International Journal of Theoretical Physics (2024) 63:5
https://doi.org/10.1007/s10773-023-05529-z

RESEARCH

Heisenberg versus the Covariant String

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Received: 18 September 2023 / Accepted: 13 December 2023 / Published online: 4 January 2024

Abstract
A Poincaré multiplet of mass eigenstates $\left(P^2 - m^2\right)\Psi = 0$ cannot be a subspace of a space with a D-vector position operator $X = (X_0, \dots X_{D-1})$: the Heisenberg algebra $[P^m, X_n] = \mathrm{i}\delta^m{}_n$ implies by a simple argument that each Poincaré multiplet of definite mass vanishes. The same conclusion follows from the Stone-von Neumann theorem. In a quantum theory the constraint of an absolutely continuous spectrum to a lower dimensional submanifold yields zero even if Dirac's treatment of the corresponding classical constraint defines a symplectic submanifold with a consistent corresponding quantum model. Its Hilbert space is not a subspace of the unconstrained theory. Hence the operator relations of the unconstrained model need not carry over to the constrained model. Our argument excludes quantized worldline models of relativistic particles and the physical states of the covariant quantum string. We correct misconceptions about the generators of Lorentz transformations acting on particles.

Keywords Heisenberg algebra · Stone-von Neumann theorem · Covariant string · Mass shell condition · Constrained system · Continuous spectrum

1 Introduction

The momentum $P = (P^0, \dots P^{D-1})$ generates the unitary representation $U_a = \mathrm{e}^{\mathrm{i}\,P\,a}$ of translations in spacetime. This makes models tempting which contain in addition a spacetime position operator $X = (X_0, \dots X_{D-1})$, which Lorentz transforms as a D-vector and which is translated,

$$\mathrm{e}^{\mathrm{i}\,a\,P}\, X_n\, \mathrm{e}^{-\mathrm{i}\,a\,P} = X_n - a_n\ ,\ a \in \mathbb{R}^D\ . \tag{1}$$

Functions of X such as $V_b(X) = \mathrm{e}^{\mathrm{i}\,b\,X} =: V_b$ are shifted, $U_a \mathrm{e}^{\mathrm{i}\,b\,X} U_a{}^{-1} = \mathrm{e}^{\mathrm{i}\,b\,(X-a)}$. These are the Weyl relations

$$U_a V_b = V_b U_a\, \mathrm{e}^{-\mathrm{i}\,a\,b}\ ,\ U_a U_b = U_{a+b}\ ,\ V_a V_b = V_{a+b}\ . \tag{2}$$

On the dense subspace of smooth, rapidly decreasing wave functions, the Schwartz space $\mathcal{S}(\mathbb{R}^D) \subset L^2(\mathbb{R}^D)$, their generators constitute an algebra and represent the Heisenberg Lie

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algebra

$$[P^m, P^n] = 0 = [X_m, X_n]\ ,\ [P^n, X_m] = \mathrm{i}\,\delta^n{}_m\ ,\ m, n \in \{0, 1, \ldots D-1\}\ . \tag{3}$$

Differentiating $V_b U_a V_b{}^{-1} = U_a\, \mathrm{e}^{\mathrm{i}ab}$ with respect to a_m at $a = 0$ shows

$$\mathrm{e}^{\mathrm{i}\,b\,X}\, P^m \mathrm{e}^{-\mathrm{i}\,b\,X} = P^m + b^m\ ,\ b \in \mathbb{R}^D\ . \tag{4}$$

Thus by (1) and (4) the operators X_n and P^m are unitarily equivalent to the shifted operators. So their spectra are invariant under shifts and consist of the continuum $\mathbb{R}^D$.

However, the possible momenta of relativistic particles do not fill a D-dimensional continuum but are restricted to mass shells

$$\mathcal{M}_m = \left\{ p \in \mathbb{R}^D : p^0 = \sqrt{m^2 + \mathbf{p}^2} \right\}\ . \tag{5}$$

The particle states constitute Poincaré multiplets with discrete masses.[1] Though this discrepancy in covariant quantum string theory, which contains (3) was observed [2, 3, 8] it was not considered a severe fault. Lecture notes as e.g. [1, 10] and books claimed “The physical space of allowed string states is a subspace of the complete Fock space” [7, p.76] with a unitary representation of the Weyl relations.

We show: This is untenable. The spacetime Heisenberg Lie algebra (3) excludes any subspace with a definite mass m whether m vanishes or not.

To constrain in canonically quantized world line models the continuous momentum spectrum to mass shells yields zero, because the volume measure of a lower dimensional submanifold vanishes. This is very much different from constraints which select from discrete possibilities.

In particular by this reason of vanishing measure the Stone-von Neumann theorem excludes mass shells in the continuous momentum spectrum.

If theories, different from the covariantly quantized worldline models or the covariant quantum string, contain only the *spatial* part of the Heisenberg Lie algebra then this is consistent with massive particles, $m > 0$. This is compatible with Lorentz covariance, even though of an unusual kind. Covariance does not require $\mathbf{X}$ be the spatial components of a D-vector.

Because Lorentz generators, which are constructed using (3), do *not* act on particles with a definite mass, we specify the generators which do.

[1] A Poincaré multiplet $\mathcal{V}$ is a space with a definite scalar product and an irreducible unitary representation of the Poincaré group. By a unitary change of the basis all its states Φ and Ψ can be chosen to be square integrable functions $\Phi, \Psi : \mathcal{M}_m \to \mathbb{C}^d$ of the mass shell with scalar product (up to its sign)

$$\langle \Phi | \Psi \rangle = \int_{\mathcal{M}_m} \frac{\mathrm{d}^{D-1}}{p^0}\, \Phi^{*T}(p) \Psi(p)\ .$$

We do not need these details but use only that the momenta P^m are self-adjoint operators which map their domain in $\mathcal{V}$ to itself, that $\mathcal{V}$ and its orthogonal complement span the complete space and that all states in $\mathcal{V}$ are eigenstates of P^2 with the common eigenvalue m^2.

2 Absence of Mass Multiplets

Lemma *A Poincaré multiplet $\mathcal{V}$ of states Ψ of a definite mass m*

$$(P^2 - m^2)\Psi = 0\ ,\ \ P^2 = (P^0)^2 - \sum_{i=1}^{D-1} (P^i)^2\ , \tag{6}$$

cannot be a subspace of a space with a nondegenerate scalar product in which Heisenberg pairs $(i, j \in \{1, \dots D-1\})$ satisfy

$$[P^i, P^j] = 0 = [X^i, X^j]\ ,\ \ [P^i, X^j] = -\mathrm{i}\,\delta^{ij}\ , \tag{7}$$

and commute with P^0

$$[P^0, X^i] \stackrel{?}{=} 0. \tag{8}$$

Proof The space is the orthogonal sum of the mass multiplet and its complement. All arbitrarily chosen states Ψ and Φ of the multiplet are orthogonal to the complement and have a vanishing matrix element of the commutator $[(P^2 - m^2), X^1] = 2\mathrm{i}P^1$

$$\begin{aligned}\langle \Phi \,|\, [(P^2 - m^2), X^1]\, \Psi\rangle &= \langle\, (P^2 - m^2)\Phi \,|\, X^1 \Psi\rangle - \langle \Phi \,|\, X^1\, (P^2 - m^2)\Psi\rangle \\ &= 0 - 0 = 2\mathrm{i}\langle \Phi \,|\, P^1 \Psi\rangle\end{aligned} \tag{9}$$

$P^1\Psi$ is in the Poincaré multiplet $\mathcal{V}$ and therefore orthogonal to the complement of the multiplet. By (9) it is also orthogonal to the multiplet. So all scalar products of $P^1\Psi$ vanish. But the scalar product is nondegenerate, hence

$$P^1\Psi = 0\ . \tag{10}$$

Exchanging Φ and Ψ in the argument, one also has $P^1\Phi = 0$. As P^1 is hermitian their matrix element of the commutator $[P^1, X^1] = -\mathrm{i}$ vanishes

$$\langle \Phi \,|\, [P^1, X^1]\, \Psi\rangle = \langle P^1\Phi \,|\, X^1\Psi\rangle - \langle \Phi \,|\, X^1 P^1 \Psi\rangle = 0 - 0 = -\mathrm{i}\langle \Phi | \Psi\rangle\ . \tag{11}$$

All scalar products of Ψ vanish, thus

$$\Psi = 0\ . \tag{12}$$

The state Ψ was arbitrarily chosen from the mass multiplet, so there is none.

The argument needs only the nondegeneracy of the scalar product, not its positivity, and does not even require X^1 be hermitian. It needs no assumptions about wavefunctions, which represent a state, nor their explicit scalar product. For our conclusion it is sufficient that they exist.

Our lemma excludes quantized models [1, 9, 13] of free relativistic particles which classically traverse worldlines $t \mapsto x(t)$ with action given by their length. Canonical quantization yields the Heisenberg algebra. The mass shell condition $(P^2 - m^2)\Psi = 0$ arises as constraint because of the reparametrization invariance $t \mapsto t'(t)$. But, no matter how suggestive, aesthetical and geometrical a classical system may be, this does not guarantee that canonical quantization yields a quantum model in which the constraint has a nonvanishing solution.

These models contain the algebra (3) and declare to contain a multiplet of definite mass. They lay claim to the name 'relativistic particle' but contain none. This justifies to rename them 'worldline models' to avoid the confusing statement that relativistic particles do not exist. Experimentally, relativistic particles are verified beyond doubt, but the covariantly quantized worldline models fail to describe them whatever their denomination pretends. Our lemma applies also to the covariant quantum string which postulates (3).[2] Its Hilbert space is claimed to be an orthogonal sum of mass multiplets – the physical states – and a complement. Our lemma excludes any multiplet of definite mass. The covariant quantum string has no physical states.[3]

In particular, one cannot constrain the absolutely continuous spectrum of a quantum model to a lower dimensional submanifold: each integral of a projection valued measure on a set of vanishing measure yields zero. This is what vanishing projection measure means.

In a quantum model there is no solutions to a constraint, which restricts a continuous spectrum to a submannifold, even if the solutions of the corresponding classical constraint define a symplectic submanifold with Dirac brackets [4] and with a consistent corresponding quantum model. Its Hilbert space is not a subspace of the unconstrained model. In the constrained model the operator relations of the unconstrained model need not hold.

If one does not want to give up reason altogether then we see only two options to evade our lemma as it excludes also the idea that some of the operators X^n are not self-adjoint. In this case the Stone-von Neumann theorem would not apply. But our lemma holds also if some X^n are not self-adjoint.

As first option one can give up the interpretation that X^n and P^m are operators in a space with a nondegenerate scalar product. Thereby one gives up the interpretation that they act on quantum particles. For example, they could act on off-shell Feynman graphs. These are not elements of a space of quantum states though one speaks about them in terms of virtual particles. In diagrams with loops virtual particles turn out to have the momentum spectrum $\mathbb{R}^D$.

The second option gives up the algebra (3). This is the only option in theories which allow for multiplets of relativistic particles.

3 The Stone-von Neumann Theorem

By the Stone-von Neumann theorem [12, Theorem XI.84][4] each unitary representation of the Weyl relations (2) is unitarily equivalent to the one in a Hilbert space $L^2(\mathbb{R}^D) \times \mathcal{N}$ of states $\Psi : p \mapsto \Psi(p)$ which map $p \in \mathbb{R}^D$ almost everywhere to $\Psi(p)$ in some Hilbert space $\mathcal{N}$. The unitary representation acts multiplicatively and by translation

$$
\begin{gathered}
(U_a \Psi)(p) = \mathrm{e}^{\mathrm{i}\, a\, p}\, \Psi(p)\,,\ (V_b \Psi)(p) = \Psi(p+b)\,, \\
\langle \Phi | \Psi \rangle = \int \mathrm{d}^D p\, \langle \Phi(p) | \Psi(p) \rangle_{\mathcal{N}}\,.
\end{gathered}
\tag{13}
$$

By the theorem one is not free to choose a different scalar product which integrates not over $\mathbb{R}^D$ but only over a mass shell with measure $\mathrm{d}^{D-1}\, p / \sqrt{m^2 + \mathbf{p}^2}$.

The scalar product of $L^2(\mathbb{R}^D)$ implies that each multiplet of definite mass vanishes: $L^2(\mathbb{R}^D)$ is the space of equivalence classes of square integrable wave functions, which are

[2] We reserve the name 'covariant string' to string models with the algebra (3). This article does not deal with the light cone string, which employs only a subalgebra, nor with the path integral quantization of worldlines [15].

[3] The result is unchanged by identifying states which differ by spurious states.

[4] There for $D = 1$. The result carries over to finite D [14, Notes 8.10].

equivalent if the D-dimensional measure of the support of their difference vanishes,

$$\Psi = 0 \Leftrightarrow \forall \Phi : \int \mathrm{d}^D p \, \langle \Phi(p) | \Psi(p) \rangle_{\mathcal{N}} = 0 \, . \tag{14}$$

All wave functions $\Psi_{\text{phys}} \in L^2(\mathbb{R}^D)$ with definite mass m

$$\Psi_{\text{phys}}(p) = 0 \;\text{ if }\; p^0 \neq \sqrt{m^2 + \mathbf{p}^2} \tag{15}$$

only have a $(D-1)$-dimensional support of vanishing D-dimensional measure. They are equivalent to 0 and vanish.

Physical states are not elements of $L^2(\mathbb{R}^{D-1})$ obtained from $L^2(\mathbb{R}^D)$ by restriction to the mass shell: restriction of equivalence classes is a linear map, it vanishes if applied to 0. On $L^2(\mathbb{R}^D)$ restriction to a mass shell vanishes altogether. To realize the algebra in a space with a different measure is excluded by the Stone-von Neumann theorem.

If it needed another argument: the Heisenberg algebra (3) is represented by the hermitian operators

$$P^m \Psi(p) = p^m \Psi(p) \, , \; X_n \Psi(p) = -\mathrm{i} \, \partial_{p^n} \Psi(p) \, . \tag{16}$$

They generate an algebra which is defined on and maps to itself the Schwartz space $\mathcal{S}(\mathbb{R}^D, \mathcal{N})$ of smooth functions $\Psi : \mathbb{R}^D \to \mathcal{N}$ which together with each of their derivatives decrease rapidly [14]. The only smooth function of $\mathbb{R}^D$ which vanishes outside mass shells is $\Psi = 0$.

4 Consistent Spatial Position Operator

The disastrous, innocent looking relation $[X^i, P^0] \stackrel{?}{=} 0$ (8) in covariantly quantized worldline models is incompatible with the Schrödinger equation $\mathrm{i} \, \partial_t \Psi(t) = P^0 \Psi(t)$ for the motion of a massive ($m > 0$) relativistic particle.

For its expected position $x^i(t) = \langle \Psi(t) | X^i \Psi(t) \rangle$ to change in the course of the time t by the expected velocity $\partial_t x^i = v^i = \langle \Psi | \big(P^i / P^0 \big) \Psi \rangle$ one has to have not (8) but different operators acting on $L^2(\mathcal{M}_m)$, $m > 0$, which satisfy (in slight notational abuse we denote these different operators in this section again by their conventional names)

$$[X^i, P^0] = \mathrm{i} \frac{P^i}{P^0} \Leftrightarrow [X^i, P^2 - m^2] = 0 \, . \tag{17}$$

It is this value which the commutator of X^i with P^0 must have in order to commute with the mass shell condition. Moreover, (17) is required to justify the denomination 'position operator'. It entails the notion that in the course of time the position of a particle changes with its velocity.

Poincaré covariance does not require the position operator $\mathbf{X}$ be the spatial part of a D-vector: $\mathbf{X} = \mathbf{X}_{\underline{u}}$ is the position operator used by an observer at rest with four-velocity $\underline{u} = (1, 0, \ldots)$. Under spacetime translations a and rotations R it transforms linear inhomogeneously[5]

$$\mathrm{e}^{\mathrm{i} a P} \, \mathbf{X} \, \mathrm{e}^{-\mathrm{i} a P} = \mathbf{X} + \mathbf{a} + a^0 \frac{\mathbf{P}}{P^0} \, , \; U_R \mathbf{X} {U_R}^{-1} = R^{-1} \mathbf{X} \, , \tag{18}$$

[5] In an orthonormal basis our metric is $\eta = \mathrm{diag}(1, -1, \ldots, -1)$.

where U_R represents the rotation R in Hilbert space. Observers boosted by

$$L_u = \begin{pmatrix} \sqrt{1+\mathbf{u}^2} & \mathbf{u}^{\mathrm{T}} \\ \mathbf{u} & \mathbf{1} + \frac{\mathbf{u}\,\mathbf{u}^{\mathrm{T}}}{1+\sqrt{1+\mathbf{u}^2}} \end{pmatrix} , \tag{19}$$

to four-velocity u, measure position with

$$\mathbf{X}_u = U_{L_u} \mathbf{X}_{\underline{u}} {U_{L_u}}^{-1} . \tag{20}$$

Under Lorentz transformations Λ the position operators transform by Wigner rotation

$$U_\Lambda \mathbf{X}_u {U_\Lambda}^{-1} = W^{-1}(\Lambda, u)\mathbf{X}_{\Lambda u} \ , \ W(\Lambda, u) = {L_{\Lambda u}}^{-1} \Lambda L_u \in \mathrm{SO}(D-1) \ , \tag{21}$$

in a Poincaré covariant way, even though $\mathbf{X}$ is not the spatial part of a D-vector.

5 Lorentz Generators of Particles

In terms of the algebra (3) one can specify generators $M_\omega = \omega^{mn} M_{mn}/2$, $M_{mn} = -M_{nm}$, of Lorentz transformations $U_{\mathrm{e}^\omega} = \mathrm{e}^{-\mathrm{i} M_\omega}$ [10],

$$-\,\mathrm{i} M^{mn} \Psi \stackrel{?}{=} \mathrm{i}(P^m X^n - P^n X^m)\Psi + \Gamma^{mn}\Psi \ , \tag{22}$$

where Γ^{mn} are skew hermitian matrices which commute with X and P and represent the Lorentz Lie algebra

$$[\Gamma^{mn}, \Gamma^{rs}] = -\eta^{mr}\Gamma^{ns} + \eta^{ms}\Gamma^{nr} + \eta^{nr}\Gamma^{ms} - \eta^{ns}\Gamma^{mr} \tag{23}$$

as do the operators $l^{mn} = \mathrm{i}(P^m X^n - P^n X^m)$.

Nonvanishing Γ^{mn} can occur only in case the scalar product in spin space is indefinite, otherwise there are no finite dimensional, skew hermitian matrices which generate the Lorentz group.

However as our lemma shows, the operators P and X do not act on particles with a definite mass but in a space in which P has the continuous spectrum $\mathbb{R}^D$ (4) with unbounded and also negative energies. This is not the space of relativistic particles.

To show that X^n and P^m are not required for the construction of Lorentz generators we specify the ones which act on massive one-particle states. These physical states, to which the generators can can be applied, correspond to rapidly decreasing momentum wave functions $\Psi : \mathcal{M}_m \to \mathbb{C}^d$ which map the massive shell $\mathcal{M}_m$, $m > 0$, (5) smoothly to some space $\mathbb{C}^d$, in which skew hermitian matrices $\Gamma_{ij} = -\Gamma_{ji}$ generate a d-dimensional unitary representation of $\mathrm{SO}(D-1)$, $(i, j, k, l \in \{1, \ldots, D-1\})$,

$$[\Gamma_{ij}, \Gamma_{kl}] = \delta_{ik}\Gamma_{jl} - \delta_{jk}\Gamma_{il} - \delta_{il}\Gamma_{jk} + \delta_{jl}\Gamma_{ik} \ . \tag{24}$$

The generators of the Poincaré group [11] and the position operator map by[6]

$$
\begin{aligned}
(-\mathrm{i}\, P^n \Psi)(p) &= -\mathrm{i}\, p^n \Psi(p)\,, \\
\big(-\mathrm{i} M_{ij}\Psi\big)(p) &= -\big(p^i \partial_{p^j} - p^j \partial_{p^i}\big)\Psi(p) + \Gamma_{ij}\Psi(p)\,, \\
\big(-\mathrm{i} M_{0i}\Psi\big)(p) &= p^0\, \partial_{p^i}\Psi(p) + \Gamma_{ij}\frac{p^j}{p^0+m}\Psi(p)\,, \\
\big(-\mathrm{i} X^i\Psi\big)(p) &= \partial_{p^i}\Psi(p) + \frac{p^i}{2(p^0)^2}\Psi(p)\,,
\end{aligned}
\tag{25}
$$

this Schwartz space $\mathcal{S}(\mathcal{M}_m, \mathbb{C}^d) \subset L^2(\mathcal{M}_m, \mathbb{C}^d)$ of smooth states of rapid decrease to itself. The generators are skew hermitian with respect to the Lorentz invariant measure $(\mathrm{d}^{D-1}p)/p^0$.

They are equivariant: observers, Lorentz boosted by L_u (19) to four-velocity u, use the generators (20, 21) and

$$
U_{L_u} M_{mn} U_{L_u}{}^{-1} = \big(L_u\big)^r{}_m \big(L_u\big)^s{}_n M_{rs}\,. \tag{26}
$$

The massless case is *not* obtained by simply specifying $m = 0$ in the energy $p^0 = \sqrt{\mathbf{p}^2}$. Its inverse $1/p^0$ and therefore the invariant measure $(\mathrm{d}^{D-1}p)/p^0$, the generators M_{0i} and X^i are singular at $\mathbf{p} = 0$. There the energy is only continuous, not smooth. The distinguished momentum $p = 0$ is a fixed point of Lorentz transformations and *not* invariant under translations. This excludes generators X^i of such translations [6]: massless states do not allow the spatial Heisenberg algebra (7).

The Lorentz generators, acting on massless, physical states turn out not to act on smooth functions of $\mathbb{R}^{D-1}$ but on smooth sections of a vector bundle over $S^{D-2} \times \mathbb{R}$ which carries a representation of $\mathrm{SO}(D-2)$ with generating matrices Γ_{ij}.

In the coordinate patch $\mathcal{U}_N = \Big\{p : p^0 = \sqrt{\mathbf{p}^2}, |\mathbf{p}| + p_z > 0\Big\}$ the sections are smooth functions Ψ_N which the generators map to ($p_z := p^{D-1}, i, j, k \in \{1, \dots D-2\}$)[7]

$$
\begin{aligned}
\big(-\mathrm{i} M_{ij}\Psi\big)_N(p) &= -\big(p^i \partial_{p^j} - p^j \partial_{p^i}\big)\Psi_N(p) + \Gamma_{ij}\,\Psi_N(p)\,, \\
\big(-\mathrm{i} M_{zi}\Psi\big)_N(p) &= -\big(p_z \partial_{p^i} - p^i \partial_{p_z}\big)\Psi_N(p) + \Gamma_{ik}\frac{p^k}{|\mathbf{p}| + p_z}\,\Psi_N(p)\,, \\
\big(-\mathrm{i} M_{0i}\Psi\big)_N(p) &= |\mathbf{p}|\partial_{p^i}\Psi_N(p) + \Gamma_{ik}\frac{p^k}{|\mathbf{p}| + p_z}\,\Psi_N(p)\,, \\
\big(-\mathrm{i} M_{0z}\Psi\big)_N(p) &= |\mathbf{p}|\partial_{p_z}\Psi_N(p)\,.
\end{aligned}
\tag{27}
$$

The detailed discussion of massless states, e.g. the relation $\Psi_S(p) = h_{SN}(p)\Psi_N(p)$ with $h_{SN}(p) = \Big(\frac{p_x + \mathrm{i}p_y}{\sqrt{p_x^2 + p_y^2}}\Big)^{2h}$ where Ψ_S is smooth in $\mathcal{U}_S = \Big\{p : p^0 = \sqrt{\mathbf{p}^2}, |\mathbf{p}| - p_z > 0\Big\}$ will be given elsewhere [5]. Here it is only important that in contrast to the covariantly quantized worldline particles they and their Poincaré transformations exist.

[6] We use matrix notation and suppress indices of the components of Ψ and Γ_{ij}. Checking the Lorentz algebra observe $\sum_{i=1}^{D-1} p^i p^i = (p^0)^2 - m^2 = (p^0 + m)(p^0 - m)$.

[7] Checking the Lorentz algebra observe $\sum_{i=1}^{D-2} p^i p^i = |\mathbf{p}|^2 - (p_z)^2 = (|\mathbf{p}| + p_z)(|\mathbf{p}| - p_z)$.

6 Conclusions

The spacetime Heisenberg Lie algebra excludes relativistic particles with a definite mass. This important result does not depend on this or that method of quantization. The lemma follows in each quantum theory by elementary algebra. The same conclusion follows from the Stone-von Neumann theorem.

More generally, each solution of a constraint which restricts an absolutely continuous spectrum to a lower dimensional submanifold vanishes.

In quantum physics not only the algebra of operators is important but also the domain on which they act. As the spacetime Heisenberg Lie algebra does not allow a subspace of relativistic particles we specify the generators of Lorentz transformations which do.

Though our lemma has far reaching implications its proof is astonishingly simple. It excludes D Heisenberg pairs as they originate from canonical quantization of classical point particles traversing worldlines which maximize a diffeomorphism invariant action. To impose the corresponding constraint, the mass shell condition, on the momenta of the states, has no solution.

Acknowledgements Norbert Dragon thanks Gleb Arutyunov, Arthur Hebecker and Hermann Nicolai for helpful email correspondence and Wilfried Buchmüller, Stefan Theisen and Sergei Kuzenko for extended, clarifying discussions.

Funding Open Access funding enabled and organized by Projekt DEAL.

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