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A Higman inequality for regular near polygons

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Abstract The inequality of Higman for generalized quadrangles of order (s,t) with $s>1$ states that $t\le s^2$. We generalize this by proving that the intersection number c_i of a regular near $2d$-gon of order (s,t) with $s>1$ satisfies the tight bound $c_i\le (s^{2i}-1)/(s^2-1)$, and we give properties in case of equality. It is known that hemisystems in generalized quadrangles meeting the Higman bound induce strongly regular subgraphs. We also generalize this by proving that a similar subset in regular near $2d$-gons meeting the bounds would induce a distance-regular graph with classical parameters $(d,b,\alpha,\beta)=(d,-q,-(q+1)/2,-((-q)^d+1)/2)$ with q an odd prime power.

Keywords Distance-regular graphs · Regular near polygons · Dual polar graphs · Hemisystems · Classical parameters

1 Introduction

We refer the reader to Sect. 2 for the definitions of, for instance, (finite) generalized polygons, near polygons, and polar spaces.

Feit and Higman [18] showed that (finite) generalized n-gons of order $(s,t)\neq(1,1)$ with $n\ge 3$ can only exist if $n\in\{3,4,6,8,12\}$; if $n=12$, then $s=1$ or $t=1$. If $s>1$, then the following inequalities must hold: if $n=4$, then $t\le s^2$ [20], if $n=6$, then $t\le s^3$ [19], and if $n=8$, then $t\le s^2$ [21]. Bose and Shrikhande [5] also proved that if $n=4$ and $t=s^2$, then for any triple of nonadjacent vertices, the number

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of vertices adjacent to all three is independent of the chosen triple, namely $s+1$. This property actually characterizes generalized quadrangles of order (s, s^2) with $s > 1$ [9].

Near polygons were introduced by Shult and Yanushka [30] and include the generalized polygons. Restrictions on the parameters of regular near polygons were obtained in, for instance, [7, 22, 23, 26], and [24]. In particular, Hiraki and Koolen [22] proved that if Γ is a regular near $2d$-gon of order (s,t) with $s>1$, then $t < s^{4d/r-1}$ for a certain integer $r \geq 1$.

We will generalize the inequality on the parameters of generalized quadrangles to regular near $2d$-gons and give a similar property in case of equality. The necessary tools will be introduced in Sect. 3, and our main result will be given in Theorems 1 and 2 in Sect. 4.

Segre [29] proved for the unique classical generalized quadrangle of order (q, q^2) with q an odd prime power that if each singular line meets a nontrivial subset of points S in exactly m points, then $m = (q+1)/2$. Such sets of points in any generalized quadrangle of order (s, s^2) are known as *hemisystems*. We will generalize this result in Sect. 5.

It was also proved in [33] (in the classical case) and in [10] (for all generalized quadrangles of order (s, s^2)) that hemisystems induce a strongly regular subgraph. We will generalize this result in Sect. 6 by proving that a similar subset of points in the regular near $2d$-gon arising from the polar space $H(2d-1, q^2)$ would induce a distance-regular subgraph of diameter d with classical parameters $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d+1)/2)$. The existence of such graphs remains an open problem for $d \geq 3$.

2 Preliminaries

2.1 Distance-regular graphs

All graphs will be assumed to be undirected, without loops or multiple edges, and with a finite nonempty set of vertices. In any connected graph Γ, we will write $d(x,y)$ for the distance between any two vertices x and y, and $\Gamma_i(x)$ will denote the set of vertices at distance i from a given vertex x. The *diameter* of a connected graph Γ is the maximum distance between its vertices, and the distance of a vertex x to a nonempty subset S is $\min\{d(x,y) \mid y \in S\}$. A *clique* in a graph is a set of mutually adjacent vertices, and a clique is *maximal* if it is not a proper subset of another clique. A *triangle* is a clique of size three. A subset of vertices with no two adjacent elements is a *coclique*. A graph is *regular* with valency k if every vertex has exactly k neighbors.

A connected graph Γ with diameter d is *distance-regular* if there are natural numbers b_i with $i \in \{0, \dots, d-1\}$ and c_i with $i \in \{1, \dots, d\}$, known as *intersection numbers*, such that $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i$ for any two vertices x and y at distance $i \in \{1, \dots, d\}$ and such that $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i$ for any two vertices x and y at distance $i \in \{0, \dots, d-1\}$. A distance-regular graph of diameter d has *classical parameters* (d, b, α, β) if

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right),$$

with $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = i$ if $b = 1$ and $\left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_b = (b^i - 1)/(b-1)$ if $b \neq 1$ (see [6, §6] for more information).

We say a graph with v vertices is *strongly regular* and write $\mathrm{srg}(v, k, \lambda, \mu)$ if it is regular with valency k and every two distinct vertices have exactly λ or μ neighbors in common, depending on whether or not these two vertices are adjacent. The strongly regular graphs $\mathrm{srg}(v, k, \lambda, \mu)$ with $k < v - 1$ and $\mu > 0$ are precisely the distance-regular graphs of diameter two.

2.2 Near polygons in general

We will only introduce near n-gons for even n. A more general discussion can be found in [6] or [14].

A graph Γ of diameter $d \geq 2$ is a *near* $2d$*-gon* if the following two axioms are satisfied:

1. A vertex not in a triangle C and adjacent to two vertices in C is adjacent to the third as well.
2. For every vertex x and every maximal clique ℓ with $x \notin \ell$, there is a unique vertex in ℓ at minimal distance from x.

Note that the first axiom implies that through any two adjacent vertices, there is a unique maximal clique, and we will refer to them as the *singular lines*. We will also refer to the vertices of a near $2d$-gon as *points*.

A *regular near* $2d$*-gon* is a distance-regular near $2d$-gon. The intersection numbers b_i and c_i of such a regular near $2d$-gon with valency k satisfy $k = b_i + sc_i$ for every $i \in \{1, \dots, d-1\}$ and $k = sc_d$ for a certain fixed parameter s (see, for instance, [6, Theorem 6.4.1]). Every singular line has size $s + 1$ in this case, and every point is on c_d singular lines. A regular near $2d$-gon is said to be of order (s, t) if the singular line size is $s + 1$ and every point is on $t + 1$ singular lines. If x and y are two points at distance i with $i \in \{1, \dots, d\}$, then there are precisely c_i singular lines through y at distance $i - 1$ from x. Similarly, if x and y are two points at distance i with $i \in \{0, \dots, d-1\}$, then there are exactly b_i/s singular lines through y at distance i from x. We will also let t_i denote $c_i - 1$ for every $i \in \{1, \dots, d\}$.

The ordinary $2d$-gons are precisely the regular near $2d$-gons of order $(1, 1)$. In the following subsections, we will discuss two important families of (regular) near polygons, generalized polygons and dual polar graphs.

2.3 Generalized polygons

Generalized $2d$-gons were introduced by Tits [37] and are near $2d$-gons with $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for any two points x and y at distance i with $1 \leq i \leq d - 1$.

Note that we consider generalized $2d$-gons as collinearity graphs, instead of as point-line geometries. A generalized $2d$-gon has order (s,t) if it is a regular near $2d$-gon of order (s,t). Some of the known conditions on its parameters were already given in Sect. 1.

The generalized 4-gons or *generalized quadrangles* are precisely the near 4-gons. The dual polar graphs of diameter $d=2$ from the next subsection will all be examples of generalized quadrangles.

Generalized 6-gons or *generalized hexagons* of order $(1,q)$, $(q,1)$, (q,q), (q^3,q), and (q,q^3) exist for every prime power q. Generalized 8-gons or *generalized octagons* of order (q,q^2) and (q^2,q) exist with q any odd power of 2, and of order $(1,q)$ or $(q,1)$ for any prime power q. Finally, generalized 12-gons or *generalized dodecagons* of order $(1,q)$ and $(q,1)$ exist for any prime power q. In all three cases, no examples of other orders $(s,t)\neq(1,1)$ are known (see [6, §6.5] for more information).

The graph with the singular lines of a generalized $2d$-gon of order (s,t) as vertices, with two adjacent when having exactly one point in common, is a generalized $2d$-gon of order (t,s) and is referred to as the *dual*.

2.4 Dual polar graphs

A classical finite *polar space* is an incidence structure consisting of the totally isotropic subspaces of a finite-dimensional vector space V over a finite field with respect to a certain nondegenerate sesquilinear or quadratic form f. The *rank* of the polar space is the dimension d of the maximal totally isotropic subspaces or simply *maximals*. Two totally isotropic subspaces of different dimension are said to be *incident* if one is included in the other. We now list all classical finite polar spaces of rank d:

- The hyperbolic quadric $Q^+(2d-1,q)$ with $V=V(2d,q)$ and f a nondegenerate quadratic form of maximal Witt index d.
- The Hermitian variety $H(2d-1,q^2)$ with $V=V(2d,q^2)$ and f a nondegenerate Hermitian form.
- The parabolic quadric $Q(2d,q)$, with $V=V(2d+1,q)$ and f a nondegenerate quadratic form.
- The symplectic space $W(2d-1,q)$, with $V=V(2d,q)$ and f a nondegenerate alternating form.
- The Hermitian variety $H(2d,q^2)$, with $V=V(2d+1,q^2)$ and f a nondegenerate Hermitian form.
- The elliptic quadric $Q^-(2d+1,q)$, with $V=V(2d+2,q)$ and f a nondegenerate quadratic form of Witt index d.

The *dual polar graph* corresponding with a classical finite polar space is the graph Γ on its maximals, where any two vertices are adjacent if and only if they intersect in a subspace of codimension one. This graph is a regular near $2d$-gon, and two vertices are at distance i if and only if they intersect in a subspace of codimension i (see [6, §9.4]). In particular, they are at maximum distance d if and only if their intersection is a trivial subspace. Table 1 provides the singular line size $s+1$ and the

Table 1 The dual polar graphs from classical finite polar spaces

		(s, t_2)
$Q^+(2d-1, q)$	$D_d(q)$	$(1, q)$
$H(2d-1, q^2)$	${}^2A_{2d-1}(q)$	(q, q^2)
$Q(2d, q)$	$B_d(q)$	(q, q)
$W(2d-1, q)$	$C_d(q)$	(q, q)
$H(2d, q^2)$	${}^2A_{2d}(q)$	(q^3, q^2)
$Q^-(2d+1, q)$	${}^2D_{d+1}(q)$	(q^2, q)

parameter $t_2 = c_2 - 1$ for the dual polar graph corresponding to all classical finite polar spaces of rank d. (The notation for the polar space in the first column is based on its embedding in a projective space, and the notation in the second is the one related to Chevalley groups.)

The parameter c_i is then equal to $(t_2^i - 1)/(t_2 - 1)$ if $1 \le i \le d$. In particular, the number of singular lines through each vertex is given by $c_d = t + 1 = (t_2^d - 1)/(t_2 - 1)$. The number of vertices is $\prod_{i=1}^{d}(s\, t_2^{i-1} + 1)$.

The dual polar graphs from $W(3, q)$ and $Q(4, q)$ are dual to each other, and so are those from $H(3, q^2)$ and $Q^-(5, q)$ (see, for instance, [28, 3.2.1 and 3.2.3]).

3 Algebraic techniques

3.1 Association schemes and Bose–Mesner algebras

As each distance-regular graph defines an association scheme, we first describe these combinatorial structures. Bose and Shimamoto [4] introduced the notion of a d-class *association scheme* on a finite set Ω as a pair $(\Omega, \mathcal{R})$ with a set $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ of symmetric (nonempty) relations on Ω such that the following axioms hold: (i) R_0 is the identity relation, (ii) $\mathcal{R}$ is a partition of Ω^2, and (iii) there are constants p_{ij}^k, known as *intersection numbers*, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k. An immediate consequence is that each relation R_i is regular, and we will denote its valency by k_i.

If Γ is a graph with vertex set Ω and diameter d, and if we denote the distance-i relation by R_i, then $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme if and only if Γ is distance-regular (see, for instance, [6, §4.1A]).

If $(\Omega, \{R_0, \ldots, R_d\})$ is an association scheme, we will always write A_i for the symmetric (0, 1)-matrix, the rows and columns of which are indexed by the elements of Ω, with $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and $(A_i)_{x,y} = 0$ if $(x, y) \notin R_i$. Axiom (iii) can be algebraically expressed as $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, and hence the vector space spanned by $\{A_0, \ldots, A_d\}$ is a commutative $(d+1)$-dimensional algebra of symmetric matrices, known as the *Bose–Mesner algebra*. It can be shown (see, for instance, [6, §2.2]) that the Bose–Mesner algebra has a unique basis of minimal idempotents $\{E_0, \ldots, E_d\}$ with $E_i E_j = \delta_{ij} E_i$, $E_0 + \cdots + E_d = I$, and $E_0 = J/|\Omega|$, where J denotes the all-one matrix. As these minimal idempotents are symmetric, they define orthogonal projections, and hence they are positive semi-definite.

If Γ is a distance-regular graph with diameter d, then the corresponding adjacency matrix A_1 has exactly $d+1$ distinct eigenvalues. Every minimal idempotent E_j corresponds to such an eigenvalue λ_j such that $A_1E_j=\lambda_jE_j$, and the column span of E_j is precisely the (right) eigenspace of A_1 for λ_j. Conversely, if any nonzero element C of the Bose–Mesner algebra satisfies $A_1C=\lambda C$, then λ must be one of the eigenvalues λ_j of A_1, and C must be a scalar multiple of E_j. We refer to [6, §4.1B and §4.1C] for proofs and much more information. Every nonzero vector in the column span of any minimal idempotent is also an eigenvector for all A_i. If for some minimal idempotent E, the corresponding eigenvalue of A_i is given by λ_i, then E can also be written, up to a positive scalar, as

$$\sum_{i=0}^{d}\frac{\lambda_i}{k_i}A_i.$$

The latter follows from the orthogonality relations between the eigenvalues of an association scheme (see, for instance, [6, Lemma 2.2.1(iv)]).

For any set Ω, we will denote by $\mathbb{R}\Omega$ the real vector space with an orthonormal basis indexed by the elements of Ω. Note that the elements of the Bose–Mesner algebra of any association scheme on Ω define endomorphisms of $\mathbb{R}\Omega$.

For any subset $S\subseteq\Omega$, the *characteristic vector* of S is the column vector χ_S with entry 1 in the positions corresponding to elements of S and zero in all others. For any two subsets S_1 and S_2, the product $(\chi_{S_1})^{\mathrm{T}}\chi_{S_2}$ is equal to $|S_1\cap S_2|$. More generally, if $(\Omega,\{R_0,\dots,R_d\})$ is an association scheme, then for any two subsets $S_1,S_2\subseteq\Omega$, the number $(\chi_{S_1})^{\mathrm{T}}A_i\chi_{S_2}=(\chi_{S_2})^{\mathrm{T}}A_i\chi_{S_1}$ is equal to $|(S_1\times S_2)\cap R_i|$.

3.2 A particular minimal idempotent for regular near $2d$-gons

We will now consider a specific minimal idempotent. The following result is in fact already implicitly given in many proofs. We will follow that of [14, Theorem 3.14].

Lemma 1 *Let Γ be a regular near $2d$-gon of order (s,t). The element $M=\sum_{i=0}^{d}(-1/s)^iA_i$ of the Bose–Mesner algebra is a minimal idempotent up to a positive scalar, and its column span is precisely the eigenspace of the eigenvalue $-(t+1)$ of A_1. The corresponding eigenvalue λ_i of A_i is given by $k_i/(-s)^i$.*

Proof Let b_i and c_i be the intersection numbers of Γ and set $b_{-1}=b_d=c_0=c_{d+1}=0$. We also define A_{-1} and A_{d+1} as zero matrices. This allows us to algebraically express the property of intersection numbers:

$$A_1A_i=b_{i-1}A_{i-1}+(k-b_i-c_i)A_i+c_{i+1}A_{i+1}\quad\forall i\in\{0,\dots,d\}.$$

We can now write

$$A_1M=\sum_{i=0}^{d}\frac{(-1)^i}{s^i}\big(b_{i-1}A_{i-1}+(k-b_i-c_i)A_i+c_{i+1}A_{i+1}\big)$$

$$= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} \left(-\frac{b_i}{s} + (k - b_i - c_i) + (-sc_i) \right) A_i$$

$$= \sum_{i=0}^{d} \frac{(-1)^i}{s^i} \left(-\frac{k}{s} \right) A_i$$

$$= -(t+1)M,$$

where we used the identities $b_i = k - sc_i$ and $k = s(t+1)$ in the last two steps. Hence, $-(t+1)$ must be an eigenvalue of A_1, and M must be a scalar multiple of the corresponding minimal idempotent E. As both $\mathrm{trace}(M) = \mathrm{trace}(A_0)$ and $\mathrm{trace}(E)$ are positive, this scalar must be positive.

Finally, as E can also be written as $\sum_{i=0}^{d} (\lambda_i / k_i) A_i$ up to a positive scalar and $\lambda_0 = k_0 = 1$, this proves the last part of the lemma. □

From now on, we will always let M denote the element $M = \sum_{i=0}^{d} (-1/s)^i A_i$ of the Bose–Mesner algebra, corresponding to a regular near $2d$-gon of order (s, t).

4 Upper bound on the intersection number c_j

We now come to the main result of this paper. It was inspired by and generalizes [2, Lemma A.1]. The proof will make implicit use of Delsarte's linear programming bound (see, for instance, [17, Formula (4.3)]).

Theorem 1 *If Γ is a regular near $2d$-gon of order (s, t) with $s > 1$, then*

$$c_j \leq \frac{s^{2j} - 1}{s^2 - 1} \quad \forall j \in \{1, \ldots, d\}.$$

For any two vertices a and b at distance j with $1 \leq j \leq d$, define $v = \alpha \chi_{\{a\}} + \beta \chi_{\{b\}} + \gamma \chi_T$ with $(\alpha, \beta, \gamma) \in \mathbb{R}^3 \backslash \{(0,0,0)\}$ and $T = \Gamma_1(a) \cap \Gamma_{j-1}(b)$. Then $Mv = 0$ if and only if $c_j = (s^{2j} - 1)/(s^2 - 1)$ and (α, β, γ) is a scalar multiple of

$$\left(s \frac{s^{2j-2} - 1}{s^2 - 1}, (-1)^j s^{j-1}, 1 \right).$$

Proof Given a and b at distance j with $1 \leq j \leq d$, there are exactly c_j points on a common singular line with a and at distance $j-1$ from b. Hence T has size c_j, and no two points in T are on the same such singular line.

We will now consider $v^{\mathrm{T}} A_i v$ for every $i \in \{0, \ldots, d\}$. Note that for any two subsets of points S_1 and S_2, the value of $(\chi_{S_1})^{\mathrm{T}} A_i \chi_{S_2} = (\chi_{S_2})^{\mathrm{T}} A_i \chi_{S_1}$ is given by the number of ordered pairs $(\omega_1, \omega_2) \in (S_1 \times S_2)$ with $d(\omega_1, \omega_2) = i$. Our assumptions immediately yield:

$$(\chi_{\{a\}})^{\mathrm{T}} A_0 \chi_{\{a\}} = (\chi_{\{b\}})^{\mathrm{T}} A_0 \chi_{\{b\}} = 1,$$

$$(\chi_{\{a\}})^{\mathrm{T}} A_i \chi_{\{a\}} = (\chi_{\{b\}})^{\mathrm{T}} A_i \chi_{\{b\}} = 0 \quad \text{if } 1 \leq i \leq d,$$

$$(\chi_{\{a\}})^{\mathrm{T}} A_i \chi_{\{b\}} = 0 \quad \text{if } i \neq j, \quad \text{and} \quad (\chi_{\{a\}})^{\mathrm{T}} A_j \chi_{\{b\}} = 1,$$
$$(\chi_{\{a\}})^{\mathrm{T}} A_i \chi_T = 0 \quad \text{if } i \neq 1, \quad \text{and} \quad (\chi_{\{a\}})^{\mathrm{T}} A_1 \chi_T = |T| = c_j,$$
$$(\chi_{\{b\}})^{\mathrm{T}} A_i \chi_T = 0 \quad \text{if } i \neq j-1, \quad \text{and} \quad (\chi_{\{b\}})^{\mathrm{T}} A_{j-1} \chi_T = |T| = c_j.$$

Finally, as every two distinct points in T are on distinct singular lines through a, they cannot be collinear, and hence they are at distance two. This yields: $(\chi_T)^{\mathrm{T}} A_0 \chi_T = |T| = c_j$, $(\chi_T)^{\mathrm{T}} A_2 \chi_T = |T|(|T|-1) = c_j(c_j-1)$, and $(\chi_T)^{\mathrm{T}} A_i \chi_T = 0$ if $i \notin \{0, 2\}$. We will now work out the following:

$$\begin{aligned} s^j\big(v^{\mathrm{T}} M v\big) &= \sum_{i=0}^{d} (-1)^i s^{j-i} \big(v^{\mathrm{T}} A_i v\big) \\ &= \sum_{i=0}^{d} (-1)^i s^{j-i} (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T)^{\mathrm{T}} A_i (\alpha\chi_{\{a\}} + \beta\chi_{\{b\}} + \gamma\chi_T). \end{aligned}$$

This is equal to

$$\begin{aligned} &s^j\big(\alpha^2 + \beta^2 + \gamma^2 c_j\big) - s^{j-1}(2\alpha\gamma)c_j + s^{j-2}\gamma^2 c_j(c_j - 1) \\ &\quad + (-1)^{j-1} s(2\beta\gamma)c_j + (-1)^j(2\alpha\beta). \end{aligned}$$

We can rewrite this as $(\alpha, \beta, \gamma)F(\alpha, \beta, \gamma)^{\mathrm{T}}$ with

$$F = \begin{pmatrix} s^j & (-1)^j & -s^{j-1}c_j \\ (-1)^j & s^j & (-1)^{j-1} s c_j \\ -s^{j-1}c_j & (-1)^{j-1} s c_j & c_j s^{j-2}(s^2 + c_j - 1) \end{pmatrix}.$$

We compute the determinant of F:

$$\operatorname{Det}(F) = c_j s^{j-2}\big(s^2 - 1\big)\big(\big(s^{2j} - 1\big) - c_j\big(s^2 - 1\big)\big).$$

We know from Lemma 1 that M is a minimal idempotent up to a positive scalar and thus positive semidefinite. Hence $v^{\mathrm{T}} M v \geq 0$ for all $\alpha, \beta, \gamma \in \mathbb{R}$. Thus F is positive semidefinite, and hence its determinant must be nonnegative, and it is positive definite if and only if this determinant is positive. We find that $c_j \leq \frac{s^{2j}-1}{s^2-1}$ since $s > 1$. We can also write

$$Mv = 0 \quad \Longleftrightarrow \quad v^{\mathrm{T}} M v = 0 \quad \Longleftrightarrow \quad (\alpha, \beta, \gamma)F(\alpha, \beta, \gamma)^{\mathrm{T}} = 0.$$

As F is positive semidefinite, the latter will hold for $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ if and only if both F is not positive definite and $F(\alpha, \beta, \gamma)^{\mathrm{T}} = 0$. This is possible if and only if $c_j = \frac{s^{2j}-1}{s^2-1}$ and (α, β, γ) is a scalar multiple of $(s\frac{s^{2j-2}-1}{s^2-1}, (-1)^j s^{j-1}, 1)$. □

We now give a property of those regular near $2d$-gons attaining one of the bounds from the previous theorem. It is in fact based on properties of outer distributions of subsets in association schemes (see [15, Theorem 3.3]).

Theorem 2 *Let Γ be a regular near $2d$-gon of order (s,t) with $s>1$. Suppose that $c_j=(s^{2j}-1)/(s^2-1)$ for some $j\in\{1,\dots,d\}$, and consider three vertices a,b,c with $d(a,b)=j$, $d(a,c)=d$, $d(b,c)=k$. The set $\Gamma_1(a)\cap\Gamma_{j-1}(b)\cap\Gamma_{d-1}(c)$ has size*

$$\frac{s^{2j-1}+(-1)^{j+k+d}s^{d-k+j}-(-1)^{j+k+d}s^{d-k+j-1}-1}{s^2-1}.$$

Proof Let T be $\Gamma_1(a)\cap\Gamma_{j-1}(b)$. We know from Theorem 1 that $v=s\frac{s^{2j-2}-1}{s^2-1}\chi_{\{a\}}+(-1)^j s^{j-1}\chi_{\{b\}}+\chi_T$ satisfies $Mv=0$. Hence we have in particular

$$0=(\chi_{\{c\}})^{\mathrm{T}}Mv=(\chi_{\{c\}})^{\mathrm{T}}\left(\sum_{i=0}^{d}\frac{(-1)^i}{s^i}A_i\right)\left(s\frac{s^{2j-2}-1}{s^2-1}\chi_{\{a\}}+(-1)^j s^{j-1}\chi_{\{b\}}+\chi_T\right).$$

As $d(a,c)=d$, all elements of T are at distance at least $d-1$ from c. Hence if x denotes $|T\cap\Gamma_{d-1}(c)|$, then $|T\cap\Gamma_d(c)|=|T|-x=c_j-x$. The assumptions now imply: $(\chi_{\{c\}})^{\mathrm{T}}A_d\chi_{\{a\}}=(\chi_{\{c\}})^{\mathrm{T}}A_k\chi_{\{b\}}=1$, $(\chi_{\{c\}})^{\mathrm{T}}A_i\chi_{\{a\}}=0$ if $i\neq d$, $(\chi_{\{c\}})^{\mathrm{T}}A_i\chi_{\{b\}}=0$ if $i\neq k$, $(\chi_{\{c\}})^{\mathrm{T}}A_{d-1}\chi_T=x$, $(\chi_{\{c\}})^{\mathrm{T}}A_d\chi_T=c_j-x$, and $(\chi_{\{c\}})^{\mathrm{T}}A_i\chi_T=0$ if $i\notin\{d-1,d\}$. Hence we obtain

$$\frac{(-1)^d}{s^d}s\frac{s^{2j-2}-1}{s^2-1}+\frac{(-1)^k}{s^k}(-1)^j s^{j-1}+\frac{(-1)^{d-1}}{s^{d-1}}x+\frac{(-1)^d}{s^d}(c_j-x)=0.$$

Since we assume $c_j=\frac{s^{2j}-1}{s^2-1}$, we can rewrite

$$s\frac{s^{2j-2}-1}{s^2-1}+(-1)^{j+k+d}s^{d-k+j-1}+\frac{s^{2j}-1}{s^2-1}=(s+1)x.$$

This yields

$$\begin{aligned}&\bigl|\bigl(\Gamma_1(a)\cap\Gamma_{j-1}(b)\bigr)\cap\Gamma_{d-1}(c)\bigr|\\&\quad=\bigl|T\cap\Gamma_{d-1}(c)\bigr|=x=\frac{s^{2j-1}+(-1)^{j+k+d}s^{d-k+j}-(-1)^{j+k+d}s^{d-k+j-1}-1}{s^2-1}.\end{aligned}$$

□

The following corollary generalizes the Higman inequality between the parameters (s,t) of generalized quadrangles and also gives a property in case of equality.

Corollary 1 *Let Γ be a regular near $2d$-gon of order (s,t) with $s>1$. Then*

$$t+1\le\frac{s^{2d}-1}{s^2-1},$$

and if equality holds, then for any triple of points a, b, and c mutually at distance d, the set $\Gamma_1(a)\cap\Gamma_{d-1}(b)\cap\Gamma_{d-1}(c)$ has size

$$\frac{(s^d-(-1)^d)(s^{d-1}+(-1)^d)}{s^2-1}.$$

Proof This follows immediately from Theorems 1 and 2 with $j = d$ and $k = d$. □

Suppose the regular near $2d$-gon is the dual polar graph arising from a classical finite polar space of rank d. Then for any two vertices a and c at distance d, there is a bijective correspondence between the one-dimensional subspaces or 1-*spaces* of c and the elements of $\Gamma_1(a) \cap \Gamma_{d-1}(c)$, as each such 1-space p in c is in a unique neighbor ω of a in the dual polar graph, which will intersect c in precisely p.

For dual polar graphs from classical finite polar spaces of diameter d and order (s,t) with $s > 1$, the bound from Corollary 1 is attained if and only if $t_2 = s^2$. It follows from Table 1 that this is the case if and only if Γ is the dual polar graph on the maximals of $H(2d-1, q^2)$, when Γ is of order $(s,t) = (q, (q^{2d}-1)/(q^2-1)-1)$. Corollary 1 then yields that for any three maximals a, b, and c mutually at maximum distance d, the size of $\Gamma_1(a) \cap \Gamma_{d-1}(b) \cap \Gamma_{d-1}(c)$ is given by $(q^d-(-1)^d)(q^{d-1}+(-1)^d)/(q^2-1)$. As these vertices are all in $\Gamma_1(a) \cap \Gamma_{d-1}(c)$, they correspond to a set of 1-spaces in c. Thas [36] already described this set of 1-spaces (instead of just determining its size) for this particular graph. For the sake of completeness, we mention the result in a somewhat different form.

Lemma 2 *Let a, b, and c be maximals in the polar space $H(2d-1, q^2)$, pairwise intersecting trivially. The set of one-dimensional subspaces p of c such that the unique neighbor in the corresponding dual polar graph of a through p also intersects b in a 1-space is precisely the set of $(q^d-(-1)^d)(q^{d-1}+(-1)^d)/(q^2-1)$ isotropic 1-spaces of an induced polar space $H(d-1, q^2)$ in c.*

Finally, we would like to remark that the property from Corollary 1 does not characterize the regular near $2d$-gons of order (s,t) meeting the bound on t. The dual polar graph arising from the polar space $W(2d-1, q)$, which is of order $(s,t) = (q, (q^d-1)/(q-1)-1)$, provides a counterexample if d is odd, as was worked out in [25, Theorem 21], although $t_2 = s \neq s^2$ in this case. We again state the result in an adapted form.

Lemma 3 *Let a, b, and c be maximals in the polar space $W(2d-1, q)$ with d odd, pairwise intersecting trivially. The set of one-dimensional subspaces p of c such that the unique neighbor in the corresponding dual polar graph of a through p also intersects b in a 1-space has size $(q^{d-1}-1)/(q-1)$ and consists of the 1-spaces in a hyperplane (if q is even) or of an induced polar space $Q(d-1, q)$ (if q is odd) in c.*

5 On m-ovoids in regular near $2d$-gons meeting the bound

An *ovoid* of a regular near $2d$-gon Γ is a set of points S such that each singular line contains a unique point of S. If Γ is of order (s,t) with set of vertices Ω, then the ovoids are precisely the cocliques of size $|\Omega|/(s+1)$.

More generally, we will say that a subset of points S in a regular near $2d$-gon is an m-*ovoid* if every singular line contains exactly m points of S. Thas [35] introduced this concept for generalized quadrangles. We first prove a fundamental algebraic property of m-ovoids.

Lemma 4 *If S is an m-ovoid of a regular near $2d$-gon Γ of order (s,t) with set of vertices Ω, then its characteristic vector χ_S can be written as $(m/(s+1))\chi_\Omega + Mw$ for some vector w.*

Proof Suppose that $\mathcal{L}$ is the set of singular lines. Let C be the incidence matrix between points and singular lines, the columns of which are indexed by the points of Γ, and the rows by the singular lines, with $C_{\ell,a} = 1$ if $a \in \ell$ and $C_{\ell,a} = 0$ if $a \notin \ell$. As each singular line contains $s+1$ points and exactly m elements of S, we can write $C\chi_\Omega = (s+1)\chi_{\mathcal{L}}$ and $C\chi_S = m\chi_{\mathcal{L}}$. We also know that two points can only be in a common singular line if they are either equal (when they are on $t+1$ common singular lines) or at distance one (when they are on a unique common singular line). This can be expressed algebraically as $C^{\mathrm{T}}C = A_1 + (t+1)A_0$, which implies

$$\begin{aligned}\big(A_1 + (t+1)A_0\big)\left(\chi_S - \frac{m}{s+1}\chi_\Omega\right) &= \big(C^{\mathrm{T}}C\big)\left(\chi_S - \frac{m}{s+1}\chi_\Omega\right)\\ &= C^{\mathrm{T}}\left((m\chi_{\mathcal{L}}) - \frac{m}{s+1}\big((s+1)\chi_{\mathcal{L}}\big)\right)\\ &= 0.\end{aligned}$$

Hence $\chi_S - \frac{m}{s+1}\chi_\Omega$ is zero or an eigenvector with eigenvalue $-(t+1)$ of A_1, and so it follows from Lemma 1 that it is in the column span of M, which is the corresponding minimal idempotent up to a positive scalar. □

Lemma 5 *If S is an m-ovoid in a regular near $2d$-gon Γ of order (s,t), then for every point $a \in S$ and every $i \in \{0,\dots,d\}$,*

$$\big|\Gamma_i(a) \cap S\big| = k_i\left(\frac{m}{s+1} + \left(-\frac{1}{s}\right)^i\left(1 - \frac{m}{s+1}\right)\right).$$

Proof We know from Lemma 4 that χ_S can be written as $(m/(s+1))\chi_\Omega + Mw$. Note that $A_i\chi_\Omega = k_i\chi_\Omega$ and $A_i(Mw) = \lambda_i(Mw)$, where λ_i denotes the eigenvalue of A_i corresponding to the column span of M (see Lemma 1). We can now write

$$\begin{aligned}\big|\Gamma_i(a) \cap S\big| &= (\chi_{\{a\}})^{\mathrm{T}} A_i \chi_S\\ &= (\chi_{\{a\}})^{\mathrm{T}} A_i\left(\frac{m}{s+1}\chi_\Omega + Mw\right)\\ &= \frac{m}{s+1}(\chi_{\{a\}})^{\mathrm{T}}(k_i\chi_\Omega) + (\chi_{\{a\}})^{\mathrm{T}}\big(\lambda_i(Mw)\big)\\ &= \frac{m}{s+1}k_i + \lambda_i(\chi_{\{a\}})^{\mathrm{T}}\left(\chi_S - \frac{m}{s+1}\chi_\Omega\right)\\ &= \frac{m}{s+1}k_i + \lambda_i\left(1 - \frac{m}{s+1}\right).\end{aligned}$$

Applying the formula $\lambda_i/k_i = (-1/s)^i$ from Lemma 1 now completes the proof. □

The technique used in the following proof is based on the concept of design-orthogonal pairs of vectors (see, for instance, [17, Theorem 6.7]).

Lemma 6 *If S is an m-ovoid in a regular near $2d$-gon of order (s,t) with $s>1$ and with $c_j=(s^{2j}-1)/(s^2-1)$ for some $j\in\{1,\dots,d\}$, and a and b are two elements of S at distance j in Γ, then*

$$\bigl|S\cap\Gamma_1(a)\cap\Gamma_{j-1}(b)\bigr| = m\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^2-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1}.$$

Proof Let T denote the subset $\Gamma_1(a)\cap\Gamma_{j-1}(b)$ and take $\alpha=s\frac{s^{2j-2}-1}{s^2-1}$ and $\beta=(-1)^j s^{j-1}$. We know from Theorem 1 that $v=\alpha\chi_{\{a\}}+\beta\chi_{\{b\}}+\chi_T$ satisfies $Mv=0$. We now consider $(\chi_S)^{\mathrm{T}}v$:

$$(\chi_S)^{\mathrm{T}}v=(\chi_S)^{\mathrm{T}}(\alpha\chi_{\{a\}}+\beta\chi_{\{b\}}+\chi_T)=\alpha+\beta+|S\cap T|.$$

On the other hand, Lemma 4 implies that χ_S can be written as $(m/(s+1))\chi_\Omega+Mw$. Hence,

$$\begin{aligned}(\chi_S)^{\mathrm{T}}v&=\left(\frac{m}{s+1}\chi_\Omega+Mw\right)^{\mathrm{T}}v\\&=\frac{m}{s+1}(\chi_\Omega)^{\mathrm{T}}v+w^t(Mv)\\&=\frac{m}{s+1}(\chi_\Omega)^{\mathrm{T}}(\alpha\chi_{\{a\}}+\beta\chi_{\{b\}}+\chi_T)\\&=\frac{m}{s+1}\bigl(\alpha+\beta+|T|\bigr)=\frac{m}{s+1}(\alpha+\beta+c_j).\end{aligned}$$

Hence we obtain

$$\bigl|S\cap\bigl(\Gamma_1(a)\cap\Gamma_{j-1}(b)\bigr)\bigr|=|S\cap T|=\frac{m}{s+1}(\alpha+\beta+c_j)-(\alpha+\beta),$$

which yields the desired result after substituting for α, β, and c_j. □

We can now severely restrict the size of m-ovoids in a regular near $2d$-gon if at least one of the nontrivial bounds from Theorem 1 is met.

Theorem 3 *If Γ is a regular near $2d$-gon of order (s,t) with $s>1$ and $c_j=(s^{2j}-1)/(s^2-1)$ for some $j\in\{2,\dots,d\}$, then m-ovoids with $0<m<s+1$ can only exist for $m=(s+1)/2$.*

Proof Suppose that S is an m-ovoid with $0<m<s+1$. Consider any point b in S. We will count the number N of pairs (p,a) of adjacent points in $(\Gamma_{j-1}(b)\cap S)\times(\Gamma_j(b)\cap S)$ in two ways. The size of $\Gamma_{j-1}(b)\cap S$ is given by Lemma 5. For each point

p in $\Gamma_{j-1}(b) \cap S$, there are b_{j-1}/s singular lines through p such that the distance from b to this singular line is $d(p,b) = j-1$. The other points on those singular lines are precisely the neighbors of p at distance j from b. Each such singular line contains exactly $m-1$ points in $S \backslash \{p\}$, all at distance j from b. Hence,

$$N = k_{j-1}\left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^{j-1}\right)\frac{b_{j-1}}{s}(m-1).$$

We also know the size of $\Gamma_j(b) \cap S$ from Lemma 5, and for each point a in that subset, the number of its neighbors in S at distance $j-1$ from b is given by Lemma 6. We find

$$\begin{aligned} N = {} & k_j\left(\frac{m}{s+1} + \left(1 - \frac{m}{s+1}\right)\left(-\frac{1}{s}\right)^{j}\right) \\ & \times \left(m\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^2-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1}\right). \end{aligned}$$

When setting $m = x(s+1)$ and using the identity $k_{j-1}b_{j-1} = k_j c_j$ and the assumption $c_j = (s^{2j}-1)/(s^2-1)$, we see that x must be a root of the following polynomial in x:

$$\begin{aligned} & \left(x + (1-x)\left(-\frac{1}{s}\right)^{j-1}\right)\frac{s^{2j}-1}{s(s^2-1)}\big(x(s+1)-1\big) - \left(x + (1-x)\left(-\frac{1}{s}\right)^{j}\right) \\ & \quad \times \left(x\frac{(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s-1} - s\frac{(s^j-(-1)^j)(s^{j-2}+(-1)^j)}{s^2-1}\right), \end{aligned}$$

which can be rewritten as

$$\frac{(-1)^j(s^j-(-1)^j)(s^{j-1}+(-1)^j)}{s^j(s-1)}(x-1)(2x-1).$$

Since we assumed that $j \geq 2$ and $0 < m < s+1$, we see that $m/(s+1) = x = 1/2$. $\square$

The $((s+1)/2)$-ovoids of generalized quadrangles of order (s, s^2) (or thus with $c_2 = (s^4-1)/(s^2-1)$) are known as *hemisystems*. For the dual polar graph Γ arising from the polar space $H(3, q^2)$, which is a generalized quadrangle of order (q, q^2), Theorem 3 was already obtained for odd q by Segre [29] and, for even q, by Bruen and Hirschfeld [8]. Segre also proved that there is a unique hemisystem (up to equivalence) if $q = 3$. A construction for hemisystems in the dual polar graph from $H(3, q^2)$ for every odd prime power q was given by Cossidente and Penttila [12]. The restriction on m was obtained for all generalized quadrangles of order (s, s^2) in [35]. A hemisystem in a nonclassical generalized quadrangle of order $(5, 5^2)$ was constructed in [1]. Very recently, it was proved in [2] that hemisystems exist in all flock generalized quadrangles (see [34] for more information on the latter).

6 Construction of an induced distance-regular graph

In the regular near $2d$-gons where every intersection number c_j meets the bound from Theorem 1, we can construct another distance-regular graph by use of a (nontrivial) m-ovoid.

In general, a hemisystem in any generalized quadrangle of order (s, s^2) induces a strongly regular graph with parameters $\mathrm{srg}((s+1)(s^3+1)/2, (s-1)(s^2+1)/2, (s-3)/2, (s-1)^2/2)$ (this was proved in [10]). For the dual polar graph from $H(3, q^2)$, this result was already obtained by Thas [33]; for $q = 3$, the induced graph on the unique hemisystem is isomorphic to the Gewirtz graph. The following lemma generalizes these facts to regular near $2d$-gons meeting the bounds from Theorem 1. It only requires assumptions on the parameters, but we will later see that for $d \geq 3$, they actually force the near polygon to be the dual polar graph from $H(2d-1, q^2)$.

Lemma 7 *Let Γ be a regular near $2d$-gon of order (s,t) with $s > 1$ and $c_j = (s^{2j}-1)/(s^2-1)$ for every $j \in \{1, \dots, d\}$. Suppose that S is an $((s+1)/2)$-ovoid. Let Γ' be the induced subgraph of Γ on S. The distance between any two vertices in Γ' is the same as in Γ, and Γ' is distance-regular with diameter d and intersection numbers*

$$b'_j = \frac{s^{2d}-s^{2j}}{2(s+1)} \quad \forall j \in \{0, \dots, d-1\};$$

$$c'_j = \frac{(s^j-(-1)^j)(s^{j-1}-(-1)^j)}{2(s+1)} \quad \forall j \in \{1, \dots, d\}.$$

Proof Consider any elements $a, b \in S$ at distance j in Γ with $j \in \{1, \dots, d\}$. Lemma 6 yields, after substituting $(s+1)/2$ for m, that

$$\left|S \cap \left(\Gamma_1(a) \cap \Gamma_{j-1}(b)\right)\right| = \frac{(s^j-(-1)^j)(s^{j-1}-(-1)^j)}{2(s+1)},$$

which is in particular at least one. Induction on j now yields that the distance between a and b in the induced subgraph is also j.

Now consider any two elements a and b of S at distance j in Γ with $0 \leq j \leq d-1$. There are precisely b_j/s singular lines through a at distance j from b. Only on these singular lines, through a, can points at distance $j+1$ from b and adjacent to a be found, and each such singular line contains exactly $(s-1)/2$ points of $S \backslash \{a\}$. Hence,

$$\left|S \cap \left(\Gamma_1(a) \cap \Gamma_{j+1}(b)\right)\right| = \frac{b_j}{s}\frac{s-1}{2} = \frac{k-sc_j}{s}\frac{s-1}{2} = (c_d-c_j)\frac{s-1}{2} = \frac{s^{2d}-s^{2j}}{2(s+1)},$$

where we let c_0 be zero. Note also that the last value is nonzero if $0 \leq j \leq d-1$, so by induction the diameter of Γ' is precisely d. □

If Γ is the dual polar graph arising from $H(2d-1, q^2)$, then it is a regular near $2d$-gon of order $(s,t) = (q, (q^{2d}-1)/(q^2-1)-1)$ with parameters $c_j = (q^{2j}-1)/(q^2-1)$ for every $j \in \{1, \dots, d\}$ and hence meeting the requirements of

Lemma 7. The following lemma characterizes these graphs as the only such regular near $2d$-gons for any $d \geq 3$.

Lemma 8 *Suppose that Γ is a regular near $2d$-gon of order (s,t) with $d \geq 3$ and $s > 1$. If $c_j = (s^{2j}-1)/(s^2-1)$ for all $j \in \{1,\dots,d\}$, then s is a prime power q, and Γ is the dual polar graph arising from the polar space $H(2d-1,q^2)$.*

Proof The assumptions imply that $b_j = s(s^{2d}-s^{2j})/(s^2-1)$ for every $j \in \{0,\dots,d-1\}$. Hence Γ has classical parameters $(d,b,\alpha,\beta) = (d,s^2,0,s)$. The regular near $2d$-gons of order (s,t) with $s>1$, $d \geq 3$, and with classical parameters $(d,b,0,\beta)$ are characterized in [6, Theorem 9.4.4] as either a dual polar graph arising from a classical finite polar space or a Hamming graph. However, since Hamming graphs have intersection numbers $c_j = j < (s^{2j}-1)/(s^2-1)$ (see, for instance, [6, Theorem 9.2.1]), we can exclude the last possibility. The condition $t_2 = c_2 - 1 = s^2$ and Table 1 now yield that Γ can only be the dual polar graph arising from $H(2d-1,q^2)$ with $q = s$. □

Remark 1 The so-called *dual intersection numbers* of the association scheme defined by the dual polar graph from $H(2d-1,q^2)$, with respect to the idempotent corresponding with M, satisfy a certain condition (see, for instance, [3, p. 315]), also known as *almost dual bipartiteness*. Terwilliger [32] proved that this condition always implies a linear dependence as the one between $M\chi_{\{a\}}$, $M\chi_{\{b\}}$, and $M\chi_T$ in Theorem 1.

Because of Lemma 8, the result from Lemma 7 comes down to the following if the diameter is at least three.

Theorem 4 *Let S be a $((q+1)/2)$-ovoid in the dual polar graph Γ from $H(2d-1,q^2)$ with q odd. The induced subgraph Γ' on S is distance-regular with classical parameters*

$$(d,b,\alpha,\beta) = \left(d, -q, -\left(\frac{q+1}{2}\right), -\left(\frac{(-q)^d+1}{2}\right)\right).$$

The distance between any two vertices in Γ' is the same as in Γ.

Proof This follows immediately from Lemma 7 and the definition of classical parameters in Sect. 2.1. □

Let Γ be the dual polar graph from $H(2d-1,q^2)$ with q an odd prime power. We first observe that any $((q+1)/2)$-ovoid in Γ would also yield a $((q+1)/2)$-ovoid in the residual graph induced on the set of vertices through a fixed one-dimensional isotropic subspace of the polar space, which is isomorphic to the dual polar graph arising from $H(2(d-1)-1,q^2)$. The case $d=2$ was already discussed at the end of Sect. 5, but even for $d=3$, no constructions are known to the author. Theorem 4 here yields that if S is a set of $(q+1)(q^3+1)(q^5+1)/2$ maximals in $H(5,q^2)$ such that every totally isotropic 2-space is in exactly $(q+1)/2$ elements of S, the induced graph Γ' on S is distance-regular with diameter three and intersection array $\{b'_0, b'_1, b'_2; c'_1, c'_2, c'_3\}$,

$$\big\{\big(q^3-1\big)\big(q^2-q+1\big)/2, q^2(q-1)\big(q^2+1\big)/2, q^4(q-1)/2; \\ 1, (q-1)^2/2, \big(q^2-q+1\big)\big(q^2+1\big)/2\big\}.$$

In general, no distance-regular graphs with the classical parameters found in Theorem 4 of diameter at least three seem to be known. Weng [39] proved that distance-regular graphs with classical parameters (d, b, α, β) with $b < -1$, $d \geq 4$, $c_2 > 1$, and with triangles must either be in one of two known families or satisfy $(d, b, \alpha, \beta) = (d, -q, -(q+1)/2, -((-q)^d + 1)/2)$ for some odd prime power q. For $q = 3$, the induced graph on a 2-ovoid in the dual polar graph from $H(2d-1, q^2)$ would be a triangle-free distance-regular graph with classical parameters $(d, b, \alpha, \beta) = (d, -3, -2, -((-3)^d + 1)/2)$, the nonexistence of which was conjectured for $d \geq 3$ in [27, Conjecture 4.11].

In a classical finite polar space of rank d, *t-designs* are defined as subsets of maximals such that each totally isotropic t-space of the polar space is included in exactly m elements of S for some m. Hence m-ovoids in dual polar graphs are precisely the $(d-1)$-designs. Algebraic characterizations for t-designs in this context (as well as in many other association schemes) are given in [31], based on Delsarte's theory of regular semilattices from [16]. Moreover, 1-designs in the dual polar graph arising from $H(5, q^2)$ with size exactly half the number of all maximals were constructed for every odd prime power q in [13]. Any *partial spread* in the polar space $H(2d-1, q^2)$ with d odd, i.e., a subset of pairwise trivially intersecting maximals, of (maximum) size $q^d + 1$ should also intersect any $((q+1)/2)$-ovoid in exactly $(q^d + 1)/2$ elements (see [38, Corollary 4.4]).

Finally, it is worth noting that in any dual polar graph, an m-ovoid S_1 and its complement S_2 yield a *regular* or *equitable partition* $\{S_1, S_2\}$: for each point p, the number of neighbors in both parts only depends on whether $p \in S_1$ or $p \in S_2$. Regular partitions of dual polar spaces were discussed in detail with many examples in [11].

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References

1. Bamberg, J., De Clerck, F., Durante, N.: A hemisystem of a nonclassical generalised quadrangle. Des. Codes Cryptogr. **51**, 157–165 (2009)
2. Bamberg, J., Giudici, M., Royle, G.: Every flock generalised quadrangle has a hemisystem. Bull. Lond. Math. Soc. **42**, 795–810 (2010)
3. Bannai, E., Ito, T.: Algebraic Combinatorics. I. Association schemes. Benjamin/Cummings, Menlo Park (1984)
4. Bose, R.C., Shimamoto, T.: Classification and analysis of partially balanced incomplete block designs with two associate classes. J. Am. Stat. Assoc. **47**, 151–184 (1952)
5. Bose, R.C., Shrikhande, S.S.: Geometric and pseudo-geometric graphs $(q^2+1, q+1, 1)$. J. Geom. **2**, 75–94 (1972)

6. Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-Regular Graphs. Springer, Berlin (1989)
7. Brouwer, A.E., Wilbrink, H.A.: The structure of near polygons with quads. Geom. Dedic. **14**, 145–176 (1983)
8. Bruen, A.A., Hirschfeld, J.W.P.: Applications of line geometry over finite fields. II. The Hermitian surface. Geom. Dedic. **7**, 333–353 (1978)
9. Cameron, P.J.: Partial quadrangles. Q. J. Math. **26**, 61–73 (1975)
10. Cameron, P.J., Delsarte, P., Goethals, J.M.: Hemisystems, orthogonal configurations, and dissipative conference matrices. Philips J. Res. **34**, 147–162 (1979)
11. Cardinali, I., De Bruyn, B.: Regular partitions of dual polar spaces. Linear Algebra Appl. **432**, 744–769 (2010)
12. Cossidente, A., Penttila, T.: Hemisystems on the Hermitian surface. J. Lond. Math. Soc. **72**, 731–741 (2005)
13. Cossidente, A., Penttila, T.: On m-regular systems on $H(5,q^2)$. J. Algebr. Comb. **29**, 437–445 (2009)
14. De Bruyn, B.: Near Polygons. Birkhäuser, Basel (2006)
15. Delsarte, P.: An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl. **10** (1973)
16. Delsarte, P.: Association schemes and t-designs in regular semilattices. J. Comb. Theory, Ser. A **20**, 230–243 (1976)
17. Delsarte, P.: Pairs of vectors in the space of an association scheme. Philips Res. Rep. **32**, 373–411 (1977)
18. Feit, W., Higman, G.: The nonexistence of certain generalized polygons. J. Algebra **1**, 114–131 (1964)
19. Haemers, W., Roos, C.: An inequality for generalized hexagons. Geom. Dedic. **10**, 219–222 (1981)
20. Higman, D.G.: Partial geometries generalized quadrangles and strongly regular graphs. In: Atti del Convegno di Geometria Combinatoria e sua Applicazioni, Perugia, pp. 263–293 (1971)
21. Higman, D.G.: Invariant relations, coherent configurations and generalized polygons. In: Combinatorics. Math Centre Tracts, vol. 57, pp. 27–43. Math Centre, Amsterdam (1974)
22. Hiraki, A., Koolen, J.: A Higman–Haemers inequality for thick regular near polygons. J. Algebr. Comb. **20**, 213–218 (2004)
23. Hiraki, A., Koolen, J.: A note on regular near polygons. Graphs Comb. **20**, 485–497 (2004)
24. Hiraki, A., Koolen, J.: A generalization of an inequality of Brouwer–Wilbrink. J. Comb. Theory, Ser. A **109**, 181–188 (2005)
25. Klein, A., Metsch, K., Storme, L.: Small maximal partial spreads in classical finite polar spaces. Adv. Geom. **10**, 379–402 (2010)
26. Neumaier, A.: Krein conditions and near polygons. J. Comb. Theory, Ser. A **54**, 201–209 (1990)
27. Pan, Y., Lu, M., Weng, C.: Triangle-free distance-regular graphs. J. Algebr. Comb. **27**, 23–34 (2008)
28. Payne, S.E., Thas, J.A.: Finite Generalized Quadrangles. European Mathematical Society (EMS), Zürich (2009)
29. Segre, B.: Forme e geometrie Hermitiane, con particolare riguardo al caso finito. Ann. Mat. Pura Appl. **70**, 1–201 (1965)
30. Shult, E., Yanushka, A.: Near n-gons and line systems. Geom. Dedic. **9**, 1–72 (1980)
31. Stanton, D.: t-designs in classical association schemes. Graphs Comb. **2**, 283–286 (1986)
32. Terwilliger, P.: Balanced sets and Q-polynomial association schemes. Graphs Comb. **4**, 87–94 (1988)
33. Thas, J.A.: Ovoids and spreads of finite classical polar spaces. Geom. Dedic. **10**, 135–143 (1981)
34. Thas, J.A.: Generalized quadrangles and flocks of cones. Eur. J. Comb. **8**, 441–452 (1987)
35. Thas, J.A.: Interesting pointsets in generalized quadrangles and partial geometries. Linear Algebra Appl. **114/115**, 103–131 (1989)
36. Thas, J.A.: Old and new results on spreads and ovoids of finite classical polar spaces. Combinatorics ’90 (Gaeta, 1990). Ann. Discrete Math. **52**, 529–544 (1992)
37. Tits, J.: Sur la trialité et certains groupes qui s’en déduisent. Publ. Math. **2**, 13–60 (1959)
38. Vanhove, F.: Antidesigns and regularity of partial spreads in dual polar graphs. J. Comb. Des. (2010). doi:10.1002/jcd.20275
39. Weng, C.: Classical distance-regular graphs of negative type. J. Comb. Theory, Ser. B **76**, 93–116 (1999)