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Association schemes from the action of PGL(2, *q*) fixing a nonsingular conic in PG(2, *q*)

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Abstract The group $\mathrm{PGL}(2, q)$ has an embedding into $\mathrm{PGL}(3, q)$ such that it acts as the group fixing a nonsingular conic in $\mathrm{PG}(2, q)$. This action affords a coherent configuration $\mathcal{R}(q)$ on the set $\mathcal{L}(q)$ of non-tangent lines of the conic. We show that the relations can be described by using the cross-ratio. Our results imply that the restrictions $\mathcal{R}_+(q)$ and $\mathcal{R}_-(q)$ of $\mathcal{R}(q)$ to the set $\mathcal{L}_+(q)$ of secant (hyperbolic) lines and to the set $\mathcal{L}_-(q)$ of exterior (elliptic) lines, respectively, are both association schemes; moreover, we show that the elliptic scheme $\mathcal{R}_-(q)$ is pseudocyclic.

We further show that the coherent configurations $\mathcal{R}(q^2)$ with q even allow certain fusions. These provide a 4-class fusion of the hyperbolic scheme $\mathcal{R}_+(q^2)$, and 3-class fusions and 2-class fusions (strongly regular graphs) of both schemes $\mathcal{R}_+(q^2)$ and $\mathcal{R}_-(q^2)$. The fusion results for the hyperbolic case are known, but our approach here as well as our results in the elliptic case are new.

Keywords Association scheme · Coherent configuration · Conic · Cross-ratio · Exterior line · Fusion · Pseudocyclic association scheme · Secant line · Strongly regular graph · Tangent line

1. Introduction

Let q be a prime power. The 2-dimensional projective linear group $\mathrm{PGL}(2, q)$ has an embedding into $\mathrm{PGL}(3, q)$ such that it acts as the group G fixing a nonsingular conic

$$\mathcal{O} = \mathcal{O}_q = \{(\xi, \xi^2, 1) \mid \xi \in \mathbf{F}_q\} \cup \{(0, 1, 0)\}$$

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in PG(2, q) setwise, see e.g. [8, p. 158]. Such a conic consists of $q + 1$ points forming an *oval*, that is, each line of PG(2, q) meets $\mathcal{O}$ in at most two points. Lines meeting the oval in two points, one point, or no points at all are called *secant* (or *hyperbolic*) lines, *tangent* lines, and *exterior* (or *elliptic*) lines, respectively. There is precisely one tangent through each point of an oval; moreover, if q is even, then all tangent lines pass through a unique point called the *nucleus* of the oval, see e.g. [8, p. 157].

It turns out that the group G acts generously transitively on both the set $\mathcal{L}_+$ of hyperbolic lines and the set $\mathcal{L}_-$ of elliptic lines. Thus we obtain two (symmetric) association schemes, one on $\mathcal{L}_+$ and the other on $\mathcal{L}_-$. We will refer to these schemes as the *hyperbolic* scheme and the *elliptic* scheme, respectively.

Our aim in this paper is to investigate these two association schemes *simultaneously*. Also investigated here is a particular fusion of these schemes when q is even. In fact, the hyperbolic and elliptic schemes are contained in the coherent configuration obtained from the action of G on the set $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$ of all non-tangent lines of the conic $\mathcal{O}$, and the fusions of the two schemes arise within a certain fusion of this coherent configuration.

These schemes as well as their fusions are not completely new, but our treatment will be new. For q even, the elliptic schemes were first introduced in [9], as a family of *pseudocyclic* association schemes on non-prime-power number of points. The hyperbolic schemes, and the particular fusion discussed here for q an even square, turn out to be the same as the schemes investigated in [3]. The fact that the particular fusion in the hyperbolic case again produces association schemes has been proved by direct computations in [6], by geometric arguments in [5], and by using character theory in [14]. The fusion schemes for q an even square in the elliptic case seem to be new.

The contents of this paper are as follows. In Section 2 we introduce the definitions and notations that are used in this paper. Then, in Section 3 we introduce the embedding of PGL(2, q) as the subgroup $G = G(\mathcal{O})$ of PGL(3, q) fixing the conic $\mathcal{O}$ in PG(2, q).

With each non-tangent line we can associate a pair of points, representing its intersection with $\mathcal{O}$ in the hyperbolic case, or its intersection with the extension $\mathcal{O}_{q^2}$ of $\mathcal{O}$ to a conic in PG(2, q^2) in the elliptic case. In Section 4 we show that the orbits of G on pairs of non-tangent lines can be described with the aid of the cross-ratio of the two pairs of points associated with the lines. These results are then used to give (new) proofs of the fact that the group action indeed affords association schemes on both $\mathcal{L}_+$ and $\mathcal{L}_-$. Moreover, these results establish the connection between the hyperbolic scheme and the scheme investigated in [3].

In Section 5 we develop an expression to determine the orbit to which a given pair of lines belongs in terms of their homogeneous coordinates.

From Section 6 on we only consider the case where q is even. In Section 6 we derive expressions for the intersection parameters of the coherent configuration $\mathcal{R}(q)$ on the non-tangent lines $\mathcal{L}$ of the conic $\mathcal{O}$; so in particular we obtain expressions for the intersection parameters of both the hyperbolic and elliptic association schemes *simultaneously*. We also include a proof of the result from [9] that the elliptic schemes are pseudocyclic. In [11] we will prove that the schemes obtained from the elliptic scheme by fusion with the aid of the Frobenius automorphism of the underlying finite field $\mathbf{F}_q$ for $q = 2^r$ with r prime are also pseudocyclic.

Then in Section 7 we define a particular fusion of the coherent configuration. The results of the previous section are used to show that this fusion is in fact again a coherent configuration, affording a four-class scheme on the set of hyperbolic lines and a three-class scheme on the set of elliptic lines. The parameters show that the restriction of these schemes to one of the classes produces in fact a *strongly regular graph*, with the same parameters as the Brouwer-Wilbrink graphs (see [2]) in the hyperbolic case and as the Metz graphs (e.g., [2]) in the elliptic case. This will be discussed in Section 8. In fact, the graphs are *isomorphic* to the Brouwer-Wilbrink graphs (in the hyperbolic case) and the Metz graphs (in the elliptic case). For the hyperbolic case, this was conjectured in [3] and proved in [5]; for the elliptic case, this was conjectured for $q = 4$ in [9], and will be proved for general even q in [10].

2. Definitions and notation

2.1. Coherent configurations

As a general reference for the material in this section, see, e.g., [1, 4, 7]. A *coherent configuration* is a pair $(X, \mathcal{R})$ where X is a finite set and $\mathcal{R}$ is a collection $\{R_0, \ldots, R_n\}$ of subsets of $X \times X$ satisfying the following conditions:

1. $\mathcal{R}$ is a partition of $X \times X$;
2. there is a subset $\mathcal{R}_{\mathrm{diag}}$ of $\mathcal{R}$ which is a partition of the diagonal $\{(x, x) \mid x \in X\}$;
3. for each R in $\mathcal{R}$, its transpose $R^\top = \{(y, x) \mid (x, y) \in R\}$ is again in $\mathcal{R}$;
4. there are integers p_{ij}^k, for $0 \le i, j, k \le n$, such that for all $(x, y) \in R_k$,

$$|\{z \in X \mid (x, z) \in R_i \quad \text{and} \quad (z, y) \in R_j\}| = p_{ij}^k.$$

The numbers p_{ij}^k are called the *intersection numbers* of the coherent configuration.

Each relation R_i can be represented by its *adjacency matrix* A_i, a matrix whose rows and columns are both indexed by X and

$$A_i(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R_i; \\ 0, & \text{otherwise.} \end{cases}$$

In terms of these matrices, and with I, J denoting the identity matrix and the all-one matrix, respectively, the axioms can be expressed in the following form:

1. $A_0 + A_1 + \cdots + A_n = J$;
2. $\sum_{i=0}^m A_i = I$, where $R_{\mathrm{diag}} = \{R_0, \ldots, R_m\}$;
3. for each i, there exists i^* such that $A_i^\top = A_{i^*}$;
4. for each $i, j \in \{0, 1, \ldots, n\}$, we have

$$A_i A_j = \sum_{k=0}^n p_{ij}^k A_k.$$

As a consequence of Properties 2 and 4, the span of the matrices $A_0, A_1, \ldots, A_n$ over the complex numbers is an algebra. It follows from Property 3 that this algebra is semi-simple, and so is isomorphic to a direct sum of full matrix algebras over the complex numbers.

The sets $Y \subseteq X$ such that $\{(y, y) \mid y \in Y\} \in \mathcal{R}$ are called the *fibres* of $\mathcal{R}$; according to Property 2, they form a partition of X. The coherent configuration is called *homogeneous* if there is only one fibre. In that case one usually numbers the relations of $\mathcal{R}$ such that R_0 is the diagonal relation.

Remark 1. The existence of the numbers $p^k_{d,k}$ and $p^k_{k,d}$ for all diagonal relations $R_d \in \mathcal{R}_{\text{diag}}$ implies that for each relation $R_k \in \mathcal{R}$ there are fibres Y, Z such that $R_k \subseteq Y \times Z$.

A coherent configuration is called *symmetric* if all the relations are symmetric. As a consequence of the above remark, a symmetric coherent configuration is homogeneous. Usually, a symmetric coherent configuration is called a *(symmetric) association scheme*. In this paper, we will call a coherent configuration *weakly symmetric* if the restriction of the coherent configuration to each of its fibres is symmetric, that is, each of its fibres carries an association scheme.

A *fusion* of a coherent configuration $\mathcal{R}$ on X is a coherent configuration $\mathcal{S}$ on X where each relation $S \in \mathcal{S}$ is a union of relations from $\mathcal{R}$.

As a typical example of coherent configuration, if G is a permutation group on a finite set X, then the orbits of the induced action of G on $X \times X$ form a coherent configuration; it is homogeneous precisely when G is *transitive*, and an association scheme if and only if G acts *generously transitively* on X, that is, for all $x, y \in X$, there exists $g \in G$ such that $g(x) = y$ and $g(y) = x$. The coherent configuration is weakly symmetric precisely when G is generously transitive on each of its orbits on X.

2.2. Association schemes

In the case of an association scheme, Properties 2 and 3 are replaced by the stronger properties:

2′. $A_0 = I$; and
3′. each A_i is symmetric.

As a consequence of these properties, the matrices $A_0 = I, A_1, \ldots, A_n$ span an algebra $\mathcal{A}$ over the reals (which is called the *Bose-Mesner algebra* of the scheme). This algebra has a basis $E_0, E_1, \ldots, E_n$ consisting of primitive idempotents, one of which is $\frac{1}{|X|}J$. So we may assume that $E_0 = \frac{1}{|X|}J$. Let $\mu_i = \text{rank } E_i$. Then

$$\mu_0 = 1, \ \ \mu_0 + \mu_1 + \cdots + \mu_n = |X|.$$

The numbers $\mu_0, \mu_1, \ldots, \mu_n$ are called the *multiplicities* of the scheme.

Define $P = (P_j(i))_{0 \le i, j \le n}$ (the *first* eigenmatrix) and $Q = (Q_j(i))_{0 \le i, j \le n}$ (the *second* eigenmatrix) as the $(n + 1) \times (n + 1)$ matrices with rows and columns indexed by $0, 1, 2, \ldots, n$ such that

$$(A_0, A_1, \ldots, A_n) = (E_0, E_1, \ldots, E_n)P,$$

and

$$|X|(E_0, E_1, \ldots, E_n) = (A_0, A_1, \ldots, A_n)Q.$$

Of course, we have

$$P = |X|Q^{-1}, \quad Q = |X|P^{-1}.$$

Note that $\{P_j(i) \mid 0 \le i \le n\}$ is the set of eigenvalues of A_j and the zeroth row and column of P and Q are as indicated below.

$$P = \begin{pmatrix} 1 & v_1 & \cdots & v_n \\ 1 & & & \\ \vdots & & & \\ 1 & & & \end{pmatrix}, Q = \begin{pmatrix} 1 & \mu_1 & \cdots & \mu_n \\ 1 & & & \\ \vdots & & & \\ 1 & & & \end{pmatrix}$$

The numbers $v_0, v_1, \ldots, v_n$ are called the *valencies (or degrees)* of the scheme.

Example 2.1. We consider *cyclotomic schemes* defined as follows. Let q be a prime power and let $q - 1 = ef$ with $e \ge 1$. Let C_0 be the subgroup of the multiplicative group of $\mathbf{F}_q$ of index e, and let $C_0, C_1, \ldots, C_{e-1}$ be the cosets of C_0. We require $-1 \in C_0$. Define $R_0 = \{(x, x) : x \in \mathbf{F}_q\}$, and for $i \in \{1, 2, \ldots, e\}$, define $R_i = \{(x, y) \mid x, y \in \mathbf{F}_q, x - y \in C_{i-1}\}$. Then $(\mathbf{F}_q, \{R_i\}_{0 \le i \le e})$ is an e-class symmetric association scheme. The intersection parameters of the cyclotomic scheme are related to the cyclotomic numbers ([13, p. 25]). Namely, for $i, j, k \in \{1, 2, \ldots, e\}$, given $(x, y) \in R_k$,

$$p_{ij}^k = |\{z \in \mathbf{F}_q \mid x - z \in C_{i-1}, y - z \in C_{j-1}\}| = |\{z \in C_{i-k} \mid 1 + z \in C_{j-k}\}|. \quad (1)$$

The first eigenmatrix P of this scheme is the following $(e + 1)$ by $(e + 1)$ matrix (with the rows of P arranged in a certain way)

$$P = \begin{pmatrix} 1 & f & \cdots & f \\ 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{pmatrix}$$

with $P_0 = \sum_{i=1}^{e} \eta_i C^i$, where C is the e by e matrix:

$$C = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}$$

and $\eta_i = \sum_{\beta \in C_i} \psi(\beta)$, $1 \leq i \leq e$, for a fixed nontrivial additive character ψ of $\mathbf{F}_q$.

Next we introduce the notion of a pseudocyclic association scheme.

Definition 2.2. Let $(X, \{R_i\}_{0 \leq i \leq n})$ be an association scheme. We say that $(X, \{R_i\}_{0 \leq i \leq n})$ is *pseudocyclic* if there exists an integer t such that $\mu_i = t$ for all $i \in \{1, \ldots, n\}$.

The following theorem gives combinatorial characterizations for an association scheme to be pseudocyclic.

Theorem 2.3. *Let $(X, \{R_i\}_{0 \leq i \leq n})$ be an association scheme, and for $x \in X$ and $1 \leq i \leq n$, let $R_i(x) = \{y \mid (x, y) \in R_i\}$. Then the following are equivalent.*

(1) *$(X, \{R_i\}_{0 \leq i \leq n})$ is pseudocyclic.*
(2) *For some constant t, we have $v_j = t$ and $\sum_{k=1}^{n} p_{kj}^k = t - 1$, for $1 \leq j \leq n$.*
(3) *$(X, \mathcal{B})$ is a $2 - (v, t, t-1)$ design, where $\mathcal{B} = \{R_i(x) \mid x \in X, 1 \leq i \leq n\}$.*

For a proof of this theorem, we refer the reader to [1, p. 48] and [9, p. 84]. Part (2) in the above theorem is very useful. For example, we may use it to prove the well-known fact that the cyclotomic scheme in Example 2.1 is pseudocyclic. The proof goes as follows. First, the nontrivial valencies of the cyclotomic scheme are all equal to f. Second, by (1) and noting that $-1 \in C_0$, we have

$$\begin{aligned} \sum_{k=1}^{e} p_{kj}^k &= \sum_{k=1}^{e} |\{z \in C_0 \mid 1 + z \in C_{j-k}\}| \\ &= |C_0| - 1 = f - 1 \end{aligned}$$

Pseudocyclic schemes can be used to construct strongly regular graphs and distance regular graphs of diameter 3 ([1, p. 388]). In view of this, it is of interest to construct pseudocyclic association schemes. For $e > 1$, the cyclotomic schemes discussed above are nontrivial examples of pseudocyclic association schemes on prime power number of points. Very few examples of pseudocyclic association schemes on non-prime-power number of points are currently known (see [9, 12], and [1, p. 390]). The examples from [9] can be found in Section 6. More examples of such association schemes will be given in [11].

3. The group PGL(2, q) as the subgroup of PGL(3, q) fixing a nonsingular conic in PG(2, q)

Through the usual identification of $\mathbf{F}_q \cup \{\infty\}$ with PG(1, q) given by

$$x \leftrightarrow (x, 1)^\top, \qquad \infty \leftrightarrow (1, 0)^\top,$$

the 2-dimensional projective linear group PGL(2, q) acts on $\mathbf{F}_q \cup \{\infty\}$, with action given by

$$\forall A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, q), \quad \text{and} \quad \forall x \in \mathbf{F}_q \cup \{\infty\},\ A \cdot x = A(x) := \frac{ax + b}{cx + d} \tag{2}$$

For any four-tuple $(\alpha, \beta, \gamma, \delta)$ in $(\mathbf{F}_q \cup \{\infty\})^4$ with no three of $\alpha, \beta, \gamma, \delta$ equal, we define the *cross-ratio* $\rho(\alpha, \beta, \gamma, \delta)$ by

$$\rho(\alpha, \beta, \gamma, \delta) = \frac{(\alpha - \gamma)(\beta - \delta)}{(\alpha - \delta)(\beta - \gamma)},$$

with obvious interpretation if one or two of $\alpha, \beta, \gamma, \delta$ are equal to ∞. For example, if $\alpha = \infty$, then we define

$$\rho(\infty, \beta, \gamma, \delta) = \begin{cases} \dfrac{\beta - \delta}{\beta - \gamma}, & \text{if} \quad \beta, \gamma, \delta \neq \infty; \\ 1, & \text{if} \quad \beta = \infty \text{ (so } \gamma, \delta \neq \infty); \\ 0, & \text{if} \quad \gamma = \infty \text{ (so } \beta, \delta \neq \infty); \\ \infty, & \text{if} \quad \delta = \infty \text{ (so } \beta, \gamma \neq \infty). \end{cases}$$

(We will return to this interpretation later on.) Note that the cross-ratio is contained in $\mathbf{F}_q \cup \{\infty\}$; moreover, note that

$$\rho(\alpha, \beta, \delta, \gamma) = \rho(\beta, \alpha, \gamma, \delta) = 1/\rho(\alpha, \beta, \gamma, \delta). \tag{3}$$

Also, it is easily verified that

$$\rho(\alpha, \beta, \gamma, \delta) = 1 \text{ if and only if } \alpha = \beta \text{ or } \gamma = \delta. \tag{4}$$

Observe that, with the above identification of $\mathbf{F}_q \cup \{\infty\}$ with PG(1, q), if $v_\alpha = (\alpha_0, \alpha_1)^\top$, $v_\beta = (\beta_0, \beta_1)^\top$, $v_\gamma = (\gamma_0, \gamma_1)^\top$, and $v_\delta = (\delta_0, \delta_1)^\top$ are the four points in PG(1, q) corresponding to α, β, γ and δ in $\mathbf{F}_q \cup \{\infty\}$, respectively, then $\rho(\alpha, \beta, \gamma, \delta)$ can be identified with the point

$$((\alpha_0\gamma_1 - \alpha_1\gamma_0)(\beta_0\delta_1 - \beta_1\delta_0), (\alpha_0\delta_1 - \alpha_1\delta_0)(\beta_0\gamma_1 - \beta_1\gamma_0))^\top, \tag{5}$$

of PG(1, q), which can be more conveniently written as

$$\begin{pmatrix} \det(v_\alpha, v_\gamma)\det(v_\beta, v_\delta) \\ \det(v_\alpha, v_\delta)\det(v_\beta, v_\gamma) \end{pmatrix} \tag{6}$$

Note that the expression in (6) is equal to the zero vector only if three of the four vectors $v_\alpha, v_\beta, v_\gamma, v_\delta$ are equal, which we have excluded. Therefore, (6) allows us to interpret the value of the cross-ratio as an element in PG(1, q).

We will need several well-known properties concerning the above action of PGL(2, q) and its relation to the cross-ratio. A proof of the following theorem can be found in [8, Section 6.1]. But to make the paper self-contained, we give a quick sketch of the proof here.

Theorem 3.1. (i) *The action of* PGL(2, q) *on* $\mathbf{F}_q \cup \{\infty\}$ *defined in* (2) *is sharply 3-transitive.*

(ii) *The group* PGL(2, q) *leaves the cross-ratio on* $\mathbf{F}_q \cup \{\infty\}$ *invariant, that is, if* $A \in \mathrm{PGL}(2, q)$*, then* $\rho(A(\alpha), A(\beta), A(\gamma), A(\delta)) = \rho(\alpha, \beta, \gamma, \delta)$ *for all* $\alpha, \beta, \gamma, \delta \in \mathbf{F}_q \cup \{\infty\}$ *with no three of* $\alpha, \beta, \gamma, \delta$ *equal.*

(iii) *Moreover, if* $\Omega_+ = \{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbf{F}_q \cup \{\infty\}, \alpha \neq \beta\}$*, then the action of* PGL(2, q) *on* $\Omega_+ \times \Omega_+$ *has orbits*

$$O_{\mathrm{diag}} = \{(\{\alpha, \beta\}, \{\alpha, \beta\}) \mid \{\alpha, \beta\} \in \Omega_+\},$$

and

$$O_{\{r, r^{-1}\}} = \{(\{\alpha, \beta\}, \{\gamma, \delta\}) \mid \{\alpha, \beta\}, \{\gamma, \delta\} \in \Omega_+, \{\alpha, \beta\} \neq \{\gamma, \delta\}, \rho(\alpha, \beta, \gamma, \delta) \in \{r, r^{-1}\}\},$$

for $r \in (\mathbf{F}_q \cup \{\infty\}) \backslash \{1\}$.

Proof: (Sketch) It is easily proved that the triple $(\infty, 0, 1)$ can be mapped to any other triple (α, β, γ) with α, β, γ all distinct. So PGL(2, q) acts 3-transitively on $\mathbf{F}_q \cup \{\infty\}$. Since PGL(2, q) has size $(q^2 - 1)(q^2 - q)/(q - 1) = (q + 1)q(q - 1)$, part (i) follows.

From the representation (6) of the cross-ratio, we immediately see that PGL(2, q) indeed leaves the cross-ratio invariant, so part (ii) holds.

We have that $\rho(\infty, 0, 1, \delta) = \delta$ for all $\delta \in \mathbf{F}_q \cup \{\infty\}$. Also, for $\{\alpha, \beta\}, \{\gamma, \delta\} \in \Omega_+$, we have that $\rho(\alpha, \beta, \gamma, \delta) \in \{0, \infty\}$ if (and only if) $\{\alpha, \beta\} \cap \{\gamma, \delta\} \neq \emptyset$. These observations are sufficient to conclude that ρ takes on all values in $\mathbf{F}_q \cup \{\infty\} \backslash \{1\}$ and that the orbits are indeed as stated in part (iii). □

For any element ξ in some extension field $\mathbf{F}_{q^m}$ of $\mathbf{F}_q$, we define a point P_ξ in PG(2, q^m) by

$$P_\xi = (\xi, \xi^2, 1)^\top;$$

furthermore, we define

$$P_\infty = (0, 1, 0)^\top$$

and

$$P_{\mathrm{Nuc}} = (1, 0, 0)^\top.$$

We will denote by $\mathcal{O}_{q^m}$ the subset of size $q^m + 1$ of $\mathrm{PG}(2, q^m)$ consisting of the points P_ξ, where $\xi \in \mathbf{F}_{q^m} \cup \{\infty\}$. It is easily verified that for each m, the set $\mathcal{O}_{q^m}$ is a nonsingular conic in $\mathrm{PG}(2, q^m)$, and constitutes an oval. We will mostly write $\mathcal{O}$ to denote $\mathcal{O}_q$ and $\bar{\mathcal{O}}$ to denote $\mathcal{O}_{q^2}$. For each $\xi \in \mathbf{F}_{q^m}$, there is a unique tangent line through P_ξ given by

$$t_\xi = (-2\xi, 1, \xi^2)^\perp \tag{7}$$

if $\xi \neq \infty$, and

$$t_\infty = (0, 0, 1)^\perp. \tag{8}$$

Note that t_ξ is contained in $\mathrm{PG}(2, q)$ if and only if $\xi \in \mathbf{F}_q \cup \{\infty\}$. Also note that if q is even, then the point P_{Nuc} is the *nucleus* of the conic, that is, all tangent lines to $\mathcal{O}$ meet at the point P_{Nuc}.

The group $\mathrm{PGL}(2, q)$ can be embedded as a subgroup G of $\mathrm{PGL}(3, q)$ fixing $\mathcal{O}$ setwise, by letting

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} ad + bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix}. \tag{9}$$

Indeed, we have the following.

Theorem 3.2. *Under the embedding* (9)*, the group* $\mathrm{PGL}(2, q)$ *fixes* $\mathcal{O}_{q^m}$ *setwise for each m; in particular, an element* $A \in \mathrm{PGL}(2, q)$ *maps a point* P_ξ *on* $\mathcal{O}_{q^m}$ *to the point* $P_{A(\xi)}$*, where* $A(\xi)$ *is defined as in* (2)*.*

Proof: It is easily verified that the image of A (which we will again denote by A) maps any point $P_\xi = (\xi, \xi^2, 1)^\top$ to the vector $((a\xi + b)(c\xi + d), (a\xi + b)^2, (c\xi + d)^2)^\top$, which represents the point $P_{A(\xi)}$. So indeed G fixes $\mathcal{O}_{q^m}$ setwise. □

Remark 2. If we identify $\mathcal{O}_{q^m}$ with $\mathbf{F}_{q^m} \cup \{\infty\}$ by letting

$$P_\xi \leftrightarrow \xi,$$

then G acts on $\mathcal{O}_{q^m}$ in exactly the same way as $\mathrm{PGL}(2, q)$ acts on $\mathbf{F}_{q^m} \cup \{\infty\}$ with the action given in (2). In fact, it turns out that G is the full subgroup $G(\mathcal{O})$ of $\mathrm{PGL}(3, q)$

fixing $\mathcal{O}$ setwise, see e.g. [8, p. 158]. This can be easily verified along the following line. Assume that a matrix A in PGL(3, q) fixes $\mathcal{O}$ setwise. Then for each x in $\mathbf{F}_q \cup \{\infty\}$ the image AP_x is on $\mathcal{O}$, hence satisfies the equation $X^2 = YZ$. Working out this condition results in a polynomial of degree three that has all $x \in \mathbf{F}_q$ as its roots. Therefore, for $q > 3$ all coefficients of the polynomial have to be zero, implying that A must have the form as described above. For $q = 2, 3$, the claim can be easily verified directly.

4. A coherent configuration containing two association schemes

The action of the subgroup $G = G(\mathcal{O})$ of PGL(3, q) fixing the conic $\mathcal{O}$ as described in the previous section produces a coherent configuration $\mathcal{R} = \mathcal{R}(q)$ on the set $\mathcal{L}$ of non-tangent lines of $\mathcal{O}$ in PG(2, q). Here we will determine the orbits of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$, and show that we obtain association schemes on both the set $\mathcal{L}_+$ of hyperbolic lines and the set $\mathcal{L}_-$ of elliptic lines. First, we need some preparation.

In what follows, we will repeatedly consider "projective objects" over a base field as a subset of similar projective objects over an extension field. (For example, we will consider PG(2, q) as a subset of PG(2, q^2) and PGL(2, q) as a subset of PGL(2, q^2).) In such situations it is crucial to be able to determine whether a given projective object over the extension field is actually an object over the base field. The next theorem addresses this question.

Theorem 4.1. *Let* $\mathbf{F}$ *be a field and let* $\mathbf{E}$ *be a Galois extension of* $\mathbf{F}$*, with Galois group* Gal($\mathbf{E}/\mathbf{F}$)*. Let* A *be an* $n \times m$ *matrix with entries from* $\mathbf{E}$*. Then there exists some* $\lambda \in \mathbf{E}\backslash\{0\}$ *such that* λA *has all its entries in* $\mathbf{F}$ *if and only if for all* $\sigma \in$ Gal($\mathbf{E}/\mathbf{F}$) *there exists some* μ_σ *in* $\mathbf{E}$ *such that* $A^\sigma = \mu_\sigma A$.

Proof: Note that given $x \in \mathbf{E}$, we have $x \in \mathbf{F}$ if and only if $x^\sigma = x$ for all $\sigma \in$ Gal($\mathbf{E}/\mathbf{F}$).

(i) If $\lambda \in \mathbf{E}\backslash\{0\}$ such that λA has all its entries in $\mathbf{F}$, then for $\sigma \in$ Gal($\mathbf{E}/\mathbf{F}$), we have $\lambda^\sigma A^\sigma = \lambda A$, hence with $\mu_\sigma = \lambda/\lambda^\sigma$, we have $A^\sigma = \mu_\sigma A$.

(ii) Conversely, suppose that $A^\sigma = \mu_\sigma A$ for every $\sigma \in$ Gal($\mathbf{E}/\mathbf{F}$). If $A = 0$, then we can take $\lambda = 1$. Otherwise, let a be some nonzero entry of A. Since $A^\sigma = \mu_\sigma A$, we have that $a^\sigma = \mu_\sigma a$. Set $\lambda = a^{-1}$. Then $\lambda^\sigma = (a^{-1})^\sigma = (a^\sigma)^{-1}$, hence $\lambda = \lambda^\sigma \mu_\sigma$. As a consequence, $(\lambda A)^\sigma = \lambda^\sigma A^\sigma = (\lambda/\mu_\sigma)\mu_\sigma A = \lambda A$. Since this holds for all $\sigma \in$ Gal($\mathbf{E}/\mathbf{F}$), we conclude that λA has all its entries in $\mathbf{F}$.

□

Remark 3. The usual method to prove that some scalar multiple λA of a matrix A has all entries in the base field is to take $\lambda = a^{-1}$, for some nonzero entry a of A. (It is easy to see that if such a scalar exists then this choice must work.) However, this approach often requires a similar but distinct argument for each entry of A separately. The above theorem can be used to avoid such cumbersome case distinction, and therefore deserves to be better known. Although the result is unlikely to be new, we do not have a reference.

Consider a point $P = (x, y, z)^\top$ in PG$(2, q^2)$. If some nonzero multiple λP has all its coordinates in $\mathbf{F}_q$, then we may regard P as actually belonging to PG$(2, q)$. Let us call such points *real*, and the remaining points in PG$(2, q^2)$ *virtual*. Similarly, we will call a line in PG$(2, q^2)$ *real* if it contains at least two real points, and *virtual* otherwise. It is not difficult to see that each real line $\ell = (a, b, c)^\perp$ in fact contains $q + 1$ real points and that ℓ is real if and only if some nonzero multiple $\lambda(a, b, c)$ has all its entries in $\mathbf{F}_q$. As a consequence, the real points in PG$(2, q^2)$ together with the real lines in PG$(2, q^2)$ constitute the plane PG$(2, q)$, a Baer subplane in PG$(2, q^2)$.

Now let $\ell \in \mathcal{L}$ be any non-tangent line to $\mathcal{O}$ in PG$(2, q)$. Then ℓ extends to a real line in PG$(2, q^2)$ (which by abuse of notation we shall again denote by ℓ). By inspection of (7) and (8), we see that all tangent lines t_ξ to $\bar{\mathcal{O}} = \mathcal{O}_{q^2}$ in PG$(2, q^2)$ are either virtual tangent lines (if $\xi \in \mathbf{F}_{q^2} \backslash \mathbf{F}_q$) or real tangent lines in PG$(2, q)$ (if $\xi \in \mathbf{F}_q \cup \{\infty\}$). Therefore ℓ cannot be a tangent to $\bar{\mathcal{O}}$, hence it must intersect $\bar{\mathcal{O}}$ in two points, P_α and P_β, say. In fact it is easily seen that either $\alpha, \beta \in \mathbf{F}_q \cup \{\infty\}$ (if ℓ is hyperbolic), or $\beta = \alpha^q$ with $\alpha \in \mathbf{F}_{q^2} \backslash \mathbf{F}_q$ (if ℓ is elliptic). We will let $\mathcal{L}_+$ and $\mathcal{L}_-$ denote the set of hyperbolic and elliptic lines, respectively, and we will say that a line in $\mathcal{L}_+$ (respectively $\mathcal{L}_-$) is of *hyperbolic type* (respectively, of *elliptic type*). Also, we define

$$\Omega_+ = \{\{\alpha, \beta\} \mid \alpha, \beta \in \mathbf{F}_q \cup \{\infty\},\ \alpha \neq \beta\},\ \ \Omega_- = \{\{\alpha, \beta\} \mid \beta = \alpha^q,\ \alpha \in \mathbf{F}_{q^2} \backslash \mathbf{F}_q\},$$

and

$$\Omega = \Omega_+ \cup \Omega_-.$$

Note that according to the above remarks, for $\epsilon \in \{-, +\}$ there is a one-to-one correspondence between lines in $\mathcal{L}_\epsilon$ and pairs in Ω_ϵ such that $\ell \in \mathcal{L}_\epsilon$ corresponds to $\{\alpha, \beta\} \in \Omega_\epsilon$ if $\ell \cap \bar{\mathcal{O}} = \{P_\alpha, P_\beta\}$. Also note that if ℓ and m are two lines in $\mathcal{L}$, with corresponding pairs $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ in Ω, respectively, and if g_A is an element of $G(\mathcal{O})$ corresponding to $A \in$ PGL$(2, q)$, then g_A maps ℓ to m precisely when A maps $\{\alpha, \beta\}$ to $\{\gamma, \delta\}$, that is, if $\{\gamma, \delta\} = \{A(\alpha), A(\beta)\}$. So the action of $G(\mathcal{O})$ on $\mathcal{L}$ and that of PGL$(2, q)$ on Ω are equivalent.

Definition 4.2. Let ℓ, m be two non-tangent lines in PG$(2, q)$, and suppose that $\ell \cap \bar{\mathcal{O}} = \{P_\alpha, P_\beta\}$ and $m \cap \bar{\mathcal{O}} = \{P_\gamma, P_\delta\}$. We define the *cross-ratio* $\rho(\ell, m)$ of the lines ℓ and m as $\rho(\ell, m) = \{r, r^{-1}\}$, where $r \in \mathbf{F}_{q^2} \cup \{\infty\}$ is defined by

$$r = \rho(\alpha, \beta, \gamma, \delta).$$

We will now show that the cross-ratio essentially determines the orbits of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$. The precise result is the following:

Theorem 4.3. *Given two ordered pairs of non-tangent lines* (ℓ, m) *and* (ℓ', m') *with* $\ell \neq m$ *and* $\ell' \neq m'$, *there exists an element of* $G(\mathcal{O})$ *mapping* (ℓ, m) *to* (ℓ', m') *if and only if*

(i) ℓ *and* ℓ' *are of the same type,*
(ii) m *and* m' *are of the same type, and*

(iii) $\rho(\ell, m) = \rho(\ell', m')$.

Proof: We first show that (i), (ii) and (iii) are necessary. Let $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \in \mathbf{F}_{q^2} \cup \{\infty\}$ be such that

$$\ell \cap \bar{\mathcal{O}} = \{P_\alpha, P_\beta\},\ m \cap \bar{\mathcal{O}} = \{P_\gamma, P_\delta\},\ \ell' \cap \bar{\mathcal{O}} = \{P_{\alpha'}, P_{\beta'}\},\ m' \cap \bar{\mathcal{O}} = \{P_{\gamma'}, P_{\delta'}\}.$$

As already remarked above, there exists some element $g_A \in G(\mathcal{O})$ mapping ℓ to ℓ' and m to m' if and only if, under the action as in (2), the associated matrix $A \in \mathrm{PGL}(2, q)$ maps $\{\alpha, \beta\}$ to $\{\alpha', \beta'\}$ and $\{\gamma, \delta\}$ to $\{\gamma', \delta'\}$. Now any element of $G(\mathcal{O})$ obviously maps a hyperbolic line to a hyperbolic line and an elliptic line to an elliptic line, hence (i) and (ii) are indeed necessary; and by Theorem 3.1, part (ii), after interchanging γ' and δ' if necessary, we have $\rho(\alpha, \beta, \gamma, \delta) = \rho(\alpha', \beta', \gamma', \delta')$. So we see that (iii) is also necessary.

Conversely, assume that the conditions (i), (ii) and (iii) hold. By applying Theorem 3.1, part (iii), with q^2 in place of q, we conclude from condition (iii) that (after interchanging γ' and δ' if necessary) there exists a (unique) matrix $A \in \mathrm{PGL}(2, q^2)$ mapping α to α', β to β', γ to γ', and δ to δ'. We have to show that actually $A \in \mathrm{PGL}(2, q)$, that is, some nonzero multiple λA of A has all its entries in $\mathbf{F}_q$. So let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

According to our assumptions, we first have that A maps (α, β) to (α', β'), that is,

$$\frac{a\alpha + b}{c\alpha + d} = \alpha', \qquad \frac{a\beta + b}{c\beta + d} = \beta'. \tag{10}$$

We distinguish two cases.

If both $\alpha, \alpha' \in \mathbf{F}_q \cup \{\infty\}$, then also $\beta, \beta' \in \mathbf{F}_q \cup \{\infty\}$. Now from (10) we conclude that

$$\frac{a^q\alpha + b^q}{c^q\alpha + d^q} = \frac{a\alpha + b}{c\alpha + d}.$$

Hence α is a zero of the polynomial

$$\begin{aligned} F_A(x) &= (a^q x + b^q)(cx + d) - (ax + b)(c^q x + d^q) \\ &= (a^q c - ac^q)x^2 + (a^q d - ad^q + b^q c - bc^q)x + (b^q d - bd^q). \end{aligned}$$

Note that this also holds for $\alpha = \infty$ if we adopt the convention that a polynomial of degree at most two has ∞ as a zero if and only if the polynomial has actually degree at most one. Indeed, F_A has ∞ as its zero if and only if $a/c = a^q/c^q$, and $\alpha' = A(\infty) = a/c$. So we conclude that if ℓ and ℓ' are both hyperbolic, then α, and by a similar reasoning also β, are zeroes of the polynomial $F_A(x)$.

On the other hand, if both $\alpha, \alpha' \in \mathbf{F}_{q^2} \backslash \mathbf{F}_q$, then also $\beta = \alpha^q$ and $\beta' = \alpha'^q$ are in $\mathbf{F}_{q^2} \backslash \mathbf{F}_q$. By raising the second equation in (10) to the q-th power, we again conclude that

$$\frac{a^q\alpha + b^q}{c^q\alpha + d^q} = \frac{a\alpha + b}{c\alpha + d},$$

hence again we have that α, and similarly α^q, is a zero of the polynomial $F_A(x)$.

In summary, if A maps (α, β) to (α', β'), we can conclude that both α and β are zeroes of F_A; hence according to our assumptions all four of $\alpha, \beta, \gamma, \delta$ determined by the lines ℓ and m are zeroes of the polynomial $F_A(x)$. Now since $\ell \neq m$, we have $|\{\alpha, \beta, \gamma, \delta\}| \geq 3$. Consequently $F_A(x)$ is the zero polynomial, that is,

$$ac^q \in \mathbf{F}_q, \qquad bd^q \in \mathbf{F}_q, \qquad a^q d - bc^q = ad^q - b^q c \in \mathbf{F}_q. \tag{11}$$

Now we want to apply Theorem 4.1. With

$$\Phi = a^q d - bc^q = ad^q - b^q c, \qquad \Delta = \det(A) = ad - bc \neq 0,$$

we have that

$$\begin{aligned} a\Phi &= a(a^q d - bc^q) = a^{q+1}d - ba^q c = a^q\Delta;\\ b\Phi &= b(ad^q - b^q c) = ab^q d - b^{q+1}c = b^q\Delta;\\ c\Phi &= c(a^q d - bc^q) = c^q ad - bc^{q+1} = c^q\Delta;\\ d\Phi &= d(ad^q - b^q c) = ad^{q+1} - d^q bc = d^q\Delta; \end{aligned}$$

hence $A^q\Delta = A\Phi$, i.e., $A^q = (\Phi/\Delta)A$. By Theorem 4.1, we may now conclude that essentially $A \in \mathrm{PGL}(2, q)$. □

Corollary 4.4. *The group $G(\mathcal{O})$ is generously transitive on both $\mathcal{L}_+$ and $\mathcal{L}_-$.*

Proof: Let ℓ, m be two distinct lines in $\mathcal{L}$. Obviously, $\rho(\ell, m) = \rho(m, \ell)$. Hence according to Theorem 4.3, there is an element in $G(\mathcal{O})$ that maps (ℓ, m) to (m, ℓ), i.e., interchanges ℓ and m, if and only if ℓ and m are of the same type. □

Our next result relates the value of the cross-ratio $\rho(\ell, m)$ of two lines ℓ and m to their types. Let us define the subsets $\mathbf{B}_0$ and $\mathbf{B}_1$ of $\mathbf{F}_{q^2} \cup \{\infty\}$ by

$$\mathbf{B}_0 = (\mathbf{F}_q \cup \{\infty\})\backslash\{1\}, \qquad \mathbf{B}_1 = \{x \in \mathbf{F}_{q^2}\backslash\{1\} \mid x^q = x^{-1}\}.$$

Note that $|\mathbf{B}_0| = |\mathbf{B}_1| = q$, also the intersection of $\mathbf{B}_0$ and $\mathbf{B}_1$ is empty if q is even, and contains only -1 if q is odd. We have the following.

Lemma 4.5. *Let ℓ, m be two distinct non-tangent lines in* PG(2, q)*, and let $\rho(\ell, m) = \{\lambda, \lambda^{-1}\}$, where $\lambda \in \mathbf{F}_{q^2} \cup \{\infty\}$. Then λ is contained in $\mathbf{B}_0$ if ℓ and m are of the* same *type, and contained in $\mathbf{B}_1$ if ℓ and m are of* different *type.*

Proof: Easy consequence of the fact that if $\alpha, \beta, \gamma, \delta \in \mathbf{F}_q \cup \{\infty\}$ and $\xi, \eta \in \mathbf{F}_{q^2} \setminus \mathbf{F}_q$, then $\rho(\alpha, \beta, \gamma, \delta)$ and $\rho(\xi, \xi^q, \eta, \eta^q)$ are both in $\mathbf{B}_0$ while $\rho(\xi, \xi^q, \gamma, \delta)$ and $\rho(\alpha, \beta, \eta, \eta^q)$ are both in $\mathbf{B}_1$. □

For $\epsilon, \phi \in \{1, -1\}$ and for $\lambda \in \mathbf{B}_0$ (if $\epsilon = \phi = 1$), or $\lambda \in \mathbf{B}_0 \setminus \{0, \infty\}$ (if $\epsilon = \phi = -1$), or $\lambda \in \mathbf{B}_1$ (if $\epsilon \neq \phi$), we define

$$\mathcal{R}_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi) = \{(\ell, m) \in \mathcal{L}_\epsilon \times \mathcal{L}_\phi, \ell \neq m \mid \rho(\ell, m) = \{\lambda, \lambda^{-1}\}\}.$$

We observed earlier that $\rho(\ell, m) \neq 1$ and $\rho(\ell, m) = \{0, \infty\}$ if and only if ℓ and m are equal or intersect on $\mathcal{O}$. Hence according to Theorem 4.3 and Lemma 4.5, each of the non-diagonal orbits of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$, that is, each non-diagonal relation of the coherent configuration $\mathcal{R}$ obtained from the action of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$, is actually of the form $\mathcal{R}_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi)$ with the restrictions on λ as given above. Moreover, since $G(\mathcal{O})$ is transitive on both $\mathcal{L}_+$ and $\mathcal{L}_-$, we have that

$$|\mathcal{R}_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi)| = |\mathcal{L}_\epsilon| v_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi),$$

where the numbers $v_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi)$ are the *valencies* of the coherent configuration $\mathcal{R}$. In order to finish our description of the orbits of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$, we will show that each of the orbits defined above is indeed nonempty.

Theorem 4.6. *We have that*

$$v_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi) = \begin{cases} 2(q-1), & \text{if } \epsilon = \phi = 1 \text{ and } \{\lambda, \lambda^{-1}\} = \{0, \infty\}; \\ (q-\epsilon)/2, & \text{if } q \text{ is odd and } \lambda = -1; \\ q - \epsilon, & \text{if } \lambda \in \mathbf{B}_{(1-\delta_{\epsilon,\phi})} \text{ and } \lambda \neq -1, 0, \infty. \end{cases}$$

(Here δ is the Kronecker delta.)

Proof: Fix a line $\ell \in \mathcal{L}_\epsilon$, and let $\ell \cap \mathcal{O}_{q^2} = \{P_\alpha, P_\beta\}$, where $\{\alpha, \beta\} \in \Omega_\epsilon$. We want to count the number of $m \in \mathcal{L}_\phi$ such that $m \cap \mathcal{O}_{q^2} = \{P_x, P_y\}$ with $\{x, y\} \in \Omega_\phi$, and $\rho(\ell, m) = \{\lambda, \lambda^{-1}\}$, where

$$\lambda = \rho(\alpha, \beta, x, y) = \frac{(\alpha - x)(\beta - y)}{(\alpha - y)(\beta - x)}. \quad (12)$$

Now we note the following. First, we have $\lambda \in \{0, \infty\}$ if and only if $\{\alpha, \beta\} \cap \{x, y\} \neq \emptyset$, that is, if and only if the corresponding lines ℓ and m intersect on $\mathcal{O}_{q^2}$. Hence

$$v_{\{0,\infty\}}(\epsilon, \phi) = \begin{cases} 2(q-1), & \text{if } \epsilon = \phi = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we have $x = y$ in (12) only if $\lambda = 1$, which is excluded. Also by interchanging x and y, the cross-ratio λ in (12) is *inverted*, and the only cases where $\lambda = \lambda^{-1}$ are $\lambda = 1$ (which is excluded) and $\lambda = -1$. As a consequence, for $\lambda \in (\mathbf{B}_0 \cup \mathbf{B}_1)\backslash\{0, \infty\}$ the number $v_{\{\lambda,\lambda^{-1}\}}(\epsilon, \phi)$ equals the number of solutions (x, y) of (12) with $\{x, y\} \in \Omega_\phi$ if q is even or $\lambda \neq -1$, and is equal to half of the number of such solutions if $\lambda = -1$ and q is odd.

First, let $\epsilon = 1$. According to Theorem 4.3, we may assume without loss of generality that $\alpha = \infty$ and $\beta = 0$, so that (12) reduces to $\lambda = -y/(-x) = y/x$. If $\phi = 1$, then $x, y \in \mathbf{F}_q \cup \{\infty\}$ and $\lambda \in \mathbf{F}_q\backslash\{0, 1\}$; in that case for each $x \in \mathbf{F}_q\backslash\{0\}$ there is a unique solution $y \in \mathbf{F}_q$, so there are $q - 1$ solutions in total. Similarly, if $\phi = -1$, then $x \in \mathbf{F}_{q^2}\backslash\mathbf{F}_q$, $y = x^q$, and $\lambda \in \mathbf{B}_1$. So (12) reduces to $\lambda = x^{q-1}$, and again there are precisely $q - 1$ solutions for each $\lambda \in \mathbf{B}_1$.

If $\epsilon = -1$, then we have $\alpha \in \mathbf{F}_{q^2}\backslash\mathbf{F}_q$ and $\beta = \alpha^q$. First, if $\phi = 1$, then $x, y \in \mathbf{F}_q \cup \{\infty\}$ and $\lambda \in \mathbf{B}_1$. In that case we see immediately from (12) that $\lambda^q = 1/\lambda$. For each $\lambda \in \mathbf{B}_1$, let z and u be the unique solutions of the equations $\lambda = (\alpha^q - z)/(\alpha - z)$ and $\lambda = (\alpha - u)/(\alpha^q - u)$, respectively. Now for $y = \infty$ the unique solution of (12) is $x = u$; for $y = z$ the unique solution is $x = \infty$, and it is easily seen that for each $y \in \mathbf{F}_q\backslash\{z\}$ there is a unique solution $x \in \mathbf{F}_q$ of (12). So there are precisely $q + 1$ solutions of (12) in this case. Finally, if $\phi = -1$, then we have $x \in \mathbf{F}_{q^2}\backslash\mathbf{F}_q$, $y = x^q$, and $\lambda \in \mathbf{F}_q\backslash\{0, 1\}$. In that case, the desired solutions of (12) satisfy

$$(\alpha - x)(\alpha^q - x^q) = \lambda(\alpha - x^q)(\alpha^q - x),$$

with $x \in \mathbf{F}_{q^2}\backslash\mathbf{F}_q$ and $x \neq \alpha, \alpha^q$. For each $\lambda \in \mathbf{F}_q\backslash\{0, 1\}$, this equation has at most $q + 1$ solutions in $\mathbf{F}_{q^2}\backslash\mathbf{F}_q$. On the other hand, there are $q^2 - q - 2$ choices of x with $x \neq \alpha, \alpha^q$; consequently, the *average* number of valid solutions equals $(q^2 - q - 2)/(q - 2) = q + 1$. Since the average number of solutions equals the maximum number of solutions, there must be exactly $q + 1$ solutions for each $\lambda \in \mathbf{F}_q\backslash\{0, 1\}$, and the result stated for $\epsilon = \phi = -1$ follows. □

By combining Theorems 4.3 and 4.6 we obtain the following result.

Theorem 4.7. (i) *The action of the group $G(\mathcal{O})$ on $\mathcal{L}$ affords a weakly symmetric coherent configuration $\mathcal{R}(q)$.*

(ii) *The restriction of $\mathcal{R}(q)$ to the fibre $\mathcal{L}_+$ of hyperbolic lines constitutes an association scheme $\mathcal{R}_+(q)$ (or $\mathcal{H}(q)$), with $q/2$ classes if q is even and with $(q+1)/2$ classes if q is odd. The non-diagonal relations are precisely the sets $\mathcal{R}_{\{\lambda,\lambda^{-1}\}}(1, 1)$ with $\lambda \in \mathbf{B}_0$, with corresponding valencies $v_{\{\lambda,\lambda^{-1}\}} = v_{\{\lambda,\lambda^{-1}\}}(1, 1)$.*

(iii) *The restriction of $\mathcal{R}(q)$ to the fibre $\mathcal{L}_-$ of elliptic lines constitutes an association scheme $\mathcal{R}_-(q)$ (or $\mathcal{E}(q)$), with $q/2 - 1$ classes if q is even and with*

$(q-1)/2$ classes if q is odd. The non-diagonal relations are precisely the sets $\mathcal{R}_{\{\lambda,\lambda^{-1}\}}(-1,-1)$ with $\lambda \in \mathbf{F}_q\backslash\{0,1\}$, with corresponding valencies $v_{\{\lambda,\lambda^{-1}\}} = v_{\{\lambda,\lambda^{-1}\}}(-1,-1)$.

We will refer to the association schemes $\mathcal{H}(q)$ and $\mathcal{E}(q)$ in part (ii) and (iii) of the above theorem as the *hyperbolic* and *elliptic* scheme, respectively. The hyperbolic scheme was recently investigated in [3] as a refinement (fission) of the triangular scheme. The elliptic scheme was first described in [9] but our approach here is new.

5. An expression based on homogeneous coordinates of lines to index the relations of $\mathcal{R}(q)$

Let $\epsilon, \phi \in \{-1, 1\}$ and let $(\ell, m) \in \mathcal{L}_\epsilon \times \mathcal{L}_\phi$. In this section we will develop an expression $\hat{\rho}(\ell, m)$ that can be used to index the relation of $\mathcal{R}(q)$ containing (ℓ, m), in terms of the homogeneous coordinates of ℓ and m.

We need some preparation. Consider the function $f : \mathbf{F}_{q^2} \cup \{\infty\} \to \mathbf{F}_{q^2} \cup \{\infty\}$ defined by

$$f(x) = \begin{cases} \dfrac{1}{x + x^{-1}}, & \text{if } q \text{ is even;} \\ \dfrac{1}{4} + \dfrac{1}{-2 + x + x^{-1}}, & \text{if } q \text{ is odd,} \end{cases}$$

for $x \in \mathbf{F}_{q^2}\backslash\{0, 1\}$, $f(1) = \infty$, and $f(0) = f(\infty) = 0$ if q is even and $f(0) = f(\infty) = 1/4$ if q is odd. (Note that the values of f on ∞, 0, 1 are consistent with the general expression for $f(x)$ when we interpret $1/0 = \infty$ and handle ∞ in the usual way.) This function has a few remarkable properties. To describe these, we introduce some notation. For $q = 2^r$ and for $e \in \mathbf{F}_2$, let $\mathbf{T}_e = \mathbf{T}_e(q)$ denote the collection of elements with absolute trace e in $\mathbf{F}_q$, that is,

$$\mathbf{T}_e = \left\{x \in \mathbf{F}_q \mid \mathrm{Tr}(x) := x + x^2 + \cdots + x^{2^{r-1}} = e\right\}$$

For q odd, we let $\mathbf{T}_0$ and $\mathbf{T}_1$ denote the collection of nonzero squares and non-squares in $\mathbf{F}_q$, respectively, that is,

$$\mathbf{T}_0 = \{x^2 \mid x \in \mathbf{F}_q\}\backslash\{0\}$$

and $\mathbf{T}_1 = \mathbf{F}_q\backslash(\{0\} \cup \mathbf{T}_0)$. Note that in the case where q is even, it is well known that

$$\mathbf{T}_0 = \{x^2 + x \mid x \in \mathbf{F}_q\}$$

Lemma 5.1. *The function f has the following properties*:

(i) *$f(x) = f(y)$ if and only if $x = y$ or $x = y^{-1}$*;
(ii) *$f(x) = \infty$ if and only if $x = 1$*;

(iii) *if q is odd, then $f(x) = 0$ if and only if $x = -1$;*
(iv) *$f(x) \in \mathbf{F}_q$ if and only if $x \in \mathbf{B}_0 \cup \mathbf{B}_1$;*
(v) *if q is even and $x \in \mathbf{F}_{q^2} \backslash \{0, 1\}$, then*

$$f(x) = \frac{1}{x+1} + \frac{1}{(x+1)^2},$$

and if q is odd and $x \in \mathbf{F}_{q^2} \backslash \{0, 1, -1\}$, then

$$f(x) = \left(\frac{x + 2 + x^{-1}}{2(x - x^{-1})} \right)^2.$$

Hence for $x \in \mathbf{F}_{q^2} \cup \{\infty\}$ and $e \in \mathbf{F}_2$, we have that $f(x) \in \mathbf{T}_e$ if and only if $x \in \mathbf{B}_e \backslash \{-1\}$.

Proof: Note first that $f(x) = f(y)$ if and only if $x + 1/x = y + 1/y$; hence part (i) follows. Parts (ii) and (iii) are evident. To see (iv), first note that $f(x)^q = f(x^q)$, then use part (i) to conclude that $f(x) \in \mathbf{F}_q \cup \{\infty\}$ if and only if $x^q \in \{x, x^{-1}\}$. The expressions for $f(x)$ in part (v) are easily verified. Since $f(\infty) = 0 \in \mathbf{T}_0$ if q is even and $f(\infty) = 1/2^2 \in \mathbf{T}_0$ if q is odd, the expressions in (v) imply that $f(x) \in \mathbf{T}_0$ if and only if $x \in (\mathbf{F}_q \cup \{\infty\}) \backslash \{1, -1\}$. Now the remainder of part (v) follows from (iv). □

Next we determine the type of a line in terms of its homogeneous coordinates, and we establish relations between the homogeneous coordinates of a line and the points of intersection of this line with the conic $\mathcal{O}_{q^2}$.

Lemma 5.2. *Let ℓ be a line in* PG(2, q) *with homogeneous coordinates $\ell = (z, x, y)^\perp$, and let $\ell \cap \mathcal{O}_{q^2} = \{P_\alpha, P_\beta\}$, where $\alpha, \beta \in \mathbf{F}_{q^2} \cup \{\infty\}$ and $\alpha = \beta$ if ℓ is a tangent line. Define $\Delta(\ell) \in \mathbf{F}_q \cup \{\infty\}$ by*

$$\Delta(\ell) = \begin{cases} xy/z^2, & \text{if } q \text{ is even;} \\ 1/(z^2 - 4xy), & \text{if } q \text{ is odd.} \end{cases}$$

(i) *We have that $\ell \in \mathcal{L}_{(-1)^e}$ if and only if $\Delta(\ell) \in \mathbf{T}_e$, and ℓ is a tangent line to $\mathcal{O}$ in* PG(2, q) *if and only if $\Delta(\ell) = \infty$.*
(ii) *If $x \neq 0$, then*

$$\alpha + \beta = -z/x, \qquad \alpha\beta = y/x; \tag{13}$$

and if $x = 0$, then

$$\alpha = \infty, \qquad \beta = -y/z. \tag{14}$$

Proof: Note first that by definition α, β are the solutions in $\mathbf{F}_{q^2} \cup \{\infty\}$ of the quadratic equation

$$z\xi + x\xi^2 + y = 0.$$

(Here, by convention, $\xi = \infty$ is a solution if and only if $x = 0$.) Now (i) follows from the standard theory on solutions of quadratic equations and (ii) follows from this equation by writing it in the form $x(\xi - \alpha)(\xi - \beta) = 0$. □

Now let $(\ell, m) \in \mathcal{L}_\epsilon \times \mathcal{L}_\phi$ be a pair of distinct non-tangent lines in PG(2, q), and let $\alpha, \beta, \gamma, \delta$ be such that

$$\ell \cap \mathcal{O}_{q^2} = \{P_\alpha, P_\beta\}, \qquad m \cap \mathcal{O}_{q^2} = \{P_\gamma, P_\delta\}.$$

Furthermore, let ℓ and m have homogeneous coordinates

$$\ell = (z, x, y)^\perp, \qquad m = (\bar{z}, \bar{x}, \bar{y})^\perp.$$

In the previous section we have seen that the orbit of the action of $G(\mathcal{O})$ on $\mathcal{L} \times \mathcal{L}$ containing the pair (ℓ, m) is $\mathcal{R}_{\{\rho,\rho^{-1}\}}(\epsilon, \phi)$, where

$$\rho = \rho(\alpha, \beta, \gamma, \delta).$$

Now we define the *modified cross-ratio* $\hat{\rho}(\ell, m)$ of the lines ℓ and m by

$$\hat{\rho}(\ell, m) = f(\rho) = \begin{cases} \dfrac{1}{\rho + \rho^{-1}}, & \text{if } q \text{ is even;} \\ \dfrac{1}{4} + \dfrac{1}{-2 + \rho + \rho^{-1}}, & \text{if } q \text{ is odd.} \end{cases}$$

We will now use the previous lemma to express $\hat{\rho}(\ell, m)$ in terms of the homogeneous coordinates of ℓ and m. Let $\sigma : \mathbf{F}_q \to \mathbf{T}_0 \cup \{0\}$ be defined by

$$\sigma(x) = \begin{cases} x^2 + x, & \text{if } q \text{ is even;} \\ x^2, & \text{if } q \text{ is odd.} \end{cases}$$

Then the result is as follows.

Theorem 5.3. *If $\ell = (z, x, y)^\perp$ and $m = (\bar{z}, \bar{x}, \bar{y})^\perp$ are two non-tangent lines and if $\Delta = \Delta(\ell)$ and $\bar{\Delta} = \Delta(m)$, then*

$$\hat{\rho}(\ell, m) = \begin{cases} \dfrac{(x\bar{y} + \bar{x}y)^2 + (x\bar{z} + \bar{x}z)(y\bar{z} + \bar{y}z)}{z^2\bar{z}^2} = \sigma((x\bar{y} + \bar{x}y)/(z\bar{z})) + \Delta + \bar{\Delta}, & \text{if } q \text{ is even;} \\ (2x\bar{y} + 2\bar{x}y - z\bar{z})^2\Delta\bar{\Delta}/4 = \sigma(x\bar{y} + \bar{x}y - \frac{z\bar{z}}{2})\Delta\bar{\Delta}, & \text{if } q \text{ is odd.} \end{cases}$$

Proof: Let $\rho = \rho(\alpha, \beta, \gamma, \delta)$ with α, β, γ and δ as given above, and let $\hat{\rho} = f(\rho)$. Initially, we will assume that $x, \bar{x} \neq 0$. First we observe that

$$-2 + \rho + \rho^{-1} = \frac{(\alpha - \beta)^2(\gamma - \delta)^2}{(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)}. \tag{15}$$

Now $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (-z/x)^2 - 4y/x$, and similarly $(\gamma - \delta)^2 = (-\bar{z}/\bar{x})^2 - 4\bar{y}/\bar{x}$, hence

$$(\alpha - \beta)^2(\gamma - \delta)^2 = \begin{cases} z^2\bar{z}^2/(x^2\bar{x}^2), & \text{if } q \text{ is even;} \\ 1/(\Delta\bar{\Delta}x^2\bar{x}^2), & \text{if } q \text{ is odd.} \end{cases} \tag{16}$$

Moreover, straightforward but somewhat tedious computations show that

$$\begin{aligned} &(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) \\ &\quad = (\alpha\beta)^2 - \alpha\beta(\alpha+\beta)(\gamma + \delta) + (\alpha+\beta)^2\gamma\delta + \alpha\beta(\gamma^2 + \delta^2) - (\alpha + \beta)\gamma\delta(\gamma + \delta) + (\gamma\delta)^2 \\ &\quad = ((x\bar{y} - \bar{x}y)^2 + (x\bar{z} - \bar{x}z)(y\bar{z} - \bar{y}z))/(x^2\bar{x}^2). \end{aligned}$$

By combining these expressions we obtain in a straightforward way the desired expressions for $\hat{\rho}$. Finally, it is not difficult to check that the expressions for $\hat{\rho}$ are also correct in the case where one of $x, \bar{x}$ is equal to zero. □

In what follows, we will use the elements of $\mathbf{F}_q$ to index the relations of the coherent configuration $\mathcal{R} = \mathcal{R}(q)$, and the modified cross-ratio $\hat{\rho}$ to determine the relation of a given pair of distinct non-tangent lines. For $\epsilon, \phi \in \{-1, 1\}$ and $\lambda \in \mathbf{F}_q$, we define

$$\mathcal{R}_{\text{diag}}(\epsilon, \epsilon) := \{(\ell, \ell) \mid \ell \in \mathcal{L}_\epsilon\}$$

and

$$\mathcal{R}_\lambda(\epsilon, \phi) := \{(\ell, m) \in \mathcal{L}_\epsilon \times \mathcal{L}_\phi \mid \ell \neq m,\ \hat{\rho}(\ell, m) = \lambda\}.$$

Here $\mathcal{R}_{\text{diag}}(1, 1)$ and $\mathcal{R}_{\text{diag}}(-1, -1)$ are the two diagonal relations on the fibres $\mathcal{L}_+$ and $\mathcal{L}_-$. By Lemma 4.5, the types of ℓ and m alone determine whether $\hat{\rho}(\ell, m)$ is contained in $\mathbf{T}_0$ or in $\mathbf{T}_1$ (except in the case where $\hat{\rho}(\ell, m) = 0$ if q is odd). This can also be seen from Lemma 5.2 together with the expressions for $\hat{\rho}(\ell, m)$ in Theorem 5.3. In order to state the next theorem concisely, we define for $e \in \mathbf{F}_2$,

$$\mathbf{T}_e^+ = \begin{cases} \mathbf{T}_e, & \text{if } q \text{ is even,} \\ \mathbf{T}_e \cup \{0\}, & \text{if } q \text{ is odd;} \end{cases}$$

and $\mathbf{T}_0^* = \mathbf{T}_0 \backslash \{0\}$.

Now a careful inspection of Theorem 4.7 in fact shows that we have the following.

Theorem 5.4. *The non-diagonal relations $\mathcal{R}_\lambda(\epsilon, \phi)$ of the coherent configuration $\mathcal{R}(q)$ are nonempty precisely when*

(i) $\epsilon = \phi = 1$ *and* $\lambda \in \mathbf{T}_0^+$;
(ii) $\epsilon \neq \phi$ *and* $\lambda \in \mathbf{T}_1^+$; *or*
(iii) $\epsilon = \phi = -1$ *and* $\lambda \in \mathbf{T}_0^*$ *if q is even or* $\lambda \in \mathbf{T}_0^+\backslash\{1/4\}$ *if q is odd.*

6. The intersection parameters of $\mathcal{R}(q)$ in the case of even characteristic

In the rest of this paper, we always assume that q is even. Here we will determine the intersection parameters of the coherent configuration $\mathcal{R}(q)$ in the case of even characteristic. In this case, the results from the previous section can be resumed as follows. By Lemma 5.2, a line ℓ in PG$(2, q)$ is non-tangent to $\mathcal{O}$ if and only if it can be represented in homogeneous coordinates as $\ell = (1, x, y)^\perp$ with $x, y \in \mathbf{F}_q$; if $\Delta = xy \in \mathbf{T}_e$, then $\ell \in \mathcal{L}_\epsilon$ with $\epsilon = (-1)^e$. Moreover, Theorem 5.3 implies that if $\ell = (1, x, y)^\perp \in \mathcal{L}_\epsilon$ and $m = (1, z, u)^\perp \in \mathcal{L}_\phi$ are two non-tangent lines with $xy \in \mathbf{T}_e$ and $zu \in \mathbf{T}_f$, then $\epsilon = (-1)^e$, $\phi = (-1)^f$, and

$$\hat{\rho}(\ell, m) = (xu + yz)^2 + (xu + yz) + xy + zu = x^2u^2 + y^2z^2 + (x + z)(y + u). \tag{17}$$

Lemma 4.5 (or Theorem 5.4) shows that $\hat{\rho}(\ell, m)$ is contained in $\mathbf{T}_{e+f}$. The above expression for $\hat{\rho}(\ell, m)$ also implies this fact since $(xu + yz)^2 + (xu + yz) \in \mathbf{T}_0$. Furthermore, we recall that $\hat{\rho}(\ell, m) = 0$ precisely when $\ell = m$ or when ℓ and m are lines in $\mathcal{L}_1$ that intersect on $\mathcal{O}$. For later reference, we state these observations explicitly.

Lemma 6.1. *Let $c \in \mathbf{F}_q$, and $e, f \in \mathbf{F}_2$. If $(\ell, m) \in \mathcal{R}_c(\epsilon, \phi)$ with $\epsilon = (-1)^e$ and $\phi = (-1)^f$, then $c \in \mathbf{T}_{e+f}$. Moreover, if $c = 0$, then $\epsilon = \phi = 1$.*

Corollary 6.2. *Let $a, b, c \in \mathbf{F}_q$, and let $e, f, g \in \mathbf{F}_2$. Write $\epsilon = (-1)^e$, $\phi = (-1)^f$, and $\theta = (-1)^g$. If $\ell \in \mathcal{L}_\epsilon$, $m \in \mathcal{L}_\phi$, and $n \in \mathcal{L}_\theta$ with $\hat{\rho}(\ell, m) = c$, $\hat{\rho}(\ell, n) = a$, and $\hat{\rho}(n, m) = b$, then* $\mathrm{Tr}(c) = e + f$, $\mathrm{Tr}(a) = e + g$, *and* $\mathrm{Tr}(b) = f + g$*; in particular, $a + b + c \in \mathbf{T}_0$.*

In order to determine the intersection parameters of $\mathcal{R}(q)$, we need to do the following. Choose any pair $(\ell, m) \in \mathcal{R}_c(\epsilon, \phi)$, with $\epsilon = (-1)^e$ and $\phi = (-1)^f$, say, and then count the number of lines $n \in \mathcal{L}_\theta$, with $\theta = (-1)^g$, say, such that $(\ell, n) \in \mathcal{R}_a(\epsilon, \theta)$ and $(n, m) \in \mathcal{R}_b(\theta, \phi)$. Now note that according to Lemma 6.1, there are such lines n only if $\mathrm{Tr}(c) = e + f$, $\mathrm{Tr}(a) = e + g$, and $\mathrm{Tr}(b) = g + f$, and so, in particular, only if $a + b + c \in \mathbf{T}_0$. These observations motivate the following definitions.

For all $a, b, c \in \mathbf{F}_q$, for all $\epsilon, \phi \in \{-1, 1\}$, and for all lines ℓ, m with $(\ell, m) \in \mathcal{R}_c(\epsilon, \phi)$ (so $\ell \neq m$), we define

$$v_a(\ell) = v_a(\epsilon) = |\{n \in \mathcal{L}\backslash\{\ell\} \mid \hat{\rho}(\ell, n) = a\}|, \tag{18}$$

$$p_{a,b}(\ell, m) = p_{a,b}^c(\epsilon) = |\{n \in \mathcal{L}\backslash\{\ell, m\} \mid \hat{\rho}(\ell, n) = a \text{ and } \hat{\rho}(n, m) = b\}|, \tag{19}$$

and

$$\pi_{a,b}(\ell, m) = \pi^c_{a,b}(\epsilon) = |\{n \in \mathcal{L} \mid \hat{\rho}(\ell, n) = a \text{ and } \hat{\rho}(n, m) = b\}|. \tag{20}$$

The above observations show that the numbers in (18) and (19) are valencies and intersection parameters of $\mathcal{R}(q)$.

We also note that since $\hat{\rho}(\ell, \ell) = 0$ for all lines $\ell \in \mathcal{L}$, we have the following:

Lemma 6.3. *Let $a, b, c \in \mathbf{F}_q$ and let $\epsilon \in \{-1, 1\}$. Then*

$$\pi^c_{a,b}(\epsilon) = p^c_{a,b}(\epsilon) + \delta_{a,0}\delta_{b,c} + \delta_{b,0}\delta_{a,c}.$$

In what follows, we will sometimes use the symbol ∞ to indicate a diagonal relation and write $\mathcal{R}_\infty(\epsilon, \epsilon)$ instead of $\mathcal{R}_{\text{diag}}(\epsilon, \epsilon)$. Remark that the intersection parameters involving a diagonal relation are $p^\infty_{a,b}(\epsilon) = \delta_{a,b}v_a(\epsilon)$, $p^c_{\infty,b}(\epsilon) = \delta_{b,c}$, and $p^c_{a,\infty}(\epsilon) = \delta_{a,c}$.

According to Lemma 6.3, in order to obtain all intersection parameters, it is sufficient to compute the numbers $\pi^c_{a,b}(\epsilon)$ for $a, b, c \in \mathbf{F}_q$ with $a + b + c \in \mathbf{T}_0$. We begin with the following observation.

Lemma 6.4. *For all $a, b, c \in \mathbf{F}_q$, we have $\pi^c_{a,b}(\epsilon) = \pi^c_{b,a}((-1)^{\mathrm{Tr}(c)}\epsilon)$ and $p^c_{a,b}(\epsilon) = p^c_{b,a}((-1)^{\mathrm{Tr}(c)}\epsilon)$.*

Proof: The number $\pi^c_{a,b}(\epsilon)$, with $\epsilon = (-1)^e$, counts the number of lines $n \in \mathcal{L}$ such that $\hat{\rho}(\ell, n) = a$ and $\hat{\rho}(n, m) = b$, for some pair of distinct non-tangent lines ℓ, m with $\ell \in \mathcal{L}_\epsilon$ and $\hat{\rho}(\ell, m) = c$. By Corollary 6.2, we then have $m \in \mathcal{L}_\phi$ with $\phi = (-1)^f$ and $f = e + \mathrm{Tr}(c)$; hence all these lines n, and no others, contribute to the number $\pi^c_{b,a}((-1)^{e+\mathrm{Tr}(c)})$. This proves the first equality; the other equality follows from Lemma 6.3. □

For $v \in \mathbf{F}_q$, we define $\ell_v = (1, v, v)^\perp$. Note that

$$\hat{\rho}(\ell_v, \ell_{v+c}) = c^2, \qquad \Delta(\ell_v) = v^2,$$

for any $c \in \mathbf{F}_q$.

Theorem 6.5. *Let $a, b, c \in \mathbf{F}_q$ with $a + b + c \in \mathbf{T}_0$ and $c \neq 0$. For each $e \in \mathbf{F}_2$ and for each $v \in \mathbf{F}_q$ with $\mathrm{Tr}(v) = e$, we have*

$$\begin{aligned}\pi^c_{a,b}(\epsilon) &= \sum_\tau |\{z \in \mathbf{F}_q \cup \{\infty\} \mid z^2 + z = v + ac/\tau^2\}| \\ &= \begin{cases} 1 + 2|\mathbf{T}_e \cap \{ac\}|, & \text{if } a + b + c = 0; \\ 2\sum_\tau |\mathbf{T}_e \cap \{ac/\tau^2\}|, & \text{if } a + b + c \in \mathbf{T}_0^*, \end{cases}\end{aligned}$$

where $\epsilon = (-1)^e$*, and the summations are over the two elements* $\tau \in \mathbf{F}_q$ *such that* $\tau^2 + \tau = a + b + c$*. (If* $\tau = 0$*, then* $z = \infty$ *is supposed to be the only solution of the equation* $z^2 + z = v + ac/\tau^2$*.)*

Proof: We choose $\ell = \ell_{v^{1/2}}$ and $m = \ell_{(v+c)^{1/2}}$ in the definition (20) of $\pi^c_{a,b}(\epsilon)$. According to (17), we conclude that $\pi^c_{a,b}(\epsilon)$ equals the number of lines $n = (1, x, y)^\perp$ for which the following holds.

$$\begin{cases} \hat{\rho}(\ell_{v^{1/2}}, n) = (v^{1/2}(x+y))^2 + v^{1/2}(x+y) + xy + v = a \\ \hat{\rho}(n, \ell_{(v+c)^{1/2}}) = ((v+c)^{1/2}(x+y))^2 + (v+c)^{1/2}(x+y) + xy + v + c = b \end{cases} \tag{21}$$

By adding the two equations, we see that $(x, y) \in \mathbf{F}_q^2$ must satisfy

$$(c^{1/2}(x+y))^2 + c^{1/2}(x+y) = a + b + c.$$

Since $a + b + c \in \mathbf{T}_0$, there are two elements $\tau \in \mathbf{F}_q$ such that $\tau^2 + \tau = a + b + c$. Noting that $c \neq 0$ by assumption, we see that $(x, y) \in \mathbf{F}_q^2$ is a solution of (21) if and only if

$$\begin{cases} x + y = \tau/c^{1/2} \\ x^2 + x\tau/c^{1/2} + \tau v^{1/2}/c^{1/2} + v\tau^2/c = a + v \end{cases} \tag{22}$$

Now we distinguish two cases. If $\tau = 0$ (which is possible if and only if $a + b + c = 0$), then (22) reduces to $x^2 = a + v$, which has a unique solution. Otherwise, $\tau \neq 0$, then the substitution $z = xc^{1/2}/\tau + v^{1/2}c^{1/2}/\tau^{1/2}$ transforms (22) into the equation

$$z^2 + z = v + ac/\tau^2$$

with $z \in \mathbf{F}_q$. The two cases can be conveniently combined by interpreting $z = \infty$ as the only solution of the above equation when $\tau = 0$.

To obtain the last expression, note that if z runs through $\mathbf{F}_q$, then $z^2 + z + v$ runs through $\mathbf{T}_e$ twice. □

Corollary 6.6. *Let* $b, c \in \mathbf{F}_q$ *with* $b + c \in \mathbf{T}_0$ *and* $c \neq 0$*. Then for each* $e \in \mathbf{F}_2$ *we have that*

$$\pi^c_{0,b}(\epsilon) = \begin{cases} 1 + 2\delta_{\epsilon,1}, & \textit{if } b = c; \\ 4\delta_{\epsilon,1}, & \textit{if } b + c \in \mathbf{T}_0^*; \end{cases} \tag{23}$$

where $\epsilon = (-1)^e$*.*

To complete the determination of the intersection parameters, we compute the numbers $\pi^0_{a,b}(\epsilon)$. (These numbers can also be derived from knowledge of the valencies and the other intersection numbers by using relations between these numbers that are valid in any coherent configuration, but the direct approach is simple enough and more revealing.)

Theorem 6.7. *Let* $a, b \in \mathbf{F}_q$. *Then* $\pi^0_{a,b}(-1) = 0$ *and*

$$\pi^0_{a,b}(1) = \begin{cases} q+1, & \text{if } a = b = 0; \\ 1, & \text{if } a = b \neq 0; \\ 2, & \text{if } a + b \in \mathbf{T}_0^*. \end{cases}$$

Proof: It follows from Corollary 6.2 that the numbers $\pi^0_{a,b}(\epsilon)$ are nonzero only if $\epsilon = 1$ and $\mathrm{Tr}(a) = \mathrm{Tr}(b)$. Take $\ell = (1, 0, 0)^\perp$ and $m = (1, 0, 1)^\perp$. Note that $P_\infty = (0, 1, 0)^\top$ and $P_0 = (0, 0, 1)^\top$ are on ℓ and P_∞ and $P_1 = (1, 1, 1)^\top$ are on m, hence ℓ and m intersect on $\mathcal{O}$ so indeed $\hat{\rho}(\ell, m) = 0$. Now count the number of non-tangent lines $n = (1, x, y)^\perp$ such that

$$\hat{\rho}(\ell, n) = xy = a \text{ and } \hat{\rho}(m, n) = x^2 + x + xy = b,$$

or, equivalently,

$$xy = a, \qquad x^2 + x = a + b.$$

Now if $a + b \in \mathbf{T}_0$, then the equation $x^2 + x = a + b$ has two solutions x in $\mathbf{F}_q$, and for each $x \neq 0$ the first equation $xy = a$ has the unique solution $y = a/x$. Finally, for $x = 0$ (which can occur only if $a = b$), there is no solution y if $a \neq 0$ and there are q solutions y if $a = 0$.

□

By combining Lemma 6.3 with our results for the numbers $\pi^c_{a,b}(\epsilon)$ we obtain all intersection parameters. For the sake of completeness, we also state the values of the valencies. From Theorem 4.6, we obtain the following.

Theorem 6.8. *For* $a \in \mathbf{F}_q$, *we have*

$$v_a(\epsilon) = \begin{cases} 2(q-1)\delta_{\epsilon,1}, & \text{if } a = 0; \\ q - \epsilon, & \text{if } a \in \mathbf{F}_q^*. \end{cases}$$

In even characteristic the elliptic association scheme $\mathcal{E}(q)$ has all valencies equal to $q + 1$. We will use the expressions that we have derived for the intersection parameters to show that, in fact, the elliptic scheme is pseudocyclic.

Theorem 6.9. (i) *For all* $b \in \mathbf{T}_0^*$ *and* $\epsilon \in \{-1, 1\}$, *we have that*

$$\sum_{a \in \mathbf{T}_0^*} p_{a,b}^a(\epsilon) = q - 4\delta_{\epsilon,1}.$$

(ii) *The elliptic association scheme* $\mathcal{E}(q)$ *is pseudocyclic.*

Proof: (i) Let $\epsilon = (-1)^e$. Using Lemma 6.3 and Theorem 6.5, we see that for $b \in \mathbf{T}_0^*$, we have

$$\begin{aligned}\sum_{a \in \mathbf{T}_0^*} p_{a,b}^a(\epsilon) &= \sum_{a \in \mathbf{T}_0^*} \pi_{a,b}^a(\epsilon)\\ &= -4\delta_{e,0} + \sum_{a \in \mathbf{T}_0} 2 \sum_{\{\tau \mid \tau^2+\tau=b\}} |\tau^2 \mathbf{T}_e \cap \{a^2\}|\\ &= -4\delta_{e,0} + 2 \sum_{\{\tau \mid \tau^2+\tau=b\}} |\tau^2 \mathbf{T}_e \cap \mathbf{T}_0|\\ &= -4\delta_{e,0} + 2.2.q/4\\ &= q - 4\delta_{e,0}.\end{aligned}$$

(ii) The non-diagonal relations of the elliptic scheme $\mathcal{E}(q)$ are $\mathcal{R}_c(-1, -1)$ with $c \in \mathbf{T}_0^*$. So the claim follows from part (i) together with Theorem 2.3. □

Let $q = 2^r$ and let k be an integer with $\gcd(k, r) = 1$. The field automorphism $\tau_k : x \mapsto x^{2^k}$ provides in a natural way a fusion of the coherent configuration $\mathcal{R}(q)$ as follows. Let the orbits of τ_k on $\mathbf{F}_q$ be $C_0 = \{0\}, C_1, \ldots, C_n$. Define new relations

$$R_j^{\text{fus}}(\epsilon, \phi) = \cup_{c \in C_j} \mathcal{R}_c(\epsilon, \phi).$$

Now since

$$p_{a,b}^c(\epsilon) = p_{\tau_k(a),\tau_k(b)}^{\tau_k(c)}(\epsilon),$$

it follows immediately that the fusion

$$\mathcal{R}^{\text{fus}} = \left\{R_j^{\text{fus}}(\epsilon, \phi) \mid j = 0, \ldots, n, \epsilon, \phi \in \{-1, 1\}\right\}$$

is a coherent configuration, which obviously is again weakly symmetric. The valencies of this new coherent configuration are of the form

$$v_j^{\text{fus}}(\epsilon) = \sum_{c \in C_j} v_c(\epsilon).$$

Now consider two association schemes obtained from this configuration by restricting to the set of hyperbolic or elliptic lines, respectively. It is not difficult to see that such a scheme has all valencies equal precisely in the elliptic case with $q = 2^r$ and r prime.

An old conjecture from [9] states that in this case the scheme is again pseudocyclic. In a subsequent paper [11] we will prove this conjecture.

7. Fusion schemes in the case of even characteristic

In this section, we will assume that the field size is of the form q^2 with q even. Our aim is to show that a certain fusion of the coherent configuration $\mathcal{R}(q^2)$ on the set $\mathcal{L}(q^2)$ of non-tangent lines in PG(2, q^2) is again a coherent configuration. We remark that this fusion does not seem to be induced by a group action. We will write $\mathbf{S}_0$ to denote the collection of elements with absolute trace zero in $\mathbf{F}_q$, that is,

$$\mathbf{S}_0 = \mathbf{T}_0(q);$$

and let

$$\mathbf{S}_1 = \mathbf{F}_q \backslash \mathbf{S}_0$$

denote the collection of elements with absolute trace one in $\mathbf{F}_q$. Also, we will write $\mathbf{T}_0 = \mathbf{T}_0(q^2)$ and $\mathbf{T}_1$ to denote the elements in $\mathbf{F}_{q^2}$ with absolute trace equal to 0 or 1, respectively. As before, for any set U, we write U^* to denote the set $U\backslash\{0\}$ and $\tilde{U}$ to denote the set $U\backslash\{0, 1\}$.

We now define the following relations for two *distinct* lines ℓ and m in $\mathcal{L}(q^2)$.

- R_1 : $\hat{\rho}(\ell, m) \in \mathbf{S}_0^*$;
- R_2 : $\hat{\rho}(\ell, m) \in \mathbf{S}_1$;
- R_3 : $\hat{\rho}(\ell, m) \in \mathbf{T}_0\backslash\mathbf{F}_q$;
- R_4 : $\hat{\rho}(\ell, m) = 0$;
- R_5 : $\hat{\rho}(\ell, m) \in \mathbf{T}_1$.

Furthermore, we let $R_0 = \{(\ell, \ell) \mid \ell \in \mathcal{L}(q^2)\}$ denote the diagonal relation. In addition, for $\epsilon, \phi \in \{-1, 1\}$ and $k = 0, \ldots, 5$, we let $R_k(\epsilon, \phi)$ denote the restriction of R_k to $\mathcal{L}_\epsilon(q^2) \times \mathcal{L}_\phi(q^2)$. For later use, we define sets $\mathbf{R}_i$ for $i = 1, 2, \ldots, 5$ by

$$\mathbf{R}_1 = \mathbf{S}_0^*, \qquad \mathbf{R}_2 = \mathbf{S}_1, \qquad \mathbf{R}_3 = \mathbf{T}_0\backslash\mathbf{F}_q, \qquad \mathbf{R}_4 = \{0\}, \qquad \mathbf{R}_5 = \mathbf{T}_1.$$

Also, we let $r_i = |\mathbf{R}_i|$ for $i = 1, \ldots, 5$, so that

$$r_1 = (q-2)/2, \qquad r_2 = q/2, \qquad r_3 = q(q-2)/2, \qquad r_4 = 1, \qquad r_5 = q^2/2.$$

Note that for distinct lines $\ell, m \in \mathcal{L}(q^2)$ and for $k = 1, \ldots, 5$, we have that $(\ell, m) \in R_k$ if and only if $\hat{\rho}(\ell, m) \in \mathbf{R}_k$.

Not all the relations $R_k(\epsilon, \phi)$ are nonempty.

Lemma 7.1. *We have that $R_k(\epsilon, \phi)$ is nonempty only if*

(i) $k \in \{0, 1, 2, 3\}$, $\epsilon = \phi$;
(ii) $k = 4$, $\epsilon = \phi = 1$;
(iii) $k = 5$, $\epsilon \neq \phi$.

Proof: Since $\mathbf{T}_0^* = \mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_3$, $\mathbf{R}_4 = \{0\}$, and $\mathbf{R}_5 = \mathbf{T}_1$, this follows directly from Lemma 6.1. □

Each of the relations $R_k(\epsilon, \phi)$ is a fusion (a union) of relations of the coherent configuration $\mathcal{R}(q^2)$ on $\mathcal{L}(q^2)$. We want to show that this fusion, which we will denote by $R(q^2)$, in fact defines a new coherent configuration. Since this coherent configuration is again weakly symmetric, the restrictions to both $\mathcal{L}_+(q^2)$ and $\mathcal{L}_-(q^2)$ define fusions $H(q^2)$ and $E(q^2)$ of the hyperbolic and elliptic association schemes $\mathcal{H}(q^2)$ and $\mathcal{E}(q^2)$ defined earlier. In fact, the 4-class fusion $H(q^2)$ of the hyperbolic association scheme is not new: it was first conjectured to be an association scheme in [3] and later this conjecture was proved in [6] (by a direct proof), in [5] (by a geometric argument), and in [14] (using characters). The 3-class fusion $E(q^2)$ of the elliptic scheme seems to be new. We will prove these fusion results by determining the valencies and the intersection parameters of the fusion.

For all $i, j, k \in \{1, \ldots, 5\}$, for all $c \in \mathbf{R}_k$, for all $\epsilon, \phi \in \{-1, 1\}$, and for all lines ℓ, m with $(\ell, m) \in \mathcal{R}_c(\epsilon, \phi)$ (so $\ell \neq m$), we define

$$v_i(\ell) = v_i(\epsilon) = |\{n \in \mathcal{L}(q^2)\backslash\{\ell\} \mid \hat{\rho}(\ell, n) \in \mathbf{R}_i\}|, \tag{24}$$

and

$$p_{i,j}(\ell, m) = p_{i,j}^k(\epsilon) = |\{n \in \mathcal{L}(q^2)\backslash\{\ell, m\} \mid \hat{\rho}(\ell, n) \in \mathbf{R}_i \text{ and } \hat{\rho}(n, m) \in \mathbf{R}_j\}|. \tag{25}$$

Our aim in this section is to show the following.

Theorem 7.2. *The numbers $v_i(\epsilon)$ and $p_{i,j}^k(\epsilon)$ are well-defined, that is, they do not depend on the particular choice of the lines ℓ and m. As a consequence, the relations $R_k(\epsilon, \phi)$ constitute a coherent configuration $R(q^2)$ which is a fusion of $\mathcal{R}(q^2)$. The numbers $v_i(\epsilon)$ are the valencies of $R(q^2)$; their values are $v_i(\epsilon) = r_i(q^2 - \epsilon)$ if $i \neq 4$ and $v_4(\epsilon) = 2(q^2 - 1)\delta_{\epsilon,1}$. The numbers $p_{i,j}^k(\epsilon)$ are the intersection parameters of $R(q^2)$; their values are given inTables* 1 *to* 5.

We will prove this theorem through a sequence of lemmas. First note that we do not need to compute *all* the intersection parameters since we have the following:

Lemma 7.3. (i) *We have that $p_{i,0}^k(\epsilon) = p_{0,i}^k(\epsilon) = \delta_{k,i}$ and $p_{i,j}^0(\epsilon) = v_i(\epsilon)\delta_{i,j}$.*
(ii) *If one of $p_{i,j}^k(\epsilon)$ and $p_{j,i}^k(\epsilon)$ exists, then so does the other and $p_{j,i}^k(\epsilon) = p_{i,j}^k(\epsilon)$ if $k = 1, \ldots, 4$ and $p_{j,i}^5(\epsilon) = p_{i,j}^5(-\epsilon)$.*
(iii) *If four of the numbers $p_{i,j}^k(\epsilon)$ for $j = 1, \ldots, 5$ exist, then so does the fifth, and*

$$\sum_{j=0}^{5} p_{i,j}^k(\epsilon) = v_i(\epsilon).$$

Proof: Part (i) and (iii) are trivial and part (ii) follows from Lemma 6.4. □

Table 1 Intersection numbers $p^1_{i,j}(\epsilon)$

$p^1_{i,j}(\epsilon)$	1	2	3	4	5
1	$(1+\epsilon)q^2/2-(4+5\epsilon)q/2+2+4\epsilon$	$q(q-2(\epsilon+1))/2$	$(q^2-(4+\epsilon)q+4(1+\epsilon))q/2$	$2(q-3)\delta_{\epsilon,1}$	0
2	$q(q-2(\epsilon+1))/2$	$q((1+\epsilon)q-\epsilon)/2$	$q^2(q-(2+\epsilon))/2$	$2q\delta_{\epsilon,1}$	0
3	$(q^2-(4+\epsilon)q+4(1+\epsilon))q/2$	$q^2(q-(2+\epsilon))/2$	$(q^3-4q^2+(4-\epsilon)q+2\epsilon)q/2$	$2q(q-2)\delta_{\epsilon,1}$	0
4	$2(q-3)\delta_{\epsilon,1}$	$2q\delta_{\epsilon,1}$	$2q(q-2)\delta_{\epsilon,1}$	$4\delta_{\epsilon,1}$	0
5	0	0	0	0	$v_5(\epsilon)$

Table 2 Intersection numbers $p^2_{i,j}(\epsilon)$

$p^2_{i,j}(\epsilon)$	1	2	3	4	5
1	$(q/2-1)(q-2(\epsilon+1))$	$(1+\epsilon)q^2/2-(3\epsilon+2)q/2+\epsilon$	$(q^2-(4+\epsilon)q+2(\epsilon+2))q/2$	$2(q-2)\delta_{\epsilon,1}$	0
2	$(1+\epsilon)q^2/2-(3\epsilon+2)q/2+\epsilon$	$q(q/2-\epsilon)$	$(q^2-(2+\epsilon)q+2\epsilon)q/2$	$2(q-1)\delta_{\epsilon,1}$	0
3	$(q^2-(4+\epsilon)q+2(\epsilon+2))q/2$	$(q^2-(2+\epsilon)q+2\epsilon)q/2$	$(q^3-4q^2+(4-\epsilon)q+2\epsilon)q/2$	$2q(q-2)\delta_{\epsilon,1}$	0
4	$2(q-2)\delta_{\epsilon,1}$	$2(q-1)\delta_{\epsilon,1}$	$2q(q-2)\delta_{\epsilon,1}$	$4\delta_{\epsilon,1}$	0
5	0	0	0	0	$v_5(\epsilon)$

Table 3 Intersection numbers $p^3_{i,j}(\epsilon)$. Here $r = q^2 - 2q - 1$

$p^3_{i,j}(\epsilon)$	1	2	3	4	5
1	$(q^2 - (\epsilon+4)q + 4(\epsilon+1))/2$	$q(q-\epsilon-2)/2$	$(q/2-1)(q^2-2q-\epsilon)$	$2(q-2)\delta_{\epsilon,1}$	0
2	$q(q-\epsilon-2)/2$	$q(q-\epsilon)/2$	$(q^2-2q-\epsilon)q/2$	$2q\delta_{\epsilon,1}$	0
3	$(q/2-1)(q^2-2q-\epsilon)$	$(q^2-2q-\epsilon)q/2$	$(q^3-4q^2+(4-3\epsilon)q+8\epsilon)q/2$	$2r\delta_{\epsilon,1}$	0
4	$2(q-2)\delta_{\epsilon,1}$	$2q\delta_{\epsilon,1}$	$2r\delta_{\epsilon,1}$	$4\delta_{\epsilon,1}$	0
5	0	0	0	0	$v_5(\epsilon)$

Table 4 Intersection numbers $p^4_{i,j}(1)$; here $r = q^2 - 2q - 1$

$p^4_{i,j}(1)$	1	2	3	4	5
1	$(q-2)(q-3)/2$	$q(q-2)/2$	$q(q-2)^2/2$	$q-2$	0
2	$q(q-2)/2$	$q(q-1)/2$	$q^2(q-2)/2$	q	0
3	$q(q-2)^2/2$	$q^2(q-2)/2$	$q(q-2)r/2$	$q(q-2)$	0
4	$q-2$	q	$q(q-2)$	q^2-1	0
5	0	0	0	0	$v_5(1)$

Table 5 Intersection numbers $p^5_{i,j}(\epsilon)$

$p^5_{i,j}(\epsilon)$	1	2	3	4	5
1	0	0	0	0	$v_1(\epsilon)$
2	0	0	0	0	$v_2(\epsilon)$
3	0	0	0	0	$v_3(\epsilon)$
4	0	0	0	0	$v_4(\epsilon)$
5	$v_1(-\epsilon)$	$v_2(-\epsilon)$	$v_3(-\epsilon)$	$v_4(-\epsilon)$	0

Our next result justifies all the zero entries in these tables.

Lemma 7.4. *We have that $p^k_{i,j}(\epsilon) = 0$ for $i, j, k = 1, \ldots, 5$ in the following cases.*

(i) *One or three of i, j, k are equal to 5.*
(ii) $k = 4$ *and* $\epsilon = -1$.
(iii) $i = 4$ *or* $j = 4$, $k = 1, 2, 3$, *and* $\epsilon = -1$.

Proof: Direct consequence of Lemma 7.1. □

As a consequence of Lemmas 7.3 and 7.4 we immediately obtain the intersection parameters $p^k_{i,j}(\epsilon)$ for $k = 5$.

Theorem 7.5. *The numbers $p^5_{i,j}(\epsilon)$ exist and are as in Table 5.*

To determine the remaining intersection parameters, we proceed as follows. For $c \in \mathbf{F}_q$, for $\epsilon \in \{-1, 1\}$, and for $A, B \subseteq \mathbf{F}_{q^2}$, we define

$$\pi^c_{A,B}(\epsilon) = \sum_{a \in A, b \in B} \pi^c_{a,b}(\epsilon).$$

Note that by Lemma 6.3, if the numbers $p^k_{i,j}(\epsilon)$ exist, then for all $c \in \mathbf{R}_k$ we have

$$p^k_{i,j}(\epsilon) = \sum_{a \in \mathbf{R}_i, b \in \mathbf{R}_j} p^c_{a,b}(\epsilon) = \pi^c_{\mathbf{R}_i,\mathbf{R}_j}(\epsilon) - \delta_{4,i}\delta_{j,k} - \delta_{4,j}\delta_{i,k}. \tag{26}$$

So to prove our claim we have to compute the numbers $\pi^c_{\mathbf{R}_i,\mathbf{R}_j}(\epsilon)$ and show that they do not depend on the choice of c in $\mathbf{R}_k$. We need the following simple results.

Lemma 7.6. *Let $f, g \in \mathbf{F}_2$ and $\epsilon \in \{-1, 1\}$. If $A \subseteq \mathbf{T}^*_f$ and $B \subseteq \mathbf{T}^*_g$, then*

$$\pi^0_{\{0\},\{0\}}(\epsilon) = \delta_{\epsilon,1}(q^2 + 1),$$

$$\pi^0_{\{0\},B}(\epsilon) = \pi^0_{B,\{0\}}(\epsilon) = 2|B|\delta_{\epsilon,1}\delta_{g,0},$$

and

$$\pi^0_{A,B}(\epsilon) = (2|A||B| - |A \cap B|)\delta_{\epsilon,1}\delta_{f,g}.$$

Proof: Direct consequence of Theorem 6.7. □

Lemma 7.7. *Let $g, h \in \mathbf{F}_2$ and $\epsilon \in \{-1, 1\}$. If $B \subseteq \mathbf{T}^*_g$ and $c \in \mathbf{T}^*_h$, then*

$$\pi^c_{\{0\},\{0\}}(\epsilon) = 4\delta_{\epsilon,1}\delta_{h,0}$$

and

$$\pi^c_{\{0\},B}(\epsilon) = \pi^c_{B,\{0\}}((-1)^h\epsilon) = \delta_{g,h}((1 - 2\delta_{\epsilon,1})\delta_{c \in B} + 4|B|\delta_{\epsilon,1}).$$

Proof: Direct consequence of Corollary 6.6. □

Using the above results, we now determine the intersection parameters involving the relation R_4.

Theorem 7.8. *The intersection parameters $p^4_{i,j}(1)$ exist and are as in Table* 4.

Proof: According to (26) and Lemma 7.6 we have that

$$\begin{aligned} p^4_{4,j}(1) &= \pi^0_{\{0\},\mathbf{R}_j}(1) - 2\delta_{j,4} \\ &= \begin{cases} 2r_j, & \text{if } j \in \{1, 2, 3\}; \\ q^2 - 1, & \text{if } j = 4. \end{cases} \end{aligned}$$

This gives the values for $p^4_{4,j}$ as in Table 4.

Next we consider $p^4_{i,j}(1)$ with $i, j \in \{1, 2, 3, 5\}$, where we assume that either $i = j = 5$ or $i, j \in \{1, 2, 3\}$. According to (26) and Theorem 6.7, we have that

$$p^4_{i,j}(1) = \pi^0_{\mathbf{R}_i,\mathbf{R}_j}(1) = \begin{cases} 2r_i r_j, & \text{if } i \neq j; \\ r_i(2(r_i - 1) + 1) = r_i(2r_i - 1), & \text{if } i = j. \end{cases}$$

In view of Lemma 7.3, this is sufficient information to obtain the remaining values for $p^4_{i,j}(1)$ in Table 4. □

Theorem 7.9. *The intersection parameters $p^k_{4,j}(1)$ and $p^k_{i,4}(1)$ exist and are as stated in Theorem* 7.2.

Proof: If $j = k = 5$ or if $j, k \in \{1, 2, 3\}$, and if $c \in \mathbf{R}_k$, then using (26) and Lemma 7.7 we find that

$$p^k_{4,j}(1) = \pi^c_{\{0\},\mathbf{R}_j}(1) - \delta_{j,k} = \begin{cases} 4r_j, & \text{if } j \neq k; \\ 4r_j - 2, & \text{if } j = k. \end{cases}$$

This produces the values for $p^k_{4,j}(1)$ as claimed. The other values follow from Lemma 7.3.

□

To complete our determination of the intersection parameters, we compute the numbers $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$ for $i, j \in \{0, 1\}$ and $c \in \mathbf{F}^*_{q^2}$. Note that both $\mathbf{S}^*_i$ and $\mathbf{S}^*_j$ are one of $\mathbf{R}_1, \mathbf{R}_2$. Since

$$\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) = \pi^c_{\mathbf{S}^*_i,\mathbf{S}^*_j}(\epsilon) + \delta_{i,0}\pi^c_{\{0\},\mathbf{S}^*_j}(\epsilon) + \delta_{j,0}\pi^c_{\mathbf{S}^*_i,\{0\}}(\epsilon) + \delta_{i,0}\delta_{j,0}\pi^c_{\{0\},\{0\}}(\epsilon) \tag{27}$$

and since we know already the numbers $\pi^c_{\{0\},\mathbf{R}_t}(\epsilon)$, $\pi^c_{\mathbf{R}_s,\{0\}}(\epsilon)$, and $\pi^c_{\{0\},\{0\}}(\epsilon)$ with $s, t \in \{1, 2\}$, knowing $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$ will enable us to find the numbers $\pi^c_{\mathbf{R}_s,\mathbf{R}_t}(\epsilon)$.

We need some preparation. For $r \in \mathbf{F}_q$, let us define

$$\mathbf{S}_r := \{x \in \mathbf{F}_{q^2} \mid \mathrm{Tr}_{\mathbf{F}_q}(x) := x + x^2 + \cdots + x^{q/2} = r\}$$

and

$$\mathbf{G}_r := \{x \in \mathbf{F}_{q^2} \mid x^q + x = r\}.$$

We will use the following properties.

Lemma 7.10. *The above definitions of* $\mathbf{S}_0$ *and* $\mathbf{S}_1$ *coincide with the definitions of* $\mathbf{S}_0$ *and* $\mathbf{S}_1$ *given earlier. Moreover,*

(i) $\mathbf{G}_0 = \mathbf{F}_q$, *the sets* $\mathbf{G}_r$ *with* $r \in \mathbf{F}_q$ *are precisely the additive cosets of* $\mathbf{F}_q$ *in* $(\mathbf{F}_{q^2}, +)$, *and* $\mathbf{G}_r + \mathbf{G}_s = \mathbf{G}_{r+s}$ *for all* $r, s \in \mathbf{F}_q$. *We also have that* $\mathbf{G}_r = r\mathbf{G}_1$ *for* $r \in \mathbf{F}_q^*$.

(ii) *The map* $F : x \mapsto x^2 + x$ *maps* $\mathbf{G}_r$ *two-to-one onto* $\mathbf{S}_r$. *In particular, we have* $|\mathbf{S}_r| = q/2$, *and the subsets* $\mathbf{S}_r$ *with* $r \in \mathbf{F}_q$ *partition* $\mathbf{T}_0$. *Also, the subsets* $\mathbf{S}_r$ *with* $r \in \mathbf{F}_q$ *are the cosets of* $\mathbf{S}_0$ *in* $(\mathbf{T}_0, +)$; *we have that* $\mathbf{S}_r + \mathbf{S}_s = \mathbf{S}_{r+s}$ *for all* $r, s \in \mathbf{F}_q$.

(iii) *For each* $r \in \mathbf{F}_q$ *we have* $\mathbf{G}_{r^2+r} = \mathbf{S}_r \cup \mathbf{S}_{r+1}$.

Proof: Part (i) follows from the fact that the map $x \mapsto x^q + x$ from $\mathbf{F}_{q^2}$ to itself is $\mathbf{F}_q$-linear with kernel $\mathbf{F}_q$ and image $\mathbf{F}_q$.

Next, if $x \in \mathbf{F}_{q^2}$ satisfies $x^q + x = r$, then

$$\begin{aligned}\mathrm{Tr}_{\mathbf{F}_q}(x^2 + x) &= (x^2 + x) + (x^2 + x)^2 + \ldots + (x^2 + x)^{q/2} \\ &= x + x^q = r,\end{aligned}$$

hence F maps $\mathbf{G}_r$ to $\mathbf{S}_r$. Note that the image of $\mathbf{F}_{q^2}$ under F is $\mathbf{T}_0$; hence part (ii) now follows from the fact that F is $\mathbf{F}_2$-linear with kernel $\mathbf{F}_2 \subseteq \mathbf{G}_0$.

Finally, if $x^q + x = r$, then $F(x)^q + F(x) = r^2 + r$, hence $\mathbf{S}_r$ (and similarly $\mathbf{S}_{r+1}$) are subsets of $\mathbf{G}_{r^2+r}$. Since $\mathbf{S}_r$ and $\mathbf{S}_{r+1}$ are disjoint and both have size $q/2$, the result follows.

Remark that $\mathbf{G}_0 = \mathbf{F}_q$, hence $\mathbf{S}_0$ consists of the elements in $\mathbf{F}_q$ with trace zero, and since $\mathbf{G}_0 = \mathbf{S}_0 \cup \mathbf{S}_1$, we have $\mathbf{S}_1 = \mathbf{F}_q \setminus \mathbf{S}_0$. So the definitions of $\mathbf{S}_0$ and $\mathbf{S}_1$ coincide with the ones given earlier in $\mathbf{F}_q$. □

Now to compute $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$ for $\epsilon = (-1)^e$ with $e \in \mathbf{F}_2$, for $i, j \in \{0, 1\}$ and for $c \in \mathbf{F}^*_{q^2}$, we start with the expression

$$\begin{aligned}\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) &= \sum_{a\in\mathbf{S}_i}\sum_{b\in\mathbf{S}_j}\pi^c_{a,b}(\epsilon) \\ &= \sum_{a\in\mathbf{S}_i}\sum_{b\in\mathbf{S}_j}\left(\delta_{a+b+c,0} + 2\sum_{\tau}|\mathbf{T}_e \cap \{ac/\tau^2\}|\right),\end{aligned}$$

where the sum is over all $\tau \in \mathbf{F}^*_{q^2}$ such that $\tau^2 + \tau = a + b + c$. (The last equality was obtained by using Theorem 6.5.) Obviously, $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$ is nonzero only if $a + b + c \in \mathbf{T}_0$, hence only if $c \in \mathbf{T}_0$. So assume that $c \in \mathbf{S}^*_k$ with $k \in \mathbf{F}_q$. We will write $r = i + j + k$. Now in the above expression for $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$, we first sum over $b \in \mathbf{S}_j$. If b runs through $\mathbf{S}_j$, then by Lemma 7.10, part (ii), we have that $a + b + c$ runs through $\mathbf{S}_r$ and the sum is over all $\tau \in \mathbf{G}^*_r$. So we obtain that

$$\begin{aligned}\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) &= \sum_{a\in\mathbf{S}_i}\left(\delta_{r,0} + 2\sum_{\substack{\tau^2+\tau\in\mathbf{S}_r\\ \tau\neq 0}}|\mathbf{T}_e \cap \{ac/\tau^2\}|\right) \\ &= \delta_{r,0}q/2 + 2\sum_{a\in\mathbf{S}_i}\sum_{\tau\in\mathbf{G}^*_r}|\mathbf{T}_e \cap \{ac/\tau^2\}| \\ &= \delta_{r,0}q/2 + 2\sum_{\tau\in\mathbf{G}^*_r}|\mathbf{T}_e \cap \mathbf{S}_i(c/\tau^2)|.\end{aligned}$$

Now we have the following.

Lemma 7.11. *Let $i, e \in \mathbf{F}_2$ and $\lambda \in \mathbf{F}_{q^2}^*$. Then*

$$|\mathbf{T}_e \cap \mathbf{S}_i\lambda| = \begin{cases} \delta_{e,0}q/2, & \text{if } \lambda \in \mathbf{G}_0^*; \\ \delta_{e,i}q/2, & \text{if } \lambda \in \mathbf{G}_1; \\ q/4, & \text{if } \lambda \in \mathbf{G}_r \text{ with } r \in \mathbf{F}_q\backslash\{0, 1\}. \end{cases}$$

Proof: Since $i \in \mathbf{F}_2$, we have $\mathbf{S}_i \subset \mathbf{F}_q$. For any $s \in \mathbf{S}_i$, we see that $\lambda s \in \mathbf{T}_e$ precisely when $e = \mathrm{Tr}_{\mathbf{F}_{q^2}/\mathbf{F}_2}(\lambda s) = \mathrm{Tr}_{\mathbf{F}_q}(s(\lambda^q + \lambda))$, that is, when $s\mu \in \mathbf{S}_e$, where $\mu = \lambda^q + \lambda$. Now $\mu = 0$ if and only if $\lambda \in \mathbf{G}_0$, in which case for all $s \in \mathbf{S}_i$ we have that $s\mu \in \mathbf{S}_e$ precisely when $e = 0$. If $\mu \neq 0$, then the above shows that $|\mathbf{T}_e \cap \mathbf{S}_i\lambda| = |\mathbf{S}_e \cap \mathbf{S}_i\mu|$. Note that $\mathbf{S}_e$ and $\mathbf{S}_i\mu$ are both hyperplanes of $\mathbf{F}_q$ (considered as a vector space over $\mathbf{F}_2$), so the size of the intersection equals $\delta_{e,i}q/2$ if $\mu = 1$ (which occurs precisely when $\lambda \in \mathbf{G}_1$) and $q/4$ otherwise. □

In order to use this result, given $c \in \mathbf{S}_k^*$ and $r = k + i + j$ with $i, j \in \mathbf{F}_2$, we have to determine for how many $\tau \in \mathbf{G}_r^*$ we have $c/\tau^2 \in \mathbf{G}_0$, and for how many $\tau \in \mathbf{G}_r^*$, we have $c/\tau^2 \in \mathbf{G}_1$. The result is as follows.

Lemma 7.12. *Let $i, j \in \mathbf{F}_2$, $k \in \mathbf{F}_q$, $r = k + i + j$, let $\tau \in \mathbf{G}_r^*$, and let $c \in \mathbf{S}_k^*$. Write $c = \gamma^2 + \gamma$ with $\gamma \in \mathbf{G}_k$. Define τ_0, τ_1, and τ_2 by $\tau_0^2 = cr/(r+1)$, $\tau_1^2 = \gamma r$, and $\tau_2^2 = (\gamma + 1)r$. Then*

(i) *τ_0, τ_1, and τ_2 are zero if and only if $r = 0$;*
(ii) *for $r \neq 1$ we have $\tau_0 \in \mathbf{G}_r$, and for $u = 1$ or 2 we have $\tau_u \in \mathbf{G}_r$ if and only if $i = j$;*
(iii) *c/τ^2 is contained in $\mathbf{G}_0$ if and only if either $r = 0$, or $r \neq 0, 1$ and $\tau = \tau_0$;*
(iv) *c/τ^2 is contained in $\mathbf{G}_1$ if and only if $r \neq 0$ and $\tau \in \{\tau_1, \tau_2\}$.*

Proof: Obviously, since $c \neq 0$ we have $\gamma \neq 0, 1$; hence for $u = 0, 1, 2$, $\tau_u = 0$ precisely when $r = 0$. Also, $r \in \mathbf{F}_q$, and by Lemma 7.10 we have $c \in \mathbf{S}_k \subseteq \mathbf{G}_{k^2+k} = \mathbf{G}_{r^2+r}$; hence

$$\tau_0^q + \tau_0 = ((c^q + c)r/(r+1))^{1/2} = ((r^2 + r)r/(r+1))^{1/2} = r,$$

so $\tau_0 \in \mathbf{G}_r$. Also, since $\gamma \in \mathbf{G}_k$ we have that

$$\tau_1^q + \tau_1 = (r(\gamma^q + \gamma))^{1/2} = (rk)^{1/2},$$

so $\tau_1 \in \mathbf{G}_r$ if and only if $k = r$, that is, $i = j$. The same argument proves the claim for τ_2.

Finally, let $c/\tau^2 \in \mathbf{G}_s$ for some $s \in \mathbf{F}_q$. We have to determine when $s = 0$ and when $s = 1$. We saw above that $c \in \mathbf{G}_{r^2+r}$, so by definition, we have that

$$\begin{aligned} s &= (c/\tau^2)^q + c/\tau^2 \\ &= (c + r^2 + r)/(\tau + r)^2 + c/\tau^2 \\ &= (cr^2 + (r^2 + r)\tau^2)/((\tau(\tau + r))^2. \end{aligned}$$

So firstly, we have $s = 0$ if and only if either $r = 0$, or $r \neq 0, 1$ and $\tau^2 = cr/(r + 1) = \tau_0^2$. Secondly, we can have $s = 1$ only if $r \neq 0$. In that case, we have $s = 1$ if

$$cr^2 + (r^2 + r)\tau^2 = (\tau(\tau + r))^2,$$

that is, if $\tau^4 + r\tau^2 + cr^2 = 0$, i.e., if $c = (\tau^2/r)^2 + \tau^2/r$. So this happens if $\tau^2/r \in \{\gamma, \gamma + 1\}$, that is, if $\tau \in \{\tau_1, \tau_2\}$. □

Corollary 7.13. *Let $i, j \in \mathbf{F}_2$, let $c \in \mathbf{S}_k^*$ with $k \in \mathbf{F}_q$, and let $e \in \mathbf{F}_2$. Writing $\epsilon = (-1)^e$, we have that*

$$\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) = \begin{cases} q((1+\epsilon)q - \epsilon)/2, & \text{if } k = i + j; \\ q(2(2\delta_{e,i} - 1) + q)/2, & \text{if } k = 1 \text{ and } i = j; \\ q(2(2\delta_{e,i} - 1) + q + \epsilon)/2, & \text{if } k \neq 0, 1 \text{ and } i = j; \\ q^2/2, & \text{if } k = 0 \text{ and } i \neq j; \\ q(q + \epsilon)/2, & \text{if } k \neq 0, 1 \text{ and } i \neq j. \end{cases} \tag{28}$$

Proof: Let $r = i + j + k$. If we combine Lemmas 7.11 and 7.12 with the expression for $\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon)$ just before Lemma 7.11, we obtain the following. First, if $r = 0$, that is, if $k = i + j$, then

$$\pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) = q/2 + 2((q-1)\delta_{\epsilon,1}q/2) = q(1 + 2(q-1)\delta_{\epsilon,1})/2.$$

Next, if $r \neq 0$, that is, if $k \neq i + j$, then $0 \notin \mathbf{G}_r$, and we obtain that

$$\begin{aligned} \pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) &= 2(\delta_{k\neq 0,1}\delta_{\epsilon,1}q/2 + 2\delta_{i=j}\delta_{e,i}q/2 + (q - \delta_{k\neq 0,1} - 2\delta_{i=j})q/4) \\ &= (\delta_{k\neq 0,1}\epsilon + 2\delta_{i=j}(2\delta_{e,i} - 1) + q)q/2, \end{aligned}$$

from which the other expressions follow. □

Now we use (26) and (27) to compute the intersection numbers $p^t_{r,s}(\epsilon)$ with $r, s \in \{1, 2\}$. For $i, j \in \mathbf{F}_2$ and $k \in \mathbf{F}_q$, define

$$\theta(i, j, k)(\epsilon) = \delta_{i,0}\pi^c_{\{0\},\mathbf{S}_j^*}(\epsilon) + \delta_{j,0}\pi^c_{\mathbf{S}_i^*,\{0\}}(\epsilon) + \delta_{i,0}\delta_{j,0}\pi^c_{\{0\},\{0\}}(\epsilon),$$

where $c \in \mathbf{S}_k$ and $c \neq 0$.

Lemma 7.14. *Write* $\epsilon = 2\delta_{e,0} - 1 = (-1)^e$. *We have that*

$$\theta(i, j, k)(\epsilon) = -\epsilon(\delta_{i,0}\delta_{j,k} + \delta_{j,0}\delta_{i,k}) + \delta_{e,0}(2q(\delta_{i,0} + \delta_{j,0}) - 4\delta_{i,0}\delta_{j,0}).$$

Proof: Direct application of Lemma 7.7. □

Theorem 7.15. *The intersection parameters* $p^t_{r,s}(\epsilon)$ *with* $r, s \in \{1, 2\}$ *and* $t \in \{1, 2, 3\}$ *exist and are as in Tables* 1, 2, *and* 3.

Proof: Let $r, s \in \{1, 2\}$ and $t \in \{1, 2, 3\}$. Put $i = r - 1$ and $j = s - 1$ (and consider i and j as elements of $\mathbf{F}_2$). If $t = 1$, then we take $k = 0$; if $t = 2$, then we take $k = 1$; and if $t = 3$, then take k to be any element in $\mathbf{F}_q \setminus \{0, 1\}$. Finally, let $c \in \mathbf{S}^*_k$. Then according to (26) and (27), we have that

$$p^t_{r,s}(\epsilon) = \pi^c_{\mathbf{S}_i,\mathbf{S}_j}(\epsilon) - \theta(i, j, k)(\epsilon),$$

where $\epsilon = (-1)^e$. Now we can use Corollary 7.13 and Lemma 7.14, firstly to see that the expression at the right-hand side indeed only depends on t and not on the actual value of k and c, and secondly to compute the value of the intersection parameters $p^t_{r,s}(\epsilon)$. In this way, we obtain the values as announced in the theorem. □

Now we can use Lemma 7.3 to find the remaining intersection numbers in Tables 1, 2 and 3. So we have proved the following.

Theorem 7.16. *The intersection parameters* $p^t_{r,s}(\epsilon)$ *with* $t \in \{1, 2, 3\}$ *exist and are as in Tables* 1, 2, *and* 3.

This completes the proof of Theorem 7.2.

For the sake of completeness, we mention that the P- and Q-matrix of the elliptic fusion scheme are given by

$$P = \begin{pmatrix} 1 & (q-2)(q^2+1)/2 & q(q^2+1)/2 & q(q-2)(q^2+1)/2 \\ 1 & -(q-1)(q-2)/2 & -q(q-1)/2 & q(q-2) \\ 1 & -(q^2-q+2)/2 & q(q+1)/2 & -q \\ 1 & q-1 & 0 & -q \end{pmatrix} \tag{29}$$

and

$$Q = \begin{pmatrix} 1 & q(q^2+1)/2 & (q-2)(q^2+1)/2 & q(q-2)(q^2+1)/2 \\ 1 & -q(q-1)/2 & -(q^2-q+2)/2 & q(q-1) \\ 1 & -q(q-1)/2 & (q-2)(q+1)/2 & 0 \\ 1 & q & -1 & -q \end{pmatrix}. \tag{30}$$

The P-matrix of the hyperbolic fusion scheme can be found in [5].

8. Further fusions

From the values of the intersection parameters $p^k_{i,j}(\epsilon)$ as computed in the previous section we immediately see that a further fusion of the relations $R_1(\epsilon,\epsilon)$ and $R_2(\epsilon,\epsilon)$ for $\epsilon=1$ and $\epsilon=-1$ produces another weakly symmetric coherent configuration, and thus a further 3-class association scheme on the hyperbolic lines and 2-class association scheme (that is, a strongly regular graph) on the elliptic lines; the intersection parameters are given in Tables 6–9 below.

Finally, we see that a further fusion of $R_1(1,1)\cup R_2(1,1)$ with $R_4(1,1)$ again produces a weakly symmetric coherent configuration, and thus a 2-class association scheme (that is, a strongly regular graph) on the hyperbolic lines. Some of the intersection parameters of this further fusion are given in Tables 10 and 11.

Table 6 Intersection numbers $p^{\{1,2\}}_{*,*}(\epsilon)$

$p^{\{1,2\}}_{*,*}(\epsilon)$	$\{1,2\}$	3	4	5
{1,2}	$(2+\epsilon)q^2-(4+5\epsilon)q+2+4\epsilon$	$q(q^2-(\epsilon+3)q+2(1+\epsilon))$	$2(2q-3)\delta_{\epsilon,1}$	0
3	$q(q^2-(\epsilon+3)q+2(1+\epsilon))$	$(q^3-4q^2+(4-\epsilon)q+2\epsilon)q/2$	$2q(q-2)\delta_{\epsilon,1}$	0
4	$2(2q-3)\delta_{\epsilon,1}$	$2q(q-2)\delta_{\epsilon,1}$	$4\delta_{\epsilon,1}$	0
5	0	0	0	$v_5(\epsilon)$

Table 7 Intersection numbers $p^3_{*,*}(\epsilon)$. Here $r=q^2-2q-1$

$p^3_{*,*}(\epsilon)$	$\{1,2\}$	3	4	5
{1,2}	$2q^2-(2\epsilon+4)q+2(\epsilon+1)$	$q^3-3q^2-(\epsilon-2)q+\epsilon$	$4(q-1)\delta_{\epsilon,1}$	0
3	$q^3-3q^2-(\epsilon-2)q+\epsilon$	$(q^3-4q^2+(4-3\epsilon)q+8\epsilon)q/2$	$2r\delta_{\epsilon,1}$	0
4	$4(q-1)\delta_{\epsilon,1}$	$2r\delta_{\epsilon,1}$	$4\delta_{\epsilon,1}$	0
5	0	0	0	$v_5(\epsilon)$

Table 8 Intersection numbers $p^4_{*,*}(1)$. Here $r=q^2-2q-1$

$p^4_{*,*}(1)$	$\{1,2\}$	3	4	5
$\{1,2\}$	$2q^2-5q+3$	$q(q-1)(q-2)$	$2(q-1)$	0
3	$q(q-1)(q-2)$	$q(q-2)r/2$	$q(q-2)$	0
4	$2(q-1)$	$q(q-2)$	q^2-1	0
5	0	0	0	$v_5(1)$

Table 9 Intersection numbers $p^5_{*,*}(\epsilon)$

$p^5_{*,*}(\epsilon)$	$\{1,2\}$	3	4	5
$\{1,2\}$	0	0	0	$v_1(\epsilon)+v_2(\epsilon)$
3	0	0	0	$v_3(\epsilon)$
4	0	0	0	$v_4(\epsilon)$
5	$v_1(-\epsilon)+v_2(-\epsilon)$	$v_3(-\epsilon)$	$v_4(-\epsilon)$	0

Table 10 Intersection numbers $p^{\{1,2,4\}}_{*,*}(1)$. Here $r = q^2 - 2q - 1$

$p^{\{1,2,4\}}_{*,*}(1)$	$\{1, 2, 4\}$	3	5
$\{1, 2, 4\}$	$3q^2 - q - 2$	$q^2(q - 2)$	0
3	$q^2(q - 2)$	$q(q - 2)r/2$	0
5	0	0	$v_5(1)$

Table 11 Intersection numbers $p^3_{*,*}(1)$. Here $r = q^2 - 2q - 1$

$p^3_{*,*}(1)$	$\{1, 2, 4\}$	3	5
$\{1, 2, 4\}$	$2q(q + 1)$	$(q + 1)r$	0
3	$(q + 1)r$	$(q^3 - 4q^2 + q + 8)q/2$	0
5	0	0	$v_5(1)$

So in this way we obtain two strongly regular graphs, with parameters (v, k, λ, μ) where $v = q^2(q^2 + \epsilon)/2$, $k = (q^2 - \epsilon)(q + \epsilon)$, $\lambda = 2(q^2 - 1) + \epsilon q(q - 1)$, and $\mu = 2q(q + \epsilon)$. Graphs with these parameters were first described by R. Metz for $\epsilon = -1$ (the elliptic case) and by Brouwer and Wilbrink for $\epsilon = 1$ (the hyperbolic case), see [2, Section 7]. The two constructions were further generalized by Wilbrink. For $q = 4$, the "elliptic" graph was obtained and conjectured to be a Metz graph in [9, p. 83].

In a forthcoming paper [10] it will be shown among other things that the strongly regular graphs obtained above are in fact *isomorphic* to the Metz graphs (for $\epsilon = -1$) or the Brouwer-Wilbrink graphs (for $\epsilon = 1$). For $\epsilon = 1$ this has already been proved in [5]; for $\epsilon = -1$ this was first conjectured in [9] for $q = 4$, and proved for the first time in [10].

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