



The Rough with the Smooth of the Light Cone String

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Abstract

The polynomials in the generators of a unitary representation of the Poincaré group constitute an algebra which maps the dense subspace \mathcal{D} of smooth, rapidly decreasing wavefunctions to itself. This mathematical result is highly welcome to physicists, who previously just assumed their algebraic treatment of unbounded operators be justified. The smoothness, however, has the side effect that a rough operator R , which does not map a dense subspace of \mathcal{D} to itself, has to be shown to allow for some other dense domain which is mapped to itself both by R and all generators. Otherwise their algebraic product, their concatenation, is not defined. Canonical quantization of the light cone string postulates operators $-iX^1$ and $P^- = (P^0 - P^z)/2$ and as their commutator the multiplicative operator $R = P^1/(P^0 + P^z)$. This is not smooth but rough on the negative z -axis of massless momentum. Using only the commutation relations of P^m with the generators $-iM_{iz}$ of rotations in the P^i - P^z -plane we show that on massless states the operator R is inconsistent with a unitary representation of $SO(D-1)$. This makes the algebraic determination of the critical dimension, $D = 26$, of the bosonic string meaningless: if the massless states of the light cone string admit R then they do not admit a unitary representation of the subgroup $SO(D-1)$ of the Poincaré group. With analogous arguments we show: Massless multiplets are inconsistent with a translation group of the spatial momentum which is generated by a self-adjoint spatial position operator \mathbf{X} .

Keywords Domain of generators · Gårding space · Schwartz space · Rough operators · Light cone string · Spatial position operator · Massless states

1 Introduction

The seminal calculation [6] of the critical dimension $D = 26$ of the spacetime, in which the bosonic quantum light cone string acts, left two nagging doubts.

If, assuming some basic algebraic rules, formally hermitian operators $M_{mn} = -M_{nm}$, $m, n \in \{0, 1, \dots, D-1\}$, satisfy the commutation relations of the Lorentz Lie algebra,¹

$$[-iM_{mn}, -iM_{rs}] = -\eta_{mr}(-iM_{ns}) + \eta_{ms}(-iM_{nr}) + \eta_{nr}(-iM_{ms}) - \eta_{ns}(-iM_{mr}) \quad (1)$$

¹ In an orthonormal basis our metric is $\eta = \text{diag}(1, -1, \dots, -1)$.

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is this also sufficient for the operators to generate a unitary representation of the Lorentz group? Each finite dimensional matrix ω generates by the exponential map the elements of a one parameter group,

$$g_t = e^{t\omega}, \quad t \in \mathbb{R}, \quad g_t g_{t'} = g_{t+t'}. \tag{2}$$

By the Baker Campbell Hausdorff formula exponentials of sufficiently small finite dimensional matrices ω generate a corresponding Lie group G , if the matrices ω represent a Lie algebra \mathfrak{g} . But which requirements are there for unbounded operators? Is a Lorentz Lie algebra in $D = 26$ of formally skew-adjoint generators sufficient for the existence of a unitary representation of the Poincaré group?

Vice versa, if a set of operators does *not* satisfy the commutation relations of the Lorentz Lie algebra does this exclude improved operators which do so? In this case $D = 26$ would be indicated only as the dimension where an apparent but correctable anomaly vanishes. Is $D = 26$ necessary for the Lorentz invariance of the light cone string?

Neither of these questions could be addressed seriously. The cumbersome calculation of the Lorentz Lie algebra shied away all attempts to exponentiate the generators or to investigate classes of improvement terms. By common consensus, one neglected the existence problem of exponentiated generators and adopted $D = 10$, the critical dimension of the superstring, as start value for compactification schemes.

We shortly review the main results of the mathematical investigations [13] of operators which generate in Hilbert space a unitary representation of a finite dimensional Lie group. They allow us to deduce the results announced in the abstract. Moreover, we point out that the excitation operators of the light cone string $\alpha_{-l}, l \in \mathbb{N}$, which map the tachyon shell and the massless shell in a momentum local way to massive shells, cannot map smooth wavefunctions of the tachyon shell and the massless shell to smooth wavefunctions of massive shells.

Though similar and even more severe inconsistencies exist with tachyon states, we restrict our considerations mainly to problems with massless states. They persist even if one could get rid of the tachyon.

2 Smoothness of Lie Group Transformations

Let $U_g : \mathcal{H} \rightarrow \mathcal{H}$ denote a unitary representation $U_g U_{g'} = U_{g g'}$ of a Lie group G in a Hilbert space \mathcal{H} for which all maps $f_{[\Psi]} : g \mapsto U_g \Psi$ from G to \mathcal{H} are measurable.

The generators $-iM_\omega$ of one-parameter subgroups $U_{e^{t\omega}}$ are defined on the subspace of smooth states $\Psi \in \mathcal{D} \subset \mathcal{H}$ on which all $U_{e^{t\omega}}$ act differentiably,

$$-iM_\omega \Psi = \lim_{t \rightarrow 0} \frac{U_{e^{t\omega}} \Psi - \Psi}{t}. \tag{3}$$

By Stone's theorem each M_ω is essentially self-adjoint. It owns a projection valued measure² by which it generates $U_{e^{t\omega}}$ not only in \mathcal{D} but in the complete Hilbert space \mathcal{H} ,

$$M_\omega = \int dE_\lambda \lambda, \quad U_{e^{t\omega}} = \int dE_\lambda e^{-it\lambda} =: e^{-itM_\omega}. \tag{4}$$

Applied to states in \mathcal{D} the products

$$U_{g(t)} = U_{e^{t_1\omega_1}} \cdots U_{e^{t_n\omega_n}}, \quad t = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n \tag{5}$$

² A projection valued measure E_λ is a parameterized set of projectors, $\lambda \in \mathbb{R}$, with $E_\lambda E_{\lambda'} = E_{\min\{\lambda, \lambda'\}}$, $\lim_{\varepsilon \rightarrow 0+} E_{\lambda+\varepsilon} = E_\lambda, E_{-\infty} = 0$ and $E_\infty = \mathbf{1}$.

are a differentiable function of \mathbb{R}^n . So the derivatives

$$\partial_{t_1} \dots \partial_{t_n} U_{g(t_1 \dots t_n)|_{t=0}} = (-i)^n M_{\omega_1} \dots M_{\omega_n} \tag{6}$$

exist on them no matter how large n is. The concatenation of generators, their algebraic product, is defined in \mathcal{D} . It is a domain of the polynomial algebra \mathcal{A} of the generators M_ω (which restricted to \mathcal{D} are essentially self-adjoint) and invariant also under all $U_g, g \in G$. The generators represent the corresponding Lie algebra \mathfrak{g} [13]:

$$M_\omega M_{\omega'} - M_{\omega'} M_\omega - iM_{[\omega, \omega']} \tag{7}$$

is a two-sided ideal which vanishes in the algebra \mathcal{A} if multiplied from the left and the right with arbitrary polynomials $A_l, A_r \in \mathcal{A}$.

As the maps $f_{[\Psi]} : g \mapsto U_g \Psi$ are measurable for all Ψ and because integrals over measurable functions of compact support exist, therefore the Gårding space \mathcal{G} exists which is spanned by smoothened states Ψ_f which are averaged with a left invariant volume form $d\mu_g = d\mu_{g'g}$ and smooth functions $f : G \rightarrow \mathbb{C}$ of compact support

$$\Psi_f = \int_G d\mu_g f(g) U_g \Psi . \tag{8}$$

The smoothened states Ψ_f transform smoothly

$$U_g \Psi_f = \Psi_{f \circ g^{-1}} . \tag{9}$$

The Gårding space coincides with the space of smooth states, $\mathcal{G} = \mathcal{D}$ [13]. It is dense in the Hilbert space $\mathcal{H}(G)$ of square integrable functions of G . That \mathcal{G} exists and constitutes the dense and invariant domain of the polynomial algebra of the generators and the group, justifies in retrospect physicists who manipulated the unbounded generators M_ω algebraically not caring about domains.

A mass shell is the Lorentz orbit G/H of some chosen momentum \underline{p} with $p^0 > 0$ and $p^2 = m^2$. Its smooth, rapidly decreasing wavefunctions also span a dense subspace $\mathcal{D} \subset \mathcal{H}(G/H)$ which is mapped to itself by the unitary representation of the Poincaré group. Because its states are smooth, differential geometry becomes applicable to quantum physics.

Recall that a Hilbert space of square integrable functions $\Psi : p \mapsto \Psi(p)$ consists more precisely of equivalence classes of functions, where functions are equivalent, if the support of their difference has measure zero.

$$\Psi = 0 \Leftrightarrow \langle \Psi | \Psi \rangle = \int dp |\Psi(p)|^2 = 0 \tag{10}$$

So the values in a set of measure zero do not count for equivalent functions. A smooth function, however, is the only smooth function in its equivalence class. It is determined and smooth *everywhere* not only ‘almost everywhere’.

The rough which one has to take with the smooth: If in a relativistic model the commutation relations, which are postulated in canonical quantization, contain a rough operator R , which does not map all smooth states to smooth states, then one has to check, whether this is consistent with the unitary representation $U_{a,\Lambda}$ of the Poincaré group.³

³ Notation: Let $T_a : x \mapsto x + a$ denote a translation in \mathbb{R}^4 , $T_\Lambda : x \mapsto \Lambda x$ a Lorentz transformation, and $T_{a,\Lambda} = T_a T_\Lambda \in \mathfrak{P}$ a Poincaré transformation. We denote by $U_{a,\Lambda}$ its unitary representation with generators P^m and M_{mn} , $U_{a,e^\omega} = e^{iPa} e^{i\omega^{mn} M_{mn}/2}$, in a Hilbert space \mathcal{H}_1 of one-particle states.

For this consistency we require R and all its commutators with generators of the Poincaré group ($U_{g(t)}$ given by (5))

$$\text{ad}_{M_{\omega_1}} \dots \text{ad}_{M_{\omega_n}} R := i^n (\partial_{t^1} \dots \partial_{t^n})|_{t=0} U_{g(t)} R U_{g(t)}^{-1} = [M_{\omega_1}, \dots [M_{\omega_n}, R] \dots] \quad (11)$$

to exist on all states $\Psi \in \mathcal{D}$.

Earlier arguments concerning a rough operator used the assumption that R and its commutators with generators M_ω were elements of a common operator algebra, a condition which underlies implicitly all algebraic calculations and which is postulated explicitly at the very beginning of the mathematical investigation [13]. There the definition of an operator algebra postulates the existence of a dense domain on which all operators are defined and which is left invariant by all operators.

These arguments we criticized objecting that requiring an operator algebra may be a too strong assumption. Only the operator R and the algebra of the generators M_ω on its right and on its left have to exist. There could be a set of rules, though presently unknown, to justify the calculation of e.g. $D = 26$ but rule out the inconsistency which results from assuming an operator algebra of R and the generators.

Because of these objections we show the inconsistency of the example of the multiplicative operator $R = P^1 / (P^0 + P^z)$ based solely on the assumed existence of all repeated commutators of R with generators of rotations, if applied to massless, smooth states.

3 Momentum Local Maps

Specific to the gauge fixed quantum string are excitation operators α_{-l} , $l \in \mathbb{N}$, which excite states on mass shells $m^2(N) = (N - 1)\mu^2$ to states on mass shells $m^2(N + l)$ [1, 8]. The transition is momentum local such that for each momentum with $p^2 = m^2(N)$ there is an excited momentum $q = g_l(p)$ with $q^2 = m^2(N + l)$ and

$$\Phi_l(q) := (\alpha_{-l}\Psi)(q) = \sqrt{\frac{q^0}{p^0} \left| \det \frac{\partial g_l^{-1}}{\partial q} \right|} M_l(p) \Psi(p), \quad p = g_l^{-1}(q). \quad (12)$$

$M_l(p)$ is some matrix, unitary on its image, with indices which we need not depict. We have made the factor explicit, which relates the measures $\tilde{d}p \sim d^{D-1}/p^0$ and $\tilde{d}q$, because it vanishes where g_l has no inverse and is a rough multiplicative operator where g_l^{-1} does not exist. By their commutation relations and the ground state property $\alpha_l \Psi = 0$, $\alpha_l = (\alpha_{-l})^*$, $l \geq 2$, for massless states Ψ , the operators α_{-l} can be inverted on their image,

$$\Psi(p) = (\alpha_l \Phi_l)(p) = \sqrt{\frac{p^0}{q^0} \left| \det \frac{\partial g_l}{\partial p} \right|} M_l(p)^{-1} \Phi_l(q) \quad (\text{no sum over } l), \quad q = g_l(p). \quad (13)$$

These operators are potentially inconsistent with the Poincaré generators. They act on smooth functions of the massless shell $\mathcal{M}_0 = \{p : p = e^\lambda (1, \mathbf{n}), \mathbf{n} \in S^{D-2}, \lambda \in \mathbb{R}\}$ or the tachyon shell $\mathcal{M}_{\text{Tachyon}} = \{p : p = (E, \sqrt{\mu^2 + E^2} \mathbf{n}), \mathbf{n} \in S^{D-2}, E \in \mathbb{R}\}$ which have the topology of $S^{D-2} \times \mathbb{R}$. Each massive shell $\mathcal{M}_m = \{p : p^0 = \sqrt{m^2 + \mathbf{p}^2}, \mathbf{p} \in \mathbb{R}^{D-1}\}$ has the topology of \mathbb{R}^{D-1} .

For α_{-l} , $l \geq 2$, to map the smooth tachyonic and massless wave functions to smooth massive wave functions, the momentum map g_l has to map the tachyon and the massless shell smoothly and with a smooth inverse g_l^{-1} to massive shells. But the topologies of the shells are different: there is no diffeomorphism g_l of $S^{D-2} \times \mathbb{R}$ to \mathbb{R}^{D-1} .

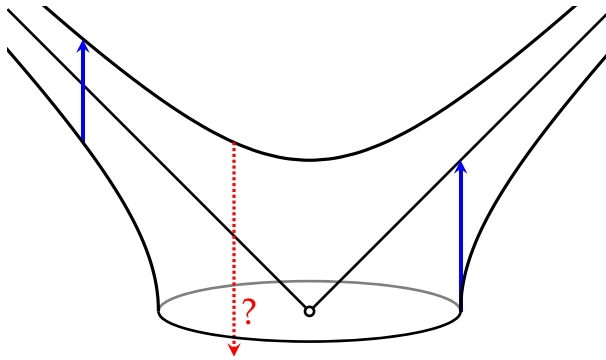


Fig. 1 Transitions between Mass Shells in the Static Gauge

Hence, in a relativistic theory it is highly questionable whether operators α_{-l} exist which excite a tachyon or massless particles in a momentum local way to massive particles.

In Fig.1 the failure of invertible transitions between the massive, massless and tachyon shell is obvious. In the static gauge of the bosonic string [9] they transfer only energy. But massive states with spatial momentum $|\mathbf{p}| < \mu$ are not related to tachyon states.

No one can do better: the different topologies exclude any smooth, invertible map between a massive shell and the massless or the tachyon shell.

4 Lie Algebra Without and with Group

The mere fact, that differential operators satisfy a Lie algebra on some space of functions does not make them generators of a representation of the corresponding group. This is demonstrated by the following operators $M_{mn} = -M_{nm}$,

$$\begin{aligned}
 (-iM_{12}\Psi)_N(p) &= -(p_x\partial_{p_y} - p_y\partial_{p_x})\Psi_N(p) - i h \Psi_N(p) , \\
 (-iM_{31}\Psi)_N(p) &= -(p_z\partial_{p_x} - p_x\partial_{p_z})\Psi_N(p) - i h \frac{p_y}{|\mathbf{p}| + p_z}\Psi_N(p) , \\
 (-iM_{32}\Psi)_N(p) &= -(p_z\partial_{p_y} - p_y\partial_{p_z})\Psi_N(p) + i h \frac{p_x}{|\mathbf{p}| + p_z}\Psi_N(p) , \\
 (-iM_{01}\Psi)_N(p) &= |\mathbf{p}|\partial_{p_x}\Psi_N(p) - i h \frac{p_y}{|\mathbf{p}| + p_z}\Psi_N(p) , \\
 (-iM_{02}\Psi)_N(p) &= |\mathbf{p}|\partial_{p_y}\Psi_N(p) + i h \frac{p_x}{|\mathbf{p}| + p_z}\Psi_N(p) , \\
 (-iM_{03}\Psi)_N(p) &= |\mathbf{p}|\partial_{p_z}\Psi_N(p) .
 \end{aligned}
 \tag{14}$$

On differentiable functions of the northern coordinate patch \mathcal{U}_N of the massless shell \mathcal{M}_0

$$\mathcal{U}_N = \{p : p^0 = \sqrt{\mathbf{p}^2}, |\mathbf{p}| + p_z > 0\} \subset \mathcal{M}_0 = \{p : p^0 = \sqrt{\mathbf{p}^2} > 0\} \subset \mathbb{R}^4 \tag{15}$$

the operators $-iM_{mn}$ satisfy the Lorentz Lie algebra⁴ (1) in $D = 4$ [2, 4, 5, 10]. The angular momentum in the direction of the momentum, the helicity h ,

$$((p_x M_{23} + p_y M_{31} + p_z M_{12})\Psi)_N(p) = h |\mathbf{p}| \Psi_N(p) \tag{16}$$

⁴ Observe $\sum_{i=1}^2 p^i p^i = |\mathbf{p}|^2 - (p_z)^2 = (|\mathbf{p}| + p_z)(|\mathbf{p}| - p_z)$.

is some real number. The Lorentz Lie algebra does not restrict $2h$ to be an integer.

The operators are formally skew-adjoint with respect to the Lorentz invariant measure $\tilde{d}p = d^3 p/|\mathbf{p}|$, formally only, because the singularities at $|\mathbf{p}| + p_z = 0$ need closer investigation.

The operators (14) cannot generate a unitary representation of the Lorentz group because the domain \mathcal{U}_N of the differentiable functions is too small: Lorentz generators act on smooth states, which have to be defined *everywhere* in the Lorentz orbit \mathcal{M}_0 . The group acts transitively on the massless shell and contains e.g. for each massless momentum p a rotation which maps \mathbf{p} to the negative z -axis

$$\mathcal{A}_- = \{p : p^0 = \sqrt{\mathbf{p}^2}, p^0 + p_z = 0\}. \tag{17}$$

For negative p_z and with $x = (\sum_{i=1}^{D-2} p_i^2)/p_z^2, p_z := p^{D-1}$, one has for each D

$$\text{for } p_z < 0 : |\mathbf{p}| + p_z = |\mathbf{p}| - |p_z| = |p_z|(\sqrt{1+x} - 1) \leq |p_z| \frac{x}{2}, \tag{18}$$

because the concave function $x \mapsto \sqrt{1+x}$ is bounded by its tangent at $x = 0$. So

$$\text{for } p_z < 0 : \frac{1}{|\mathbf{p}| + p_z} \geq \frac{2|p_z|}{r^2}, r^2 = \sum_{i=1}^{D-2} p_i^2 > 0, \tag{19}$$

diverges in a neighbourhood \mathcal{U} of $\hat{p} \in \mathcal{A}_-$ at least like the inverse square of the axial distance to \mathcal{A}_- .

If $h \Psi_N(\hat{p}) \neq 0$ then it must not be differentiable there. Otherwise the multiplicative term of $M_{31} \Psi_N$ dominates near \hat{p} where it scales as $|p_z|/r$. Its squared modulus integrated with $d^3 p/|\mathbf{p}|$ over a sufficiently small \mathcal{U} in cylindrical coordinates is bounded from below by a positive number times an r -integral $r/r^2 dr$ which diverges at the lower limit $r = 0$. The multiplicative term alone diverges.

Near \mathcal{A}_- the derivative term $D\Psi_N = -p_z \partial_{p_x} \Psi_N$ in $M_{31} \Psi_N$ has to cancel the multiplicative singularity $M\Psi_N$ up to a function χ , which is smooth. This linear inhomogeneous condition $(D + M)\Psi_N = \chi$ is solved by variation of constants $\Psi_N = f \Psi_S$ where f satisfies the two homogeneous conditions

$$|p_z|(\partial_{p_x} - 2ih \frac{p_y}{p_x^2 + p_y^2})f = 0, |p_z|(\partial_{p_y} + 2ih \frac{p_x}{p_x^2 + p_y^2})f = 0, \tag{20}$$

for both $M_{31} \Psi_N$ and $M_{32} \Psi_N$ to exist. They determine $f(p) = e^{-2ih\varphi(p)}$ up to a factor.

The function Ψ_S is smooth in the southern coordinate patch

$$\mathcal{U}_S = \{p : p^0 = \sqrt{\mathbf{p}^2}, |\mathbf{p}| - p_z > 0\} \tag{21}$$

and related in $\mathcal{U}_N \cap \mathcal{U}_S$ by the transition function $f^{-1} = h_{SN}$ to Ψ_N

$$\Psi_S(p) = h_{SN}(p) \Psi_N(p), h_{SN}(p) = e^{2ih\varphi(p)} = \left(\frac{p_x + ip_y}{\sqrt{p_x^2 + p_y^2}} \right)^{2h}. \tag{22}$$

The transition function $e^{2ih\varphi(p)}$ is defined and smooth in $\mathcal{U}_N \cap \mathcal{U}_S$ only if $2h$ is integer. This is why the helicity of a massless particle is integer or half integer.

Multiplying (14) with h_{SN} one obtains from (22)

$$\begin{aligned}
 (-iM_{12}\Psi)_S(p) &= -(p_x\partial_{p_y} - p_y\partial_{p_x})\Psi_S(p) + i h \Psi_S(p) , \\
 (-iM_{31}\Psi)_S(p) &= -(p_z\partial_{p_x} - p_x\partial_{p_z})\Psi_S(p) - i h \frac{p_y}{|\mathbf{p}| - p_z} \Psi_S(p) , \\
 (-iM_{32}\Psi)_S(p) &= -(p_z\partial_{p_y} - p_y\partial_{p_z})\Psi_S(p) + i h \frac{p_x}{|\mathbf{p}| - p_z} \Psi_S(p) , \\
 (-iM_{01}\Psi)_S(p) &= |\mathbf{p}|\partial_{p_x}\Psi_S(p) + i h \frac{p_y}{|\mathbf{p}| - p_z} \Psi_S(p) , \\
 (-iM_{02}\Psi)_S(p) &= |\mathbf{p}|\partial_{p_y}\Psi_S(p) - i h \frac{p_x}{|\mathbf{p}| - p_z} \Psi_S(p) , \\
 (-iM_{03}\Psi)_S(p) &= |\mathbf{p}|\partial_{p_z}\Psi_S(p) .
 \end{aligned}
 \tag{23}$$

Ψ_N and Ψ_S are local sections of a bundle over $S^2 \times \mathbb{R}$ with transition function h_{SN} . A massless quantum state Ψ is a section given locally in \mathcal{U}_N by Ψ_N and in \mathcal{U}_S by Ψ_S [3].

All $M_{mn}\Psi$ are square integrable, rapidly decreasing and smooth in \mathcal{M}_0 if Ψ is.

For all ω in the Lorentz algebra the operators $-iM_\omega = -i/2 \omega^{mn} M_{mn}$ are by construction [3] the derivatives of unitary one-parameter groups

$$-iM_\omega(U_{e^{t\omega}}\Psi) = \partial_t(U_{e^{t\omega}}\Psi) ,
 \tag{24}$$

which act on a dense and invariant domain \mathcal{D} of smooth states, where the transformations U_{e^ω} together with all their products represent unitarily the Lorentz group. So $-iM_\omega$ not only satisfy the Lorentz algebra but they are also skew-adjoint (by Stone’s theorem) and generate a unitary representation of the Lorentz group.

The singularities of the multiplicative terms in (14) for $h \neq 0$ on the 3-axis do not allow to conclude that the operators are rough. They can be combined with the partial derivatives to covariant derivatives

$$D_i = i |\mathbf{p}|^{-1/2} M_{0i} |\mathbf{p}|^{-1/2} = \partial_{p^i} + A_i - \frac{p^i}{2|\mathbf{p}|^2} ,
 \tag{25}$$

where the connection \mathbf{A}

$$\mathbf{A}_N(p) = \frac{i h}{|\mathbf{p}|(|\mathbf{p}| + p_z)} \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix} , \quad \mathbf{A}_S(p) = \frac{-i h}{|\mathbf{p}|(|\mathbf{p}| - p_z)} \begin{pmatrix} -p_y \\ p_x \\ 0 \end{pmatrix} .
 \tag{26}$$

In the intersection of their coordinate patches they are related by the transition function

$$D_{Si} = e^{2ih\varphi(p)} D_{Ni} e^{-2ih\varphi(p)} .
 \tag{27}$$

The covariant derivatives D_j and the momenta P^i do not constitute Heisenberg pairs as the commutators $[D_i, D_j]$ do not vanish but yield the field strength of a momentum space monopole of charge h at $p = 0$,

$$[P^i, P^j] = 0 , \quad [P^i, D_j] = -\delta^i_j , \quad [D_i, D_j] = F_{ij} = \partial_i A_j - \partial_j A_i = i h \varepsilon_{ijk} \frac{P^k}{|\mathbf{P}|^3} .
 \tag{28}$$

The geometry of the massless momentum shell of particles with nonvanishing helicity is noncommutative.

In terms of the covariant derivative the generators of Lorentz transformations (14, 23) take the smooth, rotation equivariant form ($\mathbf{p} \neq 0$)

$$-iM_{ij} = -(P^i D_j - P^j D_i) - i h \varepsilon_{ijk} \frac{P^k}{|\mathbf{P}|}, \quad -iM_{0i} = -|\mathbf{P}|^{1/2} D_i |\mathbf{P}|^{1/2}. \quad (29)$$

They satisfy the Lorentz algebra on account of (28) for arbitrary real h . However, the covariant derivative D is a skew-adjoint operator only if $2h$ is integer.

The integrand $\mathbb{F} = \frac{1}{2} d p^i d p^j F_{ij}$, the first Chern class, is a topological density: integrated on each surface \mathcal{S} which encloses the apex $p = 0$ of the cone $p^0 = |\mathbf{p}|$

$$\frac{1}{4\pi} \int_{\mathcal{S}} \mathbb{F} = i h \quad (30)$$

it yields a value which depends only on the transition functions of the bundle. The integral remains constant under smooth changes of the connection A of the covariant derivative as the Euler derivative of $\mathbb{F}(A)$ with respect to A vanishes.

5 Failing Rotational Symmetry of the Light Cone String

Canonical quantization of the light cone string [1, 8, page 23] postulates transverse Heisenberg pairs $P^i, X^j, i, j \in \{1, \dots, D - 2\}$, which commute with $P^+ = (P^0 + P^z)$ and the level operator N ,

$$[P^i, X^j] = -i\delta^{ij}, \quad [P^i, P^j] = 0, \quad [X^i, X^j] = 0, \quad [X^i, P^+] = 0, \quad [X^i, N] = 0. \quad (31)$$

By the mass shell relation

$$P^- = \frac{1}{2}(P^0 - P^z) = \frac{(N - 1)\mu^2 + \sum_{i=1}^{D-2} P^i P^i}{2P^+} \quad (32)$$

and the innocent looking relation $[X^1, P^+] = 0$ the operator $R = P^1/(P^0 + P^z)$ is in the postulated algebra,

$$-i[X^1, P^-] = \frac{P^1}{P^0 + P^z}. \quad (33)$$

In $D = 3$ and $D = 4$ this operator R diverges on each smooth massless state Ψ which does not vanish at a point $\hat{p} \in \mathcal{A}_-$ on the negative z -axis such that in some neighbourhood $\mathcal{U}_{\hat{p}}$ of this point

$$\text{for all } p \in \mathcal{U}_{\hat{p}} : |\Psi(p)|^2 > c > 0. \quad (34)$$

If $\Psi \neq 0$ does not satisfy this condition then there is a rotation ρ such that $U_\rho \Psi$ does, because $|U_\rho \Psi|(\hat{p}) = |\Psi|(\rho^{-1} \hat{p})$ and each $p = \rho^{-1} \hat{p}$, where Ψ does not vanish, can be rotated to a point \hat{p} on the negative z -axis.

Restrict for definiteness $\mathcal{U}_{\hat{p}}$ to a cylinder with $0 < a < -p^z < b$ and $0 < r < \bar{r}$, where r denotes the axial distance to the z -axis.

Proof: to determine the contribution of $\mathcal{U}_{\hat{p}}$ to $|R\Psi|^2$ we remark that in cylindrical coordinates in $D = 3$ the operator P^1 multiplies with r or $-r$, in $D = 4$ it multiplies $\Psi(r \cos \theta, r \sin \theta, p^z)$ with $r \cos \theta$. In $\mathcal{U}_{\hat{p}}$ the function $1/(p^0 + p^z)$ is larger than $2|p^z|/r^2$ (19). So in $\mathcal{U}_{\hat{p}}$ the function $|(p^1/(p^0 + p^z))\Psi(p)|^2$ is larger in $D = 3$ than $4c|p^z|^2(r/r^2)^2$ and in $D = 4$ larger than $4c|p^z|^2(r/r^2)^2 \cos^2 \theta$.

The integral of this function in $D = 4$ over the angle θ is a positive function of r and $(p^z)^2$, as the integrand vanishes as a function of the angle only in a set of spherical measure zero. Then the integral over p^z from a to b leaves a positive constant (the factor $1/p^0$ of the Lorentz invariant volume element $d\bar{p}$ is larger than $1/\sqrt{b^2 + \bar{r}^2}$) times a linearly divergent integral $\int_0^{\bar{r}} dr/r^2$ in $D = 3$, where $d^2 p = dr d p^z$, or in $D = 4$ a logarithmically divergent integral $\int_0^{\bar{r}} dr r/r^2$ because $d^3 p \sim r dr d p^z d\theta$. \square

Consider the generators $-iM_{iz}$ of rotations in the P^i - P^z -plane. They commute with P^j , $[-iM_{iz}, P^j] = 0$ if $j \neq i$ and $j \neq z$ and satisfy

$$[-iM_{iz}, P^i] = P^z \text{ (no sum over } i \text{)}, \quad [-iM_{iz}, P^z] = -P^i. \tag{35}$$

To ease our subsequent discussion note that by the chain rule (valid outside \mathcal{A}_-)

$$[-iM_{1z}, -\ln(P^0 + P^z)] = \frac{P^1}{P^0 + P^z} \tag{36}$$

the rough operator $P^1/(P^0 + P^z)$ is the commutator of $-iM_{1z}$ with $\ln(P^0 + P^z)$ which is less rough. Commutation with $-iM_{iz}$ increases roughness! Consider the n -fold commutator of $M_{1z}, M_{2z}, \dots, M_{nz}$ with $-\ln(P^0 + P^z)$ applied to Ψ ($n \leq D - 2$)

$$\Psi_n = (-i)^n [M_{nz}, \dots [M_{2z}, [M_{1z}, -\ln(P^0 + P^z)]] \dots] \Psi = (n - 1)! \frac{P^1 \dots P^n}{(P^0 + P^z)^n} \Psi \tag{37}$$

By (19,34) $|\Psi_n|^2(p)$ is in $\mathcal{U}_{\hat{p}}$ larger than

$$|\Psi_n|^2(p) > c((n - 1)!)^2 \left(\frac{r^n (2|p_z|)^n}{r^{2n}} \right)^2 |f|^2 \tag{38}$$

where f is a function of the coordinates of the sphere S^{D-3} of directions of the transverse momentum and vanishes on the sphere only in a set of spherical measure zero (the coordinate planes). So the integral of $|\Psi_n|^2(p)$ on S^{D-3} gives a positive number. The factor $1/p^0$ of the Lorentz invariant measure is larger than $1/\sqrt{b^2 + \bar{r}^2}$, the integral of $-p_z$ from a to b over this lower bound yields just a positive number. The contribution of $\mathcal{U}_{\hat{p}}$ to the norm squared of Ψ_n is therefore bounded from below by a positive number times an integral

$$\lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\bar{r}} dr \frac{r^{D-3}}{r^{2n}}. \tag{39}$$

It diverges linearly for $D = 2n + 1$ and logarithmically for $D = 2n + 2$. For each $D > 2$ there is an n , such that the $(n - 1)$ -fold commutator of the generators $M_{2z}, M_{3z}, \dots, M_{nz}$, with $R = P^1/(P^0 + P^z)$ or the n -fold commutator of $M_{1z}, M_{2z}, \dots, M_{nz}$, with $\ln(P^0 + P^z)$ cannot be applied to massless states Ψ which satisfy (34). But for each $\Psi \neq 0$ there is a rotation ρ such that $U_{\rho} \Psi$ satisfies (34) and is not in the domain of the n -fold commutator.

The commutation relation, postulated by the canonical quantization of the string in the lightcone gauge are inconsistent with the existence of a massless unitary representation of the Poincaré group. It is irrelevant that the operators, which are postulated in the canonical quantization of the lightcone string, are shown in an algebraic calculation to satisfy in $D = 26$ a Lorentz Lie algebra [6] and to contain massless states. The operator $R = P^1/(P^0 + P^z)$, which is postulated in the canonical quantization, and the generators of rotations have no common, massless domain.

Equally unimportant and possibly misleading is the observation [5, 14] that the measure of the set \mathcal{A}_- vanishes where $p^1/(p^0 + p^z)$ diverges. Essential is that $|\Psi_n(p)|^2$ (38) becomes

so large in $\mathcal{U}_{\hat{p}}$ that in $D = 2n + 2$ or $D = 2n + 1$ this function is not in the Hilbert space of square integrable states.

Though we have not worked out the details, analogous arguments should also exclude the excitation operators α_{-l} of Section 3. They map massless states to massive states but cannot do so smoothly. They multiply with a determinantal factor R (12,13) which diverges at least in a point \hat{p} of one of the involved mass shells. The Poincaré algebra always contains $(D - 1 - d)$ independent generators $-iM_\omega$ which act like a derivative along some line through the d -dimensional manifold of points \hat{p} , where R becomes singular. Applied to a state, which does not vanish in $\mathcal{U}_{\hat{p}}$, we expect these derivatives to increase the order of the divergency until the n^{th} commutator applied to Ψ gives a function Ψ_n which is not square integrable on $\mathcal{U}_{\hat{p}}$. The virtue of this arguments is, that it does not need R to be in an algebra but considers only objects linear in R as seems appropriate. After all, such a singularity R occurs in the action of α_{-l} and α_l , $l \geq 2$, only once when they make the transition between the massive and massless shells.

6 No Position Operator for Massless Particles

Similar arguments exclude a position operator \mathbf{X} which generates the group of translations of spatial momentum of massless particles

$$(e^{i\mathbf{b}\cdot\mathbf{X}}\Psi)(\mathbf{p}) = \Psi(\mathbf{p} - \mathbf{b}), \mathbf{b} \in \mathbb{R}^{D-1}. \tag{40}$$

which is unitary relative to the measure $d^{D-1}p$.⁵ It enlarges the algebra of the Poincaré generators by Heisenberg partners X^j of the spatial momenta,

$$[P^i, P^j] = 0 = [X^i, X^j], [P^i, X^j] = -i\delta^{ij}, i, j \in \{1, \dots, D - 1\}. \tag{41}$$

The Hamiltonian $H = P^0 = \sqrt{\mathbf{P}^2}$ is rough with respect to the translation group. Consider a state Ψ which in a ball $\mathcal{U}_0 = \{p : \mathbf{p}^2 < \bar{r}^2\}$ around 0 does not vanish

$$\text{for all } p \in \mathcal{U}_0 : |\Psi(p)|^2 > c > 0 \tag{42}$$

and apply to it the $n + 1$ -fold commutator, $n < (D - 2)$,

$$\Psi_n(p) = (-i)^{n+1}[X^{n+1}, \dots, [X^2, [X^1, \sqrt{\mathbf{P}^2}]] \dots]\Psi(p) = c_n \frac{p^1 p^2 \dots p^{n+1}}{\sqrt{\mathbf{p}^2}^{2n+1}} \Psi(p) \tag{43}$$

where c_n is a non-vanishing combinatorial factor. Its modulus squared is the product of a non-negative function f of the spherical angles, which vanishes on the sphere only in a set of vanishing spherical measure. So integrated over the sphere one is left as \mathcal{U}_0 's contribution to the norm squared of Ψ_n with a positive constant times an integral

$$\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^{\bar{r}} dr \frac{r^{D-2}}{r^{2n}}. \tag{44}$$

To confirm this elementarily: each commutator with X^i decreases for dimensional reasons the degree in P by one. We started with $r = \sqrt{\mathbf{p}^2}$ of degree 1 and obtain after $n + 1$ commutators degree $-n$, its modulus squared has degree $-2n$. In spherical coordinates $d^{D-1}p$ is of the form $r^{D-2} dr d\Omega_{D-2}$.

⁵ In this section we absorb a factor $1/\sqrt{p^0}$ in the wavefunction and deal with the translation invariant measure $d^{D-1}p$.

Analogous to the case of the rough operator $P^1/(P^0 + P^z)$ we conclude, that the relativistic massless Hamiltonian $H = \sqrt{\mathbf{p}^2}$ is incompatible with a Heisenberg algebra of $(D - 1)$ spatial self-adjoint Heisenberg pairs.

The transverse position operators for massless particles exclude also the light cone string in $D \geq 5$. It contains besides other operators $(D - 2)$ spatial, transversal Heisenberg pairs acting on massless states. In $D = 5$ the wavefunction

$$[-iX^3, [-iX^2, [-iX^1, \sqrt{\mathbf{P}^2}]]]\Psi(p) = 3 \frac{p^1 p^2 p^3}{\sqrt{\mathbf{p}^2}^5} \Psi(p) \tag{45}$$

has r -degree -2 , if $\Psi(p)$ satisfies (42). Its modulus squared behaves like $1/r^4$. Integrated on \mathcal{U}_0 in spherical coordinates with $d^4 p \sim r^3 dr d\Omega_3$ the r -integral diverges at $r = 0$ logarithmically. For each $\Psi \neq 0$ there is an $e^{i\mathbf{b}\cdot\mathbf{X}}\Psi$, $\mathbf{b} \in \mathbb{R}^{D-1}$, which satisfies (42).

All attempts [7, 11, 12, 15] to construct position operators \mathbf{X} for massless particles failed.

Different from massive particles the momentum spectrum (not the Lorentz orbit) of massless particles contains a Lorentz fixed point, $p = 0$. There the function $p^0 = \sqrt{\mathbf{p}^2}$ of \mathbb{R}^{D-1} is only continuous but not smooth. This single, distinguished point is sufficient to spoil the translation invariance of spatial momentum. It prevents spatial translations $e^{i\mathbf{b}\cdot\mathbf{X}}$ to enlarge on massless states the unitary action of the Poincaré group. Repeated commutators of its generators \mathbf{X} with the Hamiltonian $P^0 = \sqrt{\mathbf{P}^2}$ diverge on states Ψ , which satisfy (42).

The proposal, to use the Fourier transformed (with respect to the spatial momentum) momentum wave function as position wave function, does not work because Ψ is a section (22). The Fourier transformation of the local section Ψ_N is not locally related to the one of Ψ_S .

That there is no position operator for massless particles disappoints expectations, because we see the world and reconstruct the position of all objects by light which we receive as flow of massless quanta. But we do not see a distant photon. Rather we see massive objects by the currents of photons which they emit or scatter and which are annihilated in our retina.

Even if there is no position operator of massless particles we find the visual perception of the world sufficiently explained by the fact, that the luminosity L of intersecting beams is roughly proportional to the spacetime overlap

$$L = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1^0 p_2^0} \int d^4x |\tilde{\Psi}_1(t, \mathbf{x})|^2 |\tilde{\Psi}_2(t, \mathbf{x})|^2 \tag{46}$$

of the colliding wave packets $\tilde{\Psi}_1(x)$ and $\tilde{\Psi}_2(x)$ [3].

7 Conclusions

Our investigation does not depend on this or that method of quantization but studies the resulting quantum theory. We exploit the smoothness of Lie groups. Their generators constitute an algebra, which leaves invariant a common domain, the smooth, rapidly decreasing wavefunctions. Smoothness and rapid decrease are properties which in bracket notation $|p, i\rangle$ are usually disregarded as it indicates not the state $\Psi : (p, i) \mapsto \Psi^i(p)$ but only its arguments.

Canonical quantization of the light cone string postulates the operator $P^1/(P^0 + P^z)$. On massless states it is not smooth but diverges near the negative z -axis. Repeated commutators of this operator with the generators of rotations in the $P^i - P^z$ planes, $i = 2, \dots, n < D - 1$, are shown to diverge on states which do not vanish on the negative z -axis.

Consequently the domain of $P^1/(P^0 + P^z)$ does not contain all smooth states and is incompatible with a unitary representation of rotations.

Therefore the algebraic confirmation that in $D = 26$ canonically quantized generators of the classical light cone string satisfy the Lorentz algebra is meaningless. In no dimension is $P^1/(P^0 + P^z)$ defined on a complete Poincaré multiplet of massless states.

A related argument excludes on each massless multiplet a position operator X^i , which generates translations of the spatial momentum P^j , $i, j \in \{1, \dots, D - 1\}$. The Hamiltonian $P^0 = \sqrt{\mathbf{P}^2}$ is continuous but not continuously differentiable at $p = 0$. This momentum is not in the orbit of $\underline{p} = (1, 0, \dots, 1)$ but only a boundary point of the momentum spectrum. But it is rough enough to exclude a cohabitation on massless multiplets of the Poincaré group and the group of spatial translations of momentum.

The operators α_l which perform *momentum local* transitions between massless and massive states cannot map the natural domains of the Lorentz generators (the Schwartz spaces of their mass shells) to each other, as the topologies of the shells differ. Our analysis of rough multiplicative operators suggests that such operators cannot exist in relativistic theories.

On the other hand, the Lorentz generators acting on massless wavefunctions with non-vanishing helicity (14) show that rough multiplicative operators can combine with differential operators to smooth operators, the covariant derivatives (25).

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