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RESEARCH

Gödel's Undecidability Theorems and the Search for a Theory of Everything

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Abstract
I investigate the question whether Gödel's undecidability theorems play a crucial role in the search for a unified theory of physics. I conclude that unless the structure of space-time is fundamentally discrete we can never decide whether a given theory is the final one or not. This is relevant for both canonical quantum gravity and string theory. Slightly elaborated version of a Prize winning essay awarded by the Kurt Gödel Circle of Friends Berlin with the support of the University of Wuppertal, first published in https://kurtgoedel.de/kurt-goedel-award-2023/

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> Was mich ursprünglich interessiert hat, ist die Erklärung der Erscheinungen des Alltagslebens aus höheren Begriffen und allgemeinen Gesetzmäßigkeiten, daher Physik. (Gödel, [13], p. 81).
>
> English translation: I have been originally interested in explaining the phenomena of everyday life in terms of higher concepts and general regularities, hence physics. Gödel [13] p. 347

1 The Search for Unification

In his inaugural lecture for the Lucasian Chair of Mathematics at the University of Cambridge, the eminent theoretical physicist Stephen Hawking expressed the following vision for the future [15]:

> In this lecture I want to discuss the possibility that the goal of theoretical physics might be achieved in the not too distant future, say, by the end of the century. By this

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> I mean that we might have a complete, consistent, and unified theory of the physical interactions which would describe all possible observations.

This was in 1979. In retrospect we can say that such a unified theory of the physical interactions was not available in the year 2000, nor is it available today. The "dreams of a final theory", in the words of Steven Weinberg [33], have not yet been materialized. In more sober words, the reductionist programme of physics has not yet come to an end. Whether it will ever come to an end, is an open issue and is the topic of this essay. As we shall see, Gödel's undecidability theorems will play a crucial role in this investigation.

In view of the history of physics, the reductionist programme seems natural and straightforward. What have earlier been separate theories (and models) were later recognized as special cases of a common theory: electricity, magnetism, and geometric optics, for example, were recovered as particular limits of the theory of electrodynamics, developed in the 19th century by James Clerk Maxwell. All known effects in these areas could be deduced from one fundamental set of partial differential equations – Maxwell's equations. Another example is the partial unification of the electromagnetic and weak interactions to the electroweak interaction that together with the strong interaction forms what is today called the *Standard Model* of particle physics.

The Standard Model was constructed at the end of the 1960s and the beginning of the 1970s by Weinberg and others. It is in this context that Hawking's speech must be seen. As part of the unification attempts in the 1970s, models of *supergravity* were constructed, which aim at a unification of gravity – so far successfully described by Einstein's theory of general relativity (GR) – with the other interactions. The *super* in this word refers to a hypothetical symmetry between fermions (to which electrons and protons belong) and bosons (to which photons and gravitons belong) called supersymmetry. Hawking, in 1979, speculated that the final theory has the form of a supergravity theory. Later, after it had become clear that this hope has remained unfulfilled, he favoured string theory and M-theory (a certain extension of string theory).[1].

String theory became popular in 1984 when indications were found that properties of the Standard Model are present in this theory. So far, however, a recovery of the Standard Model from string theory remains an unfulfilled dream. String (and M-) theory contain supersymmetry as a necessary ingredient to ensure its consistency, but no sign of supersymmetry was found to date in experiments performed at the Large Hadron Collider (LHC) at Cern, Geneva, and elsewhere.

When we speculate about a unified theory, we implicitly assume that we deal with a final theory, that is, we assume that there is no deeper structure of physical theories. In Weinberg's words, a final theory is characterized as follows:

> A final theory will be final in only one sense–that it will bring to an end a certain sort of science, the ancient search for those principles that cannot be explained in terms of deeper principles. Weinberg ([33] p. 13)

One may add here that a truly final theory should also be rigid in the sense that small changes in its parameters do not change its essential structure. But can we really decide whether a given unified theory is final in this sense?

Physical theories are formulated in the language of mathematics, so the question of unification in physics is deeply related with the construction of a 'unified' mathematical language.

[1] In a talk given in 2002, however, he speculated that we shall never find an ultimate theory, and even mentioned Gödel's theorem(s) as a reason for that [16]

At the beginning of the last century, it was the mathematician David Hilbert who attempted to construct a unified mathematical language. He searched for an axiomatic foundation of geometry and finally the whole of mathematics. Hilbert belonged to what is today called the formalistic school. There, axioms are no longer 'obvious' statements in the sense of Euclid but arbitrary formal settings whose justification lies in the success of finding a unified scheme. In view of Ludwig Wittgenstein's much later philosophical investigations, one may call Hilbert's formal system a *Sprachspiel* (language-game). In mathematics, such a language-game is constrained by two important properties that will play an important role below: completeness and consistency. Completeness means that every statement that can be formulated in this formal system can be proved or disproved; consistency means that there are no logical contradictions between different statements in the formal system.

Hilbert's dream of unification was not restricted to mathematics. He intended to generalize this to physics by providing a unified mathematical language for the physical interactions. The only known fundamental interactions at the time were gravity and electrodynamics. The latter was described by Maxwell's equations, but what about the former? Here, the big achievement was Einstein's theory of general relativity (GR), completed in November 1915. Its central equations, Einstein's field equations, so far describe all known gravitational phenomena (or are at least not obviously in conflict with them). Hilbert arrived at those field equations around the same time by postulating a geometric variational principle, starting from what today is known as the Einstein–Hilbert action. But in contrast to Einstein, he envisaged to derive all field equations of physics by such a geometric variational principle. This is why his publication bears, not very modestly, the title "The Foundations of Physics" [17].

The goal of unifying gravity with electrodynamics by a geometric theory in the spirit of GR was not achieved by Hilbert. Nor was it achieved by Einstein, who spent most of his later years with attemps of finding a unified field theory. In retrospect, we can claim mainly two reasons for this failure. First, neither Hilbert nor Einstein took into account the microscopic interactions known as weak and strong interactions, which were studied from the beginning of the 1930s onward. And second, perhaps even more important, quantum theory was not addressed in these attempts, a theory not known at the time of Hilbert's 1915 article, but known and experimentally established at the time of Einstein's later work in the 1940s and 1950s. Connected with this second point is the question of the space-time continuum, used by Einstein in the traditional sense, and its fate in a unified theory encompassing quantum theory. We shall come back to this point below.

Hilbert was very optimistic towards the materializability of his axiomatic programme. In this, he was an antagonist of the physician and physiologist Emil du Bois-Reymond who had formulated his famous *Ignoramus et ignorabimus* (we do not know and will not know) in his 1872 keynote address *Über die Grenzen des Naturerkennens* (On the limits of science). Du Bois-Reymond was convinced that there were fundamental limits to our knowledge of Nature and natural laws. Already in 1900, at a major conference on mathematics in Paris,[2] Hilbert emphasized that in his opinion there is no *ignorabimus* in mathematics. Thirty years later, just one year before the publication of Gödel's undecidability theorems, he emphasized his standpoint again in strong words in a radio address:[3]

> We must not believe those, who today with philosophical bearing and a tone of superiority prophesy the downfall of culture and accept the *ignorabimus*. For us there is no

[2] It was at that conference where Hilbert presented his list of 23 important unsolved problems in mathematics.

[3] See and "hear" www.maa.org/press/periodicals/convergence/david-hilberts-radio-address, where also the German transcription and the English translation can be found.

ignorabimus, and in my opinion even none whatever in natural science. In place of the foolish *ignorabimus* let stand our slogan: We must know, We will know.[4]

So will we one day know the final theory of physics or will there be a final *ignorabimus*? In modern days, especially in particle physics, a final theory is sometimes referred to as a "theory of everything" (TOE). This wording implies that such a theory will not only provide a unified theory of physics, but – at least in principle – a theory for all possible effects in chemistry, biology, and maybe even beyond. The crucial word here is *in principle*. As we know today, even within physics we cannot, in general derive effects at an effective level from a fundamental theory. For example, nuclear physics is thought to arise as a limit from quantum chromodynamics (QCD), our fundamental theory of the strong interactions. But in practice, the formalism is so complicated that this can hardly be done; this is why models such as the nuclear shell model are still essentially used in the everyday work of a nuclear physicist. It is thus evident that this limitation holds even stronger for biology. There, we deal with even more complex systems, and concepts such as *synthetic biology* are more powerful than biological laws arising from fundamental physics [34]. Nevertheless, the important question for us is whether a final theory exists in principle, independent of these practical limitations. To address this question, it is necessary to clarify the relation between mathematics and physics, which is subject of the next section.

2 Mathematics and Physics

In a well-known article, the Nobel Prize winner Eugene Wigner speculates about the, according to his opinion, unreasonable effectiveness of mathematics in the natural sciences, in particular physics [35]. Why is it that physical phenomena can be described by mathematical equations? And why exists, it seems, a small set of equations, such as Einstein's equations, Maxwell's equations, and the Schrödinger equation, that lie at the foundation of this description? Already Galileo, in his *Il Saggiatore*, envisaged the Universe as being written in mathematical language, which for him was the language of triangles, circles, and other geometric figures. Our modern mathematical description of physics dates back to Isaac Newton, the second Lucasian Professor of Mathematics at the University of Cambridge. Wigner, in his Nobel Prize speech of 1963, emphasized the surprising discovery of Newton's age that the laws of Nature can be clearly separated in dynamical laws and initial conditions. The dynamical laws are given by differential equations up to second order in space and time. They thus leave room for initial (or, more generally, boundary) conditions, which are not fixed by the laws and which thus represent contingent features of our Universe.

But why is mathematics so effective? A full answer is elusive, but a partial answer may lie in the role symmetries play at a fundamental level. The structure of the Standard Model is given and, in fact, strongly constrained by gauge invariance. This is an internal symmetry that acts on all the quantum fields representing particles, thus connecting them in a non-trivial manner. Gravity is not contained in the Standard Model. It is described by Einstein's field equations which displays another type of symmetry (or, more properly, invariance) – the diffeomorphism invariance of space-time. Hereby is meant the mathematical exact expression

[4] German original: "Wir dürfen nicht denen glauben, die heute mit philosophischer Miene und überlegenem Tone den Kulturuntergang prophezeien und sich in dem Ignorabimus gefallen. Für uns gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. Statt des törichten Ignorabimus heiße im Gegenteil unsere Losung: Wir müssen wissen, Wir werden wissen." This slogan is engraved on his tombstone in Göttingen.

for coordinate invariance: space-time points have no meaning independent of the dynamic degrees of freedom representating geometry and matter fields.

There are quite a few theoretical physicists who regard the fundamental mathematical equations as 'beautiful'. This sense of beauty is connected with the symmetries or invariances that these equations exhibit. It only applies to the equations themselves; their solutions as well as approximations following from them may be lengthy, complicated, and ugly. In the words of ([33] p. 131): "It is when we study truly fundamental problems that we expect to find beautiful solutions." Paul Dirac, another holder of the Lucasian chair of Mathematics, who invented the equation named after him, went even further in claiming that beauty in one's equations is more important than compatibility with experiment. Most physicists would not support such a strong view because it brings the danger of formulating equations devoid of empirical content. In 1931, Dirac himself had presented an even more symmetrical version of Maxwell's equations, which in addition to the usual electric charges contains magnetic monopoles. But such monopoles have never been seen and they may, in fact, not exist at all. Still, the question remains what is the structure of the fundamental equations of a unified and final theory. String or M-theory, in its present state, does not exhibit 'beautiful' equations nor is it based on an aesthetically appealing fundamental principle. But can we say something about the possible mathematical structure of a final theory? The mathematician Kurt Gödel, too, expressed this idea of beauty:

> The beauty in the presentation of a subject lies in first giving general (abstract) concepts and possibly a theory and <u>then</u> the application to the empirical …For the same reason, the beauty in physics is the explanation of everyday phenomena. Hence also the name "knowledge". Gödel ([14] p. 229) [5]

It is most likely that also Hilbert in his search for unification in both mathematics and physics was driven by some concept of beauty. But Hilbert's programme received a severe blow when Gödel presented his undecidability theorems [10] [6]. This blow applied to mathematics; whether it also applies to physics and to the dreams of a final theory is the subject of this essay. In the words of Douglas Hofstadter, Gödel's first undecidability theorem can be paraphrased as follows:

> All consistent axiomatic formulations of number theory include undecidable propositions. Hofstadter ([18] p. 17)

Hofstadter compares this theorem with a pearl that is buried in an oyster, the oyster standing for the mathematical proof of this theorem, which makes essential use of self-referring statements. The theorem has the far-reaching consequence that in any sufficiently complex axiomatic system (complex enough to contain the arithmetics of natural numbers) there are statements that can neither be proved nor disproved. So, as Hofstadter continues to write, "provability is a *weaker* notion than truth, no matter what axiomatic system is involved." Traditionally, the assumption entertained by mathematicians always was that a certain statement within a formal system can either be proved or its negation can be proved; now there is a third option called *undecidable*.

[5] German original: Das Schöne an der Darstellung einer Sache ist, zunächst allgemeine Begriffe (abstrakte) und eventuell ihre Theorie zu geben und <u>dann</u> die Anwendung auf die *Empirie*. …Dasselbe ist der Grund dafür, dass das Schöne an der Physik die Erklärung der alltäglichen Erscheinungen ist. Daher auch der Name ≫Erkennen≪. Godel ([14] p. 70)

[6] For an English translation, see ([11] pp. 145 ff.)

At the end of his article, Gödel announces what today is called his Second Incompleteness Theorem, which can be paraphrased as "if a sufficiently complex axiomatic system containing the arithmetics of natural numbers is consistent (free of contradictions), it is impossible to prove this consistency within the system itself". Thus, Hilbert's ambitious programme of developing a complete and consistent formal scheme for the whole of mathematics cannot be accomplished.

An important application is the halting problem, which is the problem of whether there exists a general algorithm with which one can decide whether an arbitrary programme with arbitrary input will finish after a finite number of steps. This was shown by Alan Turing in 1936 to be undecidable. Apart from the halting problem, the perhaps most important example for an undecidable problem is the continuum hypothesis (CH). It goes back to the mathematician Georg Cantor and can be phrased as stating that there is no cardinal number between the set of natural numbers $\mathbb{N}$ and the set of real numbers $\mathbb{R}$. Assuming the validity of the axiom of choice, it can also be stated in the form that the cardinality of the power set $2^{\aleph_0}$ is equal to $\aleph_1$, where $\aleph_0$ is the cardinality of the natural numbers and $\aleph_1$ the cardinality of the real numbers. The continuum hypothesis is actually the first entry in Hilbert's famous list of 23 problems presented in 1900. It is a statement about numbers; the term continuum comes from the idea, mostly taken as granted, to identifying $\mathbb{R}$ with the points on a line. That such an identification can be questioned is a later insight and will be discussed below in connection with its relevance for unified theories.

The continuum hypothesis was shown to be undecidable by Paul Cohen in 1963. He could prove that it cannot be proved or disproved from the standard axioms of set theory (the Zermelo–Fraenkel axioms). Gödel, who subscribed philosophically to what one may call platonic realism, believed that the CH is true or false, even if no proof or disproof can be given within standard set theory.[7] He envisaged that there will be a future powerful axiomatic system within which a decision can be made.

The story of Gödel's theorems and their consequences for the history of mathematics has been told many times, and their is no point to repeating it here.[8] We are interested, instead, in discussing the relevance of Gödel's results for the construction of a unified physical theory (e.g. string theory), a story that so far is unduely neglected.[9]

In the last section below, we will investigate the relevance of the continuum hypothesis for mathematical models of space-time. Before doing so, we have to understand the role of quantum theory in the search for a unified theory, something not yet attempted by Hilbert and Einstein in their searches for a unified theory of physics.

3 The Role of Quantum Theory

The non-relativistic version of quantum theory was constructed ten years after Einstein's (and Hilbert's) work on general relativity, in the years 1925–27. Generalizations to field theory were started later and are not yet fully completed. The reason for this unfinished state of affairs lies in the infinitely many degrees of freedom of quantum field theory: sophisticated

7 See his article "What is Cantor's continuum problem?", first published in 1947, see [12] pp. 176–187, and later published in a revised version in 1964, see ([12] pp. 254–270).

8 See, for example, [18] or the detailed editorial notes in [11] and [12].

9 As Roger Penrose has remarked: "It is my own personal opinion that we shall find that computability issues will eventually be found to have a deep relevance to future physical theory, but only very little use of these ideas has so far been made in mathematical physics." Penrose ([29] p. 378). This essay is an attempt to fill this gap.

mathematical schemes of regularization and renormalization were invented to deal successfully with formally infinite expressions, but the problem of the infinitely small (and, thus, the continuum) remain unsolved. The Standard Model of particle physics mentioned above is such a quantum field theory.

Mathematically, the Standard Model is a gauge theory (Yang–Mills theory) and contains in particular the theory of strong interactions (QCD). An important issue is whether one can prove within this theory the observed confinement of quarks. Interestingly, this problem seems to be undecidable [4]. In fact, this problem belongs to the general class of spectral-gap problems, under which one understands the question whether there is a gap between the ground state energy of a given system and its first excited state or not. Cubitt et al. [4] were able to relate this problem to Turing's halting problem, from which the undecidability of the spectral-gap problem follows.[10] In this way, Gödel's theorems enter the Standard Model of particle physics.

In their proof, [4] make essential use of the thermodynamic limit, that is, the limit where the number of degrees of freedom tends to infinity. This limit was taken because of its relevance for quantum phase transitions: the transition from a gapless to a gapped situation (or *vice versa*) can occur at arbitrary large (and uncomputable) values for the parameter describing the thermodynamic limit.

An application to field theory was recently presented by [31] in the context of supersymmetry. His line of arguments is interesting. He poses the question whether one can prove that a certain supersymmetric model (Wess–Zumino model) can entail supersymmetry breaking. He related this problem to Hilbert's tenth problem about Diophantine equations, which is known to be undecidable, and concluded that his question about supersymmetry breaking is undecidable, too.

The notion of infinity thus plays an essential role in all these considerations. It seems that so far in physics only the cardinality $\aleph_1$ of the real numbers play a role,[11] but this is already sufficient to give rise to the problems discussed here. Early on, [25] remarked, on the basis of Gödel's theorem but without going into technical details, that the question of whether two states in quantum field theory are macroscopically distinguishable or not, is undecidable. His arguments only work for systems with infinitely many degrees of freedom, thus assuming a space-time continuum.

These issues are connected with another most important problem in quantum theory: the problem of the classical limit. A central (and perhaps its most characteristic) feature of this theory is the superposition principle – the sum of two physically allowed quantum states is again an allowed state. This immediately leads to the occurrence of weird macroscopic states such as Schrödinger's cat. The fact that such states are not observed was a perennial puzzle of the theory. One way towards its solution is the assumption of a major modification of quantum theory in the form of a wave-function collapse. Collapse models are studied in detail [1], but so far none has been experimentally established. Another way is the realistic modelling of the system's environment which can lead to the formation of correlations rendering the superposition unobservable by transfering the information about it in entanglement between system and environment. This process is called decoherence and is experimentally well established [20].

[10] A concrete construction of a Hamiltonian operator whose spectral gap is undecidable is given in [3].

[11] The cardinality of the complex numbers $\mathbb{C}$, which play a major role in quantum theory, is the same as the cardinality of $\mathbb{R}$.

Decoherence is also of relevance for an interesting discussion about the origin of consciousness. Roger Penrose, in collaboration with the anesthesiologist Stuart Hameroff, developed the idea that quantum superpositions in neural microtubules in the brain are responsible for the emergence of consciousness; this was again concluded by making a connection with the undecidability of the halting problem [28]. The brain, according to Penrose and Hameroff, works in a non-algorithmic way and can thus give rise to free will.[12]

It is of some interest to note that other scientists had speculated earlier about a connection between consciousness and quantum theory. The mathematician John von Neumann as well as the above mentioned Eugene Wigner entertained the idea that consciouness is, in fact, responsible for the occurrence of a wave-function collapse, avoiding in this way paradoxical states such as Schrödinger's cat. Wigner gave up this idea in the 1970s after the process of decoherence was discovered by the physicist Dieter Zeh.[13]

Decoherence did not only lead Wigner to change his mind, but also to render the Penrose–Hameroff scenario unlikely. As Max Tegmark has shown, the decoherence times for possible quantum superpositions in the brain are much shorter than standard time-scales used for conscious processes, thus leading to their irrelevance [32].

The question of the quantum-to-classical transition is also related to the question of where the "Heisenberg cut" can be applied. This notion goes back to discussions between Werner Heisenberg and Wolfgang Pauli in 1935 and refers to the scale of a problem in quantum theory at which a classical description can be used without invoking a conflict with experiment. In the above example of the brain, the decoherence timescale gives a lower bound for the cut, guaranteeing the validity of an effective classical description of neural processes. There may, however, be other situations in which the Heisenberg cut lies outside the range of observational scales and where thus quantum effects *can* play a role even in macroscopic situations.

Such situations may occur when the gravitational interaction becomes relevant. A theory of quantum gravity is not yet available in complete form, but it seems that such a theory is needed as a major part of a unified final theory. One reason for this belief is the incompleteness of general relativity as expressed in the singularity theorems. Geroch and Hartle [9] have argued that such a theory contains undecidable statements, at least in present formulations of the theory that make use of path integrals. In this formulation, Geroch and Hartle argue, no computer can carry out a computation of expectation values, basically because the question of whether two four-dimensional manifolds have the same topology is undecidable; in the path integral, all possible topologies are superposed, so no calculation can be performed.

Another important application of Gödel's theorems to quantum gravity is in the canonical formulation of the theory [19, 23]. The quantum-gravitational wave function Ψ is there to be determined as a solution to the Wheeler–DeWitt equation, which is of the form $H\Psi = 0$, with H being the Hamilton (energy) operator of all degrees of freedom. In the usual formalism of quantum theory, this equation only makes sense if the value 0 is contained in the discrete spectrum of H. But a decision about this question runs into the spectral-gap problem discussed above: it is undecidable whether there is indeed a gap between zero and other eigenvalues (as desired) or not. To our knowledge, this important point has not yet been addressed in the quantum gravity literature.

12 Incidentally, the question of free will is also included in the seven "world riddles" formulated by du Bois-Reymond.

13 See, for example, [24] for a discussion.

4 Can we Decide Whether a Unified Physical Theory is the Final One?

Kurt Gödel, although being mathematician, had interests ranging far beyond mathematics. In his own words:

> I am apparently neither talented nor interested in combinatorial thinking (card games and chess, and poor memory). I am apparently talented and interested in conceptual thinking. I am always interested only in how it works …(and not in the actual execution). Therefore, I should dedicate myself to the foundations of the sciences (and philosophy). This means: Not only the foundations of physics, biology and mathematics, but also sociology, psychology, history …. This means an overview of all sciences and then foundations (which is also what I am primarily interested in). Gödel ([13] p. 346) [14]

Among his main other interests was physics, for which he envisaged an underlying reality in the same sense as in mathematics. He believed "that a question not decidable now has meaning and may be decided in the future." Gödel ([12] p. 170). Can we answer the question posed in the title of this section, whether a unified theory is the final one or not?

It is certainly not an easy task to construct a candidate theory in the first place. For Gödel, the role of intuition in research was of great importance. This he had in common with Einstein, who emphasized the importance of intuition in his work at various places, for example in [7]. Whether intuition works, is of course not clear. Einstein was successful in constructing general relativity, but he failed in constructing a unified field theory. As Friedrich Dürrenmatt remarks: "While he arrived from the empirical by intuition to the a priori picture, he now tried [in his attempts for a unified field theory] to arrive by intuition from the a priori [i.e. mathematical] description to the empirical" ([6] p. 167).[15] But without connecting the mathematical formalism to experiments or observations, all efforts may be in vain.

In most attempts to construct a final theory, the underlying concept of space (or space-time) is that of a continuum. In general relativity, space-time is modelled as a (pseudo-)Riemannian manifold, which locally looks like $\mathbb{R}^4$ and thus possesses the same cardinality as $\mathbb{R}$, namely $\aleph_1$. Similar features apply to the spaces employed in other approaches, such as canonical quantum gravity or string theory. Take the latter as an example. The theory is defined on a ten- or eleven-dimensional manifold. To understand why we observe only four dimensions, one must invoke a mechanism to render the remaining dimensions unobservable. This can be achieved, for example, by compactifying them in the form of Calabi–Yau spaces.[16] But even these spaces are manifolds and thus possess an uncountable number of degrees of freedom. One thus has to face the continuum hypothesis, which we know is undecidable.

The continuum was imagined by many mathematicians to represent the real numbers in the sense of a point set. This picture turned out to be a very powerful one for the development of mathematics, but it is not the only one. In the field of non-standard analysis [30] developed

[14] German original: Kombinatorisch scheine ich weder begabt noch interessiert zu sein (Karten- und Schachspiel, und schlechtes Gedächtnis). Begrifflich scheine ich begabt und interessiert zu sein. Es interessiert mich bei allem nur, wie es …geht (nicht die tatsächliche Ausführung). Also soll ich mich den Grundlagen der Wissenschaften (und der Philosophie) widmen. Das bedeutet: Nicht nur Grundlagen der Physik, Biologie und Mathematik, sondern auch der Soziologie, Psychologie, Geschichte …. Das heißt Überblick über sämtliche Wissenschaften und dann Grundlagen (das ist auch, worauf ich mich eigentlich interessiere). Godel ([13] p. 81)

[15] The German original reads: "Gelangte er vom Empirischen durch Intuition zum Apriorischen, versuchte er nun, durch Intuition vom Apriorischen zum Empirischen zu gelangen."

[16] The number of possible compactifications was estimated to be at least of the order $10^{272,000}$ [5], which means that it is not possible to derive the Standard Model from string theory in any reasonable way.

over the last decades, consistent models of the continuum were constructed where many more numbers of different cardinality find place on the continuum. Such non-standard models exist with arbitrary high cardinality, much beyond the cardinality of the real numbers; see, for example, [27]. These numbers are also called hyperreal numbers; in such models, a number such as $0.999\ldots$ is not equal to 1 (as we learn in school), but is strictly smaller than it. These non-standard models put the notion of infinitesimals on a sound footing. One may call the continuum as being *inexhaustible*, uncomparable to the picture of a point set with the points representing real numbers. Therefore, as long as we use the notion of a continuum at a fundamental level, Gödel's undecidability theorems apply and we will never know the microstructure of space-time. The final answer about continuum or discrete space can eventually only come from the empirical.

The idea of mathematical realism is put to an extreme by Max Tegmark, who entertains the idea that our Universe, in fact, consists of mathematical structures in a realistic sense [32]. In order to avoid problems with Gödel's theorems, he makes the assumption that only computable numbers are realized in Nature. But this assumption is already in contradiction with the undecidability of the spectral-gap problem discussed above, a problem that occurs in standard quantum theory. Tegmark's world would be plagued with undecidability problems.

That there should be no infinities in our actual world was already emphasized by Hilbert. Ellis et al., too, adopt this point of view and argue that "infinity" in physics always means potential infinity in the sense of very large numbers and that actual infinity (which they call essential infinity) does not occur. In this case, all antinomies and paradoxes connected with infinities vanish. Mathematical procedures such as regularization and renormalization in quantum field are then only of preliminary nature and would become obsolete in a final theory.

A finite world would also lead to a finite number of superposed quantum states in the situations of entanglement discussed above. For a finite number of quantum degrees of freedom, it seems that the probability interpretation of quantum theory can be derived without invoking a wave-function collapse and an *ad hoc*-rule.[17] No problems connected with Heisenberg's cut would remain.

Candidates for a unified final theory usually employ a continuous picture of space-time. Some notable exceptions include Carl Friedrich von Weizsäcker's *Urtheorie* and John Wheeler's models of *It from Bit*. So far, these ideas have not led to a final theory that is both complete and empirically testable, but it is imaginable that a final theory will make use of such structures.

Even for a finite world, the number of degrees of freedom may be very large. Seth Lloyd has estimated the amount of information that the observable part of the Universe can register and arrived at the number of 10^{120} bits [26]. This gives an upper bound to the amount of possible computation, from which we shall of course stay far away in any practical application. Numbers with this order of magnitude are prevalent in cosmology and originate from the assumption of a smallest spatial scale of the order of the Planck length.[18] It is the large size of this number and the corresponding smallness of the Planck scale that allow the consistent formulation of physical theories with an underlying space-time continuum, even if our "actual" space-time is of discrete nature.

[17] See, for example, the discussion in ([24] p. 94).

[18] The Planck scale follows from combining Planck's constant, the speed of light, and the gravitational constant into a quantity with unit of length; it is of the order of 10^{-35} metres.

It thus seems that we could decide, at least in principle, whether a given theory is final or not only if the world were finite at small and large scales.[19] Long ago, Bernhard Riemann already made speculations in this direction, although he was completely unaware of later developments in physics and mathematics. In his famous *habilitation thesis*, he writes:

> The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space.…Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it. Jost ([22] p. 40) [20]

So we conclude that, unless the space-time structure is fundamentally discrete and the total number of degrees of freedom in the world is finite, the question whether a given theory is the final one or not will remain undecidable and so there will forever remain an *ignorabimus*.

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[19] Our line of arguments is different from the one in [2], where observer participation in the Universe plays an essential role.

[20] The German original reads ([21] p. 43): "Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Massverhältnisse des Raumes. …Es muss also entweder das dem Raume zu Grunde liegende Wirkliche eine discrete Mannigfaltigkeit bilden, oder der Grund der Massverhältnisse ausserhalb, in darauf wirkenden bindenden Kräften, gesucht werden." The English translation is by William Clifford. The "binding forces which act upon it" are in our picture affected by the undecidability theorems.

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