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C_4-free edge ideals

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Abstract We discuss the linearity of the minimal free resolution of a power of a monomial edge ideal.

Keywords Free resolutions · Edge ideals · Powers of ideals

Throughout this note, $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k, and G is a finite simple graph (that is, without loops and multiple edges) on vertex set $\{x_1, \ldots, x_n\}$. The monomial *edge ideal* associated to G is

$$I_G = (\, x_i x_j \mid x_i x_j \in G \,),$$

where the edge $\{x_i, x_j\}$ is denoted $x_i x_j$ for short. The homological properties of I_G depend on the combinatorial properties of G and on the *complement graph* G^c with edges $\{x_i x_j \mid x_i x_j \notin G\}$. By polarization, studying the minimal free resolutions of quadratic monomial ideals is equivalent to studying the minimal free resolutions of edge ideals. Describing all such resolutions is beyond reach since they can have very complicated structures. In fact, very little is known about the graded Betti numbers $\beta_{i,j}(I_G)$ and regularity $\operatorname{reg}(I_G) = \max\{j - i \mid \beta_{i,j}(I_G) \neq 0\}$. The following problem is wide open: find upper and lower bounds on $\operatorname{reg}(I_G)$ in terms of the combinatorial properties of the graphs G and G^c.

The simplest case is when the regularity is as minimal as possible, that is, when $\operatorname{reg}(I) = p$ for a graded ideal I generated in degree p; in this case, we say that the *minimal free resolution is linear*. Such edge ideals are characterized combinatorially

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by Fröberg's Theorem 1.1. The following definitions are helpful: we say that a simple graph T contains a *q-cycle* if there exist distinct vertices $x_{i_1}, \ldots, x_{i_q}$ such that $x_{i_q}x_{i_1} \in T$ and $x_{i_j}x_{i_{j+1}} \in T$ for all $1 \le j \le q-1$; a *chord* in the cycle is an edge in T between two non-consecutive vertices; a cycle is called *induced* if it has no chords. The following is known:

Theorem 1.1 [7] *The minimal free resolution of I_G is linear if and only if the complement graph G^c is chordal* (*that is*, *every induced cycle in G^c is a triangle*).

Suppose that the complement graph G^c is chordal. The minimal free resolution of I_G is constructed in [5] and [12] in special cases, and is constructed by Chen in [2] in the general case.

We are interested in when the higher powers of the ideal have linear minimal free resolutions; see [4]. Conca [3, Example 3.4] constructed an example of an ideal J generated by six quadratic monomials and one quadratic binomial such that J has a linear minimal free resolution and its second power J^2 has a non-linear minimal free resolution. Such flops do not happen in the monomial world by the following result of Herzog, Hibi, and Zheng.

Theorem 1.2 [10] *If G^c is chordal*, *then for every $s \ge 2$ the minimal free resolution of I_G^s is linear.*

This note focuses on a wider class of edge ideals: the C_4-free edge ideals. As we will see, such ideals arise naturally when we study edge ideals whose powers have linear minimal free resolutions. We say that I_G is *C_4-free* if the complement graph G^c has no induced 4-cycles. The algebraic meaning of this condition is the following.

Proposition 1.3 (Francisco-Hà-Van Tuyl; personal communication) *G^c has no induced 4-cycles if and only if the Betti numbers $\beta_{1,j}(I_G)$ vanish for $j > 3$* (*that is*, *I_G has only linear minimal first syzygies*).

Proof By Taylor's resolution, it follows that for any graph G we have $\beta_{1,j}(I_G) = 0$ for $j \ne 3, 4$. By [9, Theorem 3.2.4], $\beta_{1,4}(I_G) = 0$ if and only if G^c has no induced 4-cycles. □

We will show that the condition that G^c has no induced 4-cycles has strong implications for the lcm-lattice. Let M be a monomial ideal in the polynomial ring S. Let $L(M)$ be the lcm-lattice of M introduced in [8]. The atoms of the lattice are the minimal monomial generators of M. The elements in $L(M)$ are the least common multiples of the atoms ordered by divisibility; in particular the bottom element is 1, considered as the lcm of the empty set. For an open interval $(1, m)_{L(M)}$ in $L(M)$, we denote by $\Delta(1, m)_{L(M)}$ the order complex of the interval, and we set $\tilde{H}_{i-1}((1, m)_{L(M)}; k) = \tilde{H}_{i-1}(\Delta(1, m)_{L(M)}; k)$ denoting reduced homology with coefficients in k. By [8, 14], the multigraded Betti numbers of M can be computed by the homology of the open intervals in $L(M)$ as follows: if $m \notin L(M)$ then $\beta_{i,m}(M) = 0$ for every i, and if $m \in L(M)$ and $i \ge 1$ we have

$$\beta_{i,m}(M) = \dim_k \tilde{H}_{i-1}\big((1, m)_{L(M)}; k\big). \tag{1.4}$$

Phan proved in [15] that if a monomial ideal has a linear minimal free resolution, then its lcm-lattice is graded. Our first main result is the next theorem, which shows that the tools for studying a graded poset topology are applicable if G^c has no induced 4-cycle.

Theorem 1.5 *If G^c has no induced 4-cycle, then for any $s \geq 1$ the lcm-lattice $L(I_G^s)$ is graded, and except for the minimum, the rank function is given by* $\operatorname{rank}(m) = \deg(m) - 2s + 1$ *(here, m is a monomial in $L(I_G^s)$).*

Proof Let $1 \neq m, m' \in L(I_G^s)$ be monomials such that m divides m' and $\deg(m') - \deg(m) > 1$. We need to show the existence of a monomial $h \in L(I_G^s) \setminus \{m, m'\}$ such that the divisibility conditions $m \mid h \mid m'$ hold.

There exist a variable x and a non-negative integer r such that x^r divides m, x^{r+1} does not divide m, and x^{r+1} divides m' (take $r = 0$ in case $s = 1$). Let N be the set of neighbors of x in the subgraph induced by G on the set $\operatorname{supp}(m')$. Let $a \in (1, m]$ be an atom, such that x^r divides a. Let $a = \Pi_{1 \leq i \leq s} v_i u_i$, where $v_i u_i \in G$. We consider the following two cases.

Case 1: Suppose that there exists an i such that $v_i \in N$ and $u_i \neq x$. In this case, $b = \dfrac{a}{v_i u_i} \cdot (v_i x)$ is an atom in $(1, m']$, and $h = b \vee m = mx$ is a monomial of the desired type.

Case 2: Suppose that there exists no i such that $v_i \in N$ and $u_i \neq x$. Since x^{r+1} divides m', it follows that x^{r+1} divides some atom, hence $s \geq r + 1$. Therefore, there exists a j such that x does not divide $v_j u_j$. By the assumption in Case 2, it follows that $v_j u_j$ is disjoint from the set $N \cup \{x\}$.

We have

$$\sum_{w \in N} \deg_{m'}(w) \geq \deg_{m'}(x) \geq r + 1 > r = \sum_{w \in N} \deg_a(w),$$

where $\deg_g(z)$ denotes the exponent of a variable z in a monomial g. Therefore, there exists a q such that $v_q \in N$ and $\deg_a(v_q) < \deg_{m'}(v_q)$.

Since G^c has no induced 4-cycle, there exists an edge e connecting the edges $v_j u_j$ and $v_q x$. This edge must be either $e = yx$ or $e = y v_q$, with $y \in v_j u_j$. If $e = yx$, then $c = \frac{a}{v_j u_j} yx$ is an atom in $(1, m']$ and $h = c \vee m = mx$ is a monomial of the desired type. Now, suppose that $e = y v_q$. Consider $b = \frac{a}{v_j u_j} \cdot (y v_q)$, which is an atom in $(1, m']$. If b is also an atom in $(1, m]$, then apply Case 1 to the atom b. Otherwise, $h = b \vee m = m v_q$ is a monomial of the desired type. □

For a simplicial complex Γ, let $\alpha(\Gamma)$ denote the largest codimension of a non-vanishing reduced homology, and set $\alpha(\Gamma) = 0$ if Γ is acyclic. For an open interval $(1, m)_{L(M)}$, we set $\alpha(1, m)_{L(M)} = \alpha(\Delta(1, m)_{L(M)})$.

Proposition 1.6 *If G^c has no induced 4-cycle, then for any $s \geq 1$ we have*

$$\operatorname{reg}\big(I_G^s\big) = 2s + \max_{m \in L(I_G^s),\, m \neq 1} \big\{\alpha(1, m)_{L(I_G^s)}\big\}.$$

Proof Denote $M = I_G^s$. Applying (1.4) and Theorem 1.5, we get

$$\begin{aligned}
\operatorname{reg}(M) &= \max\{j - i \mid \beta_{i,j}(M) \neq 0\} \\
&= \max\{\deg(m) - i \mid \tilde{H}_{i-1}((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s - 1 + \max\{\deg(m) - 2s + 1 - i \mid \tilde{H}_{i-1}((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s - 1 + \max\{\operatorname{rank}(m) - i \mid \tilde{H}_{i-1}((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s + \max\{\operatorname{rank}(m) - 2 - p \mid \tilde{H}_p((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s + \max\{\dim \Delta(1,m)_{L(M)} - p \mid \tilde{H}_p((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s + \max\{q \mid \tilde{H}_{\dim \Delta(1,m)_{L(M)} - q}((1,m)_{L(M)}; k) \neq 0\} \\
&= 2s + \max_{m \in L(M),\, m \neq 1} \{\alpha(1,m)_{L(M)}\}.
\end{aligned}$$

□

Based on many Macaulay2 examples, computed by Francisco, the following possibility seemed reasonable:

Question 1.7 (Francisco-Hà-Van Tuyl; personal communication) Is it true that I_G^s has a linear resolution for all $s \geq 2$ if and only if G^c has no induced 4-cycles?

Francisco, Hà, and Van Tuyl proved the following result, which provides one direction of Question 1.7. We present our own short proof.

Proposition 1.8 (Francisco-Hà-Van Tuyl; non-published) *If I_G^s has a linear resolution for some $s \geq 1$, then G^c has no induced 4-cycles.*

Proof Suppose that there exist two strongly disjoint edges $x_i x_j$ and $x_p x_q$ in G, that is, $x_i x_p x_j x_q$ is a 4-cycle in G^c. By [8], the Betti numbers of I_G^s can be computed using the lcm-lattice $L(I_G^s)$ of I_G^s. The monomials $(x_i x_j)^s$ and $(x_p x_q)^s$ are atoms in the lattice. The monomial $(x_i x_j)^s (x_p x_q)^s$ covers these two atoms since the edges are strongly disjoint. Therefore, the open interval $(1, (x_i x_j)^s (x_p x_q)^s)$ consists of the two atoms. Hence,

$$\beta_{1,(x_i x_j)^s (x_p x_q)^s} = \tilde{H}_0\big((1, (x_i x_j)^s (x_p x_q)^s); k\big) = 1.$$

Since $\deg((x_i x_j)^s (x_p x_q)^s) = 4s$, we find that the graded Betti number $\beta_{1,4s} \neq 0$, so the minimal free resolution of I_G^s is not linear. □

Thus, Question 1.7 is reduced to the following:

Question 1.9 Is it true that if G^c has no induced 4-cycles then I_G^s has a linear resolution for all $s \geq 2$?

Nevo [13] proves that if G is claw-free (in particular, in the case when G^c is a q-cycle and $q \geq 5$), then I_G^2 has a linear resolution. Our second main result is a counterexample to Question 1.9 (and thus, to Question 1.7).

Counterexample 1.10 We felt that candidates for counterexamples are the graphs such that G^c has no induced 4-cycles and the clique complex of G^c is a triangulation of a sphere. If the sphere has a high dimension, then the known graphs of this type have many edges and vertices, so the examples cannot be checked with Macaulay2. However, we can construct small such graphs if the sphere is 2-dimensional.

We will define a graph Q. Let Q have vertices $a_1, a_2, a_3, a_4, a_5, a_6$, and $b_1, b_2, b_3, b_4, b_5, b_6$. Let the edges of Q^c be the edges of an icosahedron on these vertices. For those who would like to verify our computer computations, we list the edges of Q^c:

$a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_1$ (from the bottom pentagon)

$b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_5b_1$ (from the top pentagon)

$a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_1b_2, a_2b_3, a_3b_4, a_4b_5, a_5b_1$ (from the walls)

$a_6a_1, a_6a_2, a_6a_3, a_6a_4, a_6a_5$ (from the bottom cone)

$b_6b_1, b_6b_2, b_6b_3, b_6b_4, b_6b_5$ (from the top cone).

In particular, Q^c has 12 vertices and 30 edges. The graph Q^c contains no induced 4-cycles and its clique complex is a triangulation of a 2-sphere. Thus, the graph Q is of the desired type. We consider the edge ideal

$$\begin{aligned} I_Q = (&a_6b_6,\ a_6b_1,\ a_6b_2,\ a_6b_3,\ a_6b_4,\ a_6b_5,\ b_6a_1,\ b_6a_2,\ b_6a_3,\ b_6a_4,\ b_6a_5,\ b_1b_4,\\ &b_4b_2,\ b_2b_5,\ b_5b_3,\ b_3b_1,\ a_1a_4,\ a_4a_2,\ a_2a_5,\ a_5a_3,\ a_3a_1,\ a_1b_3,\ a_1b_4,\ a_1b_5,\\ &a_2b_4,\ a_2b_5,\ a_2b_1,\ a_3b_5,\ a_3b_1,\ a_3b_2,\ a_4b_1,\ a_4b_2,\ a_4b_3,\ a_5b_2,\ a_5b_3,\ a_5b_4\,). \end{aligned}$$

Computation with Macaulay2 shows that

$$\operatorname{reg}(I_Q) = 4 \quad \text{and} \quad \operatorname{reg}\big(I_Q^2\big) = 5.$$

Thus, I_Q^2 does not have a linear minimal free resolution.

M. Stillman computed with Macaulay2 that $\operatorname{reg}(I_Q^3) = 6$ and $\operatorname{reg}(I_Q^4) = 8$, so I_Q^3 and I_Q^4 have linear minimal free resolutions.

Open Problems 1.11 Based on the examples computed by Francisco using Macaulay2 and also on our own experience, we raise the following questions:

(1) The main question is if it is true that I_G is C_4-free if and only if I_G^s has a linear minimal free resolution for every $s \gg 0$?
(2) Our second question is meant to be a tool for the study of the first question. Suppose that G^c has no induced 4-cycles. Is it true that for $s \geq 1$, we have

$$\operatorname{reg}\big(I_G^{s+1}\big) \leq \max\big\{2s+2,\ \operatorname{reg}\big(I_G^s\big)+1\,\big\}?$$

Note that the inequality $2s + 2 \leq \operatorname{reg}(I_G^{s+1})$ holds since I_G^{s+1} is generated in degree $2s + 2$.

A positive answer to Question 1.11(2) will imply that the following conditions are equivalent:

(a) I_G^s has a linear resolution for some $s \geq 2$.
(b) I_G^s has a linear resolution for every $s \geq \operatorname{reg}(I_G) - 1$.

(c) I_G has only linear minimal first syzygies, that is, $\beta_{1,j}(I_G) = 0$ for $j > 3$.
(d) G^c has no induced 4-cycles.

Indeed: (c) and (d) are equivalent by Proposition 1.3; (a) implies (d) by Proposition 1.8. Obviously, (b) implies (a). Finally, (d) implies (b) if we have a positive answer to question (2).

Question 1.11(2) is open even in the case $\operatorname{reg}(I_G) = 3$: is it true that I_G^s has a linear resolution for all $s \geq 2$ if G^c has no induced 4-cycles and $\operatorname{reg}(I_G) = 3$?

Remarks 1.12 The related question about the existence of linear quotients was studied in [11]. The structure of the minimal free resolution of a C_4-free graph is only known in the special case when the complement graph G^c is a cycle of length ≥ 5. In that case, the Betti numbers are computed in [6, Proposition 3.1], and the minimal free resolution is constructed in [1].

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