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Balanced Configurations of $2n + 1$ Plane Vectors

N. RESSAYRE ressayre@math.univ-montp2.fr
Université Montpellier II, Département de Mathématiques, Case courrier 051-Place Eugène Bataillon, 34095 Montpellier Cedex 5, France

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1. Introduction

A plane configuration $\{v_1, v_2, \dots, v_m\}$ (where m is a positive integer) of vectors of $\mathbb{R}^2$ is said to be *balanced* if for any index $i \in \{1, \dots, m\}$ the multiset

$$\{\det(v_i, v_j) : j \neq i\}$$

is symmetric around the origin. A plane configuration is said to be *uniform* if every pair of vectors is linearly independent.

E. Cattani, A. Dickenstein and B. Sturmfels introduced this notion in [1, 2] for its relationship with multivariable hypergeometric functions in the sense of Gel'fand, Kapranov and Zelevinsky (see [3, 4]).

Balanced plane configurations with at most six vectors have been classified in [2]. With the help of computer calculation, E. Cattani, A. Dickenstein classified the balanced plane configurations of seven vectors in [2]. Moreover, they conjectured that any uniform balanced plane configuration is $GL_2(\mathbb{R})$-equivalent to a regular $(2n + 1)$-gon (where n is a positive integer). In this note, we prove this conjecture.

2. Statement of the result

Let m be a positive integer.

Definition 1 A configuration $\{v_1, \dots, v_m\}$ is said to be *balanced* if for all $i = 1, \dots, m$ and for all x in $\mathbb{R}$ the cardinality of the set $\{j \neq i : \det(v_i, v_j) = x\}$ equals that of the set $\{j \neq i : \det(v_i, v_j) = -x\}$.

Definition 2 A balanced configuration $\{v_1, \dots, v_m\}$ is said to be *uniform* if for any pair $i \neq j$, the vectors v_i, v_j are linearly independent.

Remark Assume $\{v_1, \dots, v_m\}$ is balanced and m even. Then, the multiset $\{\det(v_1, v_j) : j = 2, \dots, m\}$ is symmetric around 0 and of odd cardinality; so it contains 0. Then,

$\{v_1, \ldots, v_m\}$ is not uniform. From, now on we are only interested in configurations with an odd number of vectors. So, we assume that $m = 2n + 1$ for an integer n.

Let us identify $\mathbb{R}^2$ with the field $\mathbb{C}$ of complex numbers. To avoid any confusion with index-numbers, we denote by $\sqrt{-1}$ the complex number i. Denote by $\mathbb{U}_m$ the set of m th-roots of 1.

Set $\omega = e^{\frac{2\sqrt{-1}\pi}{m}}$. Then, $\mathbb{U}_m = \{w^k : k = 0, \ldots 2n\}$. For all integers k and a, we have

$$\det(\omega^k, \omega^{k+a}) = -\det(\omega^k, \omega^{k-a}). \tag{1}$$

In particular, $\mathbb{U}_m$ is a uniform balanced configuration.

One can note that the group $\mathrm{GL}_2(\mathbb{R})$ acts naturally on the set of balanced (resp. uniform balanced) configurations of m vectors. Indeed, if $g \in \mathrm{GL}_2(\mathbb{R})$ then $\det(g.v_i, g.v_j) = \det(g)\det(v_i, v_j)$.

The aim of this note is to prove the

Theorem 1 *For any odd integer m, $\mathrm{GL}_2(\mathbb{R})$ acts transitively on the set of uniform balanced configurations of m vectors.*

In other words, modulo $\mathrm{GL}_2(\mathbb{R})$, $\mathbb{U}_m$ is the only uniform balanced configuration of m vectors.

3. The proof

3.1. —

Let us fix some notation and convention. The set $\{0, \ldots, 2n\}$ is denoted by I.

Definition 3.1 Let us recall that we identify $\mathbb{R}^2$ with the field $\mathbb{C}$ of complex numbers. Let $\{v_0, \ldots, v_{m-1}\}$ be a uniform configuration of m points in $\mathbb{R}^2$. Each v_i has a unique polar form $v_i = \rho_i e^{\alpha_i}$ with ρ_i in $]0; +\infty[$ and α_i in $[0; 2\pi[$. The set $\{v_0, \ldots, v_{m-1}\}$ is said to be *labelled by increasing arguments* if

$$\alpha_0 < \alpha_1 < \cdots < \alpha_{m-1}.$$

Convention 1 Let $i \in I$. For all k in $\mathbb{Z}$ which equals i modulo m, we also denote by v_k the vector v_i.

The first step of the proof is to show that any uniform configuration satisfies equations similar to Eqs. (1). Precisely, we have:

Lemma 3.1 *Let $\mathcal{C} = \{v_0, \ldots, v_{2n}\}$ be a uniform balanced configuration labelled by increasing arguments. Then,*

$$\det(v_k, v_{k+a}) = -\det(v_k, v_{k-a}) \quad \forall\, k,\ a \in \mathbb{Z}$$

Proof: We denote by $\mathcal{P}_2(I)$ the set of pairs of elements of I. The fact that $\mathcal{C}$ is uniform balanced can be formulated as follow. For all $i \in I$, there exists a part $\mathcal{P}_2^i(I)$ of $\mathcal{P}_2(I)$ such that:

- $I - \{i\}$ is the disjoint union of the elements of $\mathcal{P}_2^i(I)$, and
- $\forall \{k, l\} \in \mathcal{P}_2^i(I) \quad \det(v_i, v_k) = -\det(v_i, v_l) \neq 0$.

For any pair $\{k, l\} \in \mathcal{P}_2(I)$, the set of vectors $v \in \mathbb{R}^2$ such that $\det(v, v_k) = -\det(v, v_l)$ is the vectorial line generated by $v_k + v_l$ (let us recall that v_k, v_l are linearly independent). In particular, since $\mathcal{C}$ is uniform there exists at most one $i \in I$ such that $\det(v_i, v_k) = -\det(v_i, v_l)$. This means that for any $i \neq j$ the set $\mathcal{P}_2^i(I) \cap \mathcal{P}_2^j(I)$ is empty.

Moreover, the cardinality of $\mathcal{P}_2^i(I)$ equals n for all $i \in I$. Then, the cardinality of $\bigcup_{i \in I} \mathcal{P}_2^i(I)$ equals nm, that is the cardinality of $\mathcal{P}_2(I)$. It follows that $\bigcup_{i \in I} \mathcal{P}_2^i(I) = \mathcal{P}_2(I)$. In other words, there exists a map

$$\phi : \mathcal{P}_2(I) \longrightarrow I,$$

such that, for all $\{k, l\} \in \mathcal{P}_2(I)$, we have:

$$\det\left(v_{\phi(\{k,l\})}, v_k\right) = -\det\left(v_{\phi(\{k,l\})}, v_l\right).$$

It is sufficient to prove the lemma for $a = -n, \ldots, -1, 1, \ldots, n$; and by symmetry for $a = 1, \ldots, n$. We prove this by decreasing induction going from $a = n$ to $a = 1$.

Assume $a = n$ and fix k. Relabeling the vectors, we may assume that $k = n + 1$. Then, we have to prove that: $\det(v_{n+1}, v_0) = -\det(v_{n+1}, v_1)$, that is, $\phi(\{0, 1\}) = n + 1$.

Note that the set of $i \in I$ such that $\det(v_0, v_i)$ is positive (that is, such that $\alpha_i - \alpha_0 < \pi$) is of cardinality n. Then, by Convention 1 $\alpha_n - \alpha_0 < \pi$.

For all $t = 0, \ldots, n - 1$, since $v_{\phi(\{t,t+1\})}$ belongs to $\mathbb{R}(v_t + v_{t+1})$, its argument $\alpha_{\phi(\{t,t+1\})}$ belongs to $]\pi + \alpha_t; \pi + \alpha_{t+1}[$. In particular, each one of the n intervals $]\pi + \alpha_t; \pi + \alpha_{t+1}[$ (for $t = 0, \ldots, n - 1$) contains one of the α_i for $i = n + 1, \ldots, 2n$. So, $\alpha_{\phi(\{0,1\})}$ is the only α_i in the interval $]\pi + \alpha_0; \pi + \alpha_1[$. It follows that $\phi(\{0, 1\}) = n + 1$.

Suppose now the proposition proved for $a = n, \ldots, n - u + 2$ (with $n \geq u \geq 2$) and prove that it is true for $a = n - u + 1$. As before, it is sufficient to prove that:

$$\begin{aligned}\phi(\{0, u\}) &= \frac{u}{2} \quad \text{if } u \text{ is even}\\ &= \frac{u+m}{2} = \frac{u+1}{2} + n \quad \text{if } u \text{ is odd}\end{aligned}$$

Since, $v_{\phi(\{0,u\})}$ belongs to $\mathbb{R}.(v_0 + v_u)$, we have:

$$\phi(\{0, u\}) \in \{1, \ldots, u - 1\} \cup \{n + 1, \ldots, n + u\}.$$

Let us assume that $u = 2v$ is even. For $w = 0, 1, \ldots v - 1$, we have $\phi(\{0, 2w + 1\}) = n + 1 + w$. But, two elements of $\mathcal{P}_2^{n+1+w}$ are disjoint. So, $\phi(\{0, u\}) \notin \{n + 1, \ldots, n + v\}$.

In the same way, for $w = 1, \dots, v-1$, we have: $\phi(\{0, 2w\}) = w$. And so, $\phi(\{0, u\}) \notin \{1, \dots, v-1\}$. For $w = 0, 1, \dots v-1$, we have $\phi(\{u, u-2w-1\}) = n+u-w$. Then, $\phi(\{0, u\}) \notin \{n+v+1, \dots, n+u\}$. For $w = 1, \dots, v-1$, we have $\phi(\{u, u-2w\}) = u-w$. Then, $\phi(\{0, u\}) \notin \{v+1, \dots, u-1\}$.

Finally, the only possible value for $\phi(\{0, u\})$ is v.

The proof is analog if $u = 2v+1$ is odd. □

Lemma 3.1 has a very useful consequence:

Lemma 3.2 *We keep notation of Lemma* 3.1. *We also use Convention* 1.
Then, for all $k = 0, \dots, 2n$ *we have*:

$$\det(v_k, v_{k+1}) = \det(v_0, v_1),$$

and

$$\det(v_k, v_{k+n}) = \det(v_0, v_n).$$

Proof: Lemma 3.1 shows that for all integers k we have $\det(v_k, v_{k+1}) = \det(v_{k+1}, v_{k+2})$. The first assertion follows immediately.

For all k, we also have $\det(v_k, v_{k+n}) = \det(v_{k+n}, v_{k+2n})$. Since n is prime with $m = 2n+1$, this implies the second assertion. □

3.2. —

Let $\mathcal{C} = \{v_0, \dots, v_{2m}\}$ be a uniform balanced configuration labelled by increasing arguments. We are going to prove

Claim 1 v_0, v_n and v_{n+1} determine $\mathcal{C}$.

Indeed, we are going to construct successively $v_1, v_{n+2}, v_2, v_{n+3}, v_3, v_{n+4} \dots$. Set $A_1 := \det(v_n, v_{n+1})$ and $A_n := \det(v_0, v_n)$. Assume that we have constructed $v_1, v_{n+2}, \dots, v_{i-1}, v_{n+i}$ (for $1 \le i \le n-1$). By Lemma 3.2, we have:

$$\det(v_{i-1}, v_i) = A_1 \quad \text{and} \quad \det(v_i, v_{n+i}) = A_n. \tag{2}$$

Then,

$$v_i = \frac{A_1}{\det(v_{i-1}, v_{n+i})} v_{n+i} + \frac{A_n}{\det(v_{i-1}, v_{n+i})} v_{i-1}.$$

But, since by Convention 1, $v_{n+i+n} = v_{i-1}$, we have: $\det(v_{i-1}, v_{n+i}) = -A_n$. Finally, we obtain:

$$v_i = \frac{A_1}{A_n} v_{n+i} - v_{i-1}.$$

In the same way, using

$$\det(v_{n+i}, v_{n+i+1}) = A_1 \quad \text{and} \quad \det(v_{n+i+1}, v_i) = A_n; \tag{3}$$

we obtain:

$$v_{n+i+1} = -\frac{A_1}{A_n} v_i - v_{n+i}.$$

Claim 1 follows.

3.3. —

Inspired by the proof of Claim 1, we define two sequences of vectors of $\mathbb{R}^2$ (with a parameter $t \in \mathbb{R}$) as follows.

Start with

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad w_0(t) = \begin{pmatrix} t \\ -1 \end{pmatrix}.$$

Set $A = \det(V, w_0) = -t$ and note that $\det(U, V) = 1$. Then we define $w_i(t)$ and $u_i(t)$ by induction:

$$\begin{cases} w_0(t) & \text{is already defined} \\ u_0(t) = U \\ u_{i+1}(t) = -t w_i(t) - u_i(t) \\ w_{i+1}(t) = t u_i(t) - w_i(t) \end{cases}$$

3.4. —

Let $\mathcal{C} = \{v_0, \ldots, v_{2m}\}$ be a uniform balanced configuration labelled by increasing arguments. Then, there exits a unique $g_{\mathcal{C}} \in \mathrm{GL}_2(\mathbb{R})$ such that $g_{\mathcal{C}}.v_0 = U$ and $g_{\mathcal{C}}.v_n = V$. Since $\det(v_0, v_n) = -\det(v_0, v_{n+1})$ (see Lemma 3.2), there exists a unique $t_{\mathcal{C}} \in \mathbb{R}$ such that $g_{\mathcal{C}}.v_{n+1} = w_0(t_{\mathcal{C}})$. Then, the proof of Claim 1 implies

Lemma 3.3 *With above notation, for all* $i = 0, \ldots, n-1$, *we have*:

$$g_{\mathcal{C}}.v_{n+i+1} = w_i(t_{\mathcal{C}}) \quad \text{and} \quad g_{\mathcal{C}}.v_i = u_i(t_{\mathcal{C}}).$$

Moreover, $w_n(t_{\mathcal{C}}) = U$ *and* $v_n(t_{\mathcal{C}}) = V$.

3.5. —

Now, we are interested in the equation $w_n(t) = U$.

Useful properties of the functions $t \mapsto u_i(t)$ and $t \mapsto w_i(t)$ are stated in

Lemma 3.4 *Denote by* (x^*, y^*) *the coordinate forms of* $\mathbb{R}^2$. *Then, for all* $i \geq 1$, *we have*:

(i) $x^*(u_i(t))$ *is an even polynomial function of degree* $2i$,
(ii) $y^*(u_i(t))$ *is an odd polynomial function of degree* $2i - 1$,
(iii) $x^*(w_i(t))$ *is an odd polynomial function of degree* $2i + 1$, *and*
(iv) $y^*(w_i(t))$ *is an even polynomial function of degree* $2i$.

In particular, the equation $w_n(t) = U$ *has at most n solutions.*

Proof: The proof of the four assumptions is an immediate induction on i.
We can note that $y^*(w_n(0)) \neq 0$. Then, by Assertion (iv), the equation $y^*(w_n(t)) = 0$ has at most $2n$ solutions: $-t_j < \cdots < -t_1 < t_1 < \cdots < t_j$ (with $j \leq n$). Since $x^*(w_n(t))$ is an odd polynomial function, at most one element of a pair $\pm t_k$ is a solution of the equation $x^*(w_n(t)) = 1$. This ends the proof of the lemma. □

3.6. —

Our goal is now to construct geometrically n solutions of the equation $w_n(t) = U$.
Let me recall that we have identified $\mathbb{R}^2$ with $\mathbb{C}$. Consider $\mathbb{U}_m = \{\omega^i : i = 0, \ldots 2n\}$. Let us fix $k \in \{1, \ldots n\}$.
Denote by g_k the element of $\mathrm{GL}_2(\mathbb{R})$ such that $g_k.1 = U$ and $g_k.\omega^k = V$. Let t_k be the unique real number such that $g_k.\omega^{-k} = w_0(t_k)$. Explicitly, $t_k = \frac{1}{\sin(2k\pi/m)}$.
For all $i \in \mathbb{Z}$, we have:

$$\det\left(\omega^{-2k(i-1)}, \omega^{-2ki}\right) = \det(\omega^k, \omega^{-k}) \quad \det\left(\omega^{-2ki}, \omega^{-2k(n+i)}\right) = \det(\omega^0, \omega^k),$$

and

$$\det\left(\omega^{-2k(n+i)}, \omega^{-2k(n+i+1)}\right) = \det(\omega^k, \omega^{-k}) \quad \det\left(\omega^{-2k(n+i+1)}, \omega^{-2ki}\right) = \det(\omega^0, \omega^k).$$

Then, the sequence $(g_k.\omega^{-2ki})_{i\in\mathbb{N}}$ satisfies Relations (2) and (3), with $A_1 = \det(V, w_0(t_k))$ and $A_n = \det(U, V)$. This implies that

$$w_i(t_k) = g_k.\omega^{-k(1+2i)}, \text{ for all } i \geq 0 \qquad (4).$$

In particular, t_k satisfies $w_n(t_k) = U$.
With Lemma 3.4, this implies the

Lemma 3.5 *We have*:

$$\{t \in \mathbb{R} : w_n(t) = U\} = \left\{\frac{1}{\sin(2k\pi/m)} : k = 1, \ldots n\right\}.$$

3.7. —

Proof of Theorem 1: Let $\mathcal{C} = \{v_0, \ldots, v_{2n}\}$ be a uniform balanced configuration labelled by increasing arguments. We define $g_{\mathcal{C}} \in \mathrm{GL}_2(\mathbb{R})$ and $t_{\mathcal{C}} \in \mathbb{R}$ as in Paragraph 3.4. Then, by Lemmas 3.3 and 3.5, there exists a unique $k_{\mathcal{C}} = 1, \ldots n$ such that $t_{\mathcal{C}} = \frac{1}{\sin(2k_{\mathcal{C}}\pi/m)}$. Let $g_{k_{\mathcal{C}}} \in \mathrm{GL}_2(\mathbb{R})$ defined as in Paragraph 3.6.

Then, by Lemma 3.3 and Equalities (4), we have:

$$\begin{aligned}
v_0 &\xmapsto{g_{\mathcal{C}}} U \xmapsto{g_{k_{\mathcal{C}}}^{-1}} 1\\
v_n &\xmapsto{g_{\mathcal{C}}} V \xmapsto{g_{k_{\mathcal{C}}}^{-1}} \omega^{k_{\mathcal{C}}}\\
v_{n+i+1} &\xmapsto{g_{\mathcal{C}}} w_i(t_{\mathcal{C}}) \xmapsto{g_{k_{\mathcal{C}}}^{-1}} \omega^{-k(1+2i)} \quad \text{for all } i = 0, \ldots, n-1\\
v_i &\xmapsto{g_{\mathcal{C}}} v_i(t_{\mathcal{C}}) \xmapsto{g_{k_{\mathcal{C}}}^{-1}} \omega^{-2ki} \quad \text{for all } i = 0, \ldots, n-1
\end{aligned}$$

Theorem 1 follows. □

References

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