2}}\sigma_{i+\frac{1}{2}} = \sigma_i p_{i+\frac{1}{2}} \sigma_i,$$

as required.

(5a), (5b) are included in the set of relations given by [3].

Next, we derive the relations given by [3] in Theorem 2.1 from the relations (1a)–(5b) above.

(1i) By the relations (1a), (1b) and (2a),

$$\sigma_{i+1}\sigma_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} \quad \text{and} \quad (\sigma_{i+\frac{1}{2}}\sigma_{i+1})^2 = 1, \quad \text{for } i = 1, \dots, k - 1.$$

Thus, writing $s_i = \sigma_{i+\frac{1}{2}}\sigma_{i+1}$, for $i = 1, \dots, k - 1$, we recover (1i).

(1ii) If $j \neq i + 1$, then, by (2b) and (2c),

$$s_i s_j = \sigma_{i+\frac{1}{2}}\sigma_{i+1}\sigma_{j+\frac{1}{2}}\sigma_{j+1} = \sigma_{j+\frac{1}{2}}\sigma_{j+1}\sigma_{i+\frac{1}{2}}\sigma_{i+1} = s_j s_i,$$

as required.

(1iii) is equivalent to (2d) with $s_i = \sigma_{i+\frac{1}{2}}\sigma_{i+1}$, for $i = 1, \dots, k - 1$.

(2i), (2ii) are identical to the relations (3a) and (3b).

(2iii) With $s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}$, the relations (3c) and (3d) give

$$s_i p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_{i+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}p_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}$$

and

$$p_{i+\frac{1}{2}}s_i = p_{i+\frac{1}{2}}\sigma_{i+1}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} = p_{i+\frac{1}{2}},$$

as required.

(2iv) With $s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}$, the relations (2a), (3e) and (3f) give

$$s_i p_i p_{i+1} = \sigma_{i+1}\sigma_{i+\frac{1}{2}}p_i p_{i+1} = p_i p_{i+1} \quad \text{and}$$

$$p_i p_{i+1} s_i = p_i p_{i+1}\sigma_{i+1}\sigma_{i+\frac{1}{2}} = p_i p_{i+1},$$

as required.

(3i)–(3iii) are identical to the relations (4a)–(4c).

(3iv) If $j \neq i, i + 1$, then the relations 4d and 4f give

$$s_i p_j = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_j = \sigma_{i+\frac{1}{2}}p_j\sigma_{i+1} = p_j\sigma_{i+\frac{1}{2}}\sigma_{i+1} = p_j s_i,$$

as required.

(3v) If $j \neq i - 1, i + 1$, then the relations (4e) and (4f) give

$$s_i p_{j+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}\sigma_{i+1}p_{j+\frac{1}{2}} = \sigma_{i+\frac{1}{2}}p_{j+\frac{1}{2}}\sigma_{i+1} = p_{j+\frac{1}{2}}\sigma_{i+\frac{1}{2}}\sigma_{i+1} = p_{j+\frac{1}{2}}s_i,$$

as required.

(3vi) From the relations (1b) and (4h),

$$s_i p_i s_i = \sigma_{i+1}\sigma_{i+\frac{1}{2}}p_i\sigma_{i+\frac{1}{2}}\sigma_{i+1} = \sigma_{i+1}^2 p_{i+1}\sigma_{i+1}^2 = p_{i+1},$$

as required.

(3vii) From the relations (4e), (4g) and (1a), (1b),

$$s_i p_{i-\frac{1}{2}} s_i = \sigma_{i+\frac{1}{2}} \sigma_{i+1} p_{i-\frac{1}{2}} \sigma_{i+1} \sigma_{i+\frac{1}{2}} = \sigma_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} \sigma_i = s_{i-1} p_{i+\frac{1}{2}} s_{i-1},$$

as required.

(4i), (4ii) are identical to the relations (5a) and (5b). □

Proposition 4.2 *If $i = 0, 1, \dots$, then $(\sigma_{i+\frac{1}{2}})^2 = 1$ and $(\sigma_{i+1})^2 = 1$.*

Proof Given that $\sigma_{i+\frac{1}{2}} s_i = s_i \sigma_{i+\frac{1}{2}} = \sigma_{i+1}$, we obtain

$$\sigma_{i+\frac{1}{2}}^2 = (s_i \sigma_{i+1})^2 = \sigma_{i+1}^2.$$

It therefore suffices to show that $\sigma_{i+1}^2 = 1$. By definition $\sigma_1 = 1$, so we proceed by induction. After taking the square of the right hand side of the definition (3.2), the proposition will follow from the relations

$$\begin{aligned} & (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)^2 \\ &= (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}); \end{aligned} \tag{4.2}$$

$$\begin{aligned} & (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}) \\ &= (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1}); \end{aligned} \tag{4.3}$$

$$\begin{aligned} & (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) \\ &= (s_{i-1} s_i \sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i); \end{aligned} \tag{4.4}$$

$$\begin{aligned} & (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) \\ &= (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_{i-1} s_i \sigma_i s_i s_{i-1}); \end{aligned} \tag{4.5}$$

$$\begin{aligned} & (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})^2 \\ &= (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}); \end{aligned} \tag{4.6}$$

$$\begin{aligned} & (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) \\ &= (s_{i-1} s_i \sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}); \end{aligned} \tag{4.7}$$

$$\begin{aligned} & (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) \\ &= (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_{i-1} s_i \sigma_i s_i s_{i-1}); \end{aligned} \tag{4.8}$$

$$\begin{aligned} & (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) \\ &= (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i); \end{aligned} \tag{4.9}$$

$$(p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}})^2 = (s_{i-1}s_i\sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}); \tag{4.10}$$

$$(s_{i-1}s_i\sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i) = (s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)^2; \tag{4.11}$$

$$(s_{i-1}s_i\sigma_i s_i s_{i-1})^2 = 1; \tag{4.12}$$

$$\begin{aligned} (s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)(p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i) \\ = (s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)(p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}); \end{aligned} \tag{4.13}$$

$$\begin{aligned} (s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}) \\ = (s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i)(s_{i-1}s_i\sigma_i s_i s_{i-1}). \end{aligned} \tag{4.14}$$

From the left hand side of (4.2), using the fact that $p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$, together with $p_{i+\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}} = p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}$, we obtain

$$\begin{aligned} (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)^2 &= p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i \\ &= p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} \\ &= p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} \\ &= p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}. \end{aligned}$$

Similarly, from the right hand side of (4.2), we obtain

$$\begin{aligned} (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}) \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}, \end{aligned}$$

which demonstrates (4.2). Now consider the left hand side of (4.3)

$$\begin{aligned} (p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i)(s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}) \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_i p_{i-1}\sigma_i p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}\sigma_i s_i p_{i-1}s_i\sigma_i p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\ = p_{i-\frac{1}{2}}p_i\sigma_i s_i p_{i-\frac{1}{2}}s_i\sigma_i p_i p_{i-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &= p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i (\sigma_{i-\frac{1}{2}})^2 p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}}.
 \end{aligned}$$

From the right hand side of (4.3), we obtain

$$\begin{aligned}
 &(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1}) \\
 &= (p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i)(s_{i-1} s_i \sigma_i s_i s_{i-1}) \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i s_i s_{i-1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_i s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_{i-1} s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i s_{i-1} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} s_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= (p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}).
 \end{aligned}$$

Now,

$$\begin{aligned}
 (p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i \sigma_i p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} p_i (\sigma_{i-\frac{1}{2}})^2 p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}}, \tag{4.15}
 \end{aligned}$$

which completes the proof of (4.3). From the left hand side of (4.4),

$$\begin{aligned}
 & (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_i s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} L_{i-1} s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} p_i (\sigma_i)^2 p_{i+\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1}.
 \end{aligned}$$

The right hand side of (4.4) gives

$$\begin{aligned}
 (s_{i-1} s_i \sigma_i s_i s_{i-1}) (s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) &= s_{i-1} s_i \sigma_i p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1} s_i \sigma_i p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1} s_i s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} s_i L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} \sigma_{i-\frac{1}{2}} s_{i-1} s_i L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_i \sigma_{i-\frac{1}{2}} (s_{i-1})^2 (s_i)^2 L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_i \sigma_{i-\frac{1}{2}} \sigma_i p_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i-\frac{1}{2}} p_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\
 &= p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1},
 \end{aligned}$$

which demonstrates (4.4). The left hand side of (4.5) gives

$$\begin{aligned}
 & (s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}) (p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i
 \end{aligned}$$

$$\begin{aligned}
 &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} p_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} \\
 &= s_{i-1} (\sigma_{i-\frac{1}{2}})^2 p_{i+\frac{1}{2}} s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1},
 \end{aligned}$$

while the right hand side of (4.5) gives

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_{i-1} s_i \sigma_i s_i s_{i-1}) \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i \sigma_i s_i s_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i s_i s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_{i+1} s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} p_i s_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} s_i s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 s_{i-1} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1},
 \end{aligned}$$

which demonstrates (4.5). The statement (4.6) is equivalent to (4.2) which has already been verified. The right hand side of (4.7) leads to

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} p_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}},
 \end{aligned}$$

while

$$\begin{aligned}
 (s_{i-1} s_i \sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}}) &= s_{i-1} s_i \sigma_i s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} s_i \sigma_i s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1} s_i \sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} s_i \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} s_i p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}},
 \end{aligned}$$

as required. Since the statement (4.8) is equivalent to (4.7), we consider (4.9). Using the relation $p_{i-\frac{1}{2}} s_i p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}} p_{i+\frac{1}{2}}$,

$$\begin{aligned}
 &(s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
 &= s_i p_{i-\frac{1}{2}} L_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}})(s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i) &= p_{i-\frac{1}{2}} L_{i-1} s_i p_{i+\frac{1}{2}} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i \\
 &= p_{i+\frac{1}{2}} (p_{i-\frac{1}{2}} L_{i-1} p_{i-\frac{1}{2}})^2 \\
 &= p_{i+\frac{1}{2}} p_{i-\frac{1}{2}}.
 \end{aligned}$$

The left hand side of (4.10) is given by (4.15), and the right hand side by

$$\begin{aligned}
 & (s_{i-1}s_i\sigma_i s_i s_{i-1})(s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}}) \\
 &= s_{i-1}s_i\sigma_i s_i s_{i-1} s_i p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1}s_i\sigma_i s_i s_{i-1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1}s_i\sigma_i s_i s_{i-1} s_i p_{i-\frac{1}{2}} s_i L_{i-1} p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1}s_i\sigma_i s_i (s_{i-1})^2 p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} s_i p_{i-1} p_{i-\frac{1}{2}} \\
 &= s_{i-1}s_i s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= s_{i-1} p_{i-\frac{1}{2}} s_i s_{i-1} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}} s_i s_{i-1} p_i p_{i-\frac{1}{2}} p_{i-1} s_i p_{i-\frac{1}{2}} \\
 &= p_{i-\frac{1}{2}},
 \end{aligned}$$

as required. Considering the left hand side of (4.11),

$$\begin{aligned}
 & (s_{i-1}s_i\sigma_i s_i s_{i-1})(p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i) \\
 &= s_{i-1}s_i\sigma_i s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1}s_i\sigma_i s_{i-1} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} s_{i-1} \sigma_i p_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1}s_i\sigma_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_{i-1} p_{i+1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i \\
 &= s_{i-1} p_{i+\frac{1}{2}} s_{i-1} \sigma_{i-\frac{1}{2}} p_{i+1} p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} (\sigma_{i-\frac{1}{2}})^2 p_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} s_i p_{i+1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} s_i \\
 &= s_i p_{i-\frac{1}{2}} s_i,
 \end{aligned}$$

which, by (4.15), is equal to the right hand side of (4.11). Since (4.9) is equivalent to (4.13), while (4.14) is equivalent to (4.4), the proof of the proposition is complete. □

We record for later reference further consequences of the presentation given in Theorem 2.1.

Proposition 4.3 *For $i = 1, 2, \dots$, the following statements hold:*

- (1) $p_{i+1}\sigma_{i+1}p_{i+1} = L_i p_{i+1}$,
- (2) $p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1} = (z - L_{i-\frac{1}{2}})p_{i+1}$,
- (3) $p_{i+\frac{3}{2}}\sigma_{i+1}p_{i+\frac{3}{2}} = p_{i+\frac{1}{2}}p_{i+\frac{3}{2}}$.

Proof (1) From Proposition 3.2, we obtain $p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = p_{i+\frac{1}{2}}L_i$. Thus

$$\begin{aligned}
 p_{i+1}\sigma_{i+1} &= p_{i+1}p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1} = p_{i+1}p_{i+\frac{1}{2}}L_i \quad \text{and} \\
 p_{i+1}\sigma_{i+1}p_{i+1} &= p_{i+1}p_{i+\frac{1}{2}}L_i p_{i+1} = L_i p_{i+1},
 \end{aligned}$$

as required.

(2) We first compute

$$\begin{aligned}
 p_{i+1}s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} &= s_i p_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} \\
 &= s_i p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} \\
 &= s_i p_i p_{i+\frac{1}{2}} \sigma_i p_{i-1} p_{i-\frac{1}{2}} p_i s_i \\
 &= s_i p_i p_{i+\frac{1}{2}} \sigma_i s_{i-1} p_i s_i \\
 &= s_i p_i p_{i+\frac{1}{2}} \sigma_{i-\frac{1}{2}} p_i s_i \\
 &= \sigma_{i-\frac{1}{2}} p_{i+1}.
 \end{aligned}$$

Now observe that $p_2\sigma_{1+\frac{1}{2}}p_2 = (z - L_{\frac{1}{2}})p_2$ and hence, by induction,

$$\begin{aligned}
 p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1} &= s_i s_{i-1} p_i \sigma_{i-\frac{1}{2}} p_i s_{i-1} s_i + p_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} s_i p_{i+1} \\
 &\quad + p_{i+1} s_i p_{i-\frac{1}{2}} L_{i-1} s_i p_{i-\frac{1}{2}} p_{i+1} \\
 &\quad - p_{i+1} p_{i-\frac{1}{2}} L_{i-1} s_{i-1} p_{i+\frac{1}{2}} p_i p_{i-\frac{1}{2}} p_{i+1} \\
 &\quad - p_{i+1} s_i p_{i-\frac{1}{2}} p_i p_{i+\frac{1}{2}} s_{i-1} L_{i-1} p_{i-\frac{1}{2}} s_i p_{i+1} \\
 &= s_{i-1} s_i p_i \sigma_{i-\frac{1}{2}} p_i s_i s_{i-1} + p_{i-\frac{1}{2}} L_{i-1} p_{i+1} + L_{i-1} p_{i-\frac{1}{2}} p_{i+1} \\
 &\quad - p_{i-\frac{1}{2}} L_{i-1} p_{i-1} p_{i-\frac{1}{2}} p_{i+1} - \sigma_{i-\frac{1}{2}} p_{i+1} \\
 &= (z - L_{i-\frac{1}{2}}) p_{i+1}.
 \end{aligned}$$

(3) Observe that $p_{2+\frac{1}{2}}\sigma_2 p_{2+\frac{1}{2}} = p_{2+\frac{1}{2}}s_1 p_{2+\frac{1}{2}} = p_{2+\frac{1}{2}}p_{1+\frac{1}{2}}$ and, by induction,

$$\begin{aligned}
 p_{i+\frac{3}{2}}s_{i-1}s_i\sigma_i s_i s_{i-1} p_{i+\frac{3}{2}} &= s_{i-1} p_{i+\frac{3}{2}} s_i \sigma_i s_i p_{i+\frac{3}{2}} s_{i-1} \\
 &= s_{i-1} s_i s_{i+1} p_{i+\frac{1}{2}} s_{i+1} \sigma_i s_{i+1} p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
 &= s_{i-1} s_i s_{i+1} p_{i+\frac{1}{2}} \sigma_i p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
 &= s_{i-1} s_i s_{i+1} p_{i-\frac{1}{2}} p_{i+\frac{1}{2}} s_i s_{i+1} s_{i-1} \\
 &= p_{i+\frac{1}{2}} p_{i+\frac{3}{2}}.
 \end{aligned}$$

Therefore, using the definition (3.2) and the fact that $p_{i-\frac{1}{2}}L_{i-1}p_{i-\frac{1}{2}} = p_{i-\frac{1}{2}}$,

$$\begin{aligned}
 p_{i+\frac{3}{2}}\sigma_{i+1}p_{i+\frac{3}{2}} &= p_{i+\frac{3}{2}}p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}}s_i p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i p_{i+\frac{3}{2}} \\
 &\quad + p_{i+\frac{3}{2}}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}p_{i+\frac{3}{2}} \\
 &\quad - p_{i+\frac{3}{2}}s_i p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}}p_{i+\frac{3}{2}} \\
 &\quad - p_{i+\frac{3}{2}}p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}s_i p_{i+\frac{3}{2}} \\
 &= p_{i+\frac{3}{2}}p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}}s_i p_{i+\frac{3}{2}}p_{i-\frac{1}{2}}L_{i-1}s_i p_{i-\frac{1}{2}}s_i \\
 &\quad + p_{i-\frac{1}{2}}L_{i-1}p_{i+\frac{3}{2}}s_i p_{i+\frac{3}{2}}p_{i-\frac{1}{2}} \\
 &\quad - p_{i+\frac{3}{2}}s_i p_{i+\frac{3}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\
 &\quad - p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{3}{2}}s_i p_{i+\frac{3}{2}} \\
 &= p_{i+\frac{3}{2}}p_{i+\frac{1}{2}} + p_{i+\frac{3}{2}}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}s_{i-1} \\
 &\quad + p_{i-\frac{1}{2}}L_{i-1}p_{i+\frac{3}{2}}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}} \\
 &\quad - p_{i+\frac{3}{2}}p_{i+\frac{1}{2}}p_{i-\frac{1}{2}}L_{i-1}s_{i-1}p_{i+\frac{1}{2}}p_i p_{i-\frac{1}{2}} \\
 &\quad - p_{i-\frac{1}{2}}p_i p_{i+\frac{1}{2}}s_{i-1}L_{i-1}p_{i-\frac{1}{2}}p_{i+\frac{1}{2}}p_{i+\frac{3}{2}} \\
 &= p_{i+\frac{3}{2}}p_{i+\frac{1}{2}},
 \end{aligned}$$

as required. □

The next statement gives an alternative recursion for the family $(L_{i+\frac{1}{2}} : i = 0, 1 \dots)$ for use in Sect. 5.

Theorem 4.4 *If $i = 1, 2, \dots$, then*

$$L_{i+\frac{1}{2}} = -L_i p_{i+\frac{1}{2}} - p_{i+\frac{1}{2}}L_i + (z - L_{i-\frac{1}{2}})p_{i+\frac{1}{2}} + s_i L_{i-\frac{1}{2}}s_i + \sigma_{i+\frac{1}{2}}.$$

Proof By Propositions 3.2, Proposition 3.4 and Proposition 4.3,

$$\begin{aligned}
 p_{i+\frac{1}{2}}L_i p_i p_{i+\frac{1}{2}} &= p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+1}p_i p_{i+\frac{1}{2}} \\
 &= p_{i+\frac{1}{2}}p_{i+1}\sigma_{i+\frac{1}{2}}p_{i+1}p_{i+\frac{1}{2}} = (z - L_{i-\frac{1}{2}})p_{i+\frac{1}{2}}.
 \end{aligned}$$

Substituting the above expression into the definition of $L_{i+\frac{1}{2}}$ given in (3.3) provides the required statement. □

5 Schur–Weyl duality

In this section we use Schur–Weyl duality to show that the family $(L_{i+\frac{1}{2}}, L_{i+1} : i = 1, 2, \dots)$, defined above, and the Jucys–Murphy elements given by Halverson and Ram [3] are in fact equal.

Let $n = 1, 2, \dots$, and V be a vector space over \mathbb{C} with basis v_1, \dots, v_n . If $r = 1, 2, \dots$, the tensor product

$$V^{\otimes r} = \underbrace{V \otimes V \otimes \dots \otimes V}_{r \text{ factors}} \quad \text{has basis } \{v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r} \mid 1 \leq i_1, \dots, i_r \leq n\},$$

and is equipped, via the inclusion $\mathfrak{S}_n \subset GL_n(\mathbb{C})$, with the diagonal \mathfrak{S}_n -action

$$w(v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r})w = v_{(i_1)w^{-1}} \otimes v_{(i_2)w^{-1}} \otimes \dots \otimes v_{(i_r)w^{-1}}, \quad \text{for } w \in \mathfrak{S}_n.$$

Let $A_r(n) = \mathcal{A}_r(z) \otimes_{\mathbb{Z}[z]} \mathbb{C}$, where z acts on \mathbb{C} as multiplication by n . The action of $A_r(n)$ on $V^{\otimes r}$ is given (Sect. 3 of [3]) by

$$u(v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r}) = v_{i_{(1)u^{-1}}} \otimes v_{i_{(2)u^{-1}}} \otimes \dots \otimes v_{i_{(r)u^{-1}}}, \quad \text{for } u \in \mathfrak{S}_r,$$

and for $k = 1, \dots, r - 1$,

$$p_{k+\frac{1}{2}}(v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r}) = \begin{cases} v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r} & \text{if } i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and for $k = 1, \dots, r$,

$$\begin{aligned} & p_k(v_{i_1} \otimes \dots \otimes v_{i_{k-1}} \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \dots \otimes v_{i_r}) \\ &= \sum_{j=1}^n v_{i_1} \otimes \dots \otimes v_{i_{k-1}} \otimes v_j \otimes v_{i_{k+1}} \otimes \dots \otimes v_{i_r}. \end{aligned}$$

The $A_{r+\frac{1}{2}}(n)$ -action on $V^{\otimes r}$ is obtained in Sect. 3 of [3] from the action of $A_{r+1}(n)$ on $V^{\otimes r+\frac{1}{2}}$ by restricting to the subspace $V^{\otimes r} \otimes v_n$ and identifying $V^{\otimes r}$ with $V^{\otimes r} \otimes v_n$. The next statement asserts that \mathfrak{S}_n and $A_r(n)$ act as commuting operators on $V^{\otimes r}$.

Theorem 5.1 (Theorem 3.22 of [3]) *Let $n, r \in \mathbb{Z}_{\geq 0}$. Let S_n^λ denote the irreducible \mathfrak{S}_n -module indexed by λ .*

(1) *As $(\mathbb{C}\mathfrak{S}_n, A_r(n))$ -bimodules*

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \hat{A}_r(n)} S_n^\lambda \otimes A_r^\lambda(n),$$

where $\lambda \in \hat{A}_r(n)$ is an indexing set for the irreducible $A_r(n)$ -modules, and the vector spaces $A_r^\lambda(n)$, for $\lambda \in \hat{A}_r(n)$, are irreducible $A_r(n)$ -modules.

(2) As $(\mathbb{C}\mathfrak{S}_{n-1}, A_{r+\frac{1}{2}}(n))$ -bimodules

$$V^{\otimes r} \cong \bigoplus_{\lambda \in \hat{A}_{r+\frac{1}{2}}(n)} S_{n-1}^\lambda \otimes A_{r+\frac{1}{2}}^\lambda(n),$$

where $\lambda \in \hat{A}_{r+\frac{1}{2}}(n)$ is an indexing set for the irreducible $A_{r+\frac{1}{2}}(n)$ -modules, and the vector spaces $A_{r+\frac{1}{2}}^\lambda(n)$, for $\lambda \in \hat{A}_{r+\frac{1}{2}}(n)$, are irreducible $A_{r+\frac{1}{2}}(n)$ -modules.

By Theorem 3.6 of [3], the homomorphism $A_r(n) \rightarrow \text{End}_{\mathfrak{S}_n}(V^{\otimes r})$ in Theorem 5.1 is an isomorphism whenever $n \geq 2r$.

If $1 \leq i, j \leq n$, let $s_{i,j} \in \mathfrak{S}_n$ denote the transposition which interchanges i and j . The next statement gives the action of the group $\langle \sigma_{i+\frac{1}{2}}, \sigma_{i+1} \mid i = 1, \dots, r-1 \rangle$ on $V^{\otimes r}$.

Proposition 5.2 *If $k = 1, 2, \dots$, and $v_{i_1} \otimes \dots \otimes v_{i_{k+1}} \in V^{\otimes k+1}$, then*

$$\sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \dots \otimes v_{i_{k+1}}) = \underbrace{s_{i_k, i_{k+1}} \otimes \dots \otimes s_{i_k, i_{k+1}}}_{k-1 \text{ factors}}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_{i_{k+1}} \tag{5.1}$$

and

$$\sigma_{k+1}(v_{i_1} \otimes \dots \otimes v_{i_{k+1}}) = \underbrace{s_{i_k, i_{k+1}} \otimes \dots \otimes s_{i_k, i_{k+1}}}_{k+1 \text{ factors}}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_{i_{k+1}}). \tag{5.2}$$

Proof The proposition is true when $k = 1$. If $k = 2, 3, \dots$, observe that the linear endomorphisms defined, for $v_{i_1} \otimes \dots \otimes v_{i_{k+1}} \in V^{\otimes k+1}$, by

$$\theta_{k+\frac{1}{2}} : v_{i_1} \otimes \dots \otimes v_{i_{k+1}} \mapsto \underbrace{s_{i_k, i_{k+1}} \otimes \dots \otimes s_{i_k, i_{k+1}}}_{k-1 \text{ factors}}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_{i_{k+1}}$$

and

$$\theta_{k+1} : v_{i_1} \otimes \dots \otimes v_{i_{k+1}} \mapsto \underbrace{s_{i_k, i_{k+1}} \otimes \dots \otimes s_{i_k, i_{k+1}}}_{k+1 \text{ factors}}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_{i_{k+1}})$$

commute with the diagonal action of \mathfrak{S}_n on $V^{\otimes k+1}$. Thus, by Theorem 5.1, $\theta_{k+\frac{1}{2}}$ and θ_{k+1} , lie in the image of the map $A_{k+1}(n) \mapsto \text{End}_{\mathfrak{S}_n}(V^{\otimes k+1})$. Observe also that the action of $\theta_{k+\frac{1}{2}}$ on $V^{\otimes k+1}$ commutes with the action of $A_{k-1}(n)$ on $V^{\otimes k+1}$, and the action of θ_{k+1} on $V^{\otimes k+1}$ commutes with the action of $A_{k-\frac{1}{2}}(n)$ on $V^{\otimes k+1}$. Since $A_{k+1}(n)$ is generated by $A_{k-\frac{1}{2}}(n)$ together with $\langle \sigma_{k+\frac{1}{2}}, \sigma_{k+1}, p_k, p_{k+\frac{1}{2}} \rangle$, to show that the map $A_{k+1}(n) \rightarrow \text{End}_{\mathfrak{S}_n}(V^{\otimes k+1})$ sends

$$\sigma_{k+\frac{1}{2}} \mapsto \theta_{k+\frac{1}{2}} \quad \text{and} \quad \sigma_{k+1} \mapsto \theta_{k+1},$$

it now suffices to verify that, as operators on $V^{\otimes k+1}$,

- (i) $\theta_{k+\frac{1}{2}}^2 = \theta_{k+1}^2 = 1$,
- (ii) $\theta_{k+\frac{1}{2}}\theta_{k+1} = \theta_{k+1}\theta_{k+\frac{1}{2}} = s_k$,
- (iii) $\theta_{k+\frac{1}{2}}p_{k+\frac{1}{2}} = p_{k+\frac{1}{2}}\theta_{k+\frac{1}{2}} = p_{k+\frac{1}{2}}$,
- (iv) $\theta_{k+\frac{1}{2}}p_k p_{k+1} = \theta_{k+1}p_k p_{k+1}$,
- (v) $p_k p_{k+1}\theta_{k+\frac{1}{2}} = p_k p_{k+1}\theta_{k+1}$,
- (vi) $\theta_{k+\frac{1}{2}}p_k\theta_{k+\frac{1}{2}} = \theta_{k+1}p_{k+1}\theta_{k+1}$,
- (vii) $\theta_{k+\frac{1}{2}}p_{k-\frac{1}{2}}\theta_{k+\frac{1}{2}} = \sigma_k p_{k+\frac{1}{2}}\sigma_k$,

where s_k in item (ii) acts on $V^{\otimes k}$ by place permutation. Since each of (i)–(vii) can be verified by inspection, the proof of the proposition is complete. \square

Proposition 5.3 *If $k = 1, 2, \dots$, and $v_{i_1} \otimes \dots \otimes v_{i_k} \in V^{\otimes k}$ then*

$$L_{k-\frac{1}{2}}(v_{i_1} \otimes \dots \otimes v_{i_k}) = nv_{i_1} \otimes \dots \otimes v_{i_k} - \sum_{j=1}^n s_{i_k,j} \otimes \dots \otimes s_{i_k,j}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_{i_k}, \tag{5.3}$$

and

$$L_k(v_{i_1} \otimes \dots \otimes v_{i_k}) = \sum_{j=1}^n s_{i_k,j} \otimes \dots \otimes s_{i_k,j}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_j. \tag{5.4}$$

Proof Identify $V^{\otimes k}$ as the subspace $V^{\otimes k} \otimes \sum_{j=1}^n v_j \subseteq V^{\otimes k+1}$. Then

$$\begin{aligned} & p_{k+1}\sigma_{k+\frac{1}{2}}p_{i+1}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_n) \\ &= p_{k+1} \sum_{j=1}^n \sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_j) \\ &= p_{k+1} \sum_{j=1}^n s_{i_k,j} \otimes \dots \otimes s_{i_k,j}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_j \\ &= \sum_{j,\ell=1}^n s_{i_k,j} \otimes \dots \otimes s_{i_k,j}(v_{i_1} \otimes \dots \otimes v_{i_{k-1}}) \otimes v_{i_k} \otimes v_\ell. \end{aligned}$$

By Proposition 4.3, as operators on $V^{\otimes k}$,

$$\begin{aligned} p_{k+1}\sigma_{k+\frac{1}{2}}p_{i+1}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_n) &= (n - L_{i-\frac{1}{2}})p_{i+1}(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_n) \\ &= \sum_{\ell=1}^n (n - L_{i-\frac{1}{2}})(v_{i_1} \otimes \dots \otimes v_{i_k} \otimes v_\ell), \end{aligned}$$

and the statement (5.3) follows. Next,

$$\begin{aligned}
 & p_{k+1}\sigma_{k+1}p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) \\
 &= p_{k+1} \sum_{j=1}^n \sigma_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_j) \\
 &= p_{k+1} \sum_{j=1}^n \sigma_{k+\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j \otimes v_{i_k}) \\
 &= p_{k+1} \sum_{j=1}^n s_{i_k, i_{k+1}} \otimes \cdots \otimes s_{i_k, i_{k+1}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j) \otimes v_{i_k} \\
 &= \sum_{j, \ell=1}^n s_{i_k, i_{k+1}} \otimes \cdots \otimes s_{i_k, i_{k+1}}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}} \otimes v_j) \otimes v_\ell.
 \end{aligned}$$

By Proposition 4.3, as operators on $V^{\otimes k}$,

$$\begin{aligned}
 p_{k+1}\sigma_{k+1}p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) &= L_k p_{k+1}(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_n) \\
 &= \sum_{\ell=1}^n L_k(v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_\ell),
 \end{aligned}$$

which yields (5.4). □

As in Sect. 3 of [3], let κ_n be the central element which is the class sum corresponding to the conjugacy class of transpositions in $\mathbb{C}\mathfrak{S}_n$,

$$\kappa_n = \sum_{1 \leq i < j \leq n} s_{i,j}.$$

For $\ell = 1, \dots, n$, we also define

$$\kappa_{n,\ell} = \sum_{\substack{1 \leq i < j \leq n \\ i, j \neq \ell}} s_{i,j},$$

so that $\kappa_{n,n} = \kappa_{n-1}$.

Proposition 5.4 *Let $n = \dim(V)$ and $r \in \mathbb{Z}_{>0}$. If $z_{r-\frac{1}{2}} \in A_{r-\frac{1}{2}}(n)$ and $z_r \in A_r(n)$ are the central elements defined by Theorem 3.10, and $v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$, then*

$$z_r(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_r}) - \left(\binom{n}{2} - rn \right) (v_{i_1} \otimes \cdots \otimes v_{i_r}), \tag{5.5}$$

and, if $r = 2, 3, \dots$, then

$$z_{r-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_r}) = \kappa_{n,i_r}(v_{i_1} \otimes \cdots \otimes v_{i_r}) - \left(\binom{n}{2} - rn + 1 \right) (v_{i_1} \otimes \cdots \otimes v_{i_r}). \tag{5.6}$$

Proof The proof is by induction on r . Let $i = 1, \dots, n$. Since $z_1 = p_1$,

$$\begin{aligned} \kappa_n v_i &= \sum_{i>j} s_{j,i} v_i + \sum_{j>i} s_{i,j} v_j + \sum_{\substack{j,\ell \\ j,\ell \neq i}} s_{j,\ell} v_i \\ &= \sum_{j=1}^n v_j + \left(\binom{n-1}{2} - 1 \right) v_i \\ &= z_1 v_i + \left(\binom{n}{2} - n \right) v_i, \end{aligned}$$

which verifies (5.5) when $r = 1$. Now observe that if $v_{i_1} \otimes \cdots \otimes v_{i_k} \in V^{\otimes r}$, then the diagonal action of κ_n and κ_{n,i_r} on $V^{\otimes k}$ allows us to write

$$\begin{aligned} \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_k}) &= \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &\quad + \sum_{\substack{j=1 \\ j \neq i_k}}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j, \end{aligned}$$

and

$$\begin{aligned} \kappa_{n,i_k}(v_{i_1} \otimes \cdots \otimes v_{i_k}) &= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \\ &\quad - \sum_{\substack{j=1 \\ j \neq i_k}}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k}. \end{aligned}$$

Assuming that (5.5) holds for $r = 1, \dots, k - 1$, we obtain

$$\begin{aligned} &z_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &= z_{k-1}(v_{i_1} \otimes \cdots \otimes v_{i_k}) + L_{k-\frac{1}{2}}(v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} - \left(\binom{n}{2} - (k-1)n \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}) \\ &\quad + n v_{i_1} \otimes \cdots \otimes v_{i_k} - \sum_{j=1}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j}(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \\ &= \kappa_n(v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} - \left(\binom{n}{2} - kn + 1 \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{\substack{j=1 \\ j \neq i_k}}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j} (v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_{i_k} \\
 & = \kappa_{n,i_k} (v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\binom{n}{2} - kn + 1 \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}),
 \end{aligned}$$

which verifies (5.6) for $r = k$. Next, since (5.6) holds for $r = 2, 3, \dots, k$, we obtain

$$\begin{aligned}
 z_k (v_{i_1} \otimes \cdots \otimes v_{i_k}) & = z_{k-\frac{1}{2}} (v_{i_1} \otimes \cdots \otimes v_{i_k}) + L_k (v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
 & = \kappa_{n,i_k} (v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\binom{n}{2} - kn + 1 \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
 & \quad + \sum_{j=1}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j} (v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j \\
 & = \kappa_{n,i_k} (v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\binom{n}{2} - kn \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}) \\
 & \quad + \sum_{\substack{j=1 \\ j \neq i_k}}^n s_{i_k,j} \otimes \cdots \otimes s_{i_k,j} (v_{i_1} \otimes \cdots \otimes v_{i_{k-1}}) \otimes v_j \\
 & = \kappa_n (v_{i_1} \otimes \cdots \otimes v_{i_k}) - \left(\binom{n}{2} - kn \right) (v_{i_1} \otimes \cdots \otimes v_{i_k}),
 \end{aligned}$$

which verifies (5.5) for $r = k$. □

Let $Z_k \in A_k(n)$ and $Z_{k+\frac{1}{2}} \in A_{k+\frac{1}{2}}(n)$ denote the central element defined by Halverson and Ram in Sect. 3 of [3]. Then the Jucys–Murphy elements of [3] are given by

$$M_{k-\frac{1}{2}} = Z_{k-\frac{1}{2}} - Z_{k-1} \quad \text{and} \quad M_k = Z_k - Z_{k-\frac{1}{2}} \quad \text{for } k = 1, 2, \dots$$

Theorem 5.5 *If $k = 1, 2, \dots$, then $M_{k+\frac{1}{2}} = L_{k+\frac{1}{2}}$ and $M_{k+1} = L_{k+1}$ as elements of $A_{k+1}(n)$.*

Proof By Theorem 3.6 of [3], the homomorphism $A_k(n) \rightarrow \text{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an isomorphism whenever $n \geq 2k$. Since the coefficients in the expansions of $Z_k, Z_{k+\frac{1}{2}}$ and $z_k, z_{k+\frac{1}{2}}$, in terms of the diagram basis for $A_{k+1}(n)$, are polynomials in n , and the map $A_k(n) \rightarrow \text{End}_{\mathfrak{S}_n}(V^{\otimes k})$ is an isomorphism for infinitely many values of n , to prove the theorem, it suffices to compare the action of z_k and Z_k (resp. $z_{k+\frac{1}{2}}$ and $Z_{k+\frac{1}{2}}$) on $V^{\otimes k}$ for an arbitrary choice of n . Identifying $V^{\otimes k}$ with the subspace $V^{\otimes k} \otimes v_n \subseteq V^{\otimes k+1}$, and κ_{n-1} with $\kappa_{n,n}$, then Theorem 3.35 of [3] states that, as

operators on $V^{\otimes k}$,

$$Z_k = \kappa_n - \left(\binom{n}{2} - kn \right) \quad \text{and} \quad Z_{k+\frac{1}{2}} = \kappa_{n-1} - \left(\binom{n}{2} - (k+1)n + 1 \right). \quad (5.7)$$

Comparing (5.7) with the action of z_k and $z_{k+\frac{1}{2}}$ on $V^{\otimes k}$ in (5.5) and (5.6) completes the proof. \square

Remark 5.6 Whereas $L_{\frac{1}{2}} = 0$ and $L_1 = p_1$, in [3], the first three Jucys–Murphy elements are $M_0 = M_{\frac{1}{2}} = \bar{1}$, and $M_1 = p_1 - 1$. Thus, although $z_1 = Z_1$ as elements of $A_1(n)$, Theorem 5.5 cannot be extended to the case $k = 0$.

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